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ANALYTICAL EXPLORATION OF NUCLEAR MATTER TRANSPORT PROPERTIES AT FINITE DENSITY

EXPLORACIÓN ANALÍTICA DE PROPIEDADES DE TRANSPORTE DE MATERIA NUCLEAR A DENSIDAD FINITA

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Resumen

En esta tesis, describiremos los avances recientes en métodos analíticos para construir soluciones exactas del modelo Skyrme que representan condensados hadrónicos no homogéneos que viven en una densidad bariónica finita. Estas novedosas herramientas analíticas se basan en la idea de generalizar el conocido ansatz del erizo esférico a situaciones (relevantes para el análisis de efectos de densidad finita) en las que ya no existe la simetría esférica. Estudiamos dos parametrizaciones; la exponencial (que llamamos *Generic Spherical Ansatz*) y la de ángulos de Euler (que llamamos *Euler Angles Ansatz*).

Además del interés matemático intrínseco para encontrar soluciones exactas con carga bariónica no nula a volumen finito, este marco abre la posibilidad de calcular cantidades físicas importantes que serían difíciles de calcular de otra manera. En particular, discutiremos las propiedades de transporte de dichos condensados hadrónicos no homogéneos del Modelo Skyrme en $(3+1)$ dimensiones a través del formalismo de Kubo.

Estos resultados contribuyen significativamente a la comprensión de la materia nuclear en el régimen de baja energía ya que, debido a su naturaleza no perturbativa, es difícil calcular sus propiedades analíticamente. Además, los resultados presentados en esta tesis están directamente relacionados con descubrimientos recientes en física nuclear sobre estructuras que presentan patrones ordenados y que coloquialmente se denominan *nuclear pasta*.

Keywords – densidad finita, skyrmiones, funciones de respuesta, pasta nuclear.

Abstract

In this thesis, we will describe recent advances in analytical methods to construct exact solutions of the Skyrme model (and its generalizations) representing inhomogeneous Hadronic condensates living at finite Baryon density. Such novel analytical tools are based on the idea to generalize the well known spherical hedgehog ansatz to situations (relevant for the analysis of finite density effects) in which there is no spherical symmetry anymore. We study two parameterizations; the exponential (which we call *Generic Spherical Ansatz*) and the Euler Angles (which we call *Euler Angles Ansatz*).

Besides the intrinsic mathematical interest to find exact solutions with non-vanishing Baryonic charge confined to a finite volume, this framework opens the possibility to compute important physical quantities, which would be difficult to compute otherwise. In particular, we will discuss the transport properties of such inhomogeneous hadronic condensates of the Skyrme Model in $(3 + 1)$ dimensions through the Kubo formalism.

These results contribute significantly to understanding nuclear matter in the low-energy regime since, due to its non-perturbative nature, it is challenging to calculate properties analytically. In addition, the results presented in this thesis are directly related to recent discoveries in nuclear physics about structures that give ordered patterns and are colloquially called *nuclear pasta*.

Keywords – finite density, skyrmions, response functions, nuclear pasta.

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Chapter 1

Introduction

Undoubtedly, one of the most significant open issues in physics is achieving a satisfactory theoretical description of the phase diagram of Quantum Chromodynamics (QCD henceforth). In this context, the most challenging part is the low temperature and finite Baryon density region of the phase diagram. The common belief is that in this regime, only refined numerical techniques can be effective (see [1], [2], [3], [4], and references therein) while analytical tools are useless. An unfortunate consequence of this fact is that until very recently, the appearance of the *nuclear pasta phase* [1] (see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], and the nice up to date review [16]), which is a very remarkable phenomenon typical of the finite Baryon density and low-temperature regime, had no theoretical "first principles" explanation. [2] Moreover, the numerical analysis of these configurations is quite challenging (see [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] and references therein) and, as the above references show, requires very high computing power.

Transport properties are fundamental properties of multi-Baryonic configurations, especially electrical conductivity, entropy, and viscosity (see [28], [29], [30], [31], [32] and references therein). It is difficult to underestimate the importance of these quantities in particle physics, nuclear physics and astrophysics. Without a proper analytic understanding of the complex structures characterizing the nuclear pasta phase, numerical simulations are the only way to compute the electric conductivity. In this work, we intend to prove that it is very difficult to extend the numerical techniques used to analyze transport properties in homogeneous condensates with very large Baryonic charge when these systems are coming out of equilibrium. On the other hand, a "theoretical dream" would be to have a proper analytic description of these multi-Baryonic systems in order to be able to apply the Green-Kubo formalism [33], [34], [35] (for a detailed pedagogical review see [36]). In this way, one would achieve a first-principle understanding of these essential properties, allowing their comparison with available experimental data.

[1] In such a phase, ordered structures appear, in which most of the Baryonic charge is contained in regular shapes like thick Baryonic layers (called *nuclear lasagna*) or thick Baryonic tubes (called *nuclear spaghetti*). This phase has many similarities with the non-homogeneous condensates, which have been discovered in integrable field theories in $(1+1)$ dimensions.

[2] Namely, a theoretical explanation that only uses the QCD Lagrangian (or its low energy limit) and the fact that there is a finite amount of Baryonic charge within a finite spatial volume.

The main goal of this thesis is to take the first steps toward the construction of a first principle understanding of the low-temperature transport properties of multi-Baryonic structures appearing in the nuclear pasta phase. Our starting point is the Skyrme theory which (at leading order in the t' Hooft expansion [37], [38], [39]) represents the low energy limit of QCD.[3]

The dynamical field of the Skyrme action [47] is a $SU(N)$-valued scalar field U (here we will consider the two-flavors case $U(x) \in SU(2)$). This action possesses both small excitations describing Pions and topological solitons describing Baryons [48], [49], [50], [51], [52]; the Baryonic charge being a topological invariant (see also [53], [54], [55], [56], [57], [58], [59] [60], [61] , [62] and references therein).

To achieve the goal of computing -via the Green-Kubo formalism- some transport properties of regular multi-Baryonic structures, we will need to generalize the analytic crystal-like solutions with high topological charge constructed in [63], [64], [65], [66], [67], [68], [69], [70], [71], and [72] (using the methods developed in [73], [74], [75], [76], [77], [78], [79], [80]).[4] It is worth emphasizing that, first of all, the plots in [63] and [72] are qualitatively very close to the ones found numerically in the analysis of spaghetti-like configurations (see the plots in [5], [6], [7], [8], [9] and [16]). Moreover, in [71] the shear modulus of lasagna configurations has been computed, the result being in good agreement with [11] and [15]. Thus, the present formalism is well equipped to analyze the nuclear pasta phase.

Now, as the analysis of the following sections will clarify, the Hamiltonian describing small Pionic fluctuations of these crystalline structures cannot be directly analyzed with the Green-Kubo formalism. An essential technical step is to allow for a more general time-dependence of these configurations and describe "Baryonic pulses". We will construct more general topologically non-trivial solutions and show that the Hamiltonian describing the small Pionic excitations is very well suited for the Green-Kubo formalism.

The manuscript is divided as follows: In chapter 2, we give a brief review of the Skyrme Model, its relation with baryons, its field equations, the energy-momentum tensor, and we define topological charge in the model. Furthermore, we define and explain the most common ansätze for the fundamental element U.

Chapter 3 summarizes previous contributions of hadronic configurations at finite density, we called these structures Skyrmions crystals. Then, we will present the metric and their ranges of coordinates.

Chapter 4 presents our analytic solution of the Skyrme Model at finite density with a non-vanishing topological charge. Besides, we will define the relevant properties of the theory, like the energy-momentum tensor and the topological current.

In chapter 5, we present recent work on transport properties of hadronic layers and tubes. We

[3] Needless to say, Skyrme's ideas have also been applied in astrophysics [40], Bose-Einstein condensates [41], nematic liquids [42], magnetic structures [43] and condensed matter physics [44] (see also [45] and [46]).

[4] In [81] and [82], numerical string shaped solutions in the Skyrme model with the mass term have been constructed. However, those configurations have vanishing topological density (and they are expected to decay into Pions). The configurations analyzed in the present thesis are topologically non-trivial and therefore can not decay into those of [81] and [82].

contrast with recent references on nuclear pasta.

In chapter 6, we show recent work related to a sector of the Yang-Mills theory that exhibits conformal symmetry. We will motivate this work with the exciting possibilities to compute and characterize nuclear matter.

Finally, chapter 7 is dedicated to conclusions and future outlook.

In our convention $c = \hbar = 1$, Greek indices run over the space-time with mostly plus signature, and Latin indices are reserved for those of the internal space. For simplicity, we denote the scalar product with $\cdot$, for example, we use $(\nabla\alpha \cdot \nabla G)$ for $(\nabla_\mu \alpha \nabla^\mu G)$.

Chapter 2

Skyrme Model

In this chapter, we present the most essential aspects of the Skyrme theory when coupled with the Maxwell theory. First, in section 2.1, we show that barions are described by skyrmions, which are solutions to the Skyrme model. Then, in section 2.2, we give the definition of the Skyrme Model coupled with the Maxwell action and the considerations for the fundamental element U of $SU(2)$. Afterwards, in section 2.3, we study the field equations of the Skyrme-Maxwell Model. Finally, in section 2.4, we compute the energy-momentum tensor; however, we analyze the energy when the fundamental field A_μ vanishes, i.e we only study the contributions of the Skyrme action.

It should be noted that in this chapter we analyze the Skyrme theory coupled with Maxwell theory; nevertheless, our work and results are focused when the field A_μ goes to 0.

2.1 Topological Solitons and Baryons

The Skyrme model is an effective field theory of Quantum Chromodynamics (QCD) at low energies proposed by T. Skyrme in 1961. The model describes baryons and mesons simultaneously by using a scalar field triplet taking values in $SU(2)$ [47]. Skyrme constructed the simplest topological action through the *skyrme term* that gives stability to its solutions called solitons (for more details, see A1). In particular, for this model, these solutions are called skyrmions.

Nevertheless, before starting with an in-depth analysis of the Skyrme model, it is convenient to highlight some aspects of QCD. This model is a theory describing the strong nuclear interaction between its fundamental constituents: the quark and gluon fields. In an analogous way to electrodynamics, a theory in which elementary particles (electrons) interact through a mediator of the force i.e. the photon, in QCD, gluons mediate strong nuclear interactions, while quarks are the elementary particles. In this sense, quarks are the fundamental constituents of known matter, everything we can interact with. These, when cooled, form hadrons such as neutrons and protons.

Hadrons are quark-bound states, which can be classified according to their electric charge or the number of quark constituents. The latter classification separates hadrons into baryons and mesons, composed of three and two quarks, respectively.

Now, when we say quarks are "cooled" we are alluding to the process in which the quark states of the theory flow into a lower energy sector such that their color charge is hidden or masked by the hadronic states. This phenomenon is called color confinement, and it explains that we do not observe color-charged particles in nature, and it is related to an interesting property of QCD: asymptotic freedom.

Asymptotic freedom is a peculiar characteristic of the fundamental constituents of QCD, which interact strongly and non-linearly as they are separated. In contrast, they interact very weakly when located relatively close to each other. This phenomenon is characteristic of the strong nuclear interaction, making it very different from other forces. Furthermore, asymptotic freedom is responsible for the fact that, at high enough energies, this theory can be studied perturbatively since the expansion parameter (coupling constant) is small in this region. Nevertheless, in the low-temperature regime, this parameter grows, which breaks the validity of the perturbation theory. Hence, for studying such regions of the phase space, we use effective theories like the Skyrme model, which describes hadron dynamics well.

The Skyrme model is related to the large N_C limit, where N_C is the number of colors (3 for QCD). This number arises from the Lie group structure of QCD, a gauge theory for the color group $SU(3)$. In this context, the fundamental parameters of the theory are the strong interaction coupling constant and the quark masses.

't Hoolf [37] is the one who studies this coefficient as one more parameter of the theory. In simple words, this parameter is very large and induces a combinatorial factor related to the Feynman diagrams of the theory. This combinatory factor goes as $1/N_C$ so that the theory can be studied in power series on this factor.

In addition to coinciding with this regime with the Skyrme Model, he also describes other phases of matter that are difficult to study since perturbation theory cannot be done.

In the particular case of the Skyrme Model, the relationship with QCD lies precisely in the large NC limit where the topological charge corresponds to the baryon number of QCD.

In conclusion, the Skyrme Model provides a satisfactory description for baryons since it can even, through quantization, a baryon itself. Among the essential features of the model is that it predicts that the baryonic charge is conserved, which is related to the stability of the neutron when it has no charge.

2.2 The Skyrme-Maxwell Model

The Skyrme model is a type of non-linear sigma model, so let's first see the explicit contribution of this term, which is defined as

$$I[U, A_\mu] = \int d^4v \left(\mathcal{L}^{\mathrm{SK}} + \mathcal{L}^{\mathrm{U(1)}}\right) , \tag{2.2.1}$$

where

$$\mathcal{L}^{\mathrm{SK}} = \frac{K}{4}\mathrm{Tr}\left\{R_\mu R^\mu + \frac{\lambda}{8}G_{\mu\nu}G^{\mu\nu}\right\} . \tag{2.2.2}$$

In this action, we recognize the first term like the non-linear sigma model that had a strong predictive character for the time in which it was created. Skyrme was interested in a theory that would allow him to have soliton-type solutions describing baryons and mesons, and wanted to provide an answer for the following question: what is the simplest term that avoids Derrick's theorem, being Lorentz covariant and having second order field equations? It turns out that his model was succesful in these points, and his work produced the so-called Skyrme term. We provide a detailed proof that this term avoids Derrick's theorem in Appendix A1. This means that the theory has finite energy static solutions, which are stabilized by the second term of (2.2.2).

$$R_\mu = U^{-1}D_\mu U = R^a_\mu t_a \ , \quad G_{\mu\nu} = [R_\mu, R_\nu] \ , \ d^4v = \sqrt{-g}d^4x \ ,$$
$$D_\mu U = \nabla_\mu U + A_\mu U\hat{O} \ , \quad \hat{O} = U^{-1}[t_3, U] \ , \tag{2.2.3}$$

where, $U(x) \in SU(2)$, g is the metric determinant, ∇_μ is the partial derivative, D_μ is the covariant derivative associated to the $U(1)$ gauge field A_μ and $t_a = i\sigma_a$ are the generators of the $SU(2)$ Lie group, being σ_a the Pauli matrices. The Skyrme couplings K and λ are positive constants that have to be fixed experimentally[1].

On the other hand, Maxwell's action is defined by the following lagrangian

$$\mathcal{L}^{U(1)} = F_{\mu\nu}F^{\mu\nu} \ , \tag{2.2.4}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.2.5}$$

Here it is worth emphasizing an important point. In the limit of vanishing gauge potential the above $U(1)$ current does not necessarily vanish:

$$\lim_{A_\mu \to 0} J_\mu \overset{def}{=} J^{(0)}_\mu \neq 0' \ . \tag{2.2.6}$$

2.3 Field Equation

Field equations for the Skyrme Model are obtained through the variation of the Skyrme Model with respect to the fundamental fields U and A_μ. This yields:

$$\nabla_\mu\left(R^\mu + \frac{\lambda}{4}[R_\nu, G^{\mu\nu}]\right) = 0 \ , \tag{2.3.1}$$
$$\nabla_\mu F^{\mu\nu} = J^\nu , \tag{2.3.2}$$

[1]The parameter K and λ are defined by the constant of decay of pions F_π and the adimensional constant that gives the stability of the skyrmions e. These has the values $F_\pi = 186MeV$ and $e = 5.45$. In this sense, the parameter $K = F_\pi^2/4$ and $\lambda = 4/(e^2F_\pi^2)$ (see the references [52] and [83]).

For the fist equation we consider the variation of the element U fulfills $\delta U = 0$ over the volume boundary of integration. While the electromagnetic current J_μ is given by

$$J_\mu = \frac{K}{2}\mathrm{Tr}\left\{\hat{O}\left(R_\mu + \frac{\lambda}{4}[R^\nu, G_{\mu\nu}]\right)\right\} \ . \tag{2.3.3}$$

In (2.3.1) we have (N^2-1) equations (where N is the group number of $SU(N)$), these equations are nonlinear and their derivation is explicited in Appendix A2. It should be noted that finding solutions of the Skyrme Model is hard, which makes the election of an intelligent Ansatz crucial for the reduction of these complicated field equations. The last part of this chapter is dedicated to the details of such Ansätze.

2.4 Energy of Solitons

The contribution to the energy-momentum tensor of the Skyrme model is given by

$$T^{\mathrm{SK}}_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}\left\{R_\mu R_\nu - \frac{1}{2}g_{\mu\nu}R^\alpha R_\alpha + \frac{\lambda}{4}\left(g^{\alpha\beta}G_{\mu\alpha}G_{\nu\beta} - \frac{1}{4}g_{\mu\nu}G_{\alpha\beta}G^{\alpha\beta}\right)\right\} \ , \tag{2.4.1}$$

while the contribution of the Maxwell theory is

$$T^{U(1)}_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \ . \tag{2.4.2}$$

We focus on the energy momentum tensor of Skyrme Model $T^{\mathrm{SK}}_{\mu\nu}$ because across this work we will use $A_\mu \to 0$.

In section 2.5, we will see with greater detail that using an in depth analysis of the energy it is possible to have a notion of how complicated it is to find a solution to the Skyrme model field equations through a Bogomol'nyi-Prasad-Sommerfield (BPS) bound in models with non trivial topological charge Q. In order to do this, it will be useful to know an expression for the energy that is obtained via the integration of the T_{00} component of (2.4.1).

$$E = E_{\mathrm{stat}} + E_{\mathrm{rot}} \ , \tag{2.4.3}$$

where E_{stat} is the static energy of the model and E_{rot} is related with the rotational energy. These are given by

$$E_{\mathrm{stat}} = -\frac{K}{2}\int d^3x \mathrm{Tr}\left\{\frac{1}{2}R_a R^a + \frac{\lambda}{16}G_{ab}G^{ab}\right\}, \tag{2.4.4}$$

$$E_{\mathrm{rot}} = -\frac{K}{2}\int d^3x \mathrm{Tr}\left\{\frac{1}{2}R_0 R^0 - \frac{\lambda}{8}G_{0a}G^{0a}\right\}. \tag{2.4.5}$$

The derivation of (2.4.3) and interpretations of these terms are given in Appendix A3.

2.5 Topological Charge like a Baryon Number

Topological charge is a property of theories that have non trivial topology and characterizes solitonic solutions of the theory. More rigurously, topological charges are conserved charges associated with discrete symmetries of the theory, which contrasts with the usual notion of Noether charges. In practice, it is very useful to know that the existance and conservation of this charge does not require to use the equations of motion.

$$B = \frac{1}{24\pi^2}\int_{\sigma} \rho_B , \tag{2.5.1}$$

where σ is a three-dimensional hypersurface at $t = \text{const.}$ and ρ_B is defined as

$$\rho_B = \rho^{\text{SK}} + \rho^{\text{U(1)}} , \tag{2.5.2}$$

where

$$\rho^{\text{SK}} = \epsilon^{abc}\text{Tr}\left\{\left(U^{-1}\partial_a U\right)\left(U^{-1}\partial_b U\right)\left(U^{-1}\partial_c U\right)\right\} , \tag{2.5.3}$$

$$\rho^{\text{U(1)}} = -\text{Tr}\left\{\partial_a\left(3A_b t_3\left(U^{-1}\partial_c U + \left(\partial_c U\right)U^{-1}\right)\right)\right\} . \tag{2.5.4}$$

Topological charge is also known as a topological invariant that is uniquely linked to the boundary conditions of the fundamental field U. One of the most important properties of this kind of charge is the fact that infinite energy is needed to change a physical state with a fixed value of the topological charge to a state with a different topological charge.

Whenever $A_\mu \to 0$ we get the topological charge associated to the Skyrme model by itself. As mentioned before, there is a bound to this charge called the BPS bound. This equation relates the static energy of a configuration of topological charge Q in such a way that it satisfies

$$E \geq |Q| . \tag{2.5.5}$$

There is a convenient case in which this bound is saturated i.e. $E = |Q|$. In this case the field equations interpolate from a system with second order equations to a first order one. This is very convenient for finding solutions to the system. Nevertheless, for the Skyrme model, this bound can not be saturated, and this path can not help us construct any novel solutions of the model. A proof of the impossibility of saturating the BPS bound in the Skyrme model is given in Appendix A4.

In what follows, we will use the terminology "Skyrmions at finite density" and "inhomogeneous Baryonic condensates" to denote analytic solutions of the field equations in Eqs. (2.3.1) and (2.3.2) with non-vanishing topological charge satisfying boundary conditions corresponding to a finite spatial volume.

2.6 Ansatz for Skyrme Field

In this section, we will show two parametrizations of the fundamental field U. The objective of this parametrization is to construct the field equations easily. Using these simplifications, we will find solutions for the field equations by considering convenient ansätze for the scalar fields in question.

2.6.1 Skyrme Generalized Spherical Ansatz SU(2)

The standard exponential form of the field $U(x)$ is

$$U = \cos(\alpha)\mathbb{1}_{2\times 2} + \sin(\alpha) n_j t^j , \tag{2.6.1}$$

where $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix and $t^i = i\sigma^i$ where σ^j are the Pauli matrices. Besides,

$$\vec{n} = (\sin(\Theta)\cos(\Phi), \sin(\Theta)\sin(\Phi), \cos(\Theta)) \ , \tag{2.6.2}$$

where $\alpha = \alpha(x^\mu)$, $\Theta = \Theta(x^\mu)$ and $\Phi = \Phi(x^\mu)$ are the three scalar degree of freedom of the $SU(2)$-valued field $U(x)$.

Then, plugging (2.6.1) and (2.6.2) into the action we get

$$\begin{aligned} I(\alpha,\Theta,\Phi) = \frac{K}{4}\int d^4v \mathrm{Tr}\Big\{ &(\nabla\alpha)^2 + \sin^2(\alpha)((\nabla\Theta)^2 + \sin^2(\Theta)(D\Phi)^2) + \frac{\lambda}{2}\big(\sin^2(\alpha)((\nabla\alpha)^2(\nabla\Theta)^2 - \\ &- (\nabla\alpha\cdot\nabla\Theta)^2) + \sin^2(\alpha)\sin^2(\Theta)((\nabla\alpha)^2(D\Phi)^2 - (\nabla\alpha\cdot D\Phi)^2) + \\ &+ \sin^4(\alpha)\sin^2(\Theta)((\nabla\Theta)^2(D\Phi)^2 - (\nabla\Theta\cdot\nabla\Phi)^2)\big)\Big\} + I^{U(1)}, \end{aligned} \tag{2.6.3}$$

where

$$D_\mu\Phi = \nabla_\mu\Phi - 2eA_\mu . \tag{2.6.4}$$

It is a trivial computation to check that the covariant derivative in terms of the three scalar degrees of freedom $\alpha(x^\mu)$, $\Theta(x^\mu)$ and $\Phi(x^\mu)$ is equivalent to the following minimal coupling rule

$$\nabla_\mu\alpha \to D_\mu\alpha = \nabla_\mu\alpha \, , \tag{2.6.5}$$

$$\nabla_\mu\Theta \to D_\mu\Theta = \nabla_\mu\Theta \, , \tag{2.6.6}$$

$$\nabla_\mu\Phi \to D_\mu\Phi = \nabla_\mu\Phi - 2eA_\mu \, . \tag{2.6.7}$$

Thus, the scalar degree of freedom Φ plays the role of the "$U(1)$ phase" of the Skyrme field since, under a gauge transformation, it transforms as

$$A_\mu \to A_\mu + \nabla_\mu A \, , \tag{2.6.8}$$

$$\Phi \to \Phi + 2e\Lambda \, . \tag{2.6.9}$$

It is also useful to write the full Skyrme-Maxwell equations in terms of the three scalar degrees of freedom α, Θ and Φ:

$$-\Box\alpha+\sin(\alpha)\cos(\alpha)\left((\nabla\Theta)^2+\sin^2(\Theta)(D\Phi)^2\right)+\lambda\Big(\sin(\alpha)\cos(\alpha)\big((\nabla\alpha)^2(\nabla\Theta)^2-(\nabla\alpha\cdot\nabla\Theta)^2\big)+$$
$$+\sin(\alpha)\cos(\alpha)\sin^2(\Theta)\big((\nabla\alpha)^2(D\Phi)^2-(\nabla\alpha\cdot D\Phi)^2\big)+2\sin^3(\alpha)\cos(\alpha)\sin^2(\Theta)\big((\nabla\Theta)^2(D\Phi)^2-$$
$$-(\nabla\Theta\cdot D\Phi)^2\big)-\nabla_\mu\big(\sin^2(\alpha)(\nabla\Theta)^2\nabla^\mu\alpha\big)+\nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha\cdot\nabla\Theta)\nabla^\mu\Theta\big)-$$
$$-\nabla_\mu\big(\sin^2(\alpha)\sin^2(\Theta)(D\Phi)^2\nabla^\mu\alpha\big)+\nabla_\mu\big(\sin^2(\alpha)\sin^2(\Theta)(\nabla\alpha\cdot D\Phi)D^\mu\Phi\big)\Big)=0\,, \tag{2.6.10}$$

$$-\sin^2(\alpha)\Box\Theta-2\sin(\alpha)\cos(\alpha)(\nabla\alpha\cdot\nabla\Theta)+\sin^2(\alpha)\sin(\Theta)\cos(\Theta)(D\Phi)^2+$$
$$+\lambda\Big(\sin^2(\alpha)\sin(\Theta)\cos(\Theta)\big((\nabla\alpha)^2(D\Phi)^2-(\nabla\alpha\cdot D\Phi)^2\big)+$$
$$+\sin^4(\alpha)\sin(\Theta)\cos(\Theta)\big((\nabla\Theta)^2(D\Phi)^2-(\nabla\Theta\cdot D\Phi)^2\big)-\nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha)^2\nabla^\mu\Theta\big)-$$
$$-\nabla_\mu\big(\sin^4(\alpha)\sin^2(\Theta)(D\Phi)^2\nabla^\mu\Theta\big)+\nabla_\mu\big(\sin^4(\alpha)\sin^2(\Theta)(\nabla\Theta\cdot D\Phi)D^\mu\Phi\big)+$$
$$+\nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha\cdot\nabla\Theta)\nabla^\mu\alpha\big)\Big)=0\,, \tag{2.6.11}$$

$$-\sin^2(\alpha)\sin^2(\Theta)\big(\Box\Phi-2e\nabla_\mu A^\mu\big)-2\sin(\alpha)\cos(\alpha)\sin^2(\Theta)(\nabla\alpha\cdot D\Phi)-$$
$$-2\sin^2(\alpha)\sin(\Theta)\cos(\Theta)(\nabla\Theta\cdot D\Phi)+\lambda\Big(-\nabla_\mu\big(\sin^2(\alpha)\sin^2(\Theta)(\nabla\alpha)^2D^\mu\Phi\big)+$$
$$+\nabla_\mu\big(\sin^2(\alpha)\sin^2(\Theta)(\nabla\alpha\cdot D\Phi)\nabla^\mu\alpha\big)-\nabla_\mu\big(\sin^4(\alpha)\sin^2(\Theta)(\nabla\Theta)^2D^\mu\Phi\big)+$$
$$+\nabla_\mu\big(\sin^4(\alpha)\sin^2(\Theta)(\nabla\Theta\cdot D\Phi)\nabla^\mu\Theta\big)\Big)=0\ , \tag{2.6.12}$$

where the electromagnetic current in the Maxwell equations (2.3.2) is

$$J_\mu=-eK\sin^2\alpha\sin^2\Theta\Big\{D_\mu\Phi+\frac{\lambda}{2}\Big((\nabla\alpha)^2D_\mu\Phi-(\nabla\alpha\cdot D\Phi)\nabla_\mu\alpha+\sin^2\alpha\big((\nabla\Theta)^2D_\mu\Phi-(\nabla\Theta\cdot\nabla\Phi)\nabla_\mu\Theta\big)\Big)\Big\}\,. \tag{2.6.13}$$

The previous parametrization is reduced to the original ansatz proposed by Skyrme in [84] when the three scalars fields are

$$\alpha=\alpha(R)\,, \tag{2.6.14}$$
$$\Theta=\theta\,, \tag{2.6.15}$$
$$\Phi=\phi\,, \tag{2.6.16}$$

if we consider a flat metric in spherical coordinates

$$ds^2=-dt^2+dR^2+R^2\left(d\theta^2+\sin^2(\theta)d\phi^2\right). \tag{2.6.17}$$

2.6.2 Skyrme Generalized Spherical Ansatz SU(3)

When the fundamental field $U(x)$ is an element of the $SU(3)$ group, Balachandran et al. [85] proposed an ansatz for describing dibaryons in flat spacetime with numerical tools. The Skyrme field is constructed by a subgroup of Gell-Mann matrices that generate $SU(3)$, the subgroup is $\{\lambda_2,-\lambda_5,\lambda_7\}$ and corresponds to the subalgebra $so(3)\subseteq su(3)$. A generalization of the spherical ansatz U is given by

$$U_B = \exp(i\psi)\mathbb{1}_{3\times 3} + i\sin(\chi)\exp\left(-\frac{i\psi}{2}\right)\mathbf{T} + \left(\cos(\chi)\exp\left(-\frac{i\psi}{2}\right) - \exp(i\psi)\right)\mathbf{T^2} \quad (2.6.18)$$

where

$$\mathbf{T} = \vec{\Lambda}\cdot\hat{n}, \quad (2.6.19)$$

$$\vec{\Lambda} = (\lambda_7, -\lambda_5, \lambda_2), \quad (2.6.20)$$

$$\hat{n} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta), \quad (2.6.21)$$

here ψ, χ, Θ and Φ are in principle arbitrary functions of the coordinates. The objective of constructing this ansatz is to rewrite de field equations to easily find their solutions.

The previous parametrization is reduced to the original Ansatz proposed by Balachandran et al, whenever

$$\psi = \psi(r), \quad (2.6.22)$$

$$\chi = \chi(r), \quad (2.6.23)$$

and the functions Θ and Φ are angular coordinates. In such a case we have

$$U = \exp(i\psi(r))\mathbb{1}_{3\times 3} + i\sin(\chi)\exp\left(-\frac{i\psi(r)}{2}\right)\vec{\Lambda}\cdot\hat{x} + \left(\cos(\chi(r))\exp\left(-\frac{i\psi(r)}{2}\right) - \exp(i\psi(r))\right)(\vec{\lambda}\cdot\hat{x})^2 \quad (2.6.24)$$

where

$$\hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \quad (2.6.25)$$

2.6.3 Euler Angles Ansatz SU(N)

The Euler angle parametrization is detailed in [86, 87]. This formula, in our case, takes the following forms for elements of $SU(2)$ and $SU(3)$, respectively

$$U_{\mathrm{E}}^{SU(2)} = \exp\left(i\frac{\alpha}{2}\sigma_3\right)\exp\left(i\frac{\beta}{2}\sigma_2\right)\exp\left(i\frac{\rho}{2}\sigma_3\right), \quad (2.6.26)$$

$$U_{\mathrm{E}}^{SU(3)} = U_1(\alpha,\beta,\rho)U_2(\Theta,\Phi)U_3(a,b,c), \quad (2.6.27)$$

where

$$U_1(\alpha,\beta,\rho) = \exp\left(i\frac{\alpha}{2}\sigma_3\right)\exp\left(i\frac{\beta}{2}\sigma_2\right)\exp\left(i\frac{\rho}{2}\sigma_3\right), \quad (2.6.28)$$

$$U_2(\Theta,\Phi) = \exp(i\Theta\lambda_5)\exp(i\Phi\lambda_8), \quad (2.6.29)$$

$$U_3(a,b,c) = \exp\left(i\frac{a}{2}\sigma_3\right)\exp\left(i\frac{b}{2}\sigma_2\right)\exp\left(i\frac{c}{2}\sigma_3\right), \quad (2.6.30)$$

α, β, ρ, a, b, c, Θ and Φ are arbitrary functions of the coordinates x^μ, $\{\sigma_i\}$, with $i = 1, \ldots, 3$ the Pauli matrices and $\{\lambda_i\}$, with $i = 1, \ldots, 8$, the Gell-Mann matrices.

For the $SU(2)$ case we chose the following element U of $SU(2)$

$$U = \exp\left(F\left(x^\mu\right) t_3\right) \exp\left(H\left(x^\mu\right) t_2\right) \exp\left(G\left(x^\mu\right) t_3\right), \tag{2.6.31}$$

where $F\left(x^\mu\right)$, $G\left(x^\mu\right)$ and $H\left(x^\mu\right)$ are the three scalar degrees of freedom (traditionally, in this parametrization the field H is called profile). It is worth emphasizing that, as it happens for the exponential representation discussed in the previous section, any group element can always be written in the Euler angle representation. We will see that in order to construct a concrete ansatz that can be adapted to the finite density situation it is convenient to explicitly write the Skyrme action in terms of the generic Euler angle parametrization in Eq. (2.6.31). A direct computation shows that

$$\begin{aligned} I(H,F,G) = &- \frac{K}{2} \int d^4 v \mathrm{Tr} \Big\{ (\nabla H)^2 + (DF)^2 + (DG)^2 + 2\cos(2H)(DF \cdot DG) - \\ &- \lambda\big(2\cos(2H)((\nabla H \cdot DF)(\nabla H \cdot DG) - (\nabla H)^2 (DF \cdot DG)) + \\ &+ 4\sin^2(H)\cos^2(H)((DF \cdot DG)^2 - (DF)^2 (DG)^2) + \\ &+ (\nabla H \cdot DF)^2 + (\nabla H \cdot DG)^2 - (\nabla H)^2 (DF)^2 - (\nabla H)^2 (DG)^2\big) \Big\} . \end{aligned} \tag{2.6.32}$$

The coupling with the Maxwell field in this parametrization is performed via

$$D_\mu F = \nabla_\mu F - eA_\mu \ , \qquad D_\mu G = \nabla_\mu G + eA_\mu \ ,$$

And the field equations in the Euler parameterization are given by

$$\begin{aligned} 0 =& \Box H + 2\sin(2H)(\nabla F \cdot \nabla G) - \lambda \Big\{ 2\sin(2H)\Big((\nabla H \cdot \nabla F)(\nabla H \cdot \nabla G) - (\nabla H)^2 (\nabla F \cdot \nabla G) - \\ &- \cos(2H)\big((\nabla F \cdot \nabla G)^2 - (\nabla F)^2 (\nabla G)^2\big)\Big) + \nabla_\mu(\cos(2H)(\nabla H \cdot \nabla G)\nabla^\mu F) + \\ &+ \nabla_\mu(\cos(2H)(\nabla H \cdot \nabla F)\nabla^\mu G) - \nabla_\mu(2\cos(2H)(\nabla G \cdot \nabla F)\nabla^\mu H) + \nabla_\mu((\nabla H \cdot \nabla F)\nabla^\mu F) + \\ &+ \nabla_\mu((\nabla H \cdot \nabla G)\nabla^\mu G) - \nabla_\mu((\nabla F)^2 \nabla^\mu H) - \nabla_\mu((\nabla G)^2 \nabla^\mu H) \Big\} , \end{aligned} \tag{2.6.33}$$

$$\begin{aligned} 0 =& \Box F - e\nabla \cdot A - 2\sin(2H)(DG \cdot \nabla H) + \cos(2H)(\Box G + e\nabla \cdot A) - \\ &- \lambda\Big(\nabla_\mu(\cos(2H)(\nabla H \cdot DG)\nabla^\mu H) - \nabla_\mu(\cos(2H)(\nabla H)^2 D^\mu G) + \\ &+ \nabla_\mu(4\sin^2(H)\cos^2(H)(DF \cdot DG)D^\mu G) - \nabla_\mu(4\sin^2(H)\cos^2(H)(DG)^2 D^\mu F) + \\ &+ \nabla_\mu((\nabla H \cdot DF)\nabla^\mu H) - \nabla_\mu((\nabla H)^2 D^\mu F)\Big) , \end{aligned} \tag{2.6.34}$$

$$\begin{aligned} 0 =& \Box G + e\nabla \cdot A - 2\sin(2H)(DF \cdot \nabla H) + \cos(2H)(\Box F - e\nabla \cdot A) - \\ &- \lambda\Big(\nabla_\mu(\cos(2H)(\nabla H \cdot DF)\nabla^\mu H) - \nabla_\mu(\cos(2H)(\nabla H)^2 D^\mu F) + \\ &+ \nabla_\mu(4\sin^2(H)\cos^2(H)(DF \cdot DG)D^\mu F) - \nabla_\mu(4\sin^2(H)\cos^2(H)(DF)^2 D^\mu G) + \\ &+ \nabla_\mu((\nabla H \cdot DG)\nabla^\mu H) - \nabla_\mu((\nabla H)^2 D^\mu G)\Big) , \end{aligned} \tag{2.6.35}$$

while the Maxwell equations are given in (2.3.2). In this case the current J_μ is very relevant for computing the electrical conductivity, and it is given by

$$J_\mu = eK\Big\{2\sin^2 H(D_\mu F - D_\mu G) + \lambda\Big(\cos(2H)\big(((\nabla H \cdot DF) - (\nabla H \cdot DG))\nabla_\mu H - (\nabla H)^2(D_\mu F - D_\mu G)\big)+$$
$$+ 4\sin^2 H\cos^2 H\big((DF \cdot DG)(D_\mu F - D_\mu G) + (DG)^2 D_\mu F - (DF)^2 D_\mu G\big)+\big((\nabla H \cdot DG)-$$
$$-(\nabla H \cdot DF)\big)\nabla_\mu H + (\nabla H)^2\big(D_\mu F - D_\mu G\big)\Big)\Big\}\ . \tag{2.6.36}$$

In the following chapters, we will discuss first these field equations in the ungauged case to show how and why effective low energy chiral conformal degrees of freedom appear.

Chapter 3

Skyrmions at Finite Density

In this chapter, we will present a brief historical introduction of Klebanov Crystal in section 3.1. Then, in 3.2, we will summarize the first example of skyrmions at finite density.

The Skyrme crystal is a hypothetical arrangement of skyrmions that has important properties for our understanding of nuclear matter. In this chapter, we will summarize these structures and the future perspective for this study.

E. Witten first proposed the concept of Skyrme crystal in 1981 [88], and later other authors such as Igor Klebanov, Manton, and others made significant contributions to understanding these structures and concluded relevant aspects in the nuclear physics context.

3.1 Klebanov Crystal

Igor Klebanov in 1985 ordered skyrmions in a cubic lattice and calculated the energy of the configuration through numerical calculations, provided the symmetry of the crystal was preserved [18]. This research inspired to coin these structures like Klebanov crystals in future works.

In this crystal, the skyrmions are arranged in a periodic array. One of the critical properties of the Klebanov crystal is that it exhibits a wide variety of topological defects and magnetic phases, which arise from the interplay between the skyrmions and the crystal lattice. In this sense, he obtains a set of equations describing the dynamics of nuclear matter in terms of the Skyrme fields.

Klebanov computes the equation of the state of nuclear matter and discusses the need to include other degrees of freedom to describe the interaction between nucleons. Then, in 1988 M. Kugler and S. Shtrikman [21] proposed a new type of skyrmion crystal with cubic symmetry, different from the previously known skyrmion crystals with triangular lattice structure, and they explained some aspects of this ferromagnetic material.

More recently, N. Manton et al. [89] contributed to finding and classifying the symmetries of the Skyrme crystal. They used numerical simulations to steady the role of the maximal symmetry group for these structures. They concluded that the maximal symmetry of the Skyrme crystal is an important tool for understanding the fundamental properties of this type of matter.

In this way, it is clear that the study of Skyrme crystals is a very active research area. In general, this type of structure has been analyzed numerically. Nevertheless, it is important to highlight that the analytical study of this phase of material can contribute significantly to understanding nuclear matter in the low energy regime.

3.2 The first example: Sine-Gordon layer

As mentioned before, the study of skyrmions at finite density has been investigated through numerical tools. However, in [90], the first analytical solution in flat space-time that describes states of skyrmions in a finite volume was built.

The authors consider the following set of expressions for the scalar fields in the generalized spherical ansatz parameterization of $SU(2)$

$$\begin{aligned}\Phi &= \frac{\gamma+\phi}{2}\,,\\ \tan\Theta &= \frac{\tan H}{\cos A}\,,\\ \tan\alpha &= \frac{\sqrt{1+\tan^2\Theta}}{tanA}\,.\end{aligned} \tag{3.2.1}$$

where, $A=(\gamma-\phi)/2$ and $H=H(t,r)$.

Then, they prove that the expression (3.2.1) in the field equations (2.6.10), (2.6.11) and (2.6.12), can be reduced to a single equation, given by

$$\begin{aligned}\Box H - \frac{\lambda}{8L^2(\lambda+2L^2)}\sin(4H) &= 0\,,\\ \frac{\partial^2}{\partial t^2}-\frac{1}{L^2}\frac{\partial^2}{\partial r^2} &= \Box\,,\end{aligned} \tag{3.2.2}$$

which is the Sine-Gordon equation (the prototype of integrable PDE), and the parameter L comes from the flat metric in $(3+1)$ dimensions.

$$ds^2 = -dt^2 + L^2(dr^2+d\phi^2+d\theta^2)\,, \tag{3.2.3}$$

where the range of coordinates are

$$0\le r\le 2\pi \quad , \quad 0\le\gamma\le 4\pi \quad , \quad 0\le\phi\le 2\pi\,. \tag{3.2.4}$$

Moreover, the topological charge is non-vanishing and it reads

$$\rho_B = 3\sin(2H)dHd\gamma d\phi\,.$$

Furthermore, when the profile H satisfies the following boundary conditions

$$H(t,0) = 0\,, \qquad H(t,2\pi) = \pm\frac{\pi}{2}\,,$$

the topological charge takes the values of $B = \pm 1$. These solutions are known as skyrmions and anti-skyrmions, respectively. These configurations represent the first analytic examples of Baryonic layers in a flat box of finite volume. In the following sections, we will discuss the generalizations of these solutions.

Chapter 4

Novel Solutions

In this chapter, we present the fundamental consideration for the objective of this work: the computation of relevant transport properties. First, in section 4.1, we study the definition of the flat metric for our case. Then, in section 4.2, we give the definition of the particular parametrization of the layer case and the ansatz that we used. Afterward, in section 4.2.1, we present a physical interpretation of the chiral degrees of freedom in the context of hadronic layers. Finally, in sections 4.3 and 4.3.1, we present the same analysis as in the previous two sections of this chapter, now in the context of hadronic tubes.

4.1 Finite Density: Metric of a Box

We are interested in analyzing the intriguing phenomena that occur when a finite amount of Baryonic charge lives within a finite spatial volume. Therefore, we consider as a starting point the metric of a box whose line element is

$$ds^2 = -dt^2 + L_r^2 dr^2 + L_\theta^2 d\theta^2 + L_\phi^2 d\phi^2 \ , \tag{4.1.1}$$

where L_i are constants representing the length of the box where the solitons are confined. The dimensionless coordinates $\{r, \theta, \phi\}$ have the following ranges

$$0 \leq r \leq 2\pi \ , \quad 0 \leq \theta \leq \pi \ , \quad 0 \leq \phi \leq 2\pi \ , \tag{4.1.2}$$

so that the volume available for the solitons is $V = 4\pi^3 L_r L_\theta L_\phi$. Notice that the coordinates are Cartesian, even if we use an angular notation to emphasize their finite ranges.

4.2 Skyrme Field in Euler Angles: Hadronic Layers

One of the most important quantities is the energy-momentum tensor, in section 5.1 we will discuss some important aspects of them. In this sense, the energy-momentum tensor of the

Skyrme model reads

$$
\begin{aligned}
T_{\mu\nu}^{SK} &= g^{\mu\nu}\mathcal{L}^{SK} + K\Big\{\nabla_\mu H\nabla_\nu H + D_\mu F D_\nu F + D_\mu G D_\nu G + \cos(2H)(D_\mu F D_\nu G + D_\nu F D_\mu G) - \\
&- \lambda\Big(\cos(2H)\big((\nabla H\cdot DG)(\nabla_\mu H D_\nu F + \nabla_\nu H D_\mu F) + (\nabla H\cdot DF)(\nabla_\mu H D_\nu G + \nabla_\nu H D_\mu G) - \\
&- 2(\nabla F\cdot DG)\nabla_\mu H\nabla_\nu H - (\nabla H)^2(D_\mu F D_\nu G + D_\nu F D_\mu G)\big) + \\
&+ 4\sin^2 H\cos^2 H\big((DF\cdot DG)(D_\mu F D_\nu G + D_\nu F D_\mu G) - (DG)^2\nabla_\mu F\nabla_\nu F - (DG)^2\nabla_\mu G\nabla_\nu F\big) + \\
&+ (\nabla H\cdot DF)(\nabla_\mu H D_\nu F + \nabla_\nu H D_\mu F) + (\nabla H\cdot DG)(\nabla_\mu H D_\nu G + \nabla_\nu H D_\mu G) - \\
&- \nabla_\mu H\nabla_\nu H((DF)^2 + (DG)^2) - (\nabla H)^2(D_\mu F D_\nu F + D_\mu G D_\nu G)\Big)\Big\}, \qquad (4.2.1)
\end{aligned}
$$

The hadronic layers-like solution can be generalized by including an arbitrary light-like function as a degree of freedom. In fact, the suitable generalization of the ansatz in [71], [75], [76], [77], [78], [79] and [80] is

$$
H = H(r)\ , \quad F = q\theta\ , \quad G = G(u)\ , \qquad u = \frac{t}{L_\phi} - \phi\ , \qquad (4.2.2)
$$

where q is an integer number. Here $G(u)$ is an arbitrary function of the light-like coordinate u. The eq. (4.2.2) explicitly avoids the no-go Derrick's theorem [91] due to the time dependence of the U field. The above ansatz keeps all the nice properties of the one in [71]. In particular, it satisfies the following identities

$$
(\nabla F\cdot\nabla G) = (\nabla H\cdot\nabla F) = (\nabla H\cdot\nabla G) = (\nabla F)^2 = 0\,, \qquad (4.2.3)
$$

which greatly simplifies the field equations. In fact, the ansatz in Eq. (4.2.2) reduces to the Skyrme field equations in Eqs. (2.6.33) to (2.6.35) to a simple linear equation

$$
\partial_r^2 H(r) = 0 \quad \Rightarrow H(r) = \kappa r + \kappa_0\ , \qquad (4.2.4)
$$

where κ_0 can be fixed to zero, the integration constant κ will be determined using appropriate boundary conditions.

Plugging the ansatz in Eqs. (2.6.31) (4.2.2) into Eq. (2.5.1), the topological charge density of the matter field reads

$$
\rho_B = -12q\sin(2H)H'\partial_\phi G,
$$

where it can be seen that the appropriate boundary conditions for the soliton profile $H(r)$ and the light-like function $G(u)$ are the following:

$$
H(r=2\pi) = \frac{\pi}{2}\ , \quad H(r=0) = 0\ , \qquad G(t,\phi=0) - G(t,\phi=2\pi) = (2\pi)p\ , \qquad (4.2.5)
$$

so that the topological charge takes the value

$$
B = pq\ . \qquad (4.2.6)
$$

Therefore, using the above boundary conditions, the profile becomes

$$H(r) = \frac{r}{4}. \tag{4.2.7}$$

On the other hand, the light-like function $G(u)$ satisfies the equations of a free scalar field in two dimensions and a first-order nonlinear equation, which leads to an emergent chiral CFT that we will discuss in the following section.

4.2.1 Physical interpretation of the chiral degrees of freedom

In order to clarify the physical meaning of the function $G(u)$ appearing in the ansatz in Eq. (4.2.2), let us consider the slightly different ansatz

$$H = \frac{r}{4}\,,\; F = q\theta\,,\; G = G(t,\phi)\,.$$

With the ansatz here above the Skyrme field equations would reduce to

$$\left(\left(\frac{\partial}{\partial t} - \frac{1}{L_\phi}\frac{\partial}{\partial \phi}\right) G\right)\left(\left(\frac{\partial}{\partial t} + \frac{1}{L_\phi}\frac{\partial}{\partial \phi}\right) G\right) = 0\,.$$

Thus, the Skyrme field equations force the choice of chirality: G can represent either left movers or right movers (but cannot represent both). Let us choose, as in Eq. (4.2.2) $G = G(u)$. Then, the boundary conditions in Eqs. (4.2.5) and (4.2.6) ensure that $G(u)$ has the following expression

$$G(u) = G_0(u) + \widetilde{G}(u)\,, \tag{4.2.8}$$

where $G_0(u) = pu$ represent the contribution without modulation and $\widetilde{G}(u)$ is periodic in the coordinates ϕ

$$\widetilde{G}(u) = \sum_N a_N \cos(Nu) + b_N \sin(Nu) \qquad , N \in \mathbb{N}\,,$$

where a_N and b_N are real coefficients and the integer N labels the chiral modes. Thus, the first term (linear in u) on the right hand side of Eq. (4.2.8) contributes to the Baryonic charge while $\widetilde{G}$ does not (being periodic in the coordinate ϕ). In order to interpret $\widetilde{G}$ it is enough to observe that, when $\widetilde{G} = 0$, the energy-momentum tensor only depends on the coordinate r (while it is homogeneous in the other two spatial coordinates).

Thus, in this case the solution represents a homogeneous Baryonic layer. On the other hand, when we turn on $\widetilde{G}$, the energy-momentum tensor depends not only on r but also on u (through the modes of $\widetilde{G}$). In this case, the plots 4.2.2 of the energy-density reveal that $\widetilde{G}$ represents modulations of the layer in the ϕ direction moving at the speed of light. Consequently, the present family of exact analytic solutions of the Skyrme field equations represents Baryonic layers dressed by modulations in the ϕ direction. Aside from the high theoretical interest to reveal the emergence of chiral conformal degrees of freedom living on Hadronic layers, the present results open unprecedented possibilities to compute analytically relevant observable quantities

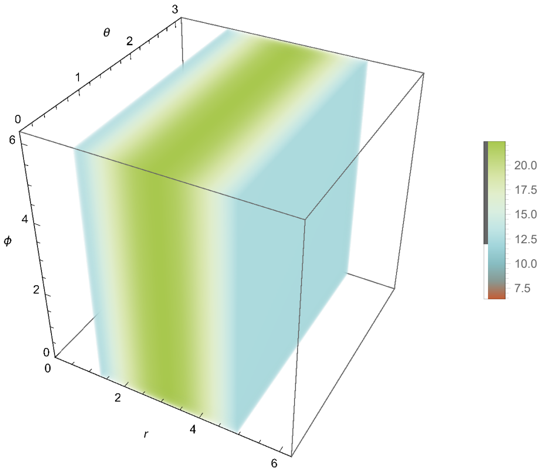

Figure 4.2.1: Energy density without modulation of hadronic layers, with topological charge $B = 4$. We have set $L_r = L_\theta = L_\phi = K = \lambda = 1$, $p = q = 2$ and $a_I = b_I = 0.1$.

as will be discussed in the next sections.

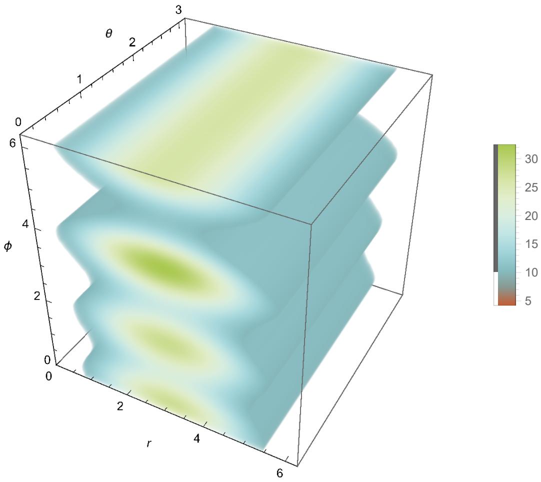

Figure 4.2.2: Energy density with modulation of hadronic layers, with topological charge $B = 4$. We have set $L_r = L_\theta = L_\phi = K = \lambda = 1$, $p = q = 2$ and, $a_1 = a_2 = b_1 = -b_3 = 0.1$.

4.3 Generalized Spherical Hedgehog: Hadronic Tubes

For computing transport properties it is necessary to know the form of the fundamental quantities of the Skyrme model. In this sense, the energy-momentum tensor related to Skyrme Model $T^{SK}_{\mu\nu}$

in the hedgehog ansatz (2.6.1) with (2.6.2) reads

$$\begin{aligned} T^{\mathrm{Sk}}_{\mu\nu} = g_{\mu\nu}\mathcal{L}^{\mathrm{SK}} - K\Big\{&\nabla_\mu\alpha\nabla_\nu\alpha + \sin^2\alpha\left(\nabla_\mu\Theta\nabla_\nu\Theta + \sin^2\Theta\, D_\mu\Phi D_\nu\Phi\right) + \\ &+ \lambda\Big[\sin^2\alpha\left((\nabla\Theta)^2\nabla_\mu\alpha\nabla_\nu\alpha + (\nabla\alpha)^2\nabla_\mu\Theta\nabla_\nu\Theta - (\nabla\alpha\cdot\nabla\Theta)\left(\nabla_\mu\alpha\nabla_\nu\Theta + \nabla_\nu\alpha\nabla_\mu\Theta\right)\right) + \\ &+ \sin^2\alpha\,\sin^2\Theta\left((D\Phi)^2\nabla_\mu\alpha\nabla_\nu\alpha + (\nabla\alpha)^2 D_\mu\Phi D_\nu\Phi - (\nabla\alpha\cdot D\Phi)\left(\nabla_\mu\alpha D_\nu\Phi + \nabla_\nu\alpha D_\mu\Phi\right)\right) + \\ &+ \sin^4\alpha\sin^2\Theta\left((D\Phi)^2\nabla_\mu\Theta\nabla_\nu\Theta + (\nabla\Theta)^2 D_\mu\Phi D_\nu\Phi - (\nabla\Theta\cdot D\Phi)\left(\nabla_\mu\Theta D_\nu\Phi + \nabla_\nu\Theta D_\mu\Phi\right)\right)\Big]\Big\}\ . \end{aligned} \tag{4.3.1}$$

While the topological charge density in Eq. (2.5.1) becomes

$$\rho_B = 12\sin^2\alpha\left(\sin\Theta\,(d\Phi + A)\wedge d\Theta + \cos\Theta\, F\right)\wedge d\alpha\ , \tag{4.3.2}$$

where $A = A_i dx^i$ and $F = \frac{1}{2}F_{ij}dx^i\wedge dx^j$. From the above expression, it follows that to have non-trivial topological configurations, we must demand that $d\Theta\wedge d\Phi\wedge d\alpha \neq 0$. This implies the necessary (but insufficient) condition that α, Θ, and Φ must be three independent functions. The existence of arbitrary topological charge of our solutions will be revealed later when appropriate boundary conditions are imposed.

To be able to apply the Green-Kubo formalism to compute some transport properties, we need to generalize the strategy introduced in [63] and [64] for the Skyrme-Maxwell case. In particular, for reasons which will be clarified in the next sections, we need more general time-dependence of the $SU(2)$-valued Skyrme field without losing the main property of such an approach which can reduce the complete set of non-linear field equations to just two integrable equations (one ODE for the Skyrmion profile and one PDE for the Maxwell potential) while also keeping alive the topological charge. In this subsection, we will not need to introduce the Maxwell field.

The suitable generalization of the ansatz in [63] and [64] is

$$\begin{aligned} &\alpha = \alpha(r), \\ &\Theta = Q\theta, \qquad Q = 2v+1\ , \quad v\in\mathbb{Z}, \\ &\Phi = G\,(u)\,, \quad u = \frac{t}{L_\phi} - \phi, \end{aligned} \tag{4.3.3}$$

where now $G\,(u)$ is an arbitrary function of the light-like coordinate u. It is easy to see that the above ansatz keeps all the nice properties of the one in [63] and [64]. Firstly, it satisfies the following identities

$$(\nabla\Phi\cdot\nabla\alpha) = (\nabla\alpha\cdot\nabla\Theta) = (\nabla\Theta\cdot\nabla\Phi) = (\nabla\Phi)^2 = 0\ , \tag{4.3.4}$$

which greatly simplifies the field equations in Eqs. (2.6.10) to (2.6.12). Moreover, Eq. (4.3.3) explicitly avoids the no-go theorem of Derrick [91] due to the time dependence of the U field. The construction in [63] and [64] also suggests the following ansatz for the A_μ (which will be needed in the following section):

$$A_\mu = (\xi, 0, 0, -L_\phi \xi) \ , \qquad \xi = \xi(u, r, \theta) \ , \tag{4.3.5}$$

$$\Rightarrow A_\mu A^\mu = 0 \quad \text{and} \quad \nabla^\mu A_\mu = 0 \ . \tag{4.3.6}$$

Replacing the ansatz introduced in Eqs. (2.6.1), (2.6.2) and (4.3.3) into the Skyrme field equations in Eqs. (2.6.10) to (2.6.12), the set of three non-linear differential equations is reduced to an ODE for the profile α, namely

$$\alpha'' + \frac{Q^2 \sin(\alpha)\cos(\alpha)(\lambda\alpha'^2 - L_r^2)}{L_\theta^2 + \lambda Q^2 \sin^2(\alpha)} = 0 \ ,$$

which can be reduced to a first-order ODE that can be conveniently written as

$$\frac{d\alpha}{\eta(\alpha, E_0)} = \pm dr \ , \quad \eta\left(\alpha, E_0\right) = \left[\frac{E_0 L_\theta^2 - \frac{1}{2} q^2 L_r^2 \cos(2\alpha)}{L_\theta^2 + \lambda Q^2 \sin^2(\alpha)}\right]^{\frac{1}{2}} \ , \tag{4.3.7}$$

E_0 being an integration constant (fixed by the boundary conditions, as we will see below).[1] On the other hand, the light-like function $G(u)$ satisfies the equations of a free scalar field in two dimensions, which leads to an emergent chiral CFT. We will discuss this important fact in the next section.

Plugging the ansatz in Eqs. (2.6.1), (2.6.2) and (4.3.3) into Eq. (2.5.1), the topological charge turns out to be

$$\rho_B = 12 q \sin(q\theta) \sin^2(\alpha) \alpha' \partial_\phi G \ ,$$

where it is clearly seen that the appropriate boundary conditions for the soliton profile $\alpha(r)$ and the light-like function $G(u)$ are the following:

$$\alpha(2\pi) - \alpha(0) = n\pi, \tag{4.3.8}$$

$$G(t, \phi = 2\pi) - G(t, \phi = 0) = (2\pi)p, \tag{4.3.9}$$

with n and p integer numbers. Therefore, using Eq. (4.3.9) and integrating with the ranges defined in Eq. (4.1.2), the topological charge takes the value

$$B = np \ , \tag{4.3.10}$$

We have used that q is an odd number, as specified in the ansatz in Eq. (4.3.3). From Eqs. (4.1.2), (4.3.7) and (4.3.9) it follows that the integration constant E_0 must satisfy

$$n \int_0^\pi \frac{d\alpha}{\eta\left(\alpha, E_0\right)} = 2\pi \ . \tag{4.3.11}$$

[1] Eq. (4.3.7) can be solved analytically in terms of Elliptic Functions; however, the explicit solution is not necessary for our purposes.

Eq. (4.3.11) is an equation for E_0 in terms of n that always has a real solution when

$$E_0 > \frac{Q^2 L_r^2}{2L_\theta^2} \ ,$$

so that, for given values of q, L_r and L_θ, the integration constant E_0 determines the value of the α profile for the boundary conditions defined in Eq. (4.3.9).

From the above condition, it is clear that for large n, the integration constant E_0 scales as n^2

$$E_0 = n^2 \xi_0, \qquad \xi_0 > 0, \tag{4.3.12}$$

where ξ_0 can also be interpreted as an integration constant and does not depend on n for large n.

4.3.1 Physical Interpretation of the chiral degrees of freedom

In the present case one can clarify the physical meaning of the function $G(u)$ appearing in the ansatz in Eq. (4.3.3) by the slightly more general ansatz

$$\begin{aligned} \alpha = \alpha(r),\ \frac{d\alpha}{\eta(\alpha, E_0)} &= \pm dr \ , \\ \eta\left(\alpha, E_0\right) &= \left[\frac{E_0 L_\theta^2 - \frac{1}{2}q^2 L_r^2 \cos(2\alpha)}{L_\theta^2 + \lambda Q^2 \sin^2(\alpha)}\right]^{\frac{1}{2}} \ , \\ \Theta = Q\theta, \qquad & Q = 2v+1 \ , \quad v \in \mathbb{Z}, \\ \Phi = G\left(t, \phi\right), \quad & \alpha(2\pi) - \alpha(0) = n\pi \ , \end{aligned}$$

where $\alpha(r)$ and $\Theta(\theta)$ are the same as in Eq. (4.3.3) but G has been taken as a generic function of t and ϕ (instead of taking G as function of a single light-like coordinate). In this way one can shed light on the true nature of G. As in the case of the Baryonic layers described in the previous sections, with the ansatz here above the Skyrme field equations reduce to

$$\left(\left(\frac{\partial}{\partial t} - \frac{1}{L_\phi}\frac{\partial}{\partial \phi}\right) G\right)\left(\left(\frac{\partial}{\partial t} + \frac{1}{L_\phi}\frac{\partial}{\partial \phi}\right) G\right) = 0 \ , \tag{4.3.13}$$

plus

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2}\frac{\partial^2}{\partial \phi^2}\right) G = 0 \, ,$$

which is a consequence of Eq. (4.3.13). Thus, G can represent either left movers or right movers (but cannot reoresent both). Let us then choose $G = G(u)$. The boundary conditions in Eq. (4.3.9) require that $G(u)$ has the following expression:

$$G\left(u\right) = G_0(u) + \widetilde{G}\left(u\right) \ , \tag{4.3.14}$$

where $G_0(u) = pu$ and $\widetilde{G}\left(u\right)$ is periodic in the coordinate ϕ:

$$\widetilde{G}\left(u\right) = \sum_N a_N \cos\left(Nu\right) + b_N \sin\left(Nu\right) \ , \ N \in \mathbb{N} \ ,$$

where a_N and b_N are real coefficients. Furthermore, in this case the first term (linear in u) on the right hand side of Eq. (4.3.14) contributes to the Baryonic charge while $\widetilde{G}$ does not (being periodic in the coordinate ϕ). Moreover, when $\widetilde{G} = 0$, the stationary energy-momentum tensor only depends on the coordinates r and θ (while it does not depend on ϕ).

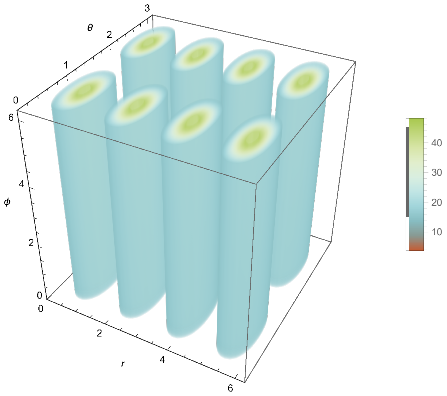

Figure 4.3.1: Energy density without modulation of hadronic tubes, with topological charge $B = 8$.We have set $L_r = L_\theta = L_\phi = K = \lambda = 1$, $p = Q = 2$, $n = 4$, and $a_I = b_I = 0$.

Thus, in this case the solution represents an ordered array of homogeneous Baryonic tubes. On the other hand, when we turn on $\widetilde{G}$, the energy-momentum tensor depends not only on r and θ but also on u (through the modes of $\widetilde{G}$). In this case, the plots 4.3.1 of the energy-density reveal that $\widetilde{G}$ describes modulations of the tubes in the ϕ direction moving at the speed of light. These analytic solutions are Baryonic tubes dressed by modulations in the ϕ direction. The fact that (both in the case of Hadronic layers and tubes) $\widetilde{G}$ behaves as a chiral conformal field in one spatial dimensions open the intriguing perspective to compute the transport coefficients associated to these modulations as it will be shortly discussed in the next chapter.

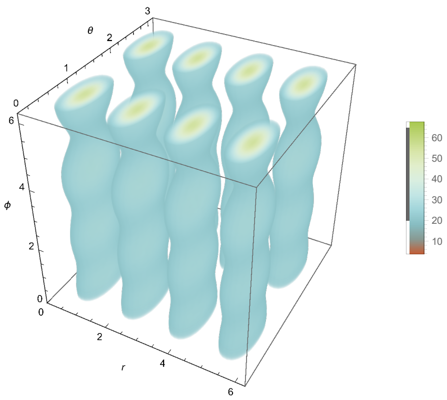

Figure 4.3.2: Energy density with modulation of hadronic tubes, with topological charge $B = 8$. We have set $L_r = L_\theta = L_\phi = K = \lambda = 1$, $p = Q = 2$, $n = 4$ and $a_1 = a_2 = b_1 = -b_3 = 0.1$.

Chapter 5

Gauged Skyrmions and applications

In this chapter, we will discuss how to generalize the modulated inhomogeneous Baryonic condensates constructed in the previous sections in the case in which the minimal coupling with Maxwell theory is taken into account. This issue is extremely important for the following reason. The numerical simulations discussing the appearance of inhomogeneous Baryonic condensates (see [5, 6, 7, 8, 9, 10, 11, 31, 13, 14, 15], and the nice up to date review [16]) usually do not consider explicitly the electromagnetic interactions of the Baryons (which often are modeled as point-like particles). The self-consistent coupling of many Baryons with the electromagnetic field generated by the Baryons themselves would make the numerical simulations considerably heavier. Therefore, to have analytic tools which can help understand the nature of the electromagnetic field naturally associated to these inhomogeneous Hadronic condensates would be extremely helpful also as a guide for numerical simulations. Here we will describe how one can "dress" the condensates introduced in the previous sections with their own electromagnetic fields. We will try to give a unified description of this construction which is valid both for Hadronic layers and tubes.

The starting point is the observation that the key property of the ansatze in the ungauged cases which allows to decouple the Skyrme field equations without killing the topological charge are the relations in Eqs. (4.2.3) and (4.3.4) for layers and tubes respectively. Thus, we have to construct an ansatz for the gauge potential A_μ which keeps Eqs. (4.2.3) and (4.3.4) alive. The answer to this question was found in [76, 77, 64], and [65].

We will first describe the method in the case of the Hadronic tube, which is easier to understand. In order to minimally couple the Hadronic tubes discussed in the previous sections with $U(1)$ Maxwell gauge field let us consider a gauge potential A_μ with the following characteristics:

$$A_\mu \partial^\mu \alpha = 0,\ A_\mu \partial^\mu \Theta = 0,\ A_\mu \partial^\mu G = 0\ , \tag{5.0.1}$$

$$A_\mu A^\mu = 0\ ,\quad \partial_\mu A^\mu = 0\ . \tag{5.0.2}$$

First of all, one observes that the above conditions for A_μ are not empty. In order to satisfy both conditions it is enough to consider a gauge potential with components only along the $t-$direction and the $\phi-$direction, then these two components, A_t and A_ϕ, must be proportional (in order to satisfy $A_\mu A^\mu = 0$) and can depend on r, θ and the same null coordinate u which enters in G. In this sense, the Eq. (4.3.5) show this point.

One can easily check that the field strength of such gauge potential in general is non-vanishing. The above conditions in Eqs. (5.0.1) and (5.0.2) complement the condition in Eq. (4.3.4) for the $SU(2)$-valued Skyrme field. From the viewpoint of the gauged Skyrme field equations, Eqs. (5.0.1) and (5.0.2) possess the very nice feature of eliminating all the terms of the gauged Skyrme field equations which could, potentially, mix the $SU(2)$ degrees of freedom with the gauge potential. Consequently, with the above ansatz for the gauge potential, the gauged Skyrme field equations remain the same as the ungauged Skyrme field equations corresponding to the ansatz in Eq. (4.3.3). On the other hand, one may wonder whether the above conditions in Eqs. (5.0.1) and (5.0.2) are too restrictive from the viewpoint of the Maxwell equations. In particular, the left hand side of the Maxwell field equations (namely $\partial^\mu F_{\mu\nu}$) only has components along the $t-$direction and the $\phi-$direction (due to the form of the gauge potential in Eq. (4.3.5)). Thus, we have to analyze the $U(1)$ Skyrme current.

The terms which could spoil the consistency of the ansatz are all the terms which are not proportional to $D_\mu \Phi$ (such as the terms proportional to $\nabla_\mu \alpha$ and $\nabla_\mu \Theta$). In fact, all these “dangerous terms" vanish (since the ansatz has the property that $\nabla\alpha \cdot D\Phi = 0 = \nabla\Theta \cdot D\Phi$). From the mathematical viewpoint, this ansatz for the gauge potential has been chosen to simplify as much as possible the coupled gauged Skyrme Maxwell system keeping alive the field strength and the interactions. From the physical viewpoint it turns out that such gauge fields belong to the important class of force free Maxwell field [68] (which are very important in astrophysics: see [92, 93, 94, 95, 96, 97] and references therein). Hence, from the physical viewpoint, the present approach disclosed a relevant property of the inhomogeneous condensates introduced in the previous sections which would have been very difficult to discover with other methods: such condensates are natural sources of force free plasmas.

As far as Hadronic layers are concerned, the story is very similar although slightly more complicated (see [76, 77], and [98]). The ansatz for the Hadronic layers

$$H = H(r) \ , \quad F = q\theta \ , \quad G = G(u) \ , \qquad u = \frac{t}{L_\phi} - \phi \ ,$$

which satisfy the useful identities

$$(\nabla F \cdot \nabla G) = (\nabla H \cdot \nabla F) = (\nabla H \cdot \nabla G) = (\nabla F)^2 = 0 \, ,$$

can also be complemented with a suitable gauge potential by requiring that the gauge potential does not spoil the solvability of the gauged Skyrme field equations. The only difference is that instead of requiring $A_\mu A^\mu = 0$ one has to require a suitable quadratic constraint on the components of A_μ (see [76, 77], and [98]) together with the usual conditions on the orthogonality of the gradients as in the case of Hadronic layers. Furthermore, final results are similar: it is possible to explicitly construct an ansatz for the gauge potential which allows the gauged

Skyrme field equations to be solved and the Maxwell equations with the $U(1)$ Skyrme current reduce consistently to a Schrodinger-like equation. The case of Hadronic layers is simpler since the Schrodinger-like equation can be reduced to the Mathieu equation which is solvable in terms of special functions. For the Hadronic layers it is also true that they are natural sources of force free plasmas (see [98] for details).

5.1 How to use the emergent Chiral Conformal Degrees of Freedom

As it has been discussed in the previous sections, the present approach does not produce just novel analytic solutions but it also provide new physical insights on relevant properties (such as the natural electromagnetic field generated by the inhomogeneous condensates). In fact, these solutions possess another characteristic which is likely to disclose intriguing observable features. As it has been already discussed, these inhomogeneous condensates can be dressed with a chiral conformal field (representing modulations along the layers and tubes) which behaves as one-dimensional (such a field was denoted as $\widetilde{G}$ in the previous sections). These "chiral CFT modes" in one spatial dimension possess the remarkable properties of being not only solutions of the linearized field equations around the inhomogeneous Baryonic condensates (defined by $\widetilde{G}=0$) but also exact solutions of the field equations. Moreover, not only the field equations for $\widetilde{G}$ are the same as in a chiral conformal field theory, but also the energy-momentum tensor (restricted to the t,ϕ directions) corresponds to an effective two-dimensional traceless energy-momentum tensor (both in the case of Hadronic layers and in the case of Hadronic tubes):

$$T_{ab}=\begin{pmatrix} T_{tt} & T_{t\phi} \\ T_{\phi t} & T_{\phi\phi} \end{pmatrix}, \qquad a,b=t,\phi \ .$$

This fact opens the exciting possibility to develop an analytic theory of transport coefficients of these inhomogeneous Baryonic condensates associated to these chiral modes. Here we will sketch in some detail this analysis which we hope to complete in the near future.

As it is well known, one can consider the usual mode quantization of $\widetilde{G}$ following the recipes of conformal field theories (see [99] and [100]). If these recipes could be implemented in a proper way in the present case one could arrive at a closed form for the shear viscosity, the thermal conductivity and the electric conductivity.

Roughly speaking, the viscosity and conductivities are defined as

$$\kappa=\langle J_\mu J'_\mu\rangle \ , \quad \eta=\langle T_{t\phi}T'_{t\phi}\rangle \ ,$$

where J_μ is the "$U(1)$" current which can be expressed in terms of the gradient $\widetilde{G}$[1]. The viscosity can be calculated directly using the well-known results of CFT [99] or [100] and the definitions of the Kubo formalism [33]. Since in the present case these inhomogeneous condensates can be dressed exactly with chiral conformal excitations which propagate in one spatial dimension, a

[1]The expression for these currents is in Eqs. (2.6.36) and (2.6.13) for the Hadronic layers and tubes, respectively.

natural guess is that the results for obtaining κ and η are the same as in a chiral conformal field theory:

$$\eta^j = \frac{T}{2}\varsigma^j \ , \ j = 1, 2 \ ,$$

where T is the temperature, $j = 1$ refers to the layers, $j = 2$ to the tubes, and the coefficients ς^j (which, in principle, can be computed explicitly) would distinguish the transport coefficients of Hadronic layers from the ones of Hadronic tubes. There are still some missing steps (in particular, the proper semi-classical quantizations of these chiral modes living on top of the condensates) but we are quite confident that such steps can be completed successfully.

Chapter 6

Yang-Mills Perspectives

The Yang-Mills model is a quantum field theory that describes the behavior of gauge bosons, particles that mediate the fundamental forces of nature. It was developed by physicists Chen Ning Yang and Robert Mills in the 1950s to generalize the Maxwell theory of electromagnetism [101].

The model is based on the concept of gauge symmetry, which is a mathematical transformation that leaves the physics of a system unchanged. The Yang-Mills model is an essential theoretical tool in particle physics used to describe the behavior of subatomic particles and the fundamental forces that govern their interactions. It has been validated by numerous experiments, including the discovery of the W and Z bosons, which mediate the weak nuclear force, and the discovery of the gluon, which mediates the strong nuclear force. Their discoverers, C. Rubbia and S. van der Meer, obtained the Nobel Prize in 1984.

In this thesis, we show the relationship with the Skyrme model in the context of a future perspective for calculating some transport properties of matter.

6.1 Yang-Mills and Conformal Symmetry

The Yang-Mills and the Skyrme Model are very different because the Yang-Mills theory is a gauge theory, while the Skyrme model only has global symmetries. However, recently, [78] found a creative relationship between Skyrme and the Yang-Mills model, despite the belief that different methods should be used for each.

However, they showed that certain sectors of these $(3 + 1)$-dimensional theories could be identified as possessing arbitrary baryonic charge and conformal symmetry. As we saw in the previous chapter, the Skyrme model has a sector with chiral conformal symmetry; however, in the case of the Yang-Mills model, in [78], they showed that it has a sector with conformal symmetry of infinite dimension. We sketch here this derivation.

The Yang-Mills theory in $(3+1)$ dimensions has the following action

$$I[A] = \frac{1}{2e^2}\int d^4x\sqrt{-g}\text{Tr}(\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}), \tag{6.1.1}$$

where e is the Yang-Mills coupling constant. The field strenght $\bar{F}_{\mu\nu}$ is defined as

$$\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{6.1.2}$$

where A_μ is the non-abelian connection, defined by

$$A = A^j_\mu t_j dx^\mu \tag{6.1.3}$$

where t_i can be written in terms of the Pauli matrices (see section 2.6.1).

The analogue of the barionic charge of the Skyrme Model in this context is the Chern-Simons charge Q^{CS}, defined by

$$Q^{CS} = \int \rho^{CS} dV, \tag{6.1.4}$$

where $\rho^{Cs} = J^{CS}_{\mu=0}$ with

$$J^{CS}_\mu = \frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}\left(A^\nu\partial^\rho A^\sigma + \frac{2}{3}A^\nu A^\rho A^\sigma\right) \tag{6.1.5}$$

The field equations can be calculated directly through variation of the action with respect to the field A_μ. These are

$$\nabla_\mu \bar{F}^{\mu\nu} + [A_\nu, \bar{F^{\mu\nu}}] = 0 \tag{6.1.6}$$

To consider finite effects, the authors need to use the flat metric (4.1.1), with ranges

$$0 \leq r \leq 4\pi \quad , \quad 0 \leq \theta \leq 2\pi \quad , \quad 0 \leq \phi \leq \pi. \tag{6.1.7}$$

Then, they consider the Euler parameterization (2.6.26) with the following scalar fields

$$\alpha = p\theta \quad , \quad \beta = 2H(t,\phi) \quad , \quad \rho = qr \tag{6.1.8}$$

And for some boundary conditions for the field $H(t,\phi)$, and some relations for the field A_μ and the element U of $SU(2)$, they find

$$H(t,\phi) = \arccos{(G)} \tag{6.1.9}$$

where G is an arbitrary function $G(t,\phi)$ that is defined in terms of

$$G(t,\phi) = \exp(3\eta)\frac{F}{\sqrt{1-\exp(4\eta)\cdot F^2}} \tag{6.1.10}$$

where η is a real number that fixes the Chern-Simons charge Q^{CS} in terms of an integer value. Using the former, F takes values depending on the boundary conditions of H. So then, from specific considerations, they conclude that.

$$\Box F \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2} \frac{\partial^2}{\partial \phi^2} \right) F = 0, \tag{6.1.11}$$

which describes the behavior of a free massless scalar field in two dimensions. Another relevant result is that they find the same behavior of the (5.1). In conclusion, they find a sector of this theory that describes the simplest bosonic CFT in two dimensions.

In the same article, they find exciting properties of the Yang-Mills-Higgs Model. This model shares the chiral properties with the Skyrme Model, unlike the Yang-Mills model.

In conclusion, the authors provide a framework that can be systematically used to characterize specific sectors of different theories, which was not expected at all due to their unequal characteristics. This simplification promises to be able to calculate relevant properties, such as transport properties, systematically.

Chapter 7

Conclusion

In this thesis, we present the formalism of the Skyrme Model coupling with the Maxwell Theory. We define the most important properties of this model, such as the topological charge and the relation with the baryonic charge of the QCD theory. Then, in chapter 3, we present a short historical introduction to Skyrme Crystals and their importance for characterizing nuclear matter at the low energy limit. In chapter 4, we construct two solutions in the context of the $SU(2)$ group; layers and tubes cases. We show the box through the line element in cartesian coordinates. In chapter 5, we expose the results and their derivations (with more details in the respective Appendices). Then, in chapter 6, we see another example where our analytical tools are necessary to compute some novel regions of the non-linear models, like the Yang-Mills. In this region, emergent an effective conformal field theory. In the case of Skyrme, in chapter 5, we see that this CFT is chiral.

We have described a proper analytic framework to construct inhomogeneous Baryonic condensates in the gauged Skyrme Maxwell theory. This approach is able not only to produce exact solutions with high Baryonic charges but also gives considerable physical insights on the nature of the configurations (such as the fact that these Hadronic layers and tubes constructed in the previous sections are natural sources of force free plasmas). Another characteristic of the present technique is that it discloses the appearance of chiral conformal degrees of freedom which describes modulations of the condensates.

Aside from the intrinsic theoretical interest of this fact, the unprecedented possibility of arriving at the analytical computation of the transport properties of these condensates using known results in conformal field theory in $(1 + 1)$ dimensions was opened by these works. There are still some missing steps to complete the proper computations of the transport coefficients. We hope to provide solutions to this issue in the near future.

There is still much work to be done; we could compute the same transport properties with the same tools while considering an ansatz for U in the $SU(3)$ group. Besides, with the recent work [72], we can extend this understanding to another useful model, like the non-linear sigma model, Yang-Mills, and the coupling with Higss and, possibly, with relativity.

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Appendix A

Basic of Skyrme Model

A1 Derrick Theorem

We consider the Skyrme Model for study static solutions that satisfied finite energy. Derrick proved this although *reductio ad absurdum*. He assumed that the static solution exist and he use a scaling of coordinates for proved that Fist we consider that the energy of the Skyrme Model is

$$E = I_{\text{NLSM}} + I_{\text{SK}}. \tag{A1.1}$$

where I_{NLSM} and I_{SK} in therms of the element $U \in SU(2)$ are

$$I_{\text{NLSM}} = \frac{K}{4}\int d^4x \text{Tr}\left\{(U^{-1}\partial_\mu U)^2\right\}, \tag{A1.2}$$

$$I_{\text{SK}} = \frac{K}{4}\int d^4x \text{Tr}\left\{\frac{\lambda}{8}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]\left[U^{-1}\partial^\mu U, U^{-1}\partial^\nu U\right]\right\}. \tag{A1.3}$$

The energy should satisfied that $E(\lambda = 1) = E_0$, where $\lambda = 1$ is an stationary point of $E(\lambda)$. We study the stationary point thought

$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=1} = 0. \tag{A1.4}$$

In particular, we have for the non-sigma model action (A1.2) after the scaling $\bar{x} = x$ for the scalar field U

$$\begin{aligned}
\bar{I}_{\text{NLSM}} &= \frac{K}{4}\int \text{Tr}\left\{\left(U(\bar{x})\partial_i U(\lambda x)\right)\left(U(\bar{x})\partial_j U(\lambda x)\right)\delta^{ij}\left(\frac{1}{\lambda}\right)^D\right\} d^D\bar{x}, \\
&= \frac{K}{4}\int \text{Tr}\left\{\left(U\left(\bar{x}\frac{\partial U}{\partial \bar{x}^i}\underbrace{\frac{\partial \bar{x}^i}{\partial x}}_{\lambda}\right)\right)\left(U\left(\bar{x}\frac{\partial U}{\partial \bar{x}^j}\underbrace{\frac{\partial \bar{x}^j}{\partial x}}_{\lambda}\right)\right)\delta^{ij}\left(\frac{1}{\lambda}\right)^D\right\} d^D\bar{x}, \\
&= \lambda^{2-D} I_{\text{NLSM}}.
\end{aligned}$$

The non sigma model action has not static solutions by finite energy in $\mathbb{R}^3$.

However, we have in the commutator term of the Skyrme term (A1.3)

$$
\begin{aligned}
&[U^{-1}\partial_i U, U^{-1}\partial_j U][U^{-1}\partial^i U, U^{-1}\partial^j U],\\
&= [U^1(\bar{x})\partial_i U(\lambda x), U^{-1}(\bar{x})\partial_j U(\lambda x)][U^1(\bar{x})\partial_k U(\lambda x), U^{-1}(\bar{x})\partial_l U(\lambda x)]\delta^{ik}\delta^{jl},\\
&= \left[U^{-1}(\bar{x})\frac{\partial U}{\partial \bar{x}^i}\underbrace{\frac{\partial \bar{x}^i}{\partial x}}_{\lambda}, U^{-1}(\bar{x})\frac{\partial U}{\partial \bar{x}^j}\underbrace{\frac{\partial \bar{x}^j}{\partial x}}_{\lambda}\right]\left[U^{-1}(\bar{x})\frac{\partial U}{\partial \bar{x}^k}\underbrace{\frac{\partial \bar{x}^k}{\partial x}}_{\lambda}, U^{-1}(\bar{x})\frac{\partial U}{\partial \bar{x}^l}\underbrace{\frac{\partial \bar{x}^l}{\partial x}}_{\lambda}\right],\\
&= \lambda^4[U^{-1}\partial_i U, U^{-1}\partial_j U][U^{-1}\partial^i U, U^{-1}\partial^j U].
\end{aligned}
$$

Thus, we find the follow expression

$$
\begin{aligned}
\bar{I}_{\mathrm{SK}} &= \frac{K}{4}\int \mathrm{Tr}\left\{\frac{\lambda}{8}[U^{-1}\partial_i U, U^{-1}\partial_j U][U^{-1}\partial^i U, U^{-1}\partial^j U]\right\}\left(\frac{1}{\lambda}\right)^D d^D\bar{x},\\
&= \lambda^{4-D} I_{\mathrm{SK}}.
\end{aligned}
$$

We concluded that the Skyrme term of the Skyrme action permit stable solution of finite energy

A2 Details of Field Equations

We want to find the field equations for for the Skyrme Action. For this we should vary the action (2.2.2) with respect the fundamental field U and we obtain

$$
\delta\mathcal{L}_{\mathrm{SK}} = \frac{K}{2}\mathrm{Tr}\left\{R^\mu \delta R_\mu + \frac{\lambda}{8}G^{\mu\nu}\delta G_{\mu\nu}\right\}. \tag{A2.1}
$$

Firstly, we consider the following relation

$$
R_\mu = U^{-1}\partial_\mu U \Rightarrow \delta U^{-1} = -U^{-1}(\delta U)U^{-1}.
$$

Then, we analyze the first term of (A2.1),

$$
\delta R_\mu = \delta(U^{-1}\partial_\mu U) = -U^{-1}(\delta U)R_\mu + R_\mu U^{-1}(\delta U) + \partial_\mu(U^{-1}\delta U).
$$

Then,

$$
\begin{aligned}
\mathrm{Tr}\big\{(\delta R_\mu R^\mu)\big\} &= \mathrm{Tr}\big\{-U^{-1}(\delta U)R_\mu + R_\mu U^{-1}(\delta U) + \partial_\mu(U^{-1}\delta U)R^\mu\big\}\\
&= -\mathrm{Tr}\big\{R^\mu U^{-1}(\delta U)R_\mu\big\} + \mathrm{Tr}\big\{R_\mu U^{-1}(\delta U)R^\mu\big\} + \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U)R^\mu\big\}\\
&= \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U)R^\mu\big\}\\
&= \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U R^\mu)\big\} - \mathrm{Tr}\big\{U^{-1}\delta U(\partial_\mu R^\mu)\big\}\\
&= \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U R^\mu)\big\} - \mathrm{Tr}\big\{(\partial_\mu R^\mu)U^{-1}\delta U\big\}.
\end{aligned} \tag{A2.2}
$$

On the other hand, for the second term of (A2.1), we have

$$\begin{aligned}\partial_\mu R_\nu &= \partial_\mu(U^{-1}\partial_\nu U)\\ &= \partial_\mu U^{-1}\partial_\nu U + U^{-1}\partial_\mu\partial_\nu U\\ &= -U^{-1}\partial_\mu U U^{-1}\partial_\nu U + U^{-1}\partial_\mu\partial_\nu U,\end{aligned}$$

then we have, the following expression

$$\partial_\mu R_\nu - \partial_\nu R_\mu = -[R_\mu, R_\nu].$$

This suggested

$$\begin{aligned}\delta[R_\mu, R_\nu] &= -\partial_\mu \delta R_\nu + \partial_\nu \delta R_\mu,\\ &= \partial_\mu(U^{-1}\delta U R_\nu - R_\nu U^{-1}\delta U) - \partial_\nu(U^{-1}\delta U R_\mu - R_\mu U^{-1}\delta U)\end{aligned}$$

Thus,

$$\begin{aligned}&\mathrm{Tr}\big\{\partial_\mu\big([U^{-1}\delta U, R_\nu]\big) - \partial_\nu\big([U^{-1}\delta U, R_\mu]\big)G^{\mu\nu}\big\}\\ &= \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U R_\nu)G^{\mu\nu} - \partial_\mu(R_\nu U^{-1}\delta U)G^{\mu\nu} - \partial_\mu(R_\nu U^{-1}\delta U)G^{\mu\nu} - \partial_\mu(U^{-1}\delta U R_\nu)G^{\mu\nu}\big\}\\ &= 2\mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U R_\nu)G^{\mu\nu} - \partial_\mu(R_\nu U^{-1}\delta U)G^{\mu\nu}\big\}\\ &= \mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U R_\nu G^{\mu\nu}) - (U^{-1}\delta U)R_\nu\partial_\mu\partial_\mu G^{\mu\nu}\\ &- \partial_\mu(R_\nu U^{-1}\delta U G^{\mu\nu}) + R_\nu U^{-1}\delta U\partial_\mu G^{\mu\nu}\big\}\\ &= 2\mathrm{Tr}\big\{\partial_\mu(U^{-1}\delta U[R_\nu, G^{\mu\nu}]) + U^{-1}\delta U[\partial_\mu G^{\mu\nu}, R_\nu]\big\}.\end{aligned} \tag{A2.3}$$

Where we have considered

$$\begin{aligned}[R_\nu, \partial_\mu G^{\mu\nu}] &= \partial_\mu([R_\nu, G^{\mu\nu}]) - \partial_\mu R_\nu G^{\mu\nu} + G^{\mu\nu}\partial_\mu R_\nu,\\ &= \partial_\mu([R_\nu, G^{\mu\nu}).\end{aligned}$$

We concluded with (A2.2) and (A2.3) that the field equation of the Skyme Action (2.2.2) are (2.3.1).

A3 Energy Derivation

From the (2.4.1) we find

$$\begin{aligned}T_{00} &= -\frac{K}{2}\mathrm{Tr}\left\{R_0R_0 - \frac{1}{2}g_{00}R^\alpha R_\alpha + \frac{\lambda}{4}\left(g^{\alpha\beta}G_{0\alpha}G_{0\beta} - \frac{1}{4}g_{00}G_{\alpha\beta}G^{\alpha\beta}\right)\right\}\\ &= -\frac{K}{2}\mathrm{Tr}\left\{\frac{1}{2}R_0R^0 - \frac{\lambda}{8}G_{0a}G^{0a}\right\} - \frac{K}{2}\mathrm{Tr}\left\{\frac{1}{2}R_aR^a + \frac{\lambda}{16}G_{ab}G^{ab}\right\},\end{aligned} \tag{A3.1}$$

where the fist term of the previous equations is the rotational energy E_{rot} and the second term is the stataic energy E_{static}.

A4 Bound BPS

We focus now on the static energy term of the system, and notice the following

$$\begin{aligned} G_{ab}G^{ab} &= [R_a, R_b][R^a, R^b] \\ &= \frac{1}{2}\epsilon^{lab}\epsilon_{lmn}[R^a, R^b][R_m, R_n]. \end{aligned} \tag{A4.1}$$

Hence, the static energy term can be rewritten as

$$\begin{aligned} E_{\text{stat}} &= -\frac{K}{4}\text{Tr}\left\{R_a R^a + \frac{\lambda}{16}\epsilon_{lab}\epsilon^{lmn}[R_a, R_b][R^m, R^n] \pm \sqrt{\lambda}\epsilon_{cab}R_c R_a R_b \mp \sqrt{\lambda}\epsilon_{cab}R_c R_a R_b\right\} \\ &= -\frac{K}{4}\text{Tr}\left\{\left(R_a \pm \frac{\sqrt{\lambda}}{4}\epsilon_{abc}[R_b, R_c]\right)^2 \mp \sqrt{\lambda}\epsilon_{abc}R_a R_b R_c\right\}. \end{aligned} \tag{A4.2}$$

From the last expression we see that the first term corresponds to the BPS equation while the second is the baryonic charge. The existance of a BPS bound is given by the fact that the first term is strictly positive

$$E_{\text{stat}} \geq \left|\text{Tr}\left\{\frac{K}{4}\sqrt{\lambda}\epsilon_{cab}R_c R_a R_b\right\}\right|. \tag{A4.3}$$

This bound gives us access to the first term, which is known as the BPS equations. These equations are of first order, which is extremely helpful for finding solutions to non linear theories where the field equations are hard to solve in general. Moreover, the BPS equations imply the field equations without the need of varying the action. In this sense, one only needs to make an analysis of the total energy of the theory.

We first need to saturate the bound

$$\left(R_a \pm \frac{\sqrt{\lambda}}{4}\epsilon_{abc}[R_b, R_c]\right) = 0\,. \tag{A4.4}$$

Then, apply a commutator from the left

$$[R_l, R_a] \pm \frac{\sqrt{\lambda}}{2}\epsilon_{abc}[R_l, [R_b, R_c]] = 0\,.$$

Furthermore, the following commutator can be obtained from the BPS equation

$$\pm\frac{2}{\sqrt{\lambda}}\epsilon_{abc}R_a = [R_b, R_c].$$

Substituting, we obtain

$$\begin{aligned}
0 = &\pm \frac{2}{\sqrt{\lambda}}\epsilon_{anl}L_n \pm \frac{\sqrt{\lambda}}{2}\epsilon_{abc}[L_l[L_b, L_c]] \\
&\mp \frac{2}{\sqrt{\lambda}}\epsilon^{ald}\epsilon_{aln}L_n \pm \frac{\sqrt{\lambda}}{2}\epsilon_{abc}\epsilon^{ald}[L_l[L_b, L_c]] \\
&\mp \frac{4}{\sqrt{\lambda}}\delta^d_n L_n \pm \frac{\sqrt{\lambda}}{2}(\delta^l_b\delta^d_c - \delta^d_b\delta^l_c)[L_l[L_b, L_c]] \\
&\mp \frac{4}{\sqrt{\lambda}}L_d \mp \sqrt{\lambda}[R_b, [R_d, R_b]]\,.
\end{aligned}$$

And with the former, we reobtain the static field equations as

$$\nabla_d \left(L_d + \frac{\lambda}{4}[L_b, G_{db}] \right) = 0\,. \tag{A4.5}$$

Appendix B

First Steps to a new Work Related to Hadronic Tubes

Here we will study the Pionic excitations on the multi-Baryonic configurations. These perturbations can be divided into two types. The first type (which is discussed in the present thesis in chapter 4) is denoted as “CFT perturbations": these are perturbations of the Hadronic tubes in which the dependence of $\Pi_j(x^\mu)$ is chosen in such a way to disclose some features of the Baryonic tubes closely related to CFT in $1+1$ dimension.

The second type will be denoted as “generic perturbations": these are perturbations (encoded in three scalar degrees of freedom $\Pi_j(x^\mu)$ with $j = 1, 2, 3$) of the Hadronic tubes in which the dependence of $\Pi_j(x^\mu)$ on the coordinates x^μ is, *a priori*, generic (in other words, no symmetry assumption on the Π_j will be made). This last type we will study in this chapter.

B1 Pionic excitations on the Baryonic tubes

The idea is quite natural: let us consider the following generic perturbations of the solutions defined above

$$\alpha(r) \to \alpha(r) + \varepsilon\Pi_1(x^\mu) \ , \quad \Theta \to Q\theta + \varepsilon\Pi_2(x^\mu) \ , \quad \Phi \to G(u) + \varepsilon\Pi_3(x^\mu) \ , \qquad |\varepsilon| \ll 1 \ . \tag{B1.1}$$

The three degrees of freedom $\Pi_j(x^\mu)$ represent excitations of the topologically non-trivial configurations. To declare that these are “Pionic excitations" we need to require that the variation of the topological charge δB to first order in ε vanishes:

$$\delta B = 0 \ .$$

The above condition defines the boundary conditions for the Pionic excitations. The simplest way to implement the condition here above is to require that either the $\Pi_J(x^\mu)$ are periodic in the spatial directions (this will be our choice in the following sections).

If we replace the expressions in Eq. (B1.1) in the Skyrme action in Eq. (2.6.3) at zeroth order in ε we obtain, of course, the on-shell action evaluated on the multi-Baryonic configurations. At first order in ε we get zero (modulo boundary terms) since the configurations in Eqs. (2.6.1), (2.6.2) and (4.3.3) are solutions of the field equations. At second order in ε we obtain the following effective action $I_{\overrightarrow{\Pi}}$,

$$I_{\overrightarrow{\Pi}} \overset{def}{=} I(U+\varepsilon\delta U)|_{\varepsilon^2} = \frac{K}{4}\int d^4v \left\{ \hat{O}^{(1)}_{\mu\nu}(\partial^\mu\Pi_i)(\partial^\nu\Pi_j)C^{ij}_{(1)} + \hat{O}^{(2)}_{\mu}(\partial^\mu\Pi_i)\Pi_j C^{ij}_{(2)} + \hat{O}^{(3)}_{ij}\Pi^i\Pi^j \right\} ,$$

where the expressions for the operators $\hat{O}^{(1)}_{\mu\nu}$,$C^{ij}_{(1)}$, $\hat{O}^{(2)}_{\mu}$, $C^{ij}_{(2)}$ and $\hat{O}^{(3)}_{ij}$ can be found in the following section.

One can see that the action for the Pionic excitations can be written as the sum of two quadratic parts. The first one is a time-independent action in which the soliton profile plays the role of an effective static potential. The second one (which can be considered as a small perturbation of the first one as long as the gradient of G is small) is a time-dependent perturbation that can be turned on and off adiabatically.

Hence, we are in the position to apply directly the Kubo formula for the electric conductivity [33] [34] [35]. The above explains why we needed the generalizations of the solutions in [63], and [64] introduced in [72]. In [72] G is an arbitrary function of the null coordinate u which can be chosen in such a way as to have a small gradient. Therefore, in this case, the Kubo theory can be applied directly.

The field equation for the Pionic excitations is a set of three coupled partial differential equations in a non-trivial background that are very difficult to deal with analytically. For this reason, in what follows we will restrict to the NLSM case, that is $\lambda \to 0$, where it is possible to derive analytic expressions for the frequencies of the excitations, as we will see immediately.

First, it is important to note that perturbation of the form (B1.1) where only Fourier modes in time are considered, leads only to trivial solutions for the perturbations. Therefore, it is not consistent to analyze perturbations Π_j which only depend on the time since the linearized field equations would imply that such perturbations are actually trivial: hence here and in the following sections the Pionic excitations will depend on, at least, one spatial coordinate.

From the above, to find non-trivial solutions of the equations system we use the following Fourier expansions for the Pionic fields

$$\Pi_1(t,r,\theta,\phi) = u_1(r,\theta)\exp i\,(\omega t + k\phi) \ , \tag{B1.2}$$

$$\Pi_2(t,r,\theta,\phi) = u_2(r,\theta)\exp i\,(\omega t + k\phi) \ , \tag{B1.3}$$

$$\Pi_3(t,r,\theta,\phi) = u_3(r,\theta)\exp i\,(\omega t + k\phi) \ . \tag{B1.4}$$

In this sector, the field equations for the Pionic excitations become

$$0=\left(-\frac{k_2^2}{L_\phi^2}+\omega^2-\frac{Q^2\cos\left(2\alpha(r)\right)}{L_\theta^2}\right)u_1(r,\theta)-\frac{Q\sin\left(2\alpha(r)\right)}{L_\theta^2}\partial_\theta u_2(r,\theta)+\frac{1}{L_\theta^2}\partial_\theta^2 u_1(r,\theta)+\frac{1}{L_r^2}\partial_r^2 u_1(r,\theta)\ , \tag{B1.5}$$

$$0=\left(-\frac{k_2^2}{L_\phi^2}+\omega^2\right)u_2(r,\theta)+\frac{1}{L_\theta^2}\partial_\theta^2 u_2(r,\theta)+2\cot\left(\alpha(r)\right)\left(\frac{Q}{L_\theta^2}\partial_\theta u_1(r,\theta)+\frac{\alpha'(r)}{L_r^2}\partial_r u_2(r,\theta)\right)+\frac{1}{L_r^2}\partial_r^2 u_2(r,\theta)\ , \tag{B1.6}$$

$$0=\left(-\frac{k_2^2}{L_\phi^2}+\omega^2\right)u_3(r,\theta)+\frac{2Q\cot\left(Q\theta\right)}{L_\theta^2}\partial_\theta u_3(r,\theta)+\frac{1}{L_\theta^2}\partial_\theta^2 u_3(r,\theta)+\frac{2\cot\left(\alpha(r)\right)\alpha'(r)}{L_r^2}\partial_r u_3(r,\theta)+\frac{1}{L_r^2}\partial_r^2 u_3(r,\theta)\ . \tag{B1.7}$$

Note that Eq. (B1.7) is decoupled equation, the one that can be lead to a Schrödinger-like equation of the form

$$-\Delta\bar{u}+V\bar{u}=\omega^2\bar{u}\ , \tag{B1.8}$$

where the new function $\bar{u}$ is

$$\bar{u}=-\sin(\alpha)\sin(q\theta)u_3\ , \tag{B1.9}$$

and where we have defined

$$\Delta\bar{u}=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\bar{u})\ ,\qquad g_{ij}=\mathrm{diag}(L_r^2,L_\theta^2)\ .$$

The potential in Eq. (B1.8) takes the form

$$V=-\frac{E_0}{L_r^2}+\frac{k^2}{L_\phi^2}-\frac{Q^2}{2L_\theta^2}+\frac{Q^2\cos(2\alpha)}{L_\theta^2}\ , \tag{B1.10}$$

where we have used the expressions for the derivatives of α in Eqs. (4.3) and (4.3.7). A sufficient condition ensuring linear stability under the Pionic excitations is the requirement $V>0$, which leads to the following constraint between the constants

$$\frac{k^2}{L_\phi^2}>\frac{3Q^2}{2L_\theta^2}+\frac{E_0}{L_r^2}\ . \tag{B1.11}$$

As per the previous ansatz, With the above condition, we notice a limit to the frequencies. This fact is notable because, with this, we will only focus on the lowest energy perturbation, i.e., CFT-type.

B2 Details of pionic perturbations and their field equations

The action I at second order in ϵ under the Pionic excitations (B1.1)

$$\begin{aligned}I = \frac{K}{4}\int d^4v\Big\{&(\partial\Pi_1)^2 + \sin^2(\alpha)(\partial\Pi_1\cdot\partial\Pi_2) + \sin^2(\alpha)\sin^2\Theta(\partial\Pi_3)^2 + \\ &+ 2\Pi_1\sin(2\alpha)(\nabla\Theta\cdot\partial\Pi_2) + \Pi_1^2\cos(2\alpha)(\nabla\Theta)^2 + \\ &+ \frac{\lambda}{2}\Big(\sin^2(\alpha)\big((\nabla\alpha)^2(\partial\Pi_2)^2 + 4(\nabla\alpha\cdot\partial\Pi_1)(\nabla\Theta\cdot\partial\Pi_2) + (\nabla\Theta)^2(\partial\Pi_1)^2 - \\ &- 2(\nabla\alpha\cdot\partial\Pi_2)(\nabla\Theta\cdot\partial\Pi_1) - (\nabla\alpha\cdot\partial\Pi_2)^2 - (\nabla\Theta\cdot\partial\Pi_1)^2\big) + \\ &+ \Pi_1\sin(2\alpha)\big(2(\nabla\alpha)^2(\nabla\Theta\cdot\partial\Pi_2) + 2(\nabla\alpha\cdot\partial\Pi_1)(\nabla\Theta)^2\big) + \\ &+ \Pi_1^2\cos(2\alpha)(\nabla\alpha)^2(\nabla\Theta)^2 + \\ &+ \sin^2(\alpha)\sin^2\Theta\big((\nabla\alpha)^2(\partial\Pi_3)^2 - (\nabla\alpha\cdot\partial\Pi_3)^2\big) + \\ &+ \sin^4\alpha\sin^2\Theta\big((\nabla\Theta)^2(\partial\Pi_3)^2 - (\nabla\Theta\cdot\partial\Pi_3)^2\big)\Big)\Big\}. \end{aligned} \tag{B2.1}$$

If we use (B1), we have the non-linear sigma term

$$\hat{O}^{(1)}_{\mu\nu}C^{ij}_{(1)} = \delta_{\mu\nu}\delta_{i1}\delta_{j1} + \sin^2(\alpha)\delta_{\mu\nu}\delta_{i1}\delta_{j2} + \sin^2(\alpha)\sin^2(\Theta)\delta_{\mu\nu}\delta_{i3}\delta_{j3}, \tag{B2.2}$$

$$\hat{O}^{(2)}_{\mu}C^{ij}_{(2)} = 2\delta_{j1}\sin(2\alpha)\left(\nabla_\mu\Theta\right)\delta_{i2}, \tag{B2.3}$$

$$\hat{O}^{(3)}_{ij} = \delta_{i1}\delta_{j1}\cos(2\alpha)\left(\nabla\Theta\right)^2. \tag{B2.4}$$

In the same way, for the Skyrme term

$$\begin{aligned}\hat{O}^{(1)}_{\mu\nu}C^{ij}_{(1)} =& \sin^2(\alpha)\left(\nabla\alpha\right)^2\delta_{\mu\nu}\delta_{j2}\delta_{j2} + 4\sin^2(\alpha)\left(\nabla_\mu\alpha\right)\left(\nabla_\nu\Theta\right)\delta_{i1}\delta_{j2} + \sin^2\alpha\left(\nabla\Theta\right)^2\delta_{\mu\nu}\delta_{i1}\delta_{j1} - \\ &- 2\sin^2(\alpha)\left(\nabla_\mu\alpha\right)\left(\nabla_\nu\Theta\right)\delta_{i2}\delta_{j1} - \sin^2(\alpha)\left(\nabla_\mu\alpha\right)\left(\nabla_\nu\alpha\right)\delta_{i2}\delta_{j2} - \sin^2(\alpha)\left(\nabla_\mu\Theta\right)\left(\nabla_\nu\Theta\right)\delta_{i1}\delta_{j1} + \\ &+ \sin^2(\alpha)\sin^2(\Theta)\left(\nabla\alpha\right)^2\delta_{\mu\nu}\delta_{i1}\delta_{j1} - \sin^2\alpha\sin^2(\Theta)\left(\nabla_\mu\alpha\right)\left(\nabla_\nu\alpha\right)\delta_{i3}\delta_{j3} + \\ &+ \sin^4\alpha\sin^2(\Theta)\left(\nabla\Theta\right)^2\delta_{\mu\nu}\delta_{i3}\delta_{j3} - \sin^4\alpha\sin^2(\Theta)\left(\nabla_\mu\Theta\right)\left(\nabla_\nu\Theta\right)\delta_{i3}\delta_{j3},\end{aligned} \tag{B2.5}$$

$$\hat{O}^{(2)}_{\mu}C^{ij}_{(2)} = 2\delta_{j1}\sin(2\alpha)\left(\nabla\alpha\right)^2\left(\nabla_\mu\Theta\right)\delta_{i2} + 2\delta_{j1}\sin(2\alpha)\left(\nabla\Theta\right)^2\left(\nabla_\mu\alpha\right)\delta_{i1}, \tag{B2.6}$$

$$\hat{O}^{(3)}_{ij} = \delta_{i1}\delta_{j1}\cos(2\alpha)\left(\nabla\alpha\right)^2\left(\nabla\Theta\right)^2. \tag{B2.7}$$

Now the field equations for the three pionic fields are

$$\begin{aligned}0 =& \Box\Pi_1 - \sin(2\alpha)(\nabla_\mu\Theta\partial^\mu\Pi_2) - \Pi_1\cos(2\alpha)(\nabla\Theta)^2 + \frac{\lambda}{2}\Big(2\nabla_\mu\big(\sin^2(\alpha)(\nabla\Theta\cdot\partial\Pi_2)\nabla^\mu\alpha\big) + \\ &+ \nabla_\mu\big(\sin^2(\alpha)(\nabla\Theta)^2\partial^\mu\Pi_1\big) - \nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha\cdot\partial\Pi_2)\nabla^\mu\Theta\big) - \nabla_\mu\big(\sin^2\alpha(\nabla\Theta\cdot\partial\Pi_1)\nabla^\mu\Theta\big) - \\ &- \sin(2\alpha)\big((\nabla\alpha)^2(\nabla\Theta\cdot\partial\Pi_2) + (\nabla\alpha\cdot\partial\Pi_1)(\nabla\Theta)^2\big) + \nabla_\mu\big(\Pi_1\sin(2\alpha)(\nabla\Theta)^2\nabla^\mu\alpha\big) - \\ &- \Pi_1\cos(2\alpha)(\nabla\alpha)^2(\nabla\Theta)^2\Big),\end{aligned} \tag{B2.8}$$

$$0 = \Box\Pi_2 \sin^2(\alpha) + \partial_\mu(\sin^2(\alpha))\partial^\mu\Pi_2 + \partial_\mu(\Pi_1 \sin(2\alpha)\nabla^\mu\Theta) + \frac{\lambda}{2}\Big(\nabla_\mu\big(\sin^2\alpha(\nabla\alpha)^2\partial^\mu\Pi_2\big) + \\ + 2\nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha\cdot\partial\Pi_1)\nabla^\mu\Theta\big) - \nabla_\mu\big(\sin^2(\alpha)(\nabla\Theta\cdot\partial\Pi_1)\nabla^\mu\alpha\big) - \nabla_\mu\big(\sin^2(\alpha)(\nabla\alpha\cdot\partial\Pi_2)\nabla^\mu\alpha\big) + \\ + \nabla_\mu\big(\Pi_1\sin(2\alpha)(\nabla\alpha)^2\nabla^\mu\Theta\big)\Big), \tag{B2.9}$$

$$0 = \Box\Pi_3 \sin^2(\alpha)\sin^2(\Theta) + \partial_\mu(\sin^2(\alpha)\sin^2\Theta)\partial^\mu\Pi_3 + \frac{\lambda}{2}\Bigg(\nabla_\mu\big(\sin^2\alpha\sin^2\Theta(\nabla\alpha)^2\partial^\mu\Pi_3\big) - \\ - \nabla_\mu\big(\sin^2\alpha\sin^2(\Theta)(\nabla\alpha\cdot\partial\Pi_3)\nabla^\mu\alpha\big) + \nabla_\mu\big(\sin^4(\alpha)\sin^2(\Theta)(\nabla\Theta)^2\partial^\mu\Pi_3\big) - \\ - \nabla_\mu\big(\sin^4\alpha\sin^2\Theta(\nabla\Theta\cdot\partial\Pi_3)\nabla^\mu\Theta\big)\Bigg). \tag{B2.10}$$

The field equations for Π_1

$$0 = -\frac{q^2\cos(2\alpha(r))\Pi_1}{L_\theta^2} + \frac{\partial_\phi^2\Pi_1}{L_\phi^2} - \frac{q\sin(2\alpha(r))\partial_\theta\Pi_2}{L_\theta^2} + \frac{\partial_\theta^2\Pi_1}{L_\theta^2} + \frac{\partial_r^2\Pi_1}{L_r} - \partial_t^2\Pi_1 + \\ + \frac{1}{2L_r^2L_\theta^2L_\phi^2}q\lambda\Big(L_\phi^2 q\Pi_1(\cos(2\alpha(r))\alpha'(r)^2 + \sin(2\alpha(r))\alpha''(r)) + \sin(2\alpha(r))\big(L_r^2 q\sin(\alpha(r))\partial_\phi^2\Pi_1 + \\ + L_\phi^2\big(2\cos(\alpha(r))\alpha'(r)^2\partial_\theta\Pi_2 + \alpha'(r)(2q\cos(\alpha(r))\partial_r\Pi_1 + \sin(\alpha(r))\partial_r\partial_\theta\Pi_2) + \\ + \sin(2\alpha(r))(2\alpha''(r)\partial_\theta\Pi_2 + q\partial_r^2\Pi_1 - L_r^2 q\partial_t^2\Pi_1)))\Big), \tag{B2.11}$$

the field equation for Π_2

$$0 = \sin(\alpha(r))\Bigg(\frac{\sin(\alpha(r))\partial_\phi^2\Pi_2}{L_\phi^2} + \frac{2q\cos(\alpha(r))\partial_\theta\Pi_1}{L_\theta^2} + \frac{\sin(\alpha(r)\partial_\theta^2\Pi_2)}{L_\theta^2} + \frac{2\cos(\alpha(r))\alpha'(r)\partial_r\Pi_2}{L_r^2} + \\ + \frac{\sin(\alpha(r))\partial_r^2\Pi_2}{L_r^2} - \sin(\alpha(r))\partial_t^2\Pi_2\Bigg) + \frac{\lambda\sin^2(\alpha(r))}{2L_r^2L_\theta^2L_\phi^2}\Bigg(-L_\phi^2\alpha(r)''\partial_\theta\Pi_1 + L_\phi^2 q\alpha'(r)\partial_r\partial_\theta\Pi_1 + \\ + \alpha'(r)^2\big(L_\theta^2\partial_\phi^2\Pi_2 + L_\phi^2(\partial_\theta^2 - L_\theta^2\partial_t^2\Pi_2)\big)\Bigg) \tag{B2.12}$$

and the field equation for Π_3 are

$$0 = \frac{\lambda\sin(q\theta)\sin^2(\alpha(r))}{2L_r^2L_\theta^2L_\phi^2}\Bigg(\sin(q\theta)(L_r^2q^2\sin^2(\alpha(r)) + L_\theta^2\alpha'(r)^2)\partial_\phi^2\Pi_3 + L_\phi^2\Big(2q^2\sin(q\theta)\sin(2\alpha(r))\alpha'(r)\partial_r\Pi_3 + \\ + q^2\sin(q\theta)\sin^2(\alpha(r))(\partial_r^2\Pi_3 - L_r^2\partial_t^2\Pi_3) + \alpha'(r)^2[2q\cos(q\theta)\partial_\theta + \sin(q\theta)(\partial_\theta^2\Pi_3 - L_\theta^2\partial_t^2\Pi_3)]\Big)\Bigg) + \\ + \frac{\sin(q\theta)\sin(\alpha(r))}{L_r^2L_\theta^2L_\phi^2}\Bigg(L_r^2L_\theta^2\sin(q\theta)\sin(\alpha(r))\partial_\phi^2\Pi_3 + L_\phi^2\Big(2L_r^2q\cos(q\theta)\sin(\alpha(r))\partial_\theta\Pi_3 + \\ + \sin(q\theta)[L_r^2\sin(\alpha(r))\partial_\theta^2\Pi_3 + L_\theta^2(2\cos(\alpha(r)))\alpha'\partial_r\Pi_3 + \sin(\alpha(r))(\partial_r^2\Pi_3 - L_r^2\partial_t^2\Pi_3)]\Big)\Bigg). \tag{B2.13}$$

To solve the system of equations, we use the following Fourier expression for the pionic fields

$$\Pi_1(t,r,\theta,\phi) = u_1(r)\exp(i(\theta k_1 + k_2\phi + \omega t)), \tag{B2.14}$$
$$\Pi_2(t,r,\theta,\phi) = u_2(r)\exp(i(\theta k_1 + k_2\phi + \omega t)), \tag{B2.15}$$
$$\Pi_3(t,r,\theta,\phi) = u_3(r)\exp(i(\theta k_1 + k_2\phi + \omega t)). \tag{B2.16}$$

The field equation for Π_1 is

$$\begin{aligned}0 = -L_r^2\Big\{&u_1(r)\left(\lambda H'^2 + L_r^2\right)\big(L_\phi^2(k_1 + L_\theta\omega)(k_1 - L_\theta\omega) + k_2^2L_\theta^2\big)+\\ &+ u_3(r)\cos(2H(r))(\lambda H'^2 + L_r^2)\big(L_\phi^2(k_1 + L_\theta\omega)(k_1 - L_\theta\omega) + k_2^2L_\theta^2\big) + L_\phi^2\big(i\lambda k_1 q u_2(r)H''(r)+\\ &+ H'(r)\big(2L_\theta^2\sin(2H(r))u_3'(r) - i\lambda k_1 q u_2'(r)\big) - \big\{L_\theta^2\left(\cos(2H(r))u_3''(r) + u_1''(r)\right)\big\}\big)\Big\},\end{aligned} \tag{B2.17}$$

then the field equation for Π_2 is

$$\begin{aligned}0 = -L_\theta^2\Big\{&- L_r^2u_2(r)\big(k_1^2L_\phi^2 + (k_2 + L_\phi\omega)(k_2 - L_\phi\omega)\big(L_\theta^2 + \lambda q^2\big)\big) + L_\phi^2\big((L_\theta^2 + \lambda q^2)u_2''(r)+\\ &+ ik_1q\big(2\lambda u_1(r)H''(r) + \lambda H'(r)\left(\cos(2H(r))u_3'(r) + u_1'(r)\right) + 2u_3(r)\big\{\lambda H''(r)\cos(2H(r))+\\ &+ \sin(2H(r))(L_r^2 - \lambda H'^2)\big\}\big)\big)\Big\},\end{aligned} \tag{B2.18}$$

and the field equation for Π_3 is

$$\begin{aligned}0 = -L_\theta^2\Big\{&- 2\lambda k_1^2L_r^2L_\phi^2u_3(r)H'^2 + 4L_r^2L_\phi^2\sin(2H(r))\big(L_\theta^2H'(r)u_1'(r) + ik_1L_r^2qu_2(r)\big)-\\ &- 2\lambda k_2^2L_\theta^2L_r^2u_3(r)H'^2 + L_\phi^2u_3''(r)\big(4\lambda L_\theta^2H'^2 + L_r^2\big(\lambda q^2\cos(4H(r)) - 2L_\theta^2 - \lambda q^2\big)\big)+\\ &+ 2\lambda L_\theta^2L_r^2L_\phi^2\omega^2u_3(r)H'^2 - 4\lambda L_r^2L_\phi^2q^2H'(r)\sin(4H(r))u_3'(r)+\\ &+ 2L_r^2\cos(2H(r))\Big(u_1(r)\big(\lambda H'^2 + L_r^2\big)\big(L_\phi^2(k_1 + L_\theta\omega)(k_1 - L_\theta\omega) + k_2^2L_\theta^2\big)+\\ &+ L_\phi^2\big\{- L_\theta^2u_1''(r) + i\lambda k_1q\big(u_2(r)H''(r) - H'(r)u_2'(r)\big)\big\}\Big) + 8\lambda L_\theta^2L_\phi^2H'(r)H''(r)u_3'(r)-\\ &- \lambda k_2^2L_r^4q^2u_3(r)\cos(4H(r)) + \lambda L_r^4L_\phi^2q^2\omega^2u_3(r)\cos(4H(r)) + 2k_1^2L_r^4L_\phi^2u_3(r) + 2k_2^2L_\theta^2L_r^4u_3(r)+\\ &+ \lambda k_2^2L_r^4q^2u_3(r) - 2L_\theta^2L_r^4L_\phi^2\omega^2u_3(r) - \lambda L_r^4L_\phi^2q^2\omega^2u_3(r)\Big\}.\end{aligned} \tag{B2.19}$$

Now in the case of $k_1 = 0$ we have the following equations.

The field equation for Π_1 is

$$\begin{aligned}0 =&\frac{1}{2L_\phi^2(L_\theta^2 + q^2\lambda\sin^2(\alpha(r))}\Big(u_1(r,\theta)\big(L_r^2(-2L_\phi^2q^2\cos(2\alpha(r)) - (k_2 - L_\phi\omega)(k_2 + L_\phi\omega)(2L_\theta^2+\\ &+ q^2\lambda\sin^2(\alpha(r)))) + \sin(2\alpha(r))\alpha''(r)\big) + 2L_\phi^2q\sin(\alpha(r))(-2L_r^2\cos(\alpha(r)) + \lambda\cos(\alpha(r))\alpha'(r)^2+\\ &+ \lambda\sin(\alpha(r))\alpha''(r))\partial_\theta u_1(r,\theta) + 2L_r^2L_\phi^2\partial_\theta^2u_1(r,\theta) + L_\phi^2q\lambda\sin(\alpha(r))\alpha'(r)(2q\cos(\alpha(r))\partial_r u_1(r,\theta)+\\ &+ \sin(\alpha(r))\partial_r\partial_\theta u_2(r,\theta))\Big) + \frac{2L_\theta^2 + q^2\lambda\sin^2(\alpha(r))\partial_r^2u_1(r,\theta)}{2L_\theta^2 + 2q^2\lambda\sin^2(\alpha)},\end{aligned} \tag{B2.20}$$

the field equation for Π_2 is

$$0 = \frac{1}{2L_\theta^2 L_\phi^2}\Big(- L_\theta^2(k_2^2 - L_\phi^2\omega^2)(2L_r^2 + \lambda\alpha'(r)^2)u_2(r,\theta) + L_\phi^2\big((\partial_\theta^2 u_1(r,\theta)(4qL_r^2\cot(\alpha(r)) - \lambda q\alpha''(r))+ + (2L_r^2 + \lambda\alpha'(r)^2)\partial_\theta^2 u_2(r,\theta)) + \alpha'(r)(4L_\theta^2\cot(\alpha(r))\partial_r u_2(r,\theta) + q\lambda\partial_r\partial_\theta u_1(r,\theta))\big)\Big) + \partial_r^2 u_2(r,\theta), \tag{B2.21}$$

the field equation for Π_3

$$0 = \frac{2L_r^2 + \lambda\alpha'(r)^2}{2L_\theta^2 + 2q^2\lambda\sin^2(\alpha(r))}\partial_\theta^2 u_3(r,\theta) + \frac{\csc(\alpha(r))}{4L_\phi^2(q^2\lambda + L_\theta^2\csc^2(\alpha(r)))}\Big((k_2^2 - L_\phi^2\omega^2)\csc(\alpha(r))u_3(r,\theta)(-4L_r^2L_\theta^2- - q^2L_r^2\lambda + q^2L_r^2\lambda\cos(2\alpha(r)))) + 4L_\phi^2(q\cot(q\theta))\csc(\alpha(r))(2L_r^2 + \lambda\alpha'(r)^2)\partial_\theta u_3 + 2(q^2\lambda\cos(\alpha(r))+ + L_\theta^2\cot(\alpha(r))\csc(\alpha(r)))\Big) + \frac{2L_\theta^2 + q^2\lambda\sin^2(\alpha(r))}{2L_\theta^2 + 2q^2\lambda\sin^2(\alpha(r))}\partial_r^2 u_3(r,\theta). \tag{B2.22}$$

Appendix C

First Steps to a New Work Related to Hadronic Layers

C1 A Future Work: Pionic excitations on the Baryonic layers

The Pionic excitations of the Hadronic layers can be divided into two types: the first type is “CFT perturbations". In contrast, the second type corresponds to “generic perturbations". In the present section, we will discuss the second type in the same form discussed in the previous chapter.

C2 Pionic excitations on the Hadronic layers

Let us consider the following generic perturbations of the solutions presented above

$$H(r) \rightarrow \kappa r + \varepsilon \Pi_2 (x^\mu) \ , \quad G \rightarrow G(u) + \varepsilon \Pi_1 (x^\mu) \ , \quad F \rightarrow q\theta + \varepsilon \Pi_3 (x^\mu) \ , \qquad |\varepsilon| \ll 1 \ . \quad \text{(C2.1)}$$

The three degrees of freedom $\Pi_J (x^\mu)$ represent Pionic excitations of the Baryonic layers configurations. Also, in this case, these excitations can be considered as “Pionic excitations" if the variation of the topological charge δB to first order in ε vanishes:

$$\delta B = 0 \ .$$

The simplest way to implement the above condition requires periodic boundary conditions for the Π_j.

If we replace the expressions in Eq. (C2.1) in the Skyrme action in Eq. (2.6.32) at zeroth order in ε we obtain the on-shell action. Once again, at first order in ε we get zero (modulo boundary terms). At second order in ε we obtain the following effective action $I_{\overrightarrow{\Pi}}$ for the Pionic excitations:

$$I_{\vec{\Pi}} = \int d^4v \Big\{ \hat{O}^{(1)}_{\mu\nu}(\partial^\mu \Pi_i)(\partial^\nu \Pi_j) C^{ij}_{(1)} + \hat{O}^{(2)}_{\mu}(\partial^\mu \Pi_i)\Pi_j C^{ij}_{(2)} + \hat{O}^{(3)}_{ij}\Pi^i\Pi^j \Big\} \ .$$

The above is the sum of two quadratic parts (see section C3 for the explicit expressions). The first is a time-dependent action in which H is an effective static potential. The second one (which can be considered as a small perturbation of the first one as long as the gradient of G is small) is a time-dependent perturbation that can be turned on and off adiabatically. Hence, we are in the position to apply the Kubo formula for the electric conductivity directly [33] [34] [35].

As in the case of the Baryonic tubes, here we will focus in the case $\lambda \to 0$, where the resulting equations can be interpreted directly.

To solve the equations system, we use the following Fourier expansions for the Pionic fields

$$\Pi_1(t, r, \theta, \phi) = u_1(r) \exp(i(k\phi + \omega t)) \ , \tag{C2.2}$$

$$\Pi_2(t, r, \theta, \phi) = u_2(r) \exp(i(+k\phi + \omega t)) \ , \tag{C2.3}$$

$$\Pi_3(t, r, \theta, \phi) = u_3(r) \exp(i(k\phi + \omega t)) \ . \tag{C2.4}$$

In this sector, the field equations for the Pionic excitations (the general equations are in section C3) become

$$0 = -L_\theta^2 L_r^2 (k_2^2 - L_\phi^2\omega^2)\left(u_1(r) + u_3(r)\cos\left(\frac{r}{2}\right)\right) + L_\theta^2 L_\phi^2 u_1''(r) + L_\theta^2 L_\phi^2 \cos\left(\frac{r}{2}\right) u_3''(r) - \frac{1}{2} L_\theta^2 L_\phi^2 \sin\left(\frac{r}{2}\right) u_3'(r) \ ,$$

$$0 = -L_\theta^2 L_r^2 (k_2^2 - L_\phi^2\omega^2)\left(u_3(r) + u_1(r)\cos\left(\frac{r}{2}\right)\right) + L_\theta^2 L_\phi^2 u_3''(r) + L_\theta^2 L_\phi^2 \cos\left(\frac{r}{2}\right) u_1''(r) - \frac{1}{2} L_\theta^2 L_\phi^2 \sin\left(\frac{r}{2}\right) u_1'(r) \ .$$

The solution of this system is in section C3. The quantization of k is related to the periodic boundary conditions in *phi*. The frequency ω turns out to be (note that at least one of the integers k and/or $\hat{n}$ must be non-vanishing as explained in the section C3.

$$\omega = \frac{\sqrt{16k_2^2 L_r^2 + L_\phi^2(\hat{n} - 1)}}{4L_r L_\phi} \ , \tag{C2.5}$$

with $\hat{n}$ an integer number that cannot be vanished. The regularity of these solutions and the constant value are explained in section C3.

C3 Details of Pionic Excitations and their Field Equations

Furthermore, the action under the Pionic excitations (C2.1) is

$$I = -\frac{K}{2}\int d^4v\Big\{(\partial\Pi_1)^2 + (\partial\Pi_2)^2 + (\partial\Pi_3)^2 + 2\cos(2H)(\partial\Pi_1\cdot\partial\Pi_3) - 4\Pi_2\sin(2H)(\nabla F\cdot\partial\Pi_3) -$$
$$-\lambda\big(2\cos(2H)\Big((\nabla H\cdot\partial\Pi_1)(\nabla H\cdot\partial\Pi_3) + (\nabla F\cdot\partial\Pi_2)(\nabla H\cdot\partial\Pi_3) - (\nabla H)^2(\partial\Pi_1\cdot\partial\Pi_3) - 2(\nabla H\cdot\partial\Pi_2)(\nabla F\cdot\partial\Pi_3) +$$
$$+ 4\Pi_2\sin(2H)\Big(\kappa^2(\nabla F\cdot\partial\Pi_3)\Big) + 4\sin^2(H)\cos^2(H)\Big((\nabla F\cdot\partial\Pi_3)^2 - \frac{1}{4}(\nabla F)^2(\partial\Pi_1)^2\Big) -$$
$$-\Pi_2\sin(4H)(\nabla F)^2(\nabla F\cdot\partial\Pi_1) - (\nabla F)^2(\partial\Pi_1)^2 + (\nabla F)^2(\partial\Pi_3)^2 + \frac{1}{4}(\nabla F)^2(\partial\Pi_2)^2 + 2(\nabla H\cdot\partial\Pi_1)(\nabla F\cdot\partial\Pi_2) +$$
$$+ (\nabla H\cdot\partial\Pi_1)^2 + (\nabla H\cdot\partial\Pi_3)^2 + (\nabla F\cdot\partial\Pi_2)^2 - 4(\nabla H\cdot\partial\Pi_2)(\nabla F\cdot\partial\Pi_1)\big)\Big\}. \tag{C3.1}$$

If we use (C2) in the non-linear sigma model term we obtain

$$\hat{O}^{(1)}_{\mu\nu}C^{ij}_{(1)} = \delta_{\mu\nu}\delta_{i1}\delta_{j1} + \delta_{\mu\nu}\delta_{i2}\delta_{j2} + \delta_{\mu\nu}\delta_{i3}\delta_{j3} + 2\cos(2H)\delta_{\mu\nu}\delta_{i1}\delta_{j3}, \tag{C3.2}$$
$$\hat{O}^{(2)}_{\mu}C^{ij}_{(2)} = -4\sin(2H)\Big(\delta_{2j}\delta_{\mu\nu}(\nabla_\nu F)\delta_{i3} + \delta_{2j}\delta_{\mu\nu}(\nabla_\nu G)\delta_{i1}\Big), \tag{C3.3}$$
$$\hat{O}^{(3)}_{ij} = 0. \tag{C3.4}$$

In the same way, for the Skyrme term, we obtain

$$\hat{O}^{(1)}_{\mu\nu}C^{ij}_{(1)} = 4\sin^2(H)\cos^2(H)\Big(2(\nabla_\mu G)(\nabla_\nu F)\delta_{i1}\delta_{j3} + (\nabla_\mu G)(\nabla_\nu G)\delta_{i1}\delta_{j1} +$$
$$+ (\nabla_\mu F)(\nabla_\nu F)\delta_{i3}\delta_{j3} - \frac{q^2}{4}\delta_{\mu\nu}\delta_{i1}\delta_{j1} - 4(\nabla_\mu G)(\nabla_\nu F)\delta_{i3}\delta_{j1}\Big) +$$
$$+ 2\cos(2H)\Big((\nabla_\mu H)(\nabla_\nu H)\delta_{i1}\delta_{j3} + (\nabla_\mu H)(\nabla_\nu G)\delta_{i1}\delta_{j2} + (\nabla_\mu F)(\nabla_\nu H)\delta_{i2}\delta_{j3} +$$
$$+ (\nabla_\mu F)(\nabla_\nu G)\delta_{i2}\delta_{j2} - \kappa^2\delta_{\mu\nu}\delta_{i1}\delta_{j3} - 2(\nabla_\mu H)(\nabla_\nu F)\delta_{i2}\delta_{j3} -$$
$$- 2(\nabla_\mu H)(\nabla_\nu G)\delta_{i2}\delta_{j1}\Big) + 2(\nabla_\mu H)(\nabla_\nu F)\delta_{i1}\delta_{j2} +$$
$$+ (\nabla_\mu H)(\nabla_\nu H)\delta_{i1}\delta_{j1} + (\nabla_\mu F)(\nabla_\nu F)\delta_{i2}\delta_{j2} + 2(\nabla_\mu H)(\nabla_\nu G)\delta_{i3}\delta_{j2} +$$
$$+ (\nabla_\mu H)(\nabla_\nu H)\delta_{i3}\delta_{j3} + (\nabla_\mu G)(\nabla_\nu G)\delta_{i2}\delta_{j2} - \kappa^2\delta_{\mu\nu}\delta_{i1}\delta_{j1} -$$
$$- 4(\nabla_\mu H)(\nabla_\nu F)\delta_{i2}\delta_{j1} + \kappa^2\delta_{\mu\nu}\delta_{i3}\delta_{j3} + \frac{q^2}{4}\delta_{\mu\nu}\delta_{i2}\delta_{j2} + 4(\nabla_\mu H)(\nabla_\nu G)\delta_{i2}\delta_{j3}, \tag{C3.5}$$

$$\hat{O}^{(2)}_{\mu}C^{ij}_{(2)} = 4\delta_{2j}\sin(2H)\Big(\kappa^2\delta_{\mu\nu}\delta_{i3}(\nabla_\nu F) + \kappa^2\delta_{\mu\nu}\delta_{i1}(\nabla_\nu G)\Big) +$$
$$+ 2\delta_{2j}\sin(4H)\Big(-2\frac{q^2}{4}\delta_{\mu\nu}\delta_{i1}(\nabla_\nu F)\Big), \tag{C3.6}$$
$$\hat{O}^{(3)}_{ij} = 0. \tag{C3.7}$$

The field equation of the Π_1 is

$$\begin{aligned}0 =&\Big\{L_\theta^2\left(L_r^2+\lambda H'(r)\right)\partial_\phi^2\Pi_1 - 2L_\theta^2 L_\phi^2 \sin\left(2H(r)\right)H'(r)\partial_r\Pi_3 + L_\phi^2\big(-q\lambda H''\partial_\theta\Pi_2 + \\ &+(L_r^2+\lambda H'(r)^2)\partial_\theta^2\Pi_1 + q\lambda H'(r)\partial_r\partial_\theta\Pi_2 + L_\theta^2\partial_r^2\Pi_1 - L_\theta^2(L_r^2+\lambda H'(r)^2)\partial_t^2\Pi_1\big)+ \\ &+\cos(2H(r))\Big(L_\theta^2(L_r^2+\lambda H'(r)^2)\partial_\phi^2\Pi_3 + L_\phi^2\big(L_\theta^2\partial_r^2\Pi_3 + (L_r^2+\lambda H'(r)^2)(\partial_\theta^2\Pi_3 - L_\theta^2\partial_t^2\Pi_3)\big)\Big)\Big\}.\end{aligned} \tag{C3.8}$$

In the same way, the field equation for Π_2 is

$$\begin{aligned}0 =& \frac{1}{L_r^2L_\theta^2L_\phi^2}\Big\{L_r^2(L_\theta^2+q^2\lambda)\partial_\phi^2\Pi_2 + L_\phi^2\big(2q\sin\left(2H(r)\right)(L_r^2-\lambda H'(r)^2)\partial_\theta\Pi_3+ \\ &+2q\lambda H''(r)(\partial_\theta\Pi_1+\cos\left(2H(r)\right)\partial_\theta\Pi_3) + q\lambda H'(r)(\partial_r\partial_\theta\Pi_1 + \cos\left(2H(r)\right)\partial_r\partial_\theta\Pi_3)+ \\ &+(L_\theta^2+q^2\lambda)\partial_r^2\Pi_2 + L_r^2\big(\partial_\theta^2\Pi_2 - (L_\theta^2+q^2\lambda)\partial_t^2\Pi_2\big)\big)\Big\},\end{aligned} \tag{C3.9}$$

and for the Π_3

$$\begin{aligned}0 =& \frac{1}{2L_r^4L_\theta^2L_\phi^2}\Big\{2L_r^2L_\theta^2\cos 2H(r)(L_r^2+\lambda H'(r)^2)\partial_\phi^2\Pi_1 + L_r^2\big(L_r^2(2L_\theta^2+q^2\lambda-\lambda\cos\left(4H(r)\right))- \\ &-2L_\theta^2H'(r)^2\big)\partial_\phi^2\Pi_3 + L_\phi^2\Big(2L_r^4\partial_\theta^2\Pi_3 - 2L_r^2\lambda H'(r)^2\partial_\theta^2\Pi_3 - 4L_r^2\sin\left(2H(r)\right)(L_r^2q\partial_\theta\Pi_2+ \\ &+L_\theta^2H'(r)\partial_r\Pi_1) + 4L_r^2q^2\lambda\sin\left(4H(r)\right)H'(r)\partial_r\Pi_3 - 8L_\theta^2\lambda H'(r)H''(r)\partial_r\Pi_3+ \\ &+2L_r^2L_\theta^2\partial_r^2\Pi_3 + L_r^2q^2\lambda\partial_r^2\Pi_3 - 4L_\theta^2\lambda H'(r)^2\partial_r^2\Pi_3+ \\ &+2L_r^2\cos\left(2H(r)\right)\big(-q\lambda H''(r)\partial_\theta\Pi_2 + (L_r^2+\lambda H'(r)^2)\partial_\theta^2\Pi_1 + q\lambda H'(r)\partial_r\partial_\theta\Pi_2+ \\ &+L_\theta^2\partial_r^2\Pi_1 - L_\theta^2(L_r^2+\lambda H'(r)^2)\partial_t^2\Pi_1\big) - 2L_r^4L_\theta^2\partial_t^2\Pi_3 - L_r^4q^2\lambda\partial_t^2\Pi_3+ \\ &+2L_r^2L_\theta^2\lambda H'(r)^2\partial_t^2\Pi_3 + L_r^2q^2\lambda\cos\left(4H(r)\right)(-\partial_r^2\Pi_3 + L_r^2\partial_t^2\Pi_3)\Big)\Big\}.\end{aligned} \tag{C3.10}$$

To solve the system of equations, we use the following Fourier expression for the pionic fields

$$\Pi_1(t,r,\theta,\phi) = u_1(r)\exp(i(\theta k_1 + k_2\phi + \omega t)), \tag{C3.11}$$

$$\Pi_2(t,r,\theta,\phi) = u_2(r)\exp(i(\theta k_1 + k_2\phi + \omega t)), \tag{C3.12}$$

$$\Pi_3(t,r,\theta,\phi) = u_3(r)\exp(i(\theta k_1 + k_2\phi + \omega t)). \tag{C3.13}$$

Now for the case of our study, when $\lambda \to 0$, we have the previous system of equations is

$$0 = -L_\theta^2L_r^2(k_2^2 - L_\phi^2\omega^2)\left(u_1(r) + u_3(r)\cos\left(\frac{r}{2}\right)\right) + L_\theta^2L_\phi^2u_1''(r) + L_\theta^2L_\phi^2\cos\left(\frac{r}{2}\right)u_3''(r) - \frac{1}{2}L_\theta^2L_\phi^2\sin\left(\frac{r}{2}\right)u_3'(r),$$
$$0 = -L_\theta^2L_r^2(k_2^2 - L_\phi^2\omega^2)\left(u_1(r)\cos\left(\frac{r}{2}\right) + u_3(r)\right) + L_\theta^2L_\phi^2u_3''(r) + L_\theta^2L_\phi^2\cos\left(\frac{r}{2}\right)u_1''(r) - \frac{1}{2}L_\theta^2L_\phi^2\sin\left(\frac{r}{2}\right)u_1'(r).$$

For solve this, we use an auxiliary variable $\Psi(r)$ defined such as

$$\Psi(r) = u_1(r) - u_3(r). \tag{C3.14}$$

Then, if we take the difference between the equations of the previous system, we have an EDO such like

$$0 = -L_r^2 L_\theta^2 (k_2 - L_\phi \omega)(k_2 + L_\phi \omega) \left(1 - \cos\left(\frac{r}{2}\right)\right) \Psi(r) + L_\theta^2 L_\phi^2 \left(\left(1 - \cos\left(\frac{r}{2}\right)\right) \Psi'(r)\right)' . \tag{C3.15}$$

Then, the solution of $\Psi(r)$ is

$$\Psi(r) = \csc\left(\frac{r}{4}\right) \left(\exp\left(\frac{r}{4}\Lambda\right) c_1 + \exp\left(-\frac{r}{4}\Lambda\right) c_2\right), \tag{C3.16}$$

where c_1 and c_2 are, in principle, complex number and Λ is defined such as

$$\Lambda = \frac{\sqrt{16 L_r^2 (k_2^2 - L_\phi^2 \omega^2) - L_\phi^2}}{L_\phi},$$

for ω to be real, we need the following constraint for Λ in the quantization of k_2

$$\Lambda - i n_6 = 0, \tag{C3.17}$$

from the above, it is clear that n_6 cannot be null (otherwise, the perturbation vanishes). Besides, for the regularity of the function $\Psi(r)$ we choose the constant c_2 so that $\Psi(r)$ remains in terms of

$$\Psi(r) = \csc\left(\frac{r}{4}\right) \sin\left(r \frac{n_6}{4}\right) c_1. \tag{C3.18}$$

With the above, the modes $u_1(r)$ and $u_3(r)$ are defined from the auxiliary function $\Psi(r)$, being also regular.