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ANÁLISIS NUMÉRICO PARA ECUACIONES DIFERENCIALES ESTOCÁSTICAS DIRIGIDAS POR MOVIMIENTOS BROWNIANOS FRACCIONARIOS

Tesis para optar al grado de Doctor
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Resumen

Esta tesis aborda el estudio de ecuaciones diferenciales estocásticas (EDE's) dirigidas por procesos multiparamétricos autosimilares con el objetivo de sentar un aporte al cálculo estocástico respecto a este tipo de procesos y así ampliar el conjunto de aplicaciones de las EDE's y los fenómenos suceptibles de ser modelados por estas. En particular se estudiaron tres tipos de EDE's dirigidas por procesos fraccionarios, analizando diferentes características y propiedades de estas. También se define la integral de Wiener con respecto a la sábana de Hermite y se ejemplifica su uso a través de una EDE.

El movimiento Browniano fraccionario (mBf) puede considerarse en muchos sentidos como la generalización natural del movimiento Browniano standard (mBs), sin embargo, las herramientas desarrolladas para el cálculo estocástico con respecto a este último dejan de ser útiles para el mBf ya que este no es una semi-martingala ni tampoco es markoviano.
Así, la primera parte de esta tesis consiste en analizar una EDE con delay dirigida por un mBf cuyo parámetro de autosimilaridad H pertenece al intervalo $(\frac{1}{2}, 1)$. A través de un método numérico se estudia una aproximación a tiempo discreto para la solución de la ecuación, se prueba la convergencia fuerte y se establece la velocidad de la misma.

Posteriormente se avanza hacia los casos multiparamétricos. Se analizó la sábana fraccionaria de Ornstein-Uhlenbeck (sfOU), la cual es definida como la solución de una ecuación de Langevin dirigida por una sábana Browniana fraccionaria (sBf), siendo este último proceso anisotrópico y para el cual se consideró la situación en que sus parámetros de autosimilaridad α y β son mayores que $\frac{1}{2}$ (*i.e.* memoria larga). Se construyó un estimador de mínimos cuadrados para el parámetro de tendencia de la sfOU, se demostró la consistencia fuerte del estimador y que este no es asintóticamente normal, esto último en contraste con el caso uniparamétrico.

Continuando con el estudio de campos aleatorios, la tercera parte de esta tesis se dedicó al estudio de una ecuación estocástica de la onda con ruido aditivo fraccionario en el tiempo y coloreado en el espacio. Se demostraron cotas óptimas para la regularidad de la solución tanto temporal como espacial, lo que posteriormente permite establecer la regularidad conjunta en función de una métrica bien definida. Esto junto con algunos conceptos de Teoría de Potencial permitió establecer cotas superiores e inferiores para las probabilidades de arrivo de la solución.

Finalmente, la última parte de esta tesis presenta un aporte en la construcción del cálculo estocástico con respecto a los procesos de Hermite, los cuales son caracterizados por el parámetro de autosimilaridad H y el parámetro q. A diferencia de los procesos estudiados previamente, los procesos de Hermite son Gaussianos solo cuando $q = 1$, caso en que se recupera el mBf.
Se define la sábana de Hermite (sH) como una integral múltiple con respecto a la sBs y se introducen las integrales de Wiener con respecto a ésta, lo que junto con otros resultados presentados previamente en esta tesis permiten analizar a modo de ejemplo una EDE de la onda con respecto a la sH, se define su solución y se demuestra la regularidad temporal, espacial y conjunta de esta. Otros resultados adicionales también son presentados.

Abstract

This thesis deals with the study of stochastic differential equations (SDE's) driven by self-similar multi-parameter processes with the aim of setting a contribution to the stochastic calculus with respect to this type of processes and thus extend the set of applications of SDE's and phenomena susceptible of being modelled by these. Specifically, three types of SDE's driven by fractional processes were studied, analysing different characteristics and properties of these. The Wiener integral with respect to the Hermite sheet is also defined and the associated SDE's are studied.

The fractional Brownian motion (fBm) can be considered in many senses as the natural generalization of the standard Brownian motion (sBm), however, the stochastic calculus used for the sBm can not be used for the fBm, mainly because is neither a semimartingale nor a Markov process. So, the first part of this thesis it's about a SDE with delay, driven by a fBm with self-similarity parameter H in the interval $(\frac{1}{2}, 1)$. By means of a numerical method a discrete time approximation for the solution of the equation is studied, the strong convergence is proved and the rate of convergence is established.

The second part of this thesis is devoted to study multi-parameter processes. The fractional Ornstein-Uhlenbeck sheet (fOUs), which is defined as the solution of a Langevin equation with respect to the fractional Brownian sheet (fBs), is studied. The fBs is an anisotropic process and it is considered the case when the self-similarity parameters α and β are greater than $\frac{1}{2}$, that is, the long memory case. A least squares estimator for the tendency parameter of the fOUs is built, the strong consistency is proved and also that is not asymptotically normal, this last is in contrast with the one parameter case.

Continuing with the study of random fields, the third part of this thesis was devoted to the study of a stochastic wave equation with additive noise, fractional in time and colored in space. Optimal bounds for the regularity of the solution were proved, in time and space, which allows to stablish the joint regularity of the solution. This along with some concepts of Potential Theory, allowed to establish upper and lower bounds for the hitting probabilities of the solution.

Finally, the last part of this thesis present a contribution to the stochastic calculus with respect to the Hermite processes, which are characterized by the self-similarity parameter H and the parameter q which is associated with the numbers of integrals involved. In contrast with the processes studied previously, the Hermite processes are Gaussian only when $q = 1$, the case of the fractional Brownian motion.

The Hermite sheet (sH) is defined as a multiple integral with respect to the standard Brownian sheet and Wiener integrals with respect to it are presented, this along with some others results presented previously in this thesis allows to analyse as an example a stochastic wave equation driven by the Hermite sheet, his solution is defined and the temporal, spatial and joint regularity are proved. Other additional results related with local time of the solution are also presented.

Contents

Introducción

Esta tesis se divide en cuatro capítulos teniendo todos en común el movimiento Browniano fraccionario y/o el uso de integrales múltiples. Por tal razón a continuación se presenta una descripción un poco mas detallada de dichas herramientas.

0.1 MOTIVACIÓN: El movimiento Browniano fraccionario

Tanto en la naturaleza como en el quehacer actual de las sociedades humanas; ya sea tecnología, producción, economía u otros tópicos; encontramos muchos fenómenos de caracter irregular o "aleatorio". Esto ha llevado a que hoy en día el estudio de este tipo de fenómenos haya ganado un espacio importante dentro del desarrollo de distintas áreas de la ciencia y la investigación académica, solo por nombrar algunas podemos considerar; Hidrología, tratamiento de imágenes, análisis de tráfico en redes y matemáticas financieras. En tal sentido, la utilización de funciones aleatorias y procesos estocásticos ha jugado un rol crucial y lo seguirá haciendo de manera progresiva. Los procesos autosimilares participan fuertemente de este creciente y diverso campo de investigación, precisamente por que la propiedad de autosimilaridad la encontramos en variados fenómenos y aplicaciones, además, esta propiedad se encuentra estrechamente relacionada con la regularidad de los procesos (Hölderiana por ejemplo), por tales razones los procesos autosimilares son una herramienta muy útil a la hora de modelar fenómenos irregulares o "aleatorios". Es precisamente en la intersección de estas características, aleatoriedad y autosimilaridad, que encontramos al movimiento Browniano fraccionario.

En 1940 Kolmogorov define las "Espirales de Wiener" (ver [60]). Es en dicho trabajo que, desde un punto de vista casi puramente teórico, se presenta por primera vez a la luz del mundo científico el movimiento Browniano fraccionario. Kolmogorov lo definió como el único proceso Gaussiano centrado $B^H = \{B_t^H, t \geq 0\}$ cuya función de covarianza es

$$R^H(s,t) = \mathbf{E}\left(B_s^H B_t^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \qquad s,t \in \mathbb{R}_+$$

donde $H \in (0,1)$.
Años mas tarde el hidrólogo británico Harold Edwind Hurst trabajando para el ministerio de Obras Públicas de Egipto con el objetivo de optimizar el uso de las aguas del río Nilo, publicó los artículos [18] y [54] (el primero junto con R. P. Black e Y. M. Simaika). En dichos artículos los

autores analizaron la capacidad de almacenamiento a largo plazo de embalses en el río Nilo, esto los llevo a encontrarse con un patrón de ”memoria larga”, a partir de ese momento el parámetro de autosimilaridad H pasa a llamarse ”parámetro de Hurst”.

El cálculo estocástico respecto al movimiento Browniano fraccionario se inicia gracias al trabajo de Mandelbrot y Van-Ness [68]. En dicho trabajo los autores presentan una representación de B^H a través de medias móviles sobre un intervalo infinito basada en el proceso de Wiener $\{W_t, t \geq 0\}$,

$$B_t^H = \frac{1}{\Gamma\left(\frac{1}{2}+H\right)} \int_{-\infty}^{t} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW_s, \qquad t \geq 0.$$

Ellos denominan a este proceso como ”movimiento Browniano fraccionario”. Cabe destacar que cuando el parámetro de autosimilaridad $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ es el movimiento Browniano standard (o proceso de Wiener).

A partir de la definición previa es posible observar que $\mathbf{E}\left(|B_s^H - B_t^H|^2\right) = |s-t|^{2H}$, luego el criterio de continuidad de Kolmogorov nos lleva a concluir que B^H admite una versión continua cuyas trayectorias son casi seguramente Hölder continuas de orden estrictamente menor que H. Esto permite deducir que mientras más pequeño es H más irregulares son las trayectorias.
Otra característica interesante de B^H es que sus incrementos son estacionarios para todo valor de H, y para $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ están correlacionados. En efecto: sea $\alpha = H - \frac{1}{2}$ y $t_1 < t_2 < t_3 < t_4$, se tiene que

$$\mathbb{E}\left(B_{t_4}^H - B_{t_3}^H\right)\left(B_{t_2}^H - B_{t_1}^H\right) = 2\alpha H \int_{t_1}^{t_2} \int_{t_3}^{t_4} (u-v)^{2\alpha-1} du dv,$$

de donde es posible observar que los incrementos están correlacionados positivamente para $H \in (0, \frac{1}{2})$ y negativamente para $H \in (\frac{1}{2}, 1)$. De aquí se desprende la propiedad conocida como ”memoria larga” para el caso $H > \frac{1}{2}$, característica que motiva aún más el estudio de los procesos fraccionarios. Esta propiedad se ilustra claramente a través de la divergencia de la serie de correlaciones,

$$\begin{aligned} \sum_{n}^{\infty} |r(n)| &:= \sum_{n}^{\infty} |\mathbf{E} B_1^H \left(B_{n+1}^H - B_n^H\right)| \\ &= \sum_{n}^{\infty} |2\alpha H \int_0^1 \int_n^{n+1} (u-v)^{2\alpha-1} du dv| \quad \sim \quad \sum_{n}^{\infty} |2\alpha H| \cdot |n|^{2\alpha-1} = \infty. \end{aligned}$$

La autosimilaridad del movimiento Browniano fraccionario de parámetro de Hust H, se expresa de la manera siguiente: Para todo $a > 0$, B^H satisface

$$\mathfrak{L}\{B_{at}^H; t \geq 0\} = \mathfrak{L}\{a^H B_t^H; t \geq 0\},$$

es decir, B^H es H-autosimilar. Esta propiedad ha generado gran interés en diferentes áreas de estudio como la modelación de superficies terrestres, el tráfico en redes de telecomunicación y la modelación de activos financieros (ver por ejemplo el libro de Mandelbrot [67]).

El movimiento Browniano fraccionario también puede ser representado en forma integral sobre el intervalo compacto $[0, T]$, en este caso se tiene (ver [38])

$$B_t^H = \int_0^t K_H(s,t)dW_s, \qquad t \in [0,T], \tag{1}$$

donde $K_H(s,t)$ es una función determinista de s y t.

Para $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ el movimiento Browniano fraccionario no es una martingala (relativa a su filtración natural) ni tampoco es un proceso markoviano.

0.1.1 Integrales estocásticas múltiples

Consideremos $\mathcal{H}$ como un espacio de Hilbert real y $(B(\varphi), \varphi \in \mathcal{H})$ un proceso Gaussiano isonormal sobre un espacio de probabilidad $(\Omega, \mathcal{A}, P)$, esto es, una familia centrada de variables aleatorias Gaussianas tales que $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Por simplicidad asumiremos que $\mathcal{H} = L^2([0,T])$. Consideramos también el conjunto $\mathscr{E}$ de funciones simples a n valores, de la forma

$$f = \sum_{i_1,\dots i_m=1}^{n} c_{i_1,\dots i_m} \mathbf{1}_{A_{i_1}} \dots \mathbf{1}_{A_{i_m}},$$

para $i_k = i_l$ los coeficientes $c_{i_1,\dots i_m}$ son nulos y $A_{i_k} \in \mathfrak{B}([0,T])$ son disjuntos dos a dos. Para una función de esta forma definimos la integral estocástica múltiple de orden n como

$$I_n(f) = \sum_{i_1,\dots,i_m=1}^{n} c_{i_1,\dots i_m} B(A_{i_1}) \dots B(A_{i_m}),$$

donde denotamos $B(A) := B(\mathbf{1}_A)$ para $A \in \mathfrak{B}([0,T])$. Observamos que para todo $n \geq 1$, I_n es una aplicación lineal continua entre $\mathscr{E}$ y $L^2(\Omega)$, además, verifica la siguiente propiedad: para toda $h \in \mathcal{H}$ tal que $\|h\|_{\mathcal{H}} = 1$, se tiene que $I_n(h^{\otimes n}) = n! H_n(B(h))$, donde H_n es el polinomio de Hermite de grado $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}$$

y $H_0(x) = 1$. De esta forma definimos el n-ésimo caos de Wiener, denotado $\mathcal{H}_n$, como la clausura en $L^2(\Omega)$ del subespacio vectorial generado por $\{H_n(B(h)); h \in \mathcal{H}, \|h\| = 1\}$. Gracias a la ortogonalidad de los polinomios de Hermite, para m, n enteros positivos se tiene que

$$\begin{aligned} \mathbf{E}\left(I_n(f) I_m(g)\right) &= n! \langle f, g \rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f) I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{2}$$

También se tiene que

$$I_n(f) = I_n(\tilde{f})$$

donde $\tilde{f}$ denota la simetrización de f definida por

$$\tilde{f}(x_1, \dots, x_x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Dado que $\mathscr{E}$ es denso en $\mathcal{H}$, I_n es una isometría entre el espacio de Hilbert $\mathcal{H}^{\odot n}$ (producto tensorial simétrico) equipado de la norma $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ y el caos de Wiener de orden n, verificando las propiedades previamente enunciadas.

Otra herramienta bastante usada para el cálculo estocástico con respecto a procesos fraccionarios es el cálculo de Malliavin. Si bien en esta tesis dicha herramienta no es mayormente utilizada, consideramos la siguiente propiedad: La derivada de Malliavin "D" actua sobre integrales múltiples $F = I_n(f)$ de la siguiente manera: para todo s

$$D_s I_n = n I_{n-1}(f(\cdot, s)),$$

donde " $\cdot$ " denota $n-1$ variables.

0.2 CAPÍTULO I: Ecuaciones diferenciales estocásticas fraccionarias con retardo

En este capítulo de la tesis, a través del uso de una aproximación numérica, se aborda el estudio de ecuaciones diferenciales estocásticas con retardo dirigidas por un movimiento Browniano fraccionario. Especificamente, se estudia un esquema numérico para la solución de la ecuación diferencial estocástica

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_{t-r})dB_t^H, \quad t \in [0,T] \\ X_s &= \phi(s) \quad s \in [-r, 0], \end{aligned} \tag{3}$$

bajo el mismo contexto expuesto en [46]. Se considera como solución de la ecuación (3) al proceso $\{X_t, t \in [-r, T]\}$ que satisface la forma integral de (3), es decir,

$$\begin{aligned} X_t &= \phi(0) + \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s-r})dB_s^H, \quad t \in [0,T] \\ X_t &= \phi(t), \qquad t \in [-r, 0]. \end{aligned} \tag{4}$$

Ecuaciones del tipo (3) fueron estudiadas por primera vez en [45], haciendo uso de Cálculo de Malliavin los autores prueban la existencia y unicidad de solución. Posteriormente los mismos autores estudian un caso mas general en [46], donde por medio de integrales fraccionarias demuestran la existencia y unicidad de solución. En [63] se utilizan integrales de Young para analizar una variante con una no-linealidad del tipo $\sigma(\{X_u;\, u \in [t-r, t]\})$. En [74] se considera el caso $1/3 < H < 1/2$ trabajado gracias a la teoria de caminos ásperos (rough paths Theory).

La motivación para el estudio de este tipo de ecuaciones viene de las múltiples aplicaciones donde se utilizan modelos dirigidos por un movimiento browniano fraccionario, por ejemplo, Biofísica [61, 94, 97], ingeniería eléctrica [40] y finanzas [25, 48, 52, 53, 105], por mencionar algunos. Un ejemplo del uso de ecuaciones con retardo es el estudio de sistemas bacteriófagos [10, 20, 21, 90].

Destacamos que en las investigaciones previamente mencionadas se consideran modelos deterministas o dirigidos por un movimiento Browniano standard.

Antes de presentar los resultados obtenidos en esta parte de la tesis se enuncia el contexto y las herramientas utilizadas.

0.2.1 Elementos de Cálculo fraccionario

A continuación se presentan los elementos básicos de cálculo fraccionario utilizados para obtener los resultados presentes en este capítulo de la tesis.

Para $0 \leq \alpha \leq 1$, denotamos por $W^{\alpha}(a,b;\mathbb{R})$ el espacio de funciones medibles $f : [a,b] \to \mathbb{R}$ tales que

$$\sup_{t\in[a,b]} \left(|f(t)| + \int_a^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty,$$

y denotamos por $C^{\alpha}(a,b;\mathbb{R})$ el espacio de funciones α-Hölder continuas $f : [a,b] \to \mathbb{R}$ dotado de la norma

$$\sup_{a\leq t\leq b} |f(t)| + \sup_{a\leq s<t\leq b} \frac{|f(t)-f(s)|}{(t-s)^{\alpha}} < \infty.$$

Se presenta ahora la noción utilizada de integral respecto del mBf B^H. Siguiendo lo realizado en [79, 106] se considera

$$\int_a^b f dB^H = \int_a^b (D^{\alpha}_{a+} f)(s)(D^{1-\alpha}_{b-} B^H_{b-})(s) ds, \tag{5}$$

donde

$$(D^{\alpha}_{a+} f)(s) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(s)}{(s-a)^{\alpha}} + \alpha \int_a^s \frac{f(s)-f(u)}{(s-u)^{1+\alpha}} du \right] I_{(a,b)}(s), \tag{6}$$

se define también la derivada fraccionaria $(D^{1-\alpha}_{b-} B^H_{b-})(s)$ como

$$\frac{\exp^{-i\pi\alpha}}{\Gamma(\alpha)} \left[\frac{B^H_{b-}(s)}{(b-s)^{1-\alpha}} + (1-\alpha) \int_s^b \frac{B^H_{b-}(s) - B^H_{b-}(u)}{(u-s)^{2-\alpha}} du \right] I_{(a,b)}(s), \tag{7}$$

con la convención

$$B^H_{b-}(s) = (B^H_s - B^H_b) I_{(a,b)}(s).$$

Si $f \in C^{\nu}(a,b;\mathbb{R})$ para $\nu + H > 1$, en [79] se demostró que la integral trayectorial fraccionaria (5) existe para todo $\alpha \in (1-H, \nu)$ y que satisface la siguiente desigualdad:

$$\left| \int_a^b f dB^H \right| \leq F_1(\omega) \left[\int_a^b \frac{|f(s)|}{(s-a)^{\alpha}} ds + \int_a^b \int_a^s \frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}} du ds \right], \tag{8}$$

donde la variable aleatoria F_1 está dada por

$$F_1(\omega) = C \cdot \sup_{a<s<b} |D^{1-\alpha}_{b-} B^H_{b-}(s)|, \tag{9}$$

y satisface $\mathbf{E}(F_1^p) < \infty$ para todo $p \in [1, \infty)$.

0.2.2 Aproximación en tiempo discreto para una ecuación diferencial estocástica con retardo dirigida por un movimiento Browniano fraccionario

En el artículo [46] los autores prueban la existencia y unicidad de solución para una ecuación un poco mas general que (3) haciendo uso de las herramientas introducidas en la sub-sección previa. Tal resultado engloba el caso estudiado en esta tesis, sin embargo, dicho trabajo no considera un tratamiento numérico de la ecuación en cuestión. A continuación se presenta la adaptación de las hipótesis y del resultado antes mencionado.

H1 $\sigma : \mathbb{R} \to \mathbb{R}$ es una función medible tal que $\sigma(x)$ es diferenciable en x y existen constantes $0 < \delta \leq 1$, y M_0 tales que para todo $N \geq 0$ existe una constante $M_N > 0$ tal que

(a) $|\sigma(x) - \sigma(y)| \leq M_0|x - y| \quad \forall x \in \mathbb{R}$.

(b) $|\sigma'(x) - \sigma'(y)| \leq M_N|x - y|^{\delta}$, $\forall |x|, |y| \leq N$.

H2 $b : \mathbb{R} \to \mathbb{R}$ es una función medible tal que para todo $N \geq 0$ existe $L_N > 0$ tal que

(a) $|b(x) - b(y)| \leq L_N|x - y|$, para todo $|x|, |y| \leq N$.

(b) $|b(x)| \leq L_0(|x| + 1)$.

H3 Exite $\alpha \in (1 - H, 1/2)$ tal que $\phi : [-r, 0] \to \mathbb{R}$ es una función medible que satisface

(a) $\phi \in W^{\alpha}(-r, 0; \mathbb{R})$,

(b) $\phi \in C^{1-\alpha}(-r, 0; \mathbb{R})$.

Theorem 1 *Consideremos un mBf B^H con parámetro de Hurst $H \in (1/2, 1)$. Sean b, σ y ϕ funciones que satisfacen las hipótesis (**H1**), (**H2**) y (**H3**) respectivamente con $\delta > (1/H) - 1$ y $\alpha_0 := \delta/(1 + \delta)$. Si $\alpha \in (1 - H, \alpha_0)$. Entonces la ecuación (4) admite una solución única X tal que*

$$X \in L^0(\Omega, \mathcal{F}, P; W^{\alpha}(-r, T; \mathbb{R}))$$

y $X \in C^{1-\alpha}(-r, T; \mathbb{R})$ casi seguramente.

Antes de enunciar el resultado principal de este capítulo de la tesis, presentamos la notación utilizada.

Se asumió que el tiempo final $T = N_0 r$ para algún entero N_0. De esta forma se define el tamaño de paso

$$\Delta_l = \frac{r}{l}, \quad l \in \{2, 3, ...\}. \tag{10}$$

y la discretización equidistante para el intervalo $[-r, T]$

$$\mathcal{T}_{\Delta_l} = \{\tau_n : n \in \{-l, -l + 1, ..., 0, 1, ..., N\}\}$$

donde

$$\tau_n = n\Delta_l, \quad \text{y} \quad N := N_0 l. \tag{11}$$

El esquema de Euler utilizado para la ecuación (4) se define recursivamente por:

$$Y_{n+1} = Y^{\Delta_l}_{\tau_{n+1}} = Y_n + b(Y_n)\Delta_l + \sigma(Y_{n-l})\Delta B^H_n \tag{12}$$

con

$$\Delta B^H_n = B^H_{\tau_{n+1}} - B^H_{\tau_n}, \quad \text{for} \;\; n = 0, 1, \cdots, N-1,$$

y valores iniciales

$$Y_i = X_{\tau_i} = \phi(\tau_i) \qquad i = -l, -l+1, \cdots, 0.$$

Con dicha notación expuesta, el resultado principal de este capítulo es el siguiente:

Theorem 2 *Sea B^H un movimiento Browniano fraccionario con $H > 1/2$ y sean σ, b y ϕ funciones que satisfacen las hipótesis **H1**, **H2** y **H3** respectivamente con $\alpha > (1/H) - 1$ y $\alpha_0 = \frac{\delta}{1+\delta}$. Para todo $\epsilon > 0$ y para $0 < \rho < (H - 1/2) \wedge (1 - H)$ suficientemente pequeño, si $\alpha \in (1 - H, (1 - H + \rho) \wedge \alpha_0)$, entonces existe $\Delta_0 > 0$ y $\tilde{\Omega} = \Omega_{\epsilon,\Delta_0,\rho}$ tal que $P(\tilde{\Omega}) > 1 - \epsilon$ y para todo $\omega \in \tilde{\Omega}$ y $\Delta_l < \Delta_0$,*

$$\sup_{-r \le t \le T} |X_t - Y_t| \le F(\omega)\Delta_l^{2H-1-2\rho}, \tag{13}$$

donde $F(\omega)$ no depende de Δ_l y ε.

Finalmente, este primer capítulo de la tesis también considera la exposición de una aproximación numérica que converge débilmente a la solución de la ecuación (3).

A partir de la representación (1) para B^H, es natural definir una aproximación del tipo

$$Z^\epsilon := \left\{ Z^\epsilon_t = \frac{1}{\epsilon} \int_0^t K_H(t,s)(-1)^{N(\frac{s}{\epsilon^2})} ds, \quad t \in [0, T] \right\} \tag{14}$$

Donde $\{N(t), t \ge 0\}$ es un proceso Poisson standard. Las investigadoras Rosario Delgado y Maria Jolis demostraron en [39] que la familia Z^ϵ converge débilmente al mBf B^H cuando $\epsilon \to 0$. Definiendo el proceso $Y = Y^\epsilon$ como la solución de la ecuación (3) dirigida por Z^ϵ se demuestra que, bajo las mismas hipótesis antes presentadas, Y^ϵ converge en ley a la solución X de la FSDDE (3), la convergencia se tiene en el espacio $C^\gamma(-r, T; \mathbb{R})$.

Este capítulo de la tesis da origen al artículo [23]

- Jorge Clarke De la Cerda, Johanna Garzón, Samy Tindel & Soledad Torres *Discrete time approximation of delay differential equations driven by fractional Brownian motion.* Preprint.

0.3 CAPÍTULO II: Estimación de parámetros para la sábana fraccionaria de Ornstein-Uhlenbeck

En esta parte de la tesis estamos interesados en un problema de estimación de parámetros para ecuaciones diferenciales estocásticas (EDE's) dirigidas por campos aleatorios. En particular estudiamos una ecuación de tipo Langevin dirigida por una sábana Browniana fraccionaria, cuya solución define el proceso fraccionario de Ornstein-Uhlenbeck bi-paramétrico.

El desarrollo del análisis estocástico para movimientos Brownianos fraccionarios conduce naturalmente al estudio de inferencia estadística para ecuaciones dirigidas por este tipo de procesos. Hoy en día podemos encontrar una amplia literatura relacionada con este tema, entre otros referimos a [15], [16], [51], [59] y [103]. Sin embargo, análisis estadístico para EDE's dirigidas por la sábana Browniana fraccionaria (sBf) ha sido menos considerado, en el artículo [91] los autores estudian un estimador de máxima verosimilitud para una SDE's con ruido aditivo dado por una sBf, también referimos a los artículos [3] y [42] para el caso de un sBs.

0.3.1 La sábana Browniana fraccionaria y la sábana de Ornstein-Uhlenbeck fraccionaria

La sBf con parámetros de Hurst $\alpha, \beta \in (0,1)$, $(B^{\alpha,\beta}_{t,s}, t, s \in [0,T]\times[0,S])$ es un proceso Gaussiano centrado cuya función de covarianza es

$$\begin{aligned} \mathbf{E}\left(B^{\alpha,\beta}_{t,s}, B^{\alpha,\beta}_{u,v}\right) &= \mathcal{R}^{\alpha}(t,u)\mathcal{R}^{\beta}(s,v) \\ &:= \frac{1}{2}\left(t^{2\alpha}+u^{2\alpha}-|t-u|^{2\alpha}\right)\frac{1}{2}\left(s^{2\beta}+v^{2\beta}-|s-v|^{2\beta}\right) \qquad (15) \end{aligned}$$

para todo $t,u \in [0,T]^2$ y $s,v \in [0,S]^2$.

Consideramos que $B^{\alpha,\beta}$ está definida en un espacio de probabilidad completo $(\Omega, \mathcal{A}, \mathbb{P})$ tal que la σ-álgebra $\mathcal{A}$ es generada por $B^{\alpha,\beta}$. Sean T y S fijos, consideramos los intervalos $[0,T]$ y $[0,S]$ y denotamos por ξ el conjunto de funciones simples a valores reales en $[0,T]\times[0,S]$. Sea $\mathcal{H}^{\alpha,\beta}$ el espacio de Hilbert definido como la clausura de ξ con respecto al producto escalar

$$\langle 1_{[0,t]\times[0,s]}, 1_{[0,u]\times[0,v]}\rangle_{\mathcal{H}^{\alpha,\beta}} = \mathcal{R}^{\alpha}(t,u)\mathcal{R}^{\beta}(s,v)$$

donde $\mathcal{R}^{\alpha}(t,u)\mathcal{R}^{\beta}(s,v)$ es la función de covarianza de la sBf dada por (15). El mapeo $1_{[0,t]\times[0,s]} \longmapsto B^{\alpha,\beta}_{t,s}$ puede ser extendido a una isometría lineal entre $\mathcal{H}^{\alpha,\beta}$ y el espacio Gaussiano $\mathcal{H}^{\alpha,\beta}_1$ generado por $B^{\alpha,\beta}$, el cual es un subespacio cerrado de $L^2(\Omega,\mathcal{A},\mathbb{P})$. Denotamos esta isometría por $\varphi \longmapsto B^{\alpha,\beta}(\varphi)$. Sean $\alpha, \beta > \frac{1}{2}$ fijos. En este caso se tiene que para toda $f, g \in \mathcal{H}^{\alpha,\beta}$ el producto escalar posee la forma

$$\langle f, g\rangle_{\mathcal{H}^{\alpha,\beta}} = c(\alpha)c(\beta)\int_0^T\int_0^S\int_0^T\int_0^S f(a,b)g(m,n)|a-m|^{2\alpha-2}|b-n|^{2\beta-2}da\ db\ dm\ dn \qquad (16)$$

y $c(r) = r(2r-1)$.

La sábana fraccionaria de Ornstein-Uhlenbeck se define como la solución de la ecuación estocástica

$$X_{t,s} = -\theta \int_0^t \int_0^s X_{v,u} dv du + B^{\alpha,\beta}_{t,s}, \qquad (t,s) \in [0,T] \times [0,S]. \tag{17}$$

Donde $B^{\alpha,\beta}$ denota la sBf con parámetros de Hurst $\alpha, \beta \in (\frac{1}{2}, 1)$. Suponemos también que $X_{0,0} = X_{t,0} = X_{0,s} = 0$ para todo t, s.
La solución de la ecuación (17) ha sido estudiada en varios artículos (ver por ejemplo [44] o [76]). Se ha demostrado que para $\theta > 0$ y $\alpha, \beta \in (\frac{1}{2}, 1)$ la ecuación 17 admite una única solución fuerte definida por

$$X_{t,s} = \int_0^T \int_0^S f(t,s,t_0,s_0) dB^{\alpha,\beta}_{t_0,s_0} \tag{18}$$

donde

$$f(t,s,t_0,s_0) = 1_{[0,t]}(t_0) 1_{[0,s]}(s_0) \sum_{n\geq 0} (-1)^n \theta^n \frac{(t-t_0)^n (s-s_0)^n}{(n!)^2}. \tag{19}$$

Es a este proceso que denominamos sábana fraccionaria de Ornstein-Uhlenbeck, el cual es un proceso Gaussiano pués está definido como una integral estocástica múltiple de orden 1 con respecto a la sBf.

0.3.2 Estimador de mínimos cuadrados para la sábana fraccionaria de Ornstein-Uhlenbeck

Inspirados en el trabajo [51] estudiamos el estimador de mínimos cuadrados para el parámetro θ en la ecuación (17), el cual está dado por la expresión

$$\widehat{\theta}_{T,S} = -\frac{\int_0^T \int_0^S X_{t,s} dX_{t,s}}{\int_0^T \int_0^S X^2_{t,s} dt ds}. \tag{20}$$

Usando (17) y (20) escribimos

$$\widehat{\theta}_{T,S} - \theta = -\frac{\int_0^T \int_0^S X_{t,s} dB^{\alpha,\beta}_{t,s}}{\int_0^T \int_0^S X^2_{t,s} dt ds}. \tag{21}$$

El primer resultado obtenido es el siguiente y trata sobre la consistencia fuerte del estimador.

Theorem 3 *Sea $\theta_{T,S}$ el estimador de mínimos cuadrados dado por (20), y sean $\alpha, \beta \in (\frac{1}{2}, \frac{5}{8})$. Entonces $\theta_{T,S}$ es fuertemente consistente para el parámetro θ, esto es,*

$$\widehat{\theta}_{T,S} \to \theta \textit{ casi seguramente cuando } \quad T, S \to \infty.$$

El resultado que presentaremos a continuación muestra que a diferencia del caso uniparamétrico estudiado en [51], el estimador de mínimos cuadrados presentado en (20) no es asintóticamente normal. De cierta forma esto no sorprende en demasía pues la sábana de Ornstein-Uhlenbeck no preserva las mismas propiedades que el proceso de Ornstein-Uhlenbeck (por ejemplo el kernel involucrado en la solución de la EDE's (18) puede tomar cualquier valor real, sin embargo, en el caso uniparametrico solo toma valores positívos). Antes de enunciar nuestro resultado presentaremos el criterio que utilizamos para analizar la normalidad asintótica, dicho criterio se encuentra en el trabajo de D. Nualart y S. Ortiz-Latorre ([78], Teorema 4)

Theorem 4 *Sea $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (con $f_k \in \mathcal{H}^{\odot n}$ para todo $k \geq 1$) una sucesión de variables aleatorias de cuadrado integrable en el n-ésimo caos de Wiener tales que $\mathbf{E}[F_k^2] \to 1$ cuando $k \to \infty$. Entonces las siguientes afirmaciones son equivalentes:*

i) La sucesión $(F_k)_{k\geq 0}$ converge en distribución a una ley normal $\mathcal{N}(0,1)$.

ii) $\|DF_k\|^2_{\mathcal{H}}$ converge a n en $L^2(\Omega)$ cuando $k \to \infty$.

Para estudiar la normalidad asintótica de nuestro estimador basta estudiar el nominador en el lado derecho de la expresión (21). Denotemos por

$$\sigma^2_{T,S} = \mathbf{E}\left(\int_0^T \int_0^S X_{t,s} dB^{\alpha,\beta}_{t,s}\right)^2 = \mathbf{E}\left(F_{T,S}\right)^2,$$

donde $F_{T,S} := I_2(f(u,v,t,s))$.

Proposition 1 *Sean $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$ y sea $F_{T,S}$ como en la expresión previa. Entonces, cuando T, S tienden a infinito,*

$$\frac{1}{\sigma_{T,S}} F_{T,S}$$

no converge en distribución a una ley normal $N(0,1)$.

Este capítulo de la tesis dió origen a la publicación [24]

- Jorge Clarke De la Cerda & Ciprian Tudor *Least squares estimator for the parameter of the fractional OrnsteinUhlenbeck sheet.* Publicada en JKSS **41**-(2012) 341-350.

0.4 CAPÍTULO III: La ecuación estocástica de la onda con ruido fraccionario coloreado

En esta sección de la tesis estudiamos una EDE de la onda con ruido fraccionario en el tiempo y coloreado en el espacio. Se demuestran cotas óptimas para la solución y posteriormente, utilizando conceptos de Teoría de Potencial, los resultados previos son aplicados para establecer cotas superiores e inferiores para la probabilidad de arrivo de la solución a un boreliano. Dichas cotas son dadas en términos de la medida de Hausdorff y la capacidad Newtoniana del boreliano.

Como ya se ha mencionado previamente en esta tesis, el reciente desarrollo del cálculo estocástico con respecto a movimientos Brownianos fraccionarios (mBf) conduce de manera natural al estudio de ecuaciones diferenciales estocásticas (EDE's) dirigidas por esto procesos, siendo la motivación para ello las múltiples aplicaciones del mBf. Por mencionar algunos trabajos de estudio teórico de este tipo de ecuaciones referimos entre otros a [50], [69], [80], [86] y [98]; ejemplos de aplicaciones prácticas es posible encontrar en Biofísica [61], series de tiempo financieras [11], ingeniería eléctrica [40] y física [26] entre otras áreas del conocimiento.

Este capítulo de la tesis es posible enmarcarlo dentro del estudio de EDE's dirigidas por mBf's pero también puede considerarse como una cotinuación de la linea de investigación iniciada por Robert C. Dalang y Eulalia Nualart en [31] la cual consiste en el estudio de Teoria de Potencial para sistemas de EDE's. Más precisamente hablando se considera un sistema de k EDE's de la onda

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) = \Delta u_i(t,x) + \dot{W}_i(t,x), \qquad t \in [0,T], x \in \mathbb{R}^d \tag{22}$$

con condición inicial $u_i(t,x) = 0$ y $\frac{\partial u_i}{\partial t}(0,x) = 0$ para todo $x \in \mathbb{R}^d$ y para todo $i = 1, ..., k$. El ruido en la ecuación proviene de un proceso Gaussiano que se comporta como un mBf en el tiempo y posee covarianza espacial dada por un kernel de Riesz, es decir

$$\mathbf{E}(W_i(t,A)W_j(s,B)) = \delta_{i,j} R_H(t,s) \int_A \int_B f(x-y)dxdy$$

para todo $t, s \in [0,T]$ y A, B Borelianos en $\mathbb{R}^d$ donde $R_H(t,s)$ es la covarianza del movimiento Browniano fraccionario, $f : \mathbb{R}^d \to \mathbb{R}_+$ es la transformada de Fourier de la medida temperada no-negativa μ sobre $\mathbb{R}^d$ cuya densidad con respecto a la medida de Lebesgue es $|\xi|^{-(d-\beta)}$, $0 < \beta < d$. En lo previo $\delta_{i,j}$ denota la delta de Kronecker.

La ecuación (22) ha sido estudiada en [7] para el caso $H > \frac{1}{2}$. Se ha demostrado que (22) admite una única solución tipo "campo aleatorio" si y solo si $\beta < 2H + 1$, lo cual extiende el resultado presentado en [30] para el caso $H = \frac{1}{2}$. El propósito de esta sección es analizar más profundamente el comportamiento de la solución de (22), especificamente, dado un Boreliano $A \subset \mathbb{R}^k$ se busca determinar cuando el proceso $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ toca al conjunto A con probabilidad positiva. Recientemente se han realizado varios trabajos en esta dirección, probabilidades de arrivo o de manera mas global, teoría de potencial para sistemas de EDE's. Referimos entre otros a las investigaciones [31], [32], [33], [35] o [72]. El estudio de probabilidades de arrivo para EDE's con ruido fraccionario en tiempo es un tema nuevo. Hasta donde sabemos, solamente los autores Eulalia Nualart y Frederi Viens han tratado este tema en el artículo [81]. En esta referencia los investigadores presentan cotas superiores e inferiores para los tiempos de arrivo de la solución de una EDE del calor sobre el circulo conducida por un ruido fraccionario en el tiempo.

Antes de presentar los resultados obtenidos en esta parte de la tesis enunciaremos las nociones básicas utilizadas, Teoría de Potencial y el espacio de Hilbert canónico asociado al ruido Gaussiano fraccionario-coloreado.

0.4.1 Teoría de Potencial

Nuestro propósito es analizar la probabilidad

$$P\left(u(I)\cap A\right)\neq\emptyset$$

donde u es la solución de (22), I es un Boreliano contenido en $[0,T]\times\mathbb{R}^d$ y A es un Boreliano en $\mathbb{R}^k$. Aquí $u(I)$ es la imagen de I a través del mapeo aleatorio $(t,x)\to u(t,x)$.

Presentamos brevemente las nociones de Teoría de Potencial que se usan en esta sección. Para todo Boreliano $F\subset\mathbb{R}^d$ se define $\mathcal{P}(F)$ como el conjunto de todas las medidas de probabilidad con soporte compacto contenido en F. Para todo $\mu\in\mathcal{P}(\mathbb{R}^d)$, denotamos por $I_\beta(\mu)$ la β-energía de la medida μ definida por

$$I_\beta(\mu)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_\beta(\|x-y\|)\mu(dx)\mu(dy) \tag{23}$$

donde

$$K_\beta(r)=\begin{cases} r^{-\beta} & \text{if } \beta>0;\\ log\left(\frac{N_0}{r}\right) & \text{if } \beta=0;\\ 1 & \text{if } \beta<0.\end{cases} \tag{24}$$

Aquí N_0 es una constante.

Para todo $\beta\in\mathbb{R}$ y $F\in\mathcal{B}(\mathbb{R}^d)$ se define la capacidad β-dimensional de F por

$$\mathrm{Cap}_\beta(F)=\left[\inf_{\mu\in\mathcal{P}(F)}I_\beta(\mu)\right]^{-1} \tag{25}$$

con la convención $1/\infty:=0$. La medida de Hausdorff β-dimensional del conjunto $F\in\mathcal{B}(\mathbb{R}^d)$ es dada por

$$\mathcal{H}_\beta(F)=\lim_{\varepsilon\to0^+}\inf\left[\sum_{i=1}^{\infty}(2r_i)^\beta;\ F\subset\bigcup_{i=1}^{\infty}B(x_i,r_i),\ \sup_{i\geq1}r_i\leq\varepsilon\right] \tag{26}$$

donde $B(x,r)$ denota la bola Euclideana de radio $r>0$ centrada en $x\in\mathbb{R}^d$. Cuando $\beta<0$, la medida de Hausdorff β-dimensional del conjunto F es infinita por definición.

0.4.2 El espacio de Hilbert canónico

Denotamos por $C_0^\infty(\mathbb{R}^{d+1})$ el espacio de funciones sobre $\mathbb{R}^{d+1}$ con soporte compacto e infinitamente diferenciables, y $\mathcal{S}(\mathbb{R}^d)$ el espacio de Schwartz de funciones C^∞ en $\mathbb{R}^d$ con decrecimiento rápido. Para $\varphi\in L^1(\mathbb{R}^d)$, denotamos $\mathcal{F}\varphi$ la transformada de Fourier de φ:

$$\mathcal{F}\varphi(\xi)=\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\varphi(x)dx.$$

Comenzaremos por introducir el contexto presentado en [30]. Sea μ una medida temperada no-negativa sobre $\mathbb{R}^d$, i.e. una medida no-negativa que satisface:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty, \quad \text{para algún } l > 0. \tag{27}$$

Dado que el integrando es no-decreciente en l, es posible asumir que $l \geq 1$ es un entero. Notar que la expresión $1 + |\xi|^2$ se comporta como una constante cuando ξ esta cerca de 0, y como $|\xi|^2$ cuando ξ se acerca a ∞, y por lo tanto (27) es equivalente a:

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty, \quad \text{y} \quad \int_{|\xi|\geq 1} \mu(d\xi)\frac{1}{|\xi|^{2l}} < \infty, \quad \text{para algún entero } l \geq 1. \tag{28}$$

Sea $f : \mathbb{R}^d \to \mathbb{R}_+$ la transformada de Fourier de μ en $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Propiedades básicas de la transformada de Fourier permiten concluir que para todo $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi).$$

Utilizando un argumento de aproximación es posible mostrar que la igualdad previa también se tiene para funciones indicadoras $\varphi = 1_A, \psi = 1_B$, con $A, B \in \mathcal{B}_b(\mathbb{R}^d)$, donde $\mathcal{B}_b(\mathbb{R}^d)$ es la familia de borelianos acotados en $\mathbb{R}^d$:

$$\int_A\int_B f(x-y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}\mu(d\xi). \tag{29}$$

Recordemos ciertas características del movimiento Browniano fraccionario $(B_t^H)_{t\in[0,T]}$ con parámetro de Hurst $H \in (0,1)$. $(B_t^H)_{t\in[0,T]}$ es el único proceso Gaussiano de media cero y covarianza

$$R_H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,T].$$

Denotamos por $\mathcal{H}$ el espacio de Hilbert canónico asociado con este proceso, el cual definimos como la clausura del espacio lineal generado por las funciones indicadoras $1_{[0,t]}, t \in [0,T]$ con respecto al producto interior

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

Es sabido que para $H > 1/2$ se tiene la expresión

$$R_H(t,s) = \alpha_H \int_0^t\int_0^s |u-v|^{2H-2}dudv \tag{30}$$

para todo $s, t \in [0,T]$ con $\alpha_H := H(2H-1)$. De forma más general, para $H > 1/2$ y todo $\psi, \phi \in \mathcal{H} = \mathcal{H}([0,T])$ se tiene que

$$\langle \psi, \phi\rangle_{\mathcal{H}} = \alpha_H \int_0^T\int_0^T \psi(u)\phi(v)|u-v|^{2H-2}dudv. \tag{31}$$

Como en [6], sobre un espacio de probabilidad completo $(\Omega, \mathcal{F}, P)$, consideramos un proceso Gaussiano de media cero $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ de covarianza:

$$\mathbf{E}(W_t(A)W_s(B)) = R_H(t,s) \int_A \int_B f(x-y)dxdy =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{HP}}. \tag{32}$$

Sea $\mathcal{E}$ el conjunto de combinaciones lineales de funciones elementales $1_{[0,t]\times A}$, $t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)$, y $\mathcal{HP}$ el espacio de Hilbert definido como la clausura de $\mathcal{E}$ con respecto a el producto interior $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$. (De manera alternativa, $\mathcal{HP}$ puede ser definido como la clausura de $C_0^\infty(\mathbb{R}^{d+1})$, con respecto al producto interior $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$; ver [6].)

El mapeo $1_{[0,t]\times A} \mapsto W_t(A)$ es una isometría entre $\mathcal{E}$ y el espacio Gaussiano H^W de W, la cual puede ser extendida a $\mathcal{HP}$. Denotamos esta extensión por:

$$\varphi \mapsto W(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) W(dt,dx).$$

En esta sección asumimos que $H > 1/2$. Por lo tanto se cumple (30). A partir de (29) y (30), se sigue que para toda $\varphi, \psi \in \mathcal{E}$,

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{HP}} &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(u,x)\psi(v,y) f(x-y) |u-v|^{2H-2} dx\, dy\, du\, dv \\ &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(u,\cdot)(\xi) \overline{\mathcal{F}\psi(v,\cdot)(\xi)} |u-v|^{2H-2} \mu(d\xi)\, du\, dv. \end{aligned}$$

Más aún, dado que para funciones indicadoras φ y ψ, el integrando es un producto entre una función de (u,v) y una función de ξ, podemos intercambiar el orden de las integrales $dudv$ y $\mu(d\xi)$. Así, para $\varphi, \psi \in \mathcal{E}$, se tiene:

$$\langle \varphi, \psi \rangle_{\mathcal{HP}} = \alpha_H \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \mathcal{F}\varphi(u,\cdot)(\xi) \overline{\mathcal{F}\psi(v,\cdot)(\xi)} |u-v|^{2H-2} du\, dv\, \mu(d\xi). \tag{33}$$

Es posible que el espacio $\mathcal{HP}$ contenga distribuciones, sin embargo también contiene al espacio $|\mathcal{HP}|$ de funciones medibles $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ tales que

$$\|\varphi\|^2_{|\mathcal{HP}|} := \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(u,x)||\varphi(v,y)| f(x-y) |u-v|^{2H-2} dx\, dy\, du\, dv < \infty.$$

0.4.3 Tiempos de Arrivo para la ecuación estocástica de la onda con ruido fraccionario en el tiempo y coloreado en el espacio

Consideramos la EDE lineal de la onda dirigida por un mBf infinito dimensional W con parámetro de Hurst $H \in (0,1)$, esto es

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{34}$$

Aquí Δ denota el Laplaciano en $\mathbb{R}^d$ y $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ es un campo Gaussiano centrado de covarianza

$$\mathbf{E}(W_t(A)W_s(B)) = R_H(t,s)\int_A\int_B f(x-y)dxdy,$$

donde R_H es la covarianza del mBf y f es el kernel de Riesz.

Sea G_1 la solución fundamental de $u_{tt} - \Delta u = 0$. Es sabido que $G_1(t,\cdot)$ es una distribución en $\mathcal{S}'(\mathbb{R}^d)$ con decrecimiento rápido, y

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{35}$$

para todo $\xi \in \mathbb{R}^d, t > 0, d \geq 1$ (ver por ejemplo [99]). En particular,

$$\begin{aligned} G_1(t,x) &= \frac{1}{2}1_{\{|x|<t\}}, \quad \text{if } d=1 \\ G_1(t,x) &= \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}}, \quad \text{if } d=2 \\ G_1(t,x) &= c_d\frac{1}{t}\sigma_t, \quad \text{if } d=3, \end{aligned}$$

donde σ_t denota la medida de superficie sobre la espera 3-dimensional de radio t.

La solución tipo "campo aleatorio" de (34) se define como el proceso de cuadrado integrable $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ definido por:

$$u(t,x) = \int_0^t\int_{\mathbb{R}^d} G_1(t-s, x-y)W(ds,dy). \tag{36}$$

Por definición, $u(t,x)$ existe si y solo si la integral estocástica anterior está bien definida, i.e. $g_{tx} := G_1(t-\cdot, x-\cdot) \in \mathcal{HP}$. En este caso, $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|^2_{\mathcal{HP}}$.

Los investigadores R. Balan & C. A. Tudor en el artículo [7] del 2010 muestran las condiciones necesarias y suficientes para asegurar la existencia de solución a la ecuación de la onda antes enunciada. Dicho resultado se presenta a continuación

Theorem 5 *La ecuación estocástica de la onda (34) admite una única solución tipo "campo aleatorio" $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ si y solo si*

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}}\mu(d\xi) < \infty. \tag{37}$$

Remark 1 *Notamos que (37) es equivalente a:*

$$\int_{|\xi|\leq 1}\mu(d\xi) < \infty, \quad \textit{and} \quad \int_{|\xi|\geq 1}\mu(d\xi)\frac{1}{|\xi|^{2H+1}} < \infty. \tag{38}$$

Como se ha mencionado al comienzo de esta sección, para este capítulo de la tésis se consideró una covarianza espacial para el ruido W dada por un kernel de Riesz particular. Esto quiere decir que la medida μ es dada por

$$d\mu(\xi) = |\xi|^{-d+\beta}d\xi \text{ con } \beta \in (0,d).$$

En cuyo caso el kernel f es dado por

$$f(\xi) = |\xi|^{-\beta} \text{ with } \beta \in (0, d).$$

Destacamos que en este caso particular de kernel de Riesz, la condición (37) es equivalente a

$$\beta \in (0, d \wedge (2H + 1)). \tag{39}$$

Remark 2 *Ya que $H > \frac{1}{2}$ y por lo tanto $2H+1 \in (2, 3)$, para dimensión $d = 1, 2$ se tiene $\beta \in (0, d)$ mientras que para $d \geq 3$ se tiene $\beta \in (0, 2H + 1)$.*

En este capítulo de la tesis se presenta como resultado principal las cotas superiores e inferiores para los tiempos de arrivo de la solución de (34) a un boreliano cualquiera. La demostración de dicho resultado se basa fuertemente en la regularidad de la solución, la que surge como efecto de las cotas óptimas obtenidas para los incrementos de la solución, tanto temporales como espaciales. Dichas cotas también forman parte de los resultados obtenidos en este capítulo de la tesis. A continuación se presentan dichos resultados, comenzando por las cotas para los incrementos temporales y la consecuente regularidad temporal de la solcuión de la ecuación de la onda (34).

Proposition 2 *Supongamos que*

$$\beta \in (2H - 1, d \wedge (2H + 1)). \tag{40}$$

Sea $t_0, M > 0$ y $x \in [-M, M]^d$ fijo. Entonces existen constantes positivas c_1, c_2 tales que para todo $s, t \in [t_0, T]$

$$c_1|t - s|^{2H+1-\beta} \leq \mathbf{E}\,|u(t, x) - u(s, x)|^2 \leq c_2|t - s|^{2H+1-\beta}.$$

La proposición (2) implica de manera casi inmediata la regularidad Hölderiana de la solución de (34).

Corollary 1 *Supongamos (40). Entonces, para todo $x \in \mathbb{R}^d$ la aplicación*

$$t \to u(t, x)$$

es casi seguramente Hölder continua de orden $\delta \in \left(0, \frac{2H+1-\beta}{2}\right)$.

Remark 3

- *Siguiendo detalladamente la demostración del Teorema 5.1 en [35] es posible mostrar que el mapeo $t \to u(t, x)$ no es Hölder continuo de orden $\frac{2H+1-\beta}{2}$.*
- *Cuando $H = \frac{1}{2}$, los resultados presentados incluyen lo obtenido en [36], [35].*

A continuación se presentan los resultados obtenidos respecto del comportamiento espacial de la solución u de la ecuación (34).

Proposition 3 *Supongamos (40), $M > 0$ fijo y $t \in [t_0, T]$. Entonces existen constantes positivas c_3, c_4 tales que para todo $x, y \in [-M, M]^d$*

$$c_3|x-y|^{2H+1-\beta} \leq \mathbf{E}\,|u(t,x)-u(t,y)|^2 \leq c_4|x-y|^{2H+1-\beta}.$$

Respecto de la regularidad espacial Hölderiana se obtuvo el siguiente resultado.

Proposition 4 *Supongamos que $\beta \in (0, d \wedge (2H+1))$. Entonces para todo $t \in [t_0, T]$ la aplicación*

$$x \to u(t,x)$$

es casi seguramente Hölder continua de orden $\delta \in \left(0, \left(\frac{2H+1-\beta}{2}\right) \wedge 1\right)$.

Remark 4

- *Cuando $H = \frac{1}{2}$, el resultado previo coincide con lo demostrado en [36], [35].*
- *En la Proposición 4 es posible distinguir dos casos: si $\beta \in (0, 2H-1)$ entonces la solución de (34) es Hölder continua de orden 1 en la variable espacial, por tanto, es Lipschitz continua en esta variable. Mientras que si $\beta \in (2H-1, d \wedge (2H+1))$ el exponente de Hölder es $\delta \in (0, \frac{2H+1-\beta}{2}) < 1$.*

Obtener la regularidad temporal y espacial para la solución de (34) permite, entre otras cosas, deducir la regularidad conjunta de la solución con respecto a una métrica bien definida.

Denotemos por Δ la siguiente métrica sobre $[0,T] \times \mathbb{R}^d$

$$\Delta\left((t,x);(s,y)\right) = |t-s|^{2H+1-\beta} + |x-y|^{2H+1-\beta}. \tag{41}$$

De las Proposiciones 2 y 3, se obtiene el resultado siguiente:

Theorem 6 *Sea $M > 0$ fijo y asumamos (40). Para todo $t, s \in [t_0, T]$ y $x, y \in [-M, M]^d$ existen constantes positivas C_1, C_2 tales que*

$$C_1\Delta\left((t,x);(s,y)\right) \leq \mathbf{E}\,|u(t,x)-u(s,y)|^2 \leq C_2\Delta\left((t,x);(s,y)\right).$$

Durante los últimos diez años un pequeño número de investigadores ha presentado diferentes criterios para el estudio de cotas a las probabilidades de arrivo. Nuestro resultado se sustenta sobre un criterio particular para procesos Gaussianos, este fue introducido por los académicos H. Biermé, C. Lacaux & Y. Xiao en el año 2009 y se enuncia a continuación.
Se considera la siguiente notación: si $V = (V(x), x \in \mathbb{R}^m)$ es un proceso estocástico a valores en $\mathbb{R}^k$, entonces $V(S)$ denota el rango del boreliano S bajo el mapeo aleatorio $x \to V(x)$.

Theorem 7 *Sea $X = X(t), t \in \mathbb{R}^N$ un proceso Gaussiano centrado a valores en $\mathbb{R}^k$ y sea $I \subset \mathbb{R}^N$ fijo. Asuma que existen constantes positivas a_1, a_2, a_3, a_4 tales que* *i.* *Para todo* $t \in I$, $\mathbf{E}\left[X(t)^2\right] \geq a_1 > 0$.

ii. *Existen* $\alpha_1, ..., \alpha_N \in (0,1)$ *tales que para todo* $t = (t_1, ..., t_N), s = (s_1, ..., s_N) \in I$ *se tiene que*

$$a_2 \sum_{j=1}^{N} |t_j - s_j|^{2\alpha_j} \leq \mathbf{E}\,|X(t) - X(s)|^2 \leq a_3 \sum_{j=1}^{N} |t_j - s_j|^{2\alpha_j}.$$

iii. *Para todo* $t = (t_1, ..., t_N), s = (s_1, ..., s_N) \in I$

$$Var(X(t)|X(s)) \geq a_4 \sum_{j=1}^{N} |t_j - s_j|^{2\alpha_j}.$$

Entonces existen constantes positivas a_5, a_6 *tales que para todo boreliano* A *en* $\mathbb{R}^k$

$$a_5 Cap_{k-Q}(A) \leq P(X(I) \cap A \neq \emptyset) \leq a_6 \mathcal{H}_{k-Q}(A)$$

donde $Q = \sum_{j=1}^{N} \frac{1}{\alpha_j}$.

Ahora se presentarán los resultados concernientes a las cotas para la probabilidad de que la solución u de la ecuación (34) arrive a un boreliano. Estas cotas están dadas en términos de la capacidad Newtoniana y la medida de Hausdorff del conjunto en cuestión (ver sub-sección 0.5.1 para las definiciones).

Theorem 8 *Asuma (40) y considere* I, J *compactos no triviales en* $[t_0, T]$ *y* $[-M, M]^d$ *respectivamente. Sea* $N > 0$ *fijo y sea* u *la solución del sistema (34). Entonces para todo boreliano* A *contenido en* $[-N, N]^k$ *se tiene que*

$$C^{-1} Cap_{k-\gamma}(A) \leq P\left(u(I \times J) \cap A \neq \emptyset\right) \leq C\mathcal{H}_{k-\gamma}(A)$$

con

$$\gamma = \frac{2(d+1)}{2H + 1 - \beta}.$$

Remark 5

- *Nuestro resultado engloba lo obtenido en [36] por los investigadores R. C. Dalang & M. Sanz-Solé para el caso* $H = \frac{1}{2}$.

- *También es posible estimar cotas para la probabilidad de que, para* t, x *fijo, los conjuntos* $u(\{t\} \times J)$ *y* $u(I \times \{x\})$ *(como en lo previo,* I, J *compactos no triviales en* $[t_0, T]$ *y en* $[-M, M]^d$ *respectivamente) toquen algún boreliano* A *contenido en* $[-N, N]^k$*. En efecto, usando argumentos de rutina es posible deducir que*

$$C^{-1} Cap_{k-\frac{2d}{2H+1-\beta}}(A) \leq P\left(u(\{t\} \times J) \cap A \neq \emptyset\right) \leq C\mathcal{H}_{k-\frac{2d}{2H+1-\beta}}(A)$$

y

$$C^{-1} Cap_{k-\frac{2}{2H+1-\beta}}(A) \leq P\left(u(I \times \{x\}) \cap A \neq \emptyset\right) \leq C\mathcal{H}_{k-\frac{2}{2H+1-\beta}}(A).$$

Este capítulo de la tesis dio origen al artículo [24],

- Jorge Clarke De la Cerda & Ciprian Tudor *Hitting times for the stochastic wave equation with fractional colored noise* (Aceptado en "Revista Iberoamericana de Matemática", Marzo 2013).

0.5 CAPÍTULO IV: La sábana de Hermite e Integrales de Wiener con respecto a ella

Este último capítulo de la tesis es dedicado a la introducción de la sábana de Hermite, definir las integrales de Wiener con respecto a esta y ejemplificar la potencialidad de la representación presentada a través de una EDE.

Debido al amplio campo de aplicabilidad que poseen, los campos aleatorios o procesos multiparamétricos han concentrado gran atención en la comunidad científica. Dentro de esta familia destacan los campos autosimilares, quienes encuentran utilidad en diversas áreas del conocimiento tales como hidrología, tratamiento de imagenes o cambio climático, por nombrar algunas (ver por ejemplo [8], [12], [65], [82] y [83]). Por otra parte, este tipo de procesos surgen también como soluciones a EDE's en varias dimensiones, lo cual los vuelve aún mas atractivos.
En tal sentido, la sábana de Hermite es un campo aleatorio autosimilar interesante, engloba varias familias de procesos (sábana Browniana fraccionaria, proceso de Rosenblatt, etc.) y considera elementos en todos los caos de Wiener.
Hasta donde sabemos, la sábana de Hermite es por primera vez definida el año 2012 en el trabajo de A. Réveillac, M. Stauch & C. A. Tudor, [87]. En dicha investigación los autores estudian las variaciones de Hermite de la sábana Browniana fraccionaria, con lo cual entre otras cosas, definen la sábana de Hermite como la integral múltiple de orden q del kernel definido como el límite en el espacio de Hilbert $\left(\mathcal{H}^{\alpha,\beta}\right)^q$ de la sucesión de cauchy dada por los kernels involucrados en las variaciones de Hermite. A partir de esta definición los autores demuestran un teorema de límite no-central, dan una expresión para la covarianza de la sábana de Hermite, prueban la autosimilaridad, la estacionareidad de los incrementos y la Hölder continuidad de las trayectorias.

En este capítulo de la tesis se presenta una definición alternativa de la sábana de Hermite, expresándola como una integral estocástica de orden q con respecto a la sábana Browniana standard. Esta definición, a diferencia de la introducida en [87], presenta el kernel involucrado en la integral múltiple de manera explícita, con lo cual se obtiene una expresión para la sábana de Hermite que posibilita una mejor comprensión y manipulación de la misma. Estas ventajas quedan en evidencia al demostrarse de manera alternativa, explotando la expresión ilustrada, las mismas propiedades probadas en [87], y también, al definir las integrales de Wiener con respecto a la sábana de Hermite.

La potencialidad de esta última definición se ejemplifica a través del estudio de una ecuación de la onda con ruido dado por la sábana de Hermite $(d+1)$-paramétrica.

Antes de presentar los resultados obtenidos en este capítulo de la tesis se enuncian la notación y ciertas definiciones necesarias para la compresión de lo expuesto posteriormente.

0.5.1 Notación y definiciones

En lo que sigue de esta sección se utiliza la siguiente notación.
Sea $d \in \mathbb{N}\backslash\{0\}$, consideramos procesos multi-paramétricos indexados en $[0,1]^d$. Para los multi-índices se usará notación marcada, i.e., $\mathbf{a} = (a_1, a_2, ..., a_d)$, $\mathbf{ab} = (a_1b_1, a_2b_2, ..., a_db_d)$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, ..., a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_i^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_i^d (a_i, b_i)$, $\sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \dots \sum_{i_d}^{N_d} a_{i_1,i_2,...,i_d}$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, y $\mathbf{a} < \mathbf{b}$ si y solo si $a_1 < b_1, a_2 < b_2, ..., a_d < b_d$ (análogamente para las otras desigualdades).

Recordamos la definición de los incrementos de un proceso d-paramétrico X sobre un rectángulo $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, ..., s_d), \mathbf{t} = (t_1, ..., t_d)$, con $\mathbf{s} \le \mathbf{t}$. Dichos incrementos se denotan por $\Delta X_{[\mathbf{s},\mathbf{t}]}$ y se definen por

$$\Delta X_{[\mathbf{s},\mathbf{t}]} = \sum_{r\in\{0,1\}^d} (-1)^{d-\sum_i r_i} X_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}.$$

Cuando $d = 1$ se obtiene $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_t - X_s$, mientras que para $d = 2$ resulta, $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}$.

Definition 1 *Un campo aleatorio* $(X_{\mathbf{t}})_{\mathbf{t}\in T}$, *donde* $T \subset \mathbb{R}^d$ *se dice auto-similar con parámetro de autosimilaridad* $\alpha = (\alpha_1, ..., \alpha_d)$ *si para todo* $\mathbf{h} = (h_1, .., h_d) > 0$ *el campo* $(\hat{X}_{\mathbf{t}})_{\mathbf{t}\in T}$ *dado por*

$$\hat{X}_{\mathbf{t}} = \mathbf{h}^\alpha X_{\frac{\mathbf{t}}{\mathbf{h}}} = h_1^{\alpha_1} ... h_d^{\alpha_d} X_{\frac{t_1}{h_1},...,\frac{t_d}{h_d}}$$

tiene la misma ley que el campo X.

Definition 2 *Un campo* $(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^d)$ *posee incrementos estacionarios si para todo* $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$ *los campos aleatorios* $(\Delta X_{[0,\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ *y* $(\Delta X_{[\mathbf{h},\mathbf{h}+\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ *tienen idénticas distribuciones finito dimensionales.*

0.5.2 Las sábanas Brownianas standard y fraccionaria d-paramétricas

La *sábana Browniana standard d-paramétrica* se define como el proceso Gaussiano $\{W_{\mathbf{t}} : \mathbf{t} = (t_1, ..., t_d) \in [0,1]^d\}$ igual a cero en los hiperplanos $\{\mathbf{t} : t_i = 0\}$, $1 \le i \le d$, y de función de covarianza dada por

$$R(\mathbf{s}, \mathbf{t}) = \mathbb{E}[W_{\mathbf{s}}, W_{\mathbf{t}}] = \prod_i^d R(s_i, t_i) = \prod_i^d s_i \wedge t_i. \qquad (42)$$

La *sábana Browniana fraccionaria* y la *sábana Browniana standard.*
La *sábana Browniana fraccionaria d-paramétrica* se define como el proceso Gaussiano centrado $\{B_{\mathbf{t}}^{\mathbf{H}} : \mathbf{t} = (t_1, ..., t_d) \in [0,1]^d\}$ con multi-índice de Hurst $\mathbf{H} = (H_1, ..., H_d) \in (0,1)^d$ igual a cero en los hiperplanos $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, y cuya función de covarianza esta dada por

$$\begin{aligned} R_{\mathbf{H}}(\mathbf{s}, \mathbf{t}) &= \mathbb{E}[B_{\mathbf{s}}^{\mathbf{H}}, B_{\mathbf{t}}^{\mathbf{H}}] \\ &= \prod_{i}^{d} R_{H_i}(s_i, t_i) = \prod_{i}^{d} \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2}. \end{aligned} \tag{43}$$

Siguiendo los conceptos introducidos en la sección ***0.2***, se definen de forma natural los espacios de Hilbert $\mathcal{H}$ y $\mathcal{H}^{\mathbf{H}}$ asociados a la sábana Browniana standard y a la sábana Browniana fraccionaria respectivamente. De la misma forma, también se consideran los espacios producto tensorial $\mathcal{H}^q$ y $\left(\mathcal{H}^{\mathbf{H}}\right)^q$. Para obtener una concepción un poco más detallada y ejemplificadora de la sábana Browniana fraccionaria y el espacio de Hilbert asociado a esta, referimos a la sub-sección ***0.4.1*** donde se ilustra el caso bi-paramétrico.

0.5.3 La sábana de Hermite e Integrales de Wiener respecto a ella.

A continuación se presentan las definiciones y resultados obtenidos en este último capítulo de la tesis.

Sea $q \geq 1, q \in \mathbb{Z}$ y consideremos el muilti-índice de Hurst $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$. La *sábana de Hermite de orden q* es definida por

$$\begin{aligned} Z_{\mathbf{H}}^q(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{t_1} \dots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \dots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\ &\quad ds_d \dots ds_1 \quad dW(y_{1,1}, ..., y_{d,1}) \dots dW(y_{1,q}, ..., y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d \cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \dots dW(\mathbf{y}_q). \end{aligned} \tag{44}$$

donde $x_+ = max(x, 0)$. Para una mejor comprensión de las integrales estocásticas múltiples referimos al lector a la sección ***0.2*** o a [77].

De la expresión previa se observa que para $q = 1$, (44) es la sábana Browniana fraccionaria con multi-índice de Hurst $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$; para $q \geq 2$ el proceso $Z_{\mathbf{H}}^q(\mathbf{t})$ no es Gaussiano y para $q = 2$ le denominamos como la *sábana de Rosenblatt*. Referimos al lector a los trabajos [1], [9], [28], [88], [95] y [100] para un estudio más acabado sobre los procesos de Rosenblatt.
Por otra parte, también es posible concluir que la función de covarianza de la sábana de Hermite satisface

$$\begin{aligned} R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= \prod_i^d \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2} \\ &= \prod_i^d R_{H_i}(s_i,t_i) = R_{\mathbf{H}}(\mathbf{s},\mathbf{t}), \end{aligned} \tag{45}$$

i.e. su función de covarianza es idéntica a la de la sábana Browniana fraccionaria.

Haciendo uso de la definición propuesta se demuestran las siguientes propiedades de la sábana de Hermite,

Proposition 5 *La sábana de Hermite es auto-similar de orden* $\mathbf{H} = (H_1, ..., H_d)$.

Proposition 6 *La sábana de Hermite* $(Z^q(\mathbf{t}))_{\mathbf{t}\geq 0}$ *posee incrementos estacionarios.*

Proposition 7 *Las trayectorias de la sábana de Hermite* $(Z^q(\mathbf{t}), \mathbf{t} \geq 0)$ *son Hölder continuas de orden* $\delta = (\delta_1, .., \delta_d) \in [0, \mathbf{H}]$ *en el siguiente sentido: para todo* $\omega \in \Omega$*, existe una constante* $C_\omega > 0$ *tal que para todo* $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d, \mathbf{s}, \mathbf{t} \geq 0$,

$$|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}| \leq C_\omega |t_1 - s_1|....|t_d - s_d|.$$

Los resultados previamente enunciados ponen de manifiesto la utilidad de la definición presentada para la sábana de Hermite, estas caracteristicas cobran más realce en la construcción de las integrales de Wiener con respecto a la sábana de Hermite d-paramétrica, la que se presenta a continuación.

Sea $\mathscr{E}$ la familia de funciones elementales en $\mathbb{R}^d$ de la forma

$$\begin{aligned} f(\mathbf{u}) &= \sum_{l=1}^n a_l 1_{(\mathbf{t}_l,\mathbf{t}_{l+1}]}(\mathbf{u}) \\ &= \sum_{l=1}^n a_l 1_{(t_{1,l},t_{1,l+1}]\times\ldots\times(t_{d,l},t_{d,l+1}]}(u_1,\ldots,u_d), \qquad \mathbf{t}_l < \mathbf{t}_{l+1}, \;\; a_l \in \mathbb{R}, \;\; l = 1,\ldots,n. \end{aligned} \tag{46}$$

Para este tipo de funciones es natural definir su integral de Wiener con respecto a la sábana de Hermite $Z^q_{\mathbf{H}}$ como,

$$\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) = \sum_{l=1}^n a_l \Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right), \tag{47}$$

donde $\Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right)$ representa los incrementos generalizados de $Z^q_{\mathbf{H}}$ sobre el rectángulo

$$\Delta_{\mathbf{t}_l} := [\mathbf{t}_l, \mathbf{t}_{l+1}] = \prod_{i=1}^{d} [t_{i,l}, t_{i,l+1}] ,$$

dados por

$$\Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right) = \sum_{\xi\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^{d}\xi_i} Z^q_{\mathbf{H}}(t_{1,l+\xi_d}, \ldots, t_{d,l+\xi_1}). \tag{48}$$

El mapeo

$$f \to \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) \tag{49}$$

es una isometría entre $\mathscr{E}$ y $L^2(\Omega)$.

Se define el funcional J sobre el conjunto de funciones $f : \mathbf{R}^d \to \mathbf{R}$ y a valores en el conjunto de funciones $f : \mathbf{R}^{d\cdot q} \to \mathbf{R}$ por

$$J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q) \;=\; c(\mathbf{H}, q) \int_{\mathbb{R}^d} f(\mathbf{u}) \prod_{j=1}^{q} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1-H}}{\mathbf{q}}\right)} d\mathbf{u}. \tag{50}$$

Esta definición permite reexpresar la definición (47) de la forma

$$\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) \;=\; \int_{\mathbb{R}^{d\cdot q}} J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q) dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q).$$

y extender de manera natural la definición de la integral al espacio

$$\mathfrak{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^{d\cdot q}} \left(J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q)\right)^2 d\mathbf{y}_1, \ldots, d\mathbf{y}_q < \infty \right\} \tag{51}$$

con la norma

$$\|f\|^2_{\mathfrak{H}} = \int_{\mathbb{R}^{d\cdot q}} \left(J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q)\right)^2 d\mathbf{y}_1, \ldots, d\mathbf{y}_q. \tag{52}$$

Se demuestra que este espacio es igual a

$$\mathfrak{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H_1-2}} d\mathbf{u} d\mathbf{v} < +\infty \right\} \tag{53}$$

con la norma

$$\|f\|^2_{\mathfrak{H}} = \mathbf{H(2H-1)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u} d\mathbf{v}. \tag{54}$$

Considerando los conceptos introducidos en la sección ***0.2*** es posible notar que para $f \in \mathfrak{H}$

$$\int_{\mathbb{R}^d} f(\mathbf{u})dZ^q_{\mathbf{H}}(\mathbf{u}) = \int_{\mathbb{R}^{d\cdot q}} J(f)(\mathbf{y}_1,\ldots,\mathbf{y}_q)dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q) = I_q\left(J(f)(\mathbf{y}_1,\ldots,\mathbf{y}_q)\right) \quad (55)$$

donde $I_q : \mathcal{H}^q \to \mathcal{H}_q$ es la integral múltiple de orden q con respecto a la sábana Browniana standard, toma valores en el q-ésimo caos de Wiener y es una isometría entre $\left(\mathcal{H}^{\odot q}, \sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes q}}\right)$ y $\mathcal{H}_q$. Es decir, la definición presentada para la integral de Wiener con respecto a la sábana de Hermite otorga una útil interpretación como integral múltiple con respecto a la sábana Browniana standard. Debido a la gran cantidad existente de fuertes resultados relacionados con integrales múltiples respecto de procesos Gaussianos (ver por ejemplo [77]), es que la interpretación (55) abre un camino para el estudio de procesos multiparamétricos no-Gaussianos posibles de definir como una integral de Wiener respecto de la sábana de Hermite.

Explotando el hecho que la estructura de covarianza de la sábana de Hermite es igual a la de la sábana Browniana fraccionaria es posible obtener resultados que se deducen de lo demostrado en el capítulo anterior de esta tesis. A continuación se presenta el contexto y los resultados logrados.

De manera análoga a lo presentado en la sección ***0.5*** para el tercer capítulo de esta tesis, se considera una ecuación lineal estocástica de la onda dirigida por la sábana de Hermite infinito dimensional $Z^q_{\mathbf{H}}$ con multi-índice de Hurst $\mathbf{H} \in (1/2,1)^{(d+1)}$, i.e.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{Z}^q_{\mathbf{H}}(t,\mathbf{x}), \quad t>0, \mathbf{x}\in[0,1]^d \quad (56)\\ u(0,x) &= 0, \quad \mathbf{x}\in[0,1]^d\\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad \mathbf{x}\in[0,1]^d,\end{aligned}$$

donde Δ es el Laplaciano en $\mathbb{R}^d$ y $Z^q_{\mathbf{H}} = \{Z^q_{\mathbf{H}}(t,\mathbf{x}); t \geq 0, \mathbf{x} \in [0,1]^d\}$ es la sábana de Hermite $(d+1)$-paramétrica de covarianza

$$\mathbb{E}\left\{\dot{Z}^q_{\mathbf{H}}(s,\mathbf{x})\dot{Z}^q_{\mathbf{H}}(t,\mathbf{y})\right\} = H(2H-1)|t-s|^{2H-2}\prod_{i=1}^{d}(H_i(2H_i-1)\cdot|x_i-y_i|^{2H_i-2}). \quad (57)$$

A partir de lo expuesto en la sub-sección ***0.5.3*** se deduce la expresión para la solución de la ecuación (56), sus carácteristicas, y las condiciones necesarias para asegurar su existencia.
En lo que sigue se presentan los resultados obtenidos respecto de la ecuación (56).

Proposition 8 *Sea $Z^q_{\mathbf{H}}(t,\mathbf{x})$ la sábana de Hermite $(d+1)$-paramétrica de orden q. Las siguientes afirmaciones son verdaderas.*

a.- *La ecuación estocástica de la onda (56) admite una única solución tipo "campo aleatorio" $(u(t,\mathbf{x}))_{t\in[0,1],\mathbf{x}\in[0,1]^d}$ si y solo si*

$$\sum_{i=1}^{d}(2H_i-1) > d-2H-1. \quad (58)$$

b.- *Sea* t_0 *y* $\mathbf{x} \in [0,1]^d$ *fijos. Existen constantes positivas* c_1, c_2*, tales que para todo* $s, t \in [t_0, 1]$

$$c_1|t-s|^{2H+1-\beta} \leq \mathbb{E}\,|u(t,\mathbf{x}) - u(s,\mathbf{x})|^2 \leq c_2|t-s|^{2H+1-\beta}.$$

Además. para todo $\mathbf{x} \in [0,1]^d$ *fijo, la aplicación*

$$t \to u(t,\mathbf{x})$$

es casi seguramente Hölder continua de orden $\delta \in \left(0, \frac{2H+1-\beta}{2}\right)$.

c.- *Sean* $t \in [t_0, T]$ *fijos. Existen constantes positivas* c_3, c_4 *tales que para todo* $\mathbf{x}, \mathbf{y} \in [0,1]^d$

$$c_3|\mathbf{x}-\mathbf{y}|^{2H+1-\beta} \leq \mathbb{E}\,|u(t,\mathbf{x}) - u(t,\mathbf{y})|^2 \leq c_4|\mathbf{x}-\mathbf{y}|^{2H+1-\beta}.$$

Además, para todo $t \in [t_0, 1]$ *la aplicación*

$$\mathbf{x} \to u(t,\mathbf{x})$$

es casi seguramente Hölder continua de orden $\delta \in \left(0, \left(\frac{2H+1-\beta}{2}\right) \wedge 1\right)$.

Theorem 9 *Sea* $M > 0$ *fijo y supongamos (58). Para todo* $t, s \in [t_0, 1]$ *y* $\mathbf{x}, \mathbf{y} \in [0,1]^d$ *existen constantes positivas* C_1, C_2 *tales que*

$$C_1\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right) \leq \mathbf{E}\,|u(t,\mathbf{x}) - u(s,\mathbf{y})|^2 \leq C_2\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right).$$

donde Δ *denota la métrica sobre* $[0,T] \times \mathbb{R}^d$ *dada por*

$$\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right) = |t-s|^{2H+1-\beta} + |\mathbf{x}-\mathbf{y}|^{2H+1-\beta}. \tag{59}$$

Los resultados obtenidos en este último capítulo de la tesis forman parte del artículo

- Jorge Clarke De la Cerda & Ciprian Tudor *Wiener integrals with respect to the Hermite random field* (Sometido en "Collectanea Mathematica").

Antes de finalizar la introducción cabe señalar que cada capítulo de la tesis es autocontenido en el sentido de que se corresponden con investigaciones aceptadas o sometidas, exepto el último que se encuentra en preparación. Por ello, la notación utilizada será la misma que en el artículo correspondiente e independiente en cada capítulo.

La tesis finaliza con las conclusiones y una breve descripción de las líneas de investigación abiertas.

Chapter 1

CAPÍTULO I: Discrete time approximation of delay stochastic differential equations driven by fractional Brownian motion

1.1 Introduction

Fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $(B_t^H)_{t\in[0,1]}$ whose covariance function can be written as

$$E\left[B_t^H\, B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

The family of processes $\{B^H;\, H \in (0,1)\}$ enjoys several nice properties:

- For $H = 1/2$, one recovers the classical Brownian motion.
- For any $H \in (0,1)$, the paths of B^H are almost surely $(H-\rho)$-Hölder continuous for any arbitrarily small $\rho > 0$. Specifically, we have

  $$|B^H(t) - B^H(s)| < F_0|t-s|^{H-\rho} \quad \text{a.s.} \quad t,s \in [0,T], \tag{1.1}$$

  where $F_0 = F_0(\omega)$ is a positive random variable such that $E(F_0^p) < \infty$ for all $p \geq 1$.
- The covariance of the increments of B^H on intervals decays asymptotically as a negative power of the distance between the intervals.
- Fractional Brownian motion is the only finite-variance process which is self-similar (with index H) and has stationary increments.

These characteristics have converted the fractional brownian family into the most natural generalization of Brownian motion among the probability community, but also for practitioners, in the recent years.

At a theoretical level, it should be noticed that the martingale type techniques used for the construction of a stochastic calculus with respect to the usual Brownian motion $B^{1/2}$ cannot be invoked anymore when $H \neq 1/2$. However, when $H \in (1/2, 1)$ one can define stochastic integrals and solutions to differential equations thanks to Young (see e.g. [47, 49] for an account on these techniques) or fractional calculus (as explained in [79, 106]) methods. The case $H \in (0, 1/2)$ is avoided in this article for sake of readability, but let us mention that one has to appeal to rough paths techniques (for which we refer to [47, 49] again) in order to solve stochastic equations in this situation.

As far as applications of differential systems are concerned, the wide range of contexts in which fBm driven models are used includes Biophysics [61, 94, 97], electrical engineering [40] and finance [25, 48, 52, 53, 105] situations. All those applications involve ordinary or Volterra type differential equations, but delayed systems can also be of huge importance. Indeed, to mention just a single biomedical example, bacteriophage systems are commonly described by delayed equations. Specifically, what we call bacteriophages are harmless viruses meant to attack bacteria involved in animal diseases according to a so-called lytic process. In short, the virus genetic material penetrates into the bacteria and uses the host replication mechanism to self-replicate. This lytic step induces a complex chain of reactions and takes about 30mn to be completed, while treatments are usually measured in hours. Thus, mathematical modelings of the treatment naturally involve delayed equations, as assessed by the recent articles [10, 20, 21, 90]. While the aforementioned references are concerned either with deterministic or Brownian driven equations for sake of simplicity, let us stress the fact that there are experimental evidences that fBm models should also be dealt with in this context.

With those motivations in mind, let us proceed to the mathematical description of the model we are dealing with. Namely, we consider the following stochastic delay differential equation driven by a fractional Brownian motion B^H (FSDDE) with Hurst parameter $H > 1/2$,

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_{t-r})dB_t^H, \quad t \in [0, T] \\ X_s &= \phi(s) \quad s \in [-r, 0] \end{aligned} \tag{1.2}$$

where ϕ is a Hölder continuous function on $[-r, 0]$ and r is a positive time delay. As a solution to this equation we shall define a process $\{X_t, t \in [-r, T]\}$ satisfying

$$\begin{aligned} X_t &= \phi(0) + \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s-r})dB_s^H, \quad t \in [0, T] \\ X_t &= \phi(t), \qquad t \in [-r, 0], \end{aligned} \tag{1.3}$$

where the integral respect to fractional Brownian motion is the generalized Riemann-Stieltjes integral introduced in [106]. It is worth mentioning that this kind of equation has been first introduced in [45] and further studied in [46]. Its analysis relies heavily on the fact that the stochastic term $\sigma(X_{s-r})$ is located strictly in the past of the process X. By means of Young integral methods, a very general nonlinearity of the form $\sigma(\{X_u;\, u \in [t-r,t]\})$ is handled in [63], while the case of $1/3 < H < 1/2$ with a nonlinearity of the form $\sigma(X_{s-r_q}, \ldots, X_{s-r_1}, X_s)$ for $0 < r_1 < \cdots < r_q \leq r$ is treated in [74] thanks to rough paths techniques. Here we stick to the original setting of [45] for sake of simplicity, and defer the analysis of the other related models to a subsequent publication.

In this context, the current article focuses on a sequence of discrete time approximations Y^l of the solution X to the FSDDE (1.3), on a compact interval $[0,T]$, following the ideas of [71]. In a natural way, this scheme will be based on a regular partition of $[0,T]$ with mesh proportional to $1/l$ and we will pursue two dual objectives:

1. At a theoretical level, we will show a strong convergence result for the sequence Y^l. Namely, under suitable assumptions on the coefficients of (1.3), we shall see that almost surely, the difference $\sup_{t\in[0,T]} |Y_t^l - X_t|$ is of order $l^{-(2H-1-2\rho)}$ for ρ arbitrarily small. This means that one can reach the same rate of convergence as in the non-delayed case, which was first obtained in [71, 73].

2. We then illustrate our theoretical results by simulations in order to depict the typical path of a delayed differential equation driven by fractional Brownian motion. In particular, the reader will observe the influence of the Hurst parameter H in terms of regularity of the path and convergence of the numerical scheme.

Let us observe that, since we are sticking to Ferrante-Rovira's setting for delay equations, we shall also analyze the convergence of our numerical scheme thanks to fractional calculus techniques as in [45, 46]. Our main technical task will then be to combine these tools with numerical analysis considerations.

On the other hand, the solution of the FSDDE can be expressed as a sequence of solutions of systems of fractional stochastic differential equations FSDE without time delay, in this way we can obtain weak approximations of the FSDDE.

This paper is organized as follows. Section 1.2 is devoted to some preliminaries related to stochastic integral with respect to fractional Brownian motion and conditions for existence and uniqueness of solution of the fractional stochastic differential delay equations. In section 1.3 we define our discrete Euler scheme for the solution of FSDDE (1.2) and we study its rate of convergence, in the space of the γ-Hölder continuous functions for $0 < \gamma < H$. In section 1.4 we express the solution of the FSDDE as solutions of systems of fractional stochastic differential equations FSDE without time delay. Finally, in section 1.5 some numerical examples are given.

1.2 Preliminaries

As mentioned in the introduction, our computations will be based on fractional calculus considerations. We proceed now to recall some basic notions on this topic, starting with the definition of some functional spaces: for $0 \leq \alpha \leq 1$, denote by $W^{\alpha}(a,b;\mathbb{R})$ the space of measurable functions $f : [a,b] \to \mathbb{R}$ such that

$$\sup_{t\in[a,b]} \left(|f(t)| + \int_a^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty,$$

and denote by $C^\alpha(a,b;\mathbb{R})$ the space of α-Hölder continuous functions $f:[a,b]\to\mathbb{R}$ equipped with the norm

$$\sup_{a\le t\le b}|f(t)|+\sup_{a\le s<t\le b}\frac{|f(t)-f(s)|}{(t-s)^\alpha}<\infty.$$

We now turn to the notion of generalized integral with respect to our fBm B^H which shall be used in the sequel: as in [79, 106] we set

$$\int_a^b f dB^H = \int_a^b (D^\alpha_{a+}f)(s)(D^{1-\alpha}_{b-}B^H_{b-})(s)ds, \tag{1.4}$$

where

$$(D^\alpha_{a+}f)(s)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(s)}{(s-a)^\alpha}+\alpha\int_a^s\frac{f(s)-f(u)}{(s-u)^{1+\alpha}}du\right]I_{(a,b)}(s), \tag{1.5}$$

and where we have also defined the fractional derivative $(D^{1-\alpha}_{b-}B^H_{b-})(s)$ as

$$\frac{\exp^{-i\pi\alpha}}{\Gamma(\alpha)}\left[\frac{B^H_{b-}(s)}{(b-s)^{1-\alpha}}+(1-\alpha)\int_s^b\frac{B^H_{b-}(s)-B^H_{b-}(u)}{(u-s)^{2-\alpha}}du\right]I_{(a,b)}(s), \tag{1.6}$$

with the convention

$$B^H_{b-}(s)=(B^H_s-B^H_b)I_{(a,b)}(s).$$

If $f\in C^\nu(a,b;\mathbb{R})$ for $\nu+H>1$, it is shown e.g. in [79] that the pathwise fractional integral (1.4) exists for any $\alpha\in(1-H,\nu)$ and that the following estimate holds true:

$$\left|\int_a^b f dB^H\right|\le F_1(\omega)\left[\int_a^b\frac{|f(s)|}{(s-a)^\alpha}ds+\int_a^b\int_a^s\frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}}duds\right], \tag{1.7}$$

where the random variable F_1 is given by

$$F_1(\omega)=C\cdot\sup_{a<s<b}|D^{1-\alpha}_{b-}B^H_{b-}(s)|, \tag{1.8}$$

and satisfies the inequality $E(F_1^p)<\infty$ for all $p\in[1,\infty)$.

Let us specify the regularity conditions we impose on the coefficients b,σ in order to get existence and uniqueness results for equation (1.3):

H1 $\sigma:\mathbb{R}\to\mathbb{R}$ is a measurable function such that $\sigma(x)$ is differentiable in x and there exist a constant $0<\delta\le 1$ and $M_0>0$ such that for every $N\ge 0$ one can find a constant $M_N>0$ such that

(a) $|\sigma(x)-\sigma(y)|\le M_0|x-y| \quad \forall x\in\mathbb{R}$.

(b) $|\sigma'(x)-\sigma'(y)|\le M_N|x-y|^\delta$, $\forall |x|,|y|\le N$.

H2 $b:\mathbb{R}\to\mathbb{R}$ is a measurable function such that for every $N\ge 0$ there exists $L_N>0$ such that

(a) $|b(x)-b(y)|\le L_N|x-y|$, for any $|x|,|y|\le N$.

(b) $|b(x)|\le L_0(|x|+1)$.

H3 There exists $\alpha \in (1-H, 1/2)$ such that $\phi : [-r, 0] \to \mathbb{R}$ is a measurable function satisfying

(a) $\phi \in W^{\alpha}(-r, 0; \mathbb{R})$,

(b) $\phi \in C^{1-\alpha}(-r, 0; \mathbb{R})$.

With these preliminaries and assumptions in mind, let us recall the following existence and uniqueness result introduced in [46] (Theorem 1.1) usefull for our delay equation:

Theorem 10 *Consider a fBm B with Hurst parameter $H \in (1/2, 1)$. Let b, σ and ϕ functions that satisfy hypothesis (**H1**), (**H2**) and (**H3**) respectively with $\delta > (1/H) - 1$ and $\alpha_0 := \delta/(1+\delta)$. If $\alpha \in (1-H, \alpha_0)$ then equation (1.3) admits a unique solution X such that*

$$X \in L^0(\Omega, \mathcal{F}, P; W^{\alpha}(-r, T; \mathbb{R}))$$

and $X \in C^{1-\alpha}(-r, T; \mathbb{R})$ a.s.

Remark 6 *By theorem 10, for any $0 < \rho < H$ such that $H - \rho < 1 - \alpha$, equation (1.3) possesses a unique solution X such that $X \in C^{H-\rho}(-r, T; \mathbb{R})$ a.s.*

1.3 The Euler scheme

We now follow the lines of [62] in order to give a numerical approximation for the solution of equation (1.3). Recall that we have assumed that $T = N_0 r$ for some given integer N_0. In a natural way, we thus define the time step size for our scheme as

$$\Delta_l = \frac{r}{l}, \quad l \in \{2, 3, ...\}. \tag{1.9}$$

and the equidistant time discretization

$$\mathcal{T}_{\Delta_l} = \{\tau_n : n \in \{-l, -l+1, ..., 0, 1, ..., N\}\}$$

of the time interval $[-r, T]$ where

$$\tau_n = n\Delta_l, \quad \text{and} \quad N := N_0 l. \tag{1.10}$$

Let us also introduce a further notation: for $s \in [-r, T]$, we set $n_s = \max\{n \geq -l : \tau_n \leq s\}$ and $t_s = \tau_{n_s}$.

With these notations in mind, here is now how our Euler scheme for Equation (1.3) is defined: we set recursively

$$Y_{n+1} = Y^{\Delta_l}_{\tau_{n+1}} = Y_n + b(Y_n)\Delta_l + \sigma(Y_{n-l})\Delta B^H_n \tag{1.11}$$

with

$$\Delta B^H_n = B^H_{\tau_{n+1}} - B^H_{\tau_n}, \quad \text{for} \;\; n = 0, 1, \cdots, N-1,$$

and with initial values

$$Y_i = X_{\tau_i} = \phi(\tau_i) \quad i = -l, -l+1, \cdots, 0.$$

We then complete the definition of our numerical scheme by continuous interpolation:

$$Y_t = Y_t^{\Delta_l} = Y_n + b(Y_n)(t-\tau_n) + \sigma(Y_{n-l})(B_t^H - B_{\tau_n}^H), \quad \text{for } t \in [\tau_n, \tau_{n+1}] \tag{1.12}$$

with initial condition $Y_t = \phi(t)$ for $t \in [-r, 0]$. Observe that this Euler type scheme can be summarized as:

$$Y_t = \begin{cases} \phi(0) + \int_0^t b(Y_{t_u})du + \int_0^t \sigma(Y_{t_u - r})dB_u^H & \text{for} \quad t \in [0, T] \\ \phi(t) & \text{for} \quad t \in [-r, 0]. \end{cases} \tag{1.13}$$

In the following lemma we obtain that the approximation (1.13) satisfies the Hölder condition of order $H-\rho$ with $H-\rho < 1-\alpha$. The proof is similar to the proof of Theorem 2.3 in [71] and we omit the details.

Lemma 1 *Under hypothesis **H1**, **H2** and **H3**, for $\epsilon > 0$ and $0 < \rho < H$, there exists $\Delta_0 > 0$ and $\tilde{\Omega} = \Omega_{\epsilon, \Delta_0, \rho}$ such that $P(\tilde{\Omega}) > 1 - \epsilon$ and for any $\omega \in \tilde{\Omega}$ and $\Delta_l < \Delta_0$, there exists a constant $F_Y = F_Y(\omega)$ does not depend on Δ_l such that*

$$|Y(s) - Y(u)| \leq F_Y(\omega)(s-u)^{H-\rho}, \quad 0 \leq u < s \leq T.$$

We can now state our main theorem in terms of numerical approximation:

Theorem 11 *Let B^H a fractional Brownian motion with $H > 1/2$ and let ϕ, b and σ functions that satisfy hypothesis **H1**, **H2** and **H3** respectively with $\alpha > (1/H) - 1$ and $\alpha_0 = \delta/(1+\delta)$. For every $\varepsilon > 0$ and for a given sufficiently small $0 < \rho < (H - 1/2) \wedge (1-H)$, if $\alpha \in (1-H, (1-H+\rho) \wedge \alpha_0)$, then there exists $\Delta_0 > 0$ and $\tilde{\Omega} = \Omega_{\epsilon, \Delta_0, \rho}$ such that $P(\tilde{\Omega}) > 1 - \epsilon$ and for any $\omega \in \tilde{\Omega}$ and $\Delta_l < \Delta_0$,*

$$\sup_{-r \leq t \leq T} |X_t - Y_t| \leq F(\omega) \Delta_l^{2H-1-2\rho}, \tag{1.14}$$

where $F(\omega)$ does not depend on Δ_l and ε.

Proof: We give the proof by recursion over $k \in \{0, 1, ..., N_0\}$ for the intervals $[(k-1)r, kr]$, namely

$$\sup_{(k-1)r \leq t \leq kr} |X_t - Y_t| \leq C(\omega) \Delta_l^{2H-1-2\rho}.$$

We now divide our study into several cases.

Case 1: $\mathbf{k = 0}$. In this case our approximation result is obvious.

Case 2: $\mathbf{k = 1}$. If $t \in [0, r]$, by (1.3) and (1.13),

$$X_t = \phi(0) + \int_0^t b(X_s)ds + \int_0^t \sigma(\phi_{s-r})dB_s^H \tag{1.15}$$

and

$$Y_t = \phi(0) + \int_0^t b(Y_{t_s})ds + \int_0^t \sigma(\phi_{t_s - r})dB_s^H. \tag{1.16}$$

Set then $Z_t^{(1)} := \sup_{0\le s\le t}|X_s - Y_s|$. Owing to hypotesis **H2** and (1.7) we get

$$Z_t^{(1)} \le \sup_{0\le s\le t}\int_0^s |b(X_u) - b(Y_{t_u})|du + \sup_{0\le s\le t}|\int_0^s [\sigma(\phi_{u-r}) - \sigma(\phi_{t_u-r})]dB_u^H| \le I_1 + I_2 + I_3 + I_4, \tag{1.17}$$

with the convention $F_1 = F_1(\omega)$ and

$$I_1 = \int_0^t |b(X_u) - b(Y_u)|du$$
$$I_2 = \int_0^t |b(Y_u) - b(Y_{t_u})|du$$
$$I_3 = F_1\int_0^t |\sigma(\phi_{u-r}) - \sigma(\phi_{t_u-r})|u^{-\alpha}du$$
$$I_4 = F_1\int_0^t\int_0^u |\sigma(\phi_{u-r}) - \sigma(\phi_{t_u-r}) - \sigma(\phi_{z-r}) + \sigma(\phi_{t_z-r})|(u-z)^{-\alpha-1}dzdu.$$

We now treat those 4 terms separately: first, it is easily seen that

$$I_1 \le L\int_0^t \sup_{-r\le s\le u}|X_s - Y_s|du = L\int_0^t \sup_{0\le s\le u}|X_u - Y_u|du \le L\int_0^t Z_u^{(1)}du. \tag{1.18}$$

Furthermore, setting $F_2 = F_2(\omega) = \sup_{-r\le t\le T}|Y_t|$, notice that F_2 is almost surely finite with bounded moments (see [63, Theorem 3.4] for a similar statement). Hence for $0 < \rho < H$, using relations (1.13), (1.1) and hypothesis **H** we get

$$|Y_u - Y_{t_u}| = |b(Y_{t_u})(u-t_u) + \sigma(Y_{t_u-r})(B_u^H - B_{t_u}^H)| \le L(|Y_s| - 1)(u-t_u) + (1+|Y_{t_u}|)F_0(u-t_u)^{H-\rho} \le F_3(u-t_u)^{H-\rho} \le F_3\Delta_l^{H-\rho}, \tag{1.19}$$

where $F_3 = F_3(\omega) = C(1+F_2F_0)$ and C is a strictly positive constant. We thus obtain

$$I_2 \le L\int_0^t \sup_{-r\le s\le u}|Y_s - Y_{t_s}|du \le CF_3\Delta_l^{H-\rho}. \tag{1.20}$$

The integral I_3 can be simply bounded as follows:

$$I_3 \le MF_1\int_0^t |\phi_{u-r} - \phi_{t_u-r}|u^{-\alpha}du \le MF_1\int_0^t |\phi_{u-r} - \phi_{t_u-r}|u^{-\alpha}du \le MCF_1\int_0^t (u-t_u)^{1-\alpha}u^{-\alpha} < CF_1\Delta_l^{1-\alpha}. \tag{1.21}$$

In order to handle the term I_4 above, notice that $t_z = t_u$ for $t_u \le z < u$. Hence

$$\begin{aligned}
I_4 &= F_1 \int_0^t \int_0^u |\sigma(\phi_{u-r}) - \sigma(\phi_{t_u-r}) - \sigma(\phi_{z-r}) + \sigma(\phi_{t_z-r})|(u-z)^{-\alpha-1} dz du \\
&\le F_1 \int_0^t \int_0^{t_u} [|\sigma(\phi_{u-r}) - \sigma(\phi_{t_u-r})| + |\sigma(\phi_{z-r}) - \sigma(\phi_{t_z-r})|](u-z)^{-\alpha-1} dz du \\
&+ F_1 \int_0^t \int_{t_u}^u [|\sigma(\phi_{u-r}) - \sigma(\phi_{z-r})|](u-z)^{-\alpha-1} dz du \\
&\le F_1 M_0 \int_0^t \int_0^{t_u} [|\phi_{u-r} - \phi_{t_u-r}| + |\phi_{z-r} - \phi_{t_z-r}|](u-z)^{-\alpha-1} dz du \\
&+ F_1 M_0 \int_0^t \int_{t_u}^u |\phi_{u-r} - \phi_{z-r}|(u-z)^{-\alpha-1} du du \\
&\le C F_1 \int_0^t \int_0^{t_u} [(u-t_u)^{H-\rho} + (z-t_z)^{H-\rho}](u-z)^{-\alpha-1} dz du \\
&+ C F_1 \int_0^t \int_{t_u}^u (u-z)^{H-\rho-\alpha-1} dz du \\
&\le C F_1 \Delta_l^{H-\rho} \int_0^t \int_0^{t_u} (u-z)^{-\alpha-1} dz du + C F_1 \int_0^t (u-t_u)^{H-\rho-\alpha} du \\
&\le C F_1 \Delta_l^{H-\rho} \int_0^t (u-t_u)^{-\alpha} du + C F_1 \Delta_1^{H-\rho-\alpha} \\
&\le C F_1 \Delta_l^{H-\rho-\alpha}. \qquad (1.22)
\end{aligned}$$

Putting now together relations (1.17) and (1.18)-(1.22), we end up with

$$Z_t^{(1)} \le L \int_0^t Z_u^{(1)} du + C F_3 \Delta_l^{H-\rho} + C F_1 \Delta_l^{1-\alpha} + C F_1 \Delta_l^{H-\rho-\alpha}.$$

Then taking into account that $\alpha < 1 - H - \rho$, $H - \rho > 2H - 1$ and invoking Gronwall's Lemma we have obtained relation (1.14) for $k = 1$, that is for all $t \in [0, r]$. It is worth mentioning at this point that the random variables F_1, F_3 above admit moments of all orders.

Case 3: General $\mathbf{k}$. Suppose relation (1.14) holds true up to $k = m-1$. We shall extend the relation to $\mathbf{k} = \mathbf{m}$ (that is $t \in [(m-1)r, mr]$) by a recursion procedure. To this aim, we shall follow the lines of [71], and we only sketch the proof for sake of conciseness.

According to (1.3) and (1.13), we have

$$X_t = X_{(m-1)r} + \int_{(m-1)r}^t b(X_s) ds + \int_{(m-1)r}^t \sigma(X_{s-r}) dB_s^H \qquad (1.23)$$

and

$$Y_t = Y_{(m-1)r} + \int_{(m-1)r}^t b(Y_{t_s}) ds + \int_{(m-1)r}^t \sigma(Y_{t_s-r}) dB_s^H. \qquad (1.24)$$

Set then $Z_t^{(m)} := \sup_{(m-1)r \le s \le t} |X_s - Y_s|$. By hypotesis **H2** and (1.7), we get

$$Z_t^{(m)} \le J_1 + J_2 + J_3 + J_4 + J_5,$$

with

$$\begin{aligned}
J_1 &= |X_{(m-1)r} - Y_{(m-1)r}| \\
J_2 &= \int_{(m-1)r}^{t} |b(X_u) - b(Y_u)| du \\
J_3 &= \int_{(m-1)r}^{t} |b(Y_u) - b(Y_{t_u})| du \\
J_4 &= \sup_{(m-1)r \le s \le mr} \Big| \int_{(m-1)r}^{s} [\sigma(X_{u-r}) - \sigma(Y_{u-r})] dB_u^H \Big| \\
J_4 &= \sup_{(m-1)r \le s \le mr} \Big| \int_{(m-1)r}^{s} [\sigma(Y_{u-r}) - \sigma(Y_{t_u - r})] dB_u^H \Big|.
\end{aligned}$$

Those terms are now handled in a similar way as in the proof of Theorem 3.2 in [71] and using the results obtained for $k = 1, 2, \cdots, (m-1)$. The details are left to the reader.

1.4 SFDDE as FSDE

In this section we express the FSDDE (1.2) as a system of fractional stochastic differential equations (FSDE in short) without time delay following the ideas of [62].

To this aim, consider the solution X to equation (1.3) on $[0, T]$, and assume (without loss of generality) that there exists a parameter $N_0 \in \mathbf{N}$ such that $N_0 = T/r$. For $k \in \{0, 1, ..., N_0\}$, $i \in \{0, 1, ..., k\}$ and $t \in [(k-1)r, kr]$, set then

$$Z_t^{i,k} := X_{t-ir}. \tag{1.25}$$

It is readily checked that for $k \in \{0, 1, ..., N_0 - 1\}$ and $i = 0, 1, ..., k$ we have

$$Z_{kr}^{i,k} = Z_{kr}^{i,k+1}. \tag{1.26}$$

Let us now give some details about the system of equations satisfied by the processes Z: consider first the case $k = 1$ and $t \in [0, r]$, for which we have

$$Z_t^{1,1} = X_{t-r} = \phi(t-r) \quad \text{and} \quad Z^{0,1} = X_t,$$

which means that (1.2) can be read as:

$$dZ_t^{0,1} = b(Z_t^{0,1})dt + \sigma(Z_t^{1,1})dB_t^H. \tag{1.27}$$

It is worth mentioning at this point that for $t \in [0, r]$, one can also recover the values of X from those of $Z^{0,1}$.

Remark 7 *Notice that the equation (1.27) has a unique solution because b and $\sigma \circ Z^{1,1}$ fulfill the conditions of existence and uniqueness of solution of FSDEs (see [79]). We shall face the same situation for the following cases and we will not mention it at each time.*

Let us handle now the case $k = 2$: for $t \in [r, 2r]$, we have

$$Z_t^{2,2} = X_{t-2r} = \phi(t-2r), \quad Z_t^{1,2} = X_{t-r}, \quad Z_t^{0,2} = X_t,$$

and one can recast equation (1.2) into

$$\begin{aligned} dZ_t^{1,2} &= b(Z_t^{1,2})dt + \sigma(Z_t^{2,2})dB_{t-r}^H \\ dZ_t^{0,2} &= b(Z_t^{0,2})dt + \sigma(Z_t^{1,2})dB_t^H. \end{aligned}$$

with initial conditions $Z_r^{1,2} = Z_r^{1,1}$ and $Z_r^{0,2} = Z_r^{0,1}$.

The reader can easily check that the previous construction can be iteratively generalized. In this way, we obtain a system of FSDEs without time delay as follows: for $k = 0, 1, ..., N_0$ and $t \in [(k-1)r, kr]$, we have $Z^{k,k} = X(t-kr) = \phi(t-kr)$ and

$$\begin{aligned} dZ_t^{k-1,k} &= b(Z_t^{k-1,k})dt + \sigma(Z_t^{k,k})dB_{t-(k-1)r}^H \\ &\vdots \\ dZ_t^{0,k} &= b(Z_t^{0,k})dt + \sigma(Z_t^{1,k})dB_t^H \end{aligned} \tag{1.28}$$

with initial conditions

$$Z_t^{k,k} = \phi(t-kr), \quad Z_{(k-1)r}^{k-1,k-1} = Z_{(k-1)r}^{k-1,k}, \quad \ldots \quad , Z_{(k-1)r}^{0,k-1} = Z_{(k-1)r}^{0,k}.$$

This provides the solution of $X_t = Z_t^{0,k}$ for $t \in [(k-1)r, kr]$.

Remark 8 *It is also possible to approximate the solution to equation (1.2) in the weak sense. Specifically, it is well known that our driving noise B^H can be represented as*

$$B_t^H = \int_0^t K_H(t,s)dW_s, \tag{1.29}$$

where W is a standard Wiener process and

$$K_H(t,s) = C_H(t-s)^{H-1/2} + C_H(1/2-H)\int_s^t (u-s)^{H-3/2}(1-(s/u))^{1/2-H}du,$$

for a normalizing constant C_H.

This induces naturally an approximating sequence of B^H as the following finite variation path:

$$Z^\epsilon := \left\{ Z_t^\epsilon = \frac{1}{\epsilon}\int_0^t K_H(t,s)(-1)^{N(\frac{s}{\epsilon^2})}ds, \quad t \in [0,T] \right\} \tag{1.30}$$

where $\{N(t), t \geq 0\}$ is a standard Poisson process. Delgado and Jolis ([39]) proved that the family Z^ϵ converges weakly to the fractional Brownian motion B^H, as $\epsilon \to 0$.

If we define now the process $Y = Y^\epsilon$ as the solution of the equation (1.2) driven by Z^ϵ, one can prove (under the same assumptions as for Theorem 11) that Y^ϵ converges in law to the the solution X of the FSDDE (1.2), where the convergence takes place in the space $C^\gamma(-r, T; \mathbb{R})$. This basically stems from the fact that the solution X to (1.2) is a continuous function of B^H when $H > 1/2$. We leave the details of this assertion to the reader for sake of conciseness.

1.5 Numerical examples

This section is devoted to an illustration of our numerical scheme with simulations. We shall focus on the particular FSDDE given by

$$\begin{aligned} dX_t &= X_t dt + 2.2 X_{t-1} dB_t^H, \quad t \in [0, 1.5] \\ X_s &= 1 \quad s \in [-1, 0]. \end{aligned} \tag{1.31}$$

Since this linear equation can be solved explicitly, comparisons of our approximation with the real solution will be easy.

Let us start by examining the Euler approximation introduced at Section 1.3. In Figure 1.1 and Figure 1.2 we show four sample paths for the Euler approximation of the solution to equation (1.31) in the cases $H = 0.75$ and $H = 0.99$ respectively. The reader might easily observe the change in smoothness for the solution.

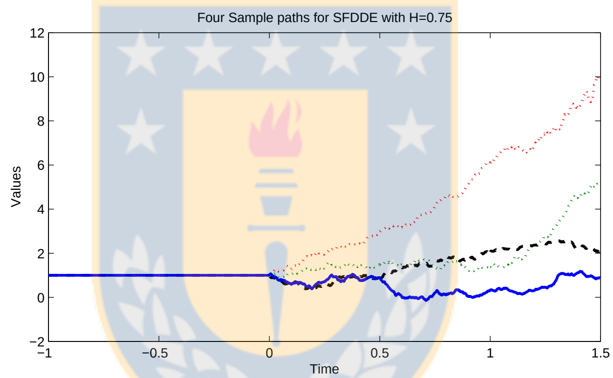

Figure 1.1: Sample paths for the solution to (1.31) driven by a fBm B^H with $H = 0.75$.

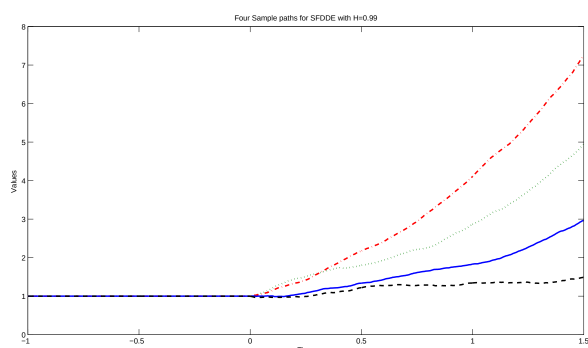

Figure 1.2: Sample paths for the solution to (1.31) driven by a fBm B^H with $H = 0.99$.

As far as convergence rate of our approximation is concerned, we have made two different experiments:

(i) First we have simulated 100 sample paths of (1.31) with $H = 0.75$ according to our approximation, and compared the result with the true value of the solution, in which the tail of the error seems to behave roughly as the absolute value of a Gaussian random variable.

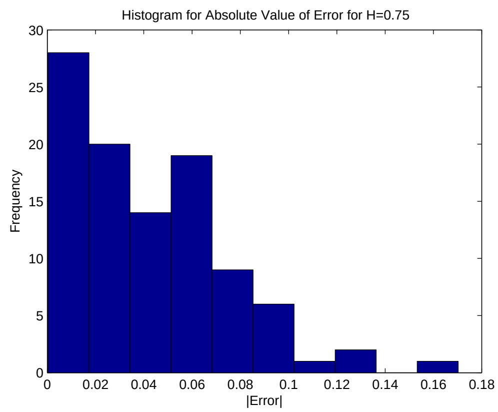

Figure 1.3: Absolute value of the error obtained on 100 paths of equation (1.31) for $H = 0.75$.

(ii) Figure 1.4 shows the empirical convergence of the approximation for different values of the Hurst parameter H. We can notice that the error becomes smaller when H is close to 1, which is consistent with Theorem 11. Notice that we have simulated 100 sample paths for each parameter.

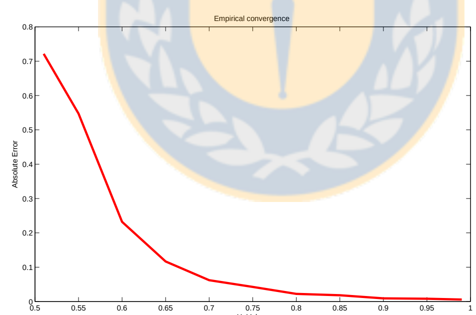

Figure 1.4: Decay of the error for the Euler scheme in terms of H.

Finally we have included two Figures corresponding to the weak approximation considered at Remark 8, in the cases $H = 0,75$ and $H = 0,99$ respectively. This is depicted at Figures 1.5 and 1.6 respectively.

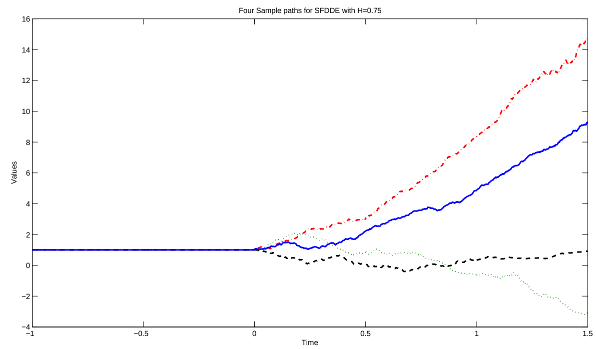

Figure 1.5: Weak approximation of the solution to (1.31) driven by a fBm B^H with $H = 0.75$.

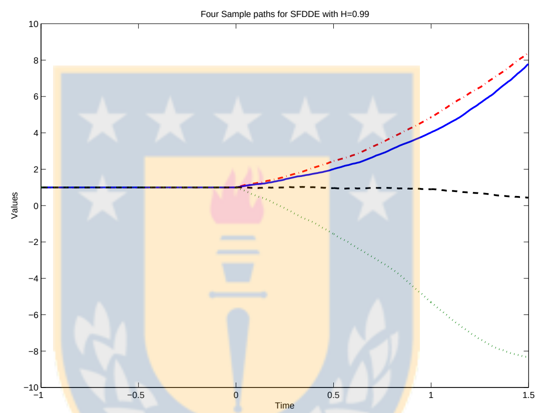

Figure 1.6: Weak approximation of the solution to (1.31) driven by a fBm B^H with $H = 0.99$.

Chapter 2

CAPÍTULO II: Least squares estimator for the parameter of the fractional Ornstein-Uhlenbeck sheet

2.1 Introduction

We consider the two-parameter fractional Ornstein-Uhlenbeck process defined as the solution of the stochastic equation

$$X_{t,s} = -\theta \int_0^t \int_0^s X_{v,u} dv du + B_{t,s}^{\alpha,\beta}, \qquad (t,s) \in [0,T] \times [0,S]. \tag{2.1}$$

Here $B^{\alpha,\beta}$ denotes a fractional Brownian sheet with Hurst parameters $\alpha, \beta \in (\frac{1}{2}, 1)$. We also suppose that $X_{0,0} = X_{t,0} = X_{0,s} = 0$ for every t, s. Our goal is to estimate the unknown parameter θ from the continuous time observation of the solution $(X_{t,s})_{(t,s)\in[0,T]\times[0,S]}$.

The development of the stochastic analysis for fractional Brownian motion (fBm) naturally led to the study of the statistical inference for stochastic equations driven by this process. There already exists an important literature related to these aspects. We refer, among others to [15], [16], [51], [59], [103]. Statistical analysis of the stochastic differential equations (SDE) driven by the fractional Brownian sheet has been less considered. We refer to the paper [91] for the study of the maximum likelihood estimator for a SDE with additive fractional Brownian sheet noise (see also [42] or [3] for the case when the noise is a standard Brownian sheet).

In this paper we propose a least square estimator for the unknown parameter θ following the approach in [51]. This estimator is obtained by formally minimizing with respect to θ the expression

$$\int_0^T \int_0^S \left| \frac{\partial^2}{\partial t \partial s} X_{t,s} + \theta X_{t,s} \right|^2 ds dt.$$

We obtain the following estimator

$$\widehat{\theta}_{T,S} = -\frac{\int_0^T \int_0^S X_{t,s} dX_{t,s}}{\int_0^T \int_0^S X_{t,s}^2 dt ds}. \tag{2.2}$$

The integral with respect to $dX_{t,s}$ is understood as the sum of the standard Lebesgue integral $-\theta\int_0^T\int_0^S dtdsX_{t,s}^2$ and of the stochastic integral $\int_0^T\int_0^S X_{t,s}dB^{\alpha,\beta}_{t,s}$ which is a divergence type integral with respect to the fractional Brownian sheet $B^{\alpha,\beta}$ (it will be defined in Section 2, we also refer to [55], [57], [58], [101], [102] for the stochastic integration with respect to $B^{\alpha,\beta}$). Using (2.1) and (2.2) we can write

$$\widehat{\theta}_{T,S}-\theta=-\frac{\int_0^T\int_0^S X_{t,s}dB^{\alpha,\beta}_{t,s}}{\int_0^T\int_0^S X_{t,s}^2dtds}. \tag{2.3}$$

We will study the asymptotic behavior of the least square estimator $\widehat{\theta}_{T,S}$ as $T,S\to\infty$. Our tools are the multiple stochastic integrals and the Malliavin calculus. Actually, the nominator and the denominator of the right hand side of (2.3) can be expressed as multiple integrals of order 2 with respect to the fractional Brownian sheet and from this we will obtain concrete estimates for their moments. We will prove that the estimator (2.2) is a strongly consistent estimator in the sense that it converges almost surely to the true value of the parameter θ. This result is similar to the one-dimensional case (see [51]), however the approach presented in [51] is not possible to be followed for the two-parameter case, instead we use among other tools, the hypercontractivity of multiples integrals. By contrary, in the two-parameter case, the least square estimator does not preserve the asymptotic normality as in the one-parameter case. This will be noticed at the end of our work by using criteria for the asymptotic normality of sequences of multiple integrals in terms of Malliavin calculus.

Our paper is structured as follows. Section 2 contains some preliminaries on multiple integrals and fractional Brownian sheet. In Section 3 we discuss the relation between the solution to (2.1) and the Bessel function of order 0. Section 4 contains the proof of the consistency of the least square estimator while Section 5 is devoted to a discussion about the asymptotic normality of the estimator.

2.2 Preliminaries

Let us introduce the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi),\varphi\in\mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega,\mathcal{A},P)$, that is, a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right)=\langle\varphi,\psi\rangle_{\mathcal{H}}$. Denote by I_n the multiple stochastic integral with respect to B (see [77]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi\in\mathcal{H}, \|\varphi\|_{\mathcal{H}}=1$ and H_n is the Hermite polynomial of degree $n\geq 1$

$$H_n(x)=\frac{(-1)^n}{n!}\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right),\qquad x\in\mathbb{R}.$$

The isometry of multiple integrals can be written as: for m,n positive integers,

$$\begin{aligned}\mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f,g\rangle_{\mathcal{H}^{\otimes n}}\quad \text{if } m=n,\\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0\quad \text{if } m\neq n.\end{aligned} \tag{2.4}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \dots, x_x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The Malliavin derivative acts on multiple integrals $F = I_n(f)$ in the following way: for every s

$$D_s I_n = n I_{n-1}(f(\cdot, s))$$

where " $\cdot$ " above denotes $n-1$ variables. We recall the following hypercontractivity property for the L^p norm of a multiple stochastic integral (see [66, Theorem 4.1])

$$\mathbf{E}\, |I_m(f)|^{2m} \leq c_m \left(\mathbf{E} I_m(f)^2\right)^m \tag{2.5}$$

where c_m is an explicit positive constant and $f \in \mathcal{H}^{\otimes m}$.

In this work we use Malliavin calculus and multiple integrals with respect to the fractional Brownian sheet (fBs). Let us define this process and its associated Hilbert space. The fBs with Hurst parameters $\alpha, \beta \in (0,1), (B^{\alpha,\beta}_{t,s}, t, s \in [0,T] \times [0,S])$ is a zero mean Gaussian process with covariance

$$\begin{aligned} \mathbf{E}\left(B^{\alpha,\beta}_{t,s}, B^{\alpha,\beta}_{u,v}\right) &= \mathcal{R}^\alpha(t,u)\mathcal{R}^\beta(s,v) \\ &:= \frac{1}{2}\left(t^{2\alpha} + u^{2\alpha} - |t-u|^{2\alpha}\right) \frac{1}{2}\left(s^{2\beta} + v^{2\beta} - |s-v|^{2\beta}\right) \end{aligned} \tag{2.6}$$

given for all $t, u \in [0,T]^2$ and $s, v \in [0,S]^2$.

We assume that $B^{\alpha,\beta}$ is defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathcal{A}$ is generated by $B^{\alpha,\beta}$. Fix a time interval $[0,T] \times [0,S]$, denote by ξ the set of real valued step functions on $[0,T] \times [0,S]$ and let $\mathcal{H}^{\alpha,\beta}$ be the Hilbert space defined as the closure of ξ with respect to the scalar product

$$\langle 1_{[0,t]\times[0,s]}, 1_{[0,u]\times[0,v]} \rangle_{\mathcal{H}^{\alpha,\beta}} = \mathcal{R}^\alpha(t,u)\mathcal{R}^\beta(s,v)$$

where $\mathcal{R}^\alpha(t,u)\mathcal{R}^\beta(s,v)$ is the covariance function of the fBs, given in (2.6). The mapping $1_{[0,t]\times[0,s]} \longmapsto B^{\alpha,\beta}_{t,s}$ can be extended to a linear isometry between $\mathcal{H}^{\alpha,\beta}$ and the Gaussian space $\mathcal{H}^{\alpha,\beta}_1$ spanned by $B^{\alpha,\beta}$ which is a closed subspace of $L^2(\Omega, \mathcal{A}, \mathbb{P})$. We denote this isometry by $\varphi \longmapsto B^{\alpha,\beta}(\varphi)$. Fix $\alpha, \beta > \frac{1}{2}$, in this case we have that for every $f, g \in \mathcal{H}^{\alpha,\beta}$ the scalar product has the form

$$\langle f, g \rangle_{\mathcal{H}^{\alpha,\beta}} = c(\alpha)c(\beta) \int_0^T \int_0^S \int_0^T \int_0^S f(a,b)g(m,n)|a-m|^{2\alpha-2}|b-n|^{2\beta-2} da db dm dn \tag{2.7}$$

and $c(\alpha) = \alpha(2\alpha - 1)$.

2.3 About the solution

The equation (2.1) has been studied in several papers (see [44], [76]). It has been showed that for $\theta > 0$ and $\alpha, \beta > \frac{1}{2}$ equation (2.1) admits an unique strong solution which can be expressed as

$$X_{t,s} = \int_0^T \int_0^S f(t,s,t_0,s_0) dB^{\alpha,\beta}_{t_0,s_0} \tag{2.8}$$

where

$$f(t,s,t_0,s_0) = 1_{[0,t]}(t_0) 1_{[0,s]}(s_0) \sum_{n\geq 0} (-1)^n \theta^n \frac{(t-t_0)^n (s-s_0)^n}{(n!)^2}. \tag{2.9}$$

We will call the process X solution to (2.1) as the fractional Ornstein-Uhlenbeck sheet. It is a Gaussian process since it is given by a multiple integral of order 1 (Wiener integral actually) with respect to the Gaussian process $B^{\alpha,\beta}$. We mention that the solution to (2.1) behaves differently that its one-dimensional counterpart which is fractional Ornstein-Uhlenbeck process introduced in [27], this will make our analysis quite different from [51]. For example, we note that the solution of some stochastic differential equations driven by the Brownian sheet or fractional Brownian sheet, which are positive in the one-parameter case, can take negative values with strictly positive probability (see [76] or [77]).

A key element of our analysis is the fact that the solution X (more precisely the kernel f of the solution) can be expressed is terms of the Bessel function of the first kind. Let us consider the Bessel function of order 0 given, for every $x \in \mathbb{R}$, by

$$J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n}$$

This Bessel function admits the integral representation, for every $x \in \mathbb{R}$

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos\left(x \sin \rho\right) d\rho.$$

The kernel f in (2.9) of the solution $(X_{t,s})_{t,s\in[0,T]\times[0,S]}$ can be expressed as

$$\begin{aligned} f(t,s,u,v) &= 1_{[0,t]}(u) 1_{[0,s]}(v) J_0\left(2\sqrt{\theta(t-u)(s-v)}\right) \\ &= 1_{[0,t]}(u) 1_{[0,s]}(v) \frac{1}{\pi} \int_0^\pi \cos\left(2\sqrt{\theta(t-u)(s-v)} \sin\rho\right) d\rho. \end{aligned} \tag{2.10}$$

Let us also recall the following property of the Bessel function (see e.g. [4]) which will play an important role for our estimates: for x large enough

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4}) \tag{2.11}$$

(the symbol $\sim$ means that the two sides have the same limit as $x \to \infty$).

2.4 Asymptotic behavior of the least square estimator

In this section we study the asymptotic behavior of the estimator $\widehat{\theta}_{T,S}$ defined in (2.2). More precisely, we will show that this estimator is strongly consistent for the parameter θ, that is, $\widehat{\theta}_{T,S}$ converges to θ almost surely as $T, S \to \infty$. To this end we will analyze separately the nominator and the denominator appearing in the right hand side of the expression (2.3). Let us start with the study of the nominator. It can be written as the double stochastic integral

$$\int_0^T \int_0^S X_{t,s} dB_{t,s}^{\alpha,\beta} := F_{T,S} := I_2\left(f(u,v,t,s)\right) \tag{2.12}$$

where the kernel f is given by (2.9) and the integral I_2 acts with respect to the variables $(u,v),(t,s)$.

We will estimate first the L^2 norm of $F_{T,S}$. We have the following result.

Proposition 9 *For every $\varepsilon > 0$ and for $\alpha, \beta \in (\frac{1}{2}, \frac{5}{8})$,*

$$\mathbf{E}\left(T^{-2\alpha+\frac{1}{4}-\varepsilon} S^{-2\beta+\frac{1}{4}-\varepsilon} \int_0^T \int_0^S X_{t,s} dB_{t,s}^{\alpha,\beta}\right)^2 \to 0 \text{ when } T, S \to \infty. \tag{2.13}$$

Moreover for T, S large enough we have

$$\mathbf{E}\left(T^{-2\alpha+\frac{1}{4}} S^{-2\beta+\frac{1}{4}} \int_0^T \int_0^S X_{t,s} dB_{t,s}^{\alpha,\beta}\right)^2 < C$$

where C is a strictly positive constant not depending on T, S.

Proof: We calculate the L^2 norm of the random variable $I_2\left(f(u,v,t,s)\right)$. By the isometry property of multiple integrals (2.4) and since $\|\tilde{f}\|_{(\mathcal{H}^{\alpha,\beta})^{\otimes 2}} \leq \|f\|_{(\mathcal{H}^{\alpha,\beta})^{\otimes 2}}$ this norm can be handles as follows

$$\begin{aligned}
I_{T,S} \leq\ & \int_{[0,T]^4} dt dt_0 du du_0 \int_{[0,S]^4} ds ds_0 dv dv_0 \\
& \times f(t,s,u,v) f(t_0,s_0,u_0,v_0) |u-u_0|^{2\alpha-2} |v-v_0|^{2\beta-2} |t-t_0|^{2\alpha-2} |s-s_0|^{2\beta-2} \\
=\ & \int_0^T dt \int_0^t du \int_0^T dt_0 \int_0^{t_0} du_0 \int_0^S ds \int_0^s dv \int_0^S ds_0 \int_0^{s_0} dv_0 \\
& \times J_0\left(2\sqrt{\theta(t-u)(s-v)}\right) J_0\left(2\sqrt{\theta(t_0-u_0)(s_0-v_0)}\right) \\
& \times |t-t_0|^{2\alpha-2} |u-u_0|^{2\alpha-2} |s-s_0|^{2\beta-2} |v-v_0|^{2\beta-2}.
\end{aligned}$$

By making the change of variables $\tilde{t} = \frac{t}{T}$, $\tilde{u} = \frac{u}{T}$ and similarly for the other variables, we obtain

$$\begin{aligned}
I_{T,S} =\ & T^{4\alpha-4} T^4 S^{4\beta-4} S^4 \int_0^1 dt \int_0^t du \int_0^1 dt_0 \int_0^{t_0} du_0 \int_0^1 ds \int_0^s dv \int_0^1 ds_0 \int_0^{s_0} dv_0 \\
& \times J_0\left(2\sqrt{\theta(t-u)(s-v)TS}\right) J_0\left(2\sqrt{\theta(t_0-u_0)(s_0-v_0)TS}\right) \\
& \times |t-t_0|^{2\alpha-2} |u-u_0|^{2\alpha-2} |s-s_0|^{2\beta-2} |v-v_0|^{2\beta-2} \\
:=\ & T^{4\alpha} S^{4\beta} U_{T,S}.
\end{aligned} \tag{2.14}$$

Using the asymptotic behavior of the Bessel function (2.11), we have that

$$\frac{J_0\left(2\sqrt{\theta(t-u)(s-v)TS}\right)}{(TS)^{-\frac{1}{4}+\varepsilon}} \to_{T,S\to\infty} 0$$

for almost every $t, u, s, v \in (0,1)$ and for every $\varepsilon > 0$. We will next apply the dominated convergence theorem. To this end, using again relation (2.11), it suffices to show that the integral

$$\begin{aligned} I \;=\; & \int_0^1 dt \int_0^t du \int_0^1 dt_0 \int_0^{t_0} du_0 \int_0^1 ds \int_0^s dv \int_0^1 ds_0 \int_0^{s_0} dv_0 \\ & ((t-u)(s-v)(t_0-u_0)(s_0-v_0))^{-\frac{1}{4}} \\ & |t-t_0|^{2\alpha-2}|u-u_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|v-v_0|^{2\beta-2} \end{aligned} \tag{2.15}$$

is finite. This is proved in the following lemma.

■

Remark 9 *In the above statement we can replace the normalization* $T^{-2\alpha+\frac{1}{4}-\varepsilon}S^{-2\beta+\frac{1}{4}-\varepsilon}$ *by* $T^{-2\alpha+\frac{1}{4}}S^{-2\beta+\frac{1}{4}}f(T,S)$ *where* $f(T,S)$ *is a deterministic function which converges to zero as* $T, S \to \infty$. *This is a consequence of the proof below.*

Lemma 2 *Let* I *be given by (2.15). Then for* $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$ *the integral* I *is finite.*

Proof: Consider the integral

$$\begin{aligned} & \int_0^{s_0} dv_0\, ((t-u)(s-v))^{-\frac{1}{4}} ((t_0-u_0)(s_0-v_0))^{-\frac{1}{4}} \\ & \times |t-t_0|^{2\alpha-2}|u-u_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|v-v_0|^{2\beta-2} \\ = \; & ((t-u)(s-v)(t_0-u_0))^{-\frac{1}{4}} |t-t_0|^{2\alpha-2}|u-u_0|^{2\alpha-2}|s-s_0|^{2\beta-2} \\ & \times \int_0^{s_0} dv_0 |v-v_0|^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}}. \end{aligned}$$

If $s_0 < v$ we have

$$\int_0^{s_0} dv_0 |v-v_0|^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}} = \int_0^{s_0} dv_0\, (v-v_0)^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}}$$

making the change of variables $z = \frac{s_0-v_0}{v-v_0}$ we get

$$\begin{aligned} & \int_0^{\frac{s_0}{v}} \left(\frac{v-s_0}{1-z}\right)^{2\beta-2} \left(\frac{z(v-s_0)}{1-z}\right)^{-\frac{1}{4}} \left(\frac{v-s_0}{(1-z)^2}\right) dz \\ & \le (v-s_0)^{2\beta-\frac{5}{4}} \int_0^1 z^{-\frac{1}{4}}(1-z)^{\frac{1}{4}-2\beta} dz = (v-s_0)^{2\beta-\frac{5}{4}} \tilde{\beta}\left(\frac{3}{4}, \frac{5}{4}-2\beta\right) \end{aligned}$$

where $\tilde{\beta}$ is the *Beta* function. The expresion above is finite for $\beta < \frac{5}{8}$.
Now, if $v \le s_0$

$$\begin{aligned} & \int_0^{s_0} dv_0 |v-v_0|^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}} \\ & = \int_0^{v} dv_0\, (v-v_0)^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}} + \int_v^{s_0} dv_0\, (v_0-v)^{2\beta-2} (s_0-v_0)^{-\frac{1}{4}}. \end{aligned}$$

For the first integral in the right hand side we make the change of variables $z = \frac{v-v_0}{s_0-v_0}$, and for the second one we make $z = \frac{s_0-v_0}{v_0-v}$ Then we get

$$\begin{aligned}
&\leq \ (s_0-v)^{2\beta-\frac{5}{4}}\left(\int_0^1 z^{2\beta-2}(1-z)^{\frac{1}{4}-2\beta}dz + \int_0^\infty z^{-\frac{1}{4}}(1+z)^{\frac{1}{4}-2\beta}dz\right)\\
&= \ (s_0-v)^{2\beta-\frac{5}{4}}\left(\tilde{\beta}\left(2\beta-1,\frac{5}{4}-2\beta\right)+\frac{\Gamma(\frac{3}{4})\Gamma(2\beta-1)}{\Gamma(2\beta-\frac{1}{4})}\ {}_2F_1(0,3/4,2\beta-1/4;0)\right)\\
&= \ (s_0-v)^{2\beta-\frac{5}{4}}\left(\tilde{\beta}\left(2\beta-1,\frac{5}{4}-2\beta\right)+\tilde{\beta}\left(\frac{3}{4},2\beta-1\right)\ {}_2F_1(0,3/4,2\beta-1/4;0)\right)
\end{aligned}$$

wich is finite for $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$. Here Γ is the *Gamma* function, ${}_2F_1$ is the *Hypergeometric* function, and we have make use of the property $\tilde{\beta}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and the formula (see [92], formula 1.6.7)

$${}_2F_1\left(a,b,c;1-x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty w^{b-1}(1+w)^{a-c}(1+wx)^{-a}dw.$$

Proceeding in a similar way with the other seven integrals we conclude that I is finite.

■

Remark 10 *We may note that the prescence of the* cos *function in the asymptotic behavior of the Bessel function does not allow to obtain a renormalization for $F_{T,S}$ in terms of the powers of T and S like in [51].*

The following proposition is a consequence of the Proposition 9 and of the hypercontractivity property of multiple stochastic integrals.

Proposition 10 *For every $\varepsilon > 0$ and for $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$ the sequence*

$$\frac{1}{\sqrt{T^{4\alpha-\frac{1}{2}+\varepsilon}S^{4\beta-\frac{1}{2}+\varepsilon}}}\int_0^T\int_0^S X_{t,s}dB^{\alpha,\beta}_{t,s}$$

converges to zero a.s. as $T, S \to \infty$.

Proof: As in [51] we can replace the couple (T,S) by a discrete sequence (T_M, S_N) such that T_M, S_N converge to infinity as $M, N \to \infty$. This is possible since the nominator and denominator in (2.3) are continuous a.s. with respect to (T,S). Indeed, the fact that the divergence integral in the nominator is continuous follows from [77], page 293 since the integrand X is regular enough, and the integral $dtds$ in the nominator is clearly continuous a.s. with respect to the couple (T,S). For simplicity, we will assume that $(T_M, S_N) = (M,N)$. Let us show that

$$A_{M,N} := M^{-2\alpha+\frac{1}{4}-\varepsilon}N^{-2\beta+\frac{1}{4}-\varepsilon}\int_0^M\int_0^N X_{t,s}dB^{\alpha,\beta}_{t,s}$$

converges to zero a.s. as M, N tend to infinity. We will use the Borel-Cantelli lemma. To do this, we will estimate $P(A_{M,N} > (MN)^{-\gamma})$ for some $\gamma > 0$. For every $p \geq 1$ we have

$$P(A_{M,N} > (MN)^{-\gamma}) \leq (MN)^{p\gamma}\,\mathbf{E}|A_{M,N}|^p$$

and since $A_{M,N}$ is a multiple integral in the second Wiener chaos, the inequality (2.5) and Proposition 9 implies that

$$\mathbf{E}|A_{M,N}|^p \leq c(p) \left(\mathbf{E}A_{M,N}^2\right)^{\frac{p}{2}} \leq c(p,\alpha,\beta)(MN)^{-\varepsilon p}.$$

Putting together the two above bounds, we get

$$\sum_{M>M_0,N>N_0} P(A_{M,N} > (MN)^{-\gamma}) \leq c(p,\alpha,\beta) \sum_{M>M_0,N>N_0} (MN)^{p(\gamma-\varepsilon)}$$

and this series is convergent when

$$(\varepsilon-\gamma)p > 1 \text{ or equivalently } \quad \gamma < \varepsilon - \frac{1}{p}.$$

For every given $\varepsilon > 0$ and for p large enough we can always chose a real number γ such that $0 < \gamma < \varepsilon - \frac{1}{p}$. This, together with the Borel-Cantelli lemma allows us to finish the proof. ■

The next step is to analyze the denominator in formula (2.3). We have the following estimate.

Proposition 11 *For any $\varepsilon > 0$ and for any $\alpha,\beta \in (\frac{1}{2},\frac{5}{8})$*

$$\frac{1}{T^{2\alpha+\frac{1}{2}-\varepsilon}S^{2\beta+\frac{1}{2}-\varepsilon}}\mathbf{E}\int_0^T dt \int_0^S ds X_{t,s}^2 \rightarrow_{T,S\rightarrow\infty} \infty.$$

Proof: By the isometry of multiple integrals (2.4), the equation (2.7) and the expression of the solution to (2.1)

$$\begin{aligned}\mathbf{E}X_{t,s}^2 &= c(\alpha,\beta)\int_0^t du \int_0^t du_0 \int_0^s dv \int_0^s dv_0 \\ &\times J_0(2\sqrt{\theta(t-u)(s-v)})J_0(2\sqrt{\theta(t-u_0)(s-v_0)})|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2},\end{aligned}$$

making the change of variables $\tilde{t} = \frac{t}{T}, \tilde{s} = \frac{s}{S}$ and similar for the other variables we can write

$$\begin{aligned}\int_0^T \int_0^S dsdt\mathbf{E}X_{t,s}^2 &= c(\alpha,\beta)T^{2\alpha+1}S^{2\beta+1}\int_0^1 dt \int_0^1 ds \int_0^t du \int_0^t du_0 \int_0^s dv \int_0^s dv_0 \\ &\times J_0(2\sqrt{\theta(t-u)(s-v)TS})J_0(2\sqrt{\theta(t-u_0)(s-v_0)TS}) \\ &\times |u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\end{aligned}$$

and thus

$$\begin{aligned}\frac{\int_0^T\int_0^S dsdt\mathbf{E}X_{t,s}^2}{T^{2\alpha+\frac{1}{2}-\varepsilon}S^{2\beta+\frac{1}{2}-\varepsilon}} &= c(\alpha,\beta)\int_0^1 dt \int_0^1 ds \int_0^t du \int_0^t du_0 \int_0^s dv \int_0^s dv_0 \\ &\times \frac{J_0(2\sqrt{\theta(t-u)(s-v)TS})}{(TS)^{-\frac{1}{4}-\frac{\varepsilon}{2}}}\frac{J_0(2\sqrt{\theta(t-u_0)(s-v_0)TS})}{(TS)^{-\frac{1}{4}-\frac{\varepsilon}{2}}} \\ &\times |u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}.\end{aligned}$$

We will use the same idea as in the proof of Proposition 9. We notice first that

$$\frac{J_0(2\sqrt{\theta(t-u)(s-v)TS})}{(TS)^{-\frac{1}{4}-\frac{\varepsilon}{2}}}$$

converges to infinity as $T, S \to \infty$ by using (2.11); also for T, S large enough

$$\frac{J_0(2\sqrt{\theta(t-u)(s-v)TS})}{(TS)^{-\frac{1}{4}}}$$

is bounded by $c\left(|t-u||s-v|\right)^{-\frac{1}{4}}$ almost everywhere s, t, u, v. Also we note that the integral

$$\int_0^1 dt \int_0^1 ds \int_0^t du \int_0^t du_0 \int_0^s dv \int_0^s dv_0 |u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}$$
$$\left(|t-u||s-v|\right)^{-\frac{1}{4}}\left(|t-u_0||s-v_0|\right)^{-\frac{1}{4}}$$

is finite for $\alpha, \beta \in (\frac{1}{2}, \frac{5}{8})$ by using the same computations as in the proof of Lemma 2.
Using Fatou's lemma (we use the following version of the Fatou's lemma: if f_n is a sequence of functions such that $f_n \geq -g$ where g is positive and integrable, then $\underline{\lim} \int f_n \geq \int \underline{\lim} f_n$) this implies that

$$\begin{aligned}
\underline{\lim}_{T,S\to\infty} \frac{\int_0^T \int_0^S dsdt \mathbf{E} X_{t,s}^2}{T^{2\alpha+\frac{1}{2}-\varepsilon} S^{2\beta+\frac{1}{2}-\varepsilon}} &\geq c(\alpha,\beta) \int_0^1 dt \int_0^1 ds \int_0^t du \int_0^t du_0 \int_0^s dv \int_0^s dv_0 \\
&\times \underline{\lim}_{T,S\to\infty} \frac{J_0(2\sqrt{\theta(t-u)(s-v)TS})}{(TS)^{-\frac{1}{4}-\frac{\varepsilon}{2}}} \frac{J_0(2\sqrt{\theta(t-u_0)(s-v_0)TS})}{(TS)^{-\frac{1}{4}-\frac{\varepsilon}{2}}} \\
&\times |u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2} \quad = \quad \infty.
\end{aligned}$$

■

At this point we will need the following auxiliary lemma.

Lemma 3 *Consider a sequence of random variables $(A_N)_N$ such that $\sum_N P(A_N > cN^{-\gamma}) < \infty$ for some $\gamma > 0$ and for every $c > 0$ (which implies $A_N \to 0$ almost surely as $N \to \infty$.) Also consider a sequence of a.s. strictly positive random variables $(B_N)_N$ such that $\mathbf{E}B_N \to \infty$ as $N \to \infty$. Then*

$$\frac{A_N}{B_N} \to_{N\to\infty} 0 \text{ almost surely}$$

Proof: We will use again the Borel-Cantelli lemma. Let $C > 0$ be arbitrary. Then

$$\begin{aligned}
\sum_N P\left(\frac{A_N}{B_N} > N^{-\gamma}\right) &= \sum_N P\left(\frac{A_N}{B_N} > N^{-\gamma}, B_N > C\right) + \sum_N P\left(\frac{A_N}{B_N} > N^{-\gamma}, B_N < C\right) \\
&\leq \sum_N P\left(\frac{A_N}{C} > N^{-\gamma}, B_N > C\right) + \sum_N P\left(\frac{A_N}{B_N} > N^{-\gamma}, B_N < C\right).
\end{aligned}$$

Using the fact that $\mathbf{E}B_N \to \infty$ as $N \to \infty$ we obtain that

$$P\left(\frac{A_N}{B_N} > N^{-\gamma}, B_N < C\right) = 0$$

for N large enough. By assumption, $\sum_N P\left(\frac{A_N}{C} > N^{-\gamma}, B_N > C\right) < \infty$. Therefore

$$\sum_N P\left(\frac{A_N}{B_N} > N^{-\gamma}\right) < \infty$$

and therefore the conclusion follows.

■

Remark 11 *Lemma 3 can be extended without difficulty to two-parameter sequences. That is, if $A_{M,N}$ is a sequence of random variables such that $\sum_{M,N\geq 0} P(A_{M,N} > c(MN)^{-\gamma}) < \infty$ and $B_{M,N}$ is a sequence of positive random variables such that $\mathbf{E}B_{M,N} \to_{M,N\to\infty} \infty$ then*

$$\frac{A_{M,N}}{B_{M,N}} \to_{M,N\to\infty} 0 \text{ almost surely.}$$

Let us state the main result of this section.

Theorem 1 *Let $\theta_{T,S}$ be the estimator given by (2.2). Suppose that $\alpha, \beta \in (\frac{1}{2}, \frac{5}{8})$. Then $\theta_{T,S}$ is a strongly consistent estimator for the parameter θ, that is,*

$$\widehat{\theta}_{T,S} \to \theta \text{ almost surely as } T, S \to \infty.$$

Proof: From relation (2.3), the difference between the estimator and the true parameter is

$$\begin{aligned} \widehat{\theta}_T - \theta &= \frac{T^{-2\alpha+\frac{1}{4}-\varepsilon} S^{-2\beta+\frac{1}{4}-\varepsilon} \int_0^T \int_0^S X_{t,s} dB_{t,s}^{\alpha,\beta}}{T^{-2\alpha+\frac{1}{4}-\varepsilon} S^{-2\beta+\frac{1}{4}-\varepsilon} \int_0^T dt \int_0^S ds X_{t,s}^2} \\ &= \frac{T^{-2\alpha+\frac{1}{4}-\varepsilon} S^{-2\beta+\frac{1}{4}-\varepsilon} \int_0^T \int_0^S X_{t,s} dB_{t,s}^{\alpha,\beta}}{T^{\frac{3}{4}-2\varepsilon} S^{\frac{3}{4}-2\varepsilon} T^{-2\alpha-\frac{1}{2}+\varepsilon} S^{-2\beta-\frac{1}{2}+\varepsilon} \int_0^T dt \int_0^S ds X_{t,s}^2}. \end{aligned}$$

The result is obtained by using Proposition 10, Proposition 11 and Lemma 3 (and the remark that follows after this lemma). ■

2.5 Asymptotic non-normality of the estimator

We proved in the previous section that the least square estimator $\widehat{\theta}_{T,S}$ is strongly consistent. This property has been proved in the one-dimensional case in ([51]). Nevertheless, we show in this paragraph that the limiting distribution of the estimator is not the same in the one-parameter and two-parameter cases. This different behavior is somehow expected since the fractional Ornstein-Uhlenbeck sheet does not keep the properties of the fractional Ornstein-Uhlenbeck process (for example, the kernel f given by (2.9) can take any real value while in the one parameter case can only take positive values since it is given by an exponential function). In order to notice the difference between the one-parameter and the two-parameter case, we will focus only on the nominator in the right hand side of (2.3). To check the asymptotic normality we use the following criterium: (see Theorem 4 in [78]).

Theorem 2 *Let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$) be a sequence of square integrable random variables in the n th Wiener chaos such that $\mathbf{E}[F_k^2] \to 1$ as $k \to \infty$. Then the following are equivalent:*

i) The sequence $(F_k)_{k\geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$.

ii) $\|DF_k\|^2_{\mathcal{H}}$ converges to n in $L^2(\Omega)$ as $k \to \infty$.

Denote by

$$\sigma^2_{T,S} = \mathbf{E}\left(\int_0^T \int_0^S X_{t,s} dB^{\alpha,\beta}_{t,s}\right)^2 = \mathbf{E}\left(F_{T,S}\right)^2.$$

Proposition 12 *Assume $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$ and let $F_{T,S}$ be given by (2.12). Then when T, S tend to infinity,*

$$\frac{1}{\sigma_{T,S}} F_{T,S}$$

does not converges in distribution to the normal law $N(0,1)$.

Proof: We will prove that $\|D\frac{1}{\sigma_{T,S}}F_{T,S}\|^2_{\mathcal{H}^{\alpha,\beta}}$ does not converges to 2 in $L^2(\Omega)$ as $T, S \to \infty$.

Let us denote by $\tilde{f}$ the symmetrization of f with respect to the variables $(t,s),(u,v)$

$$\tilde{f}\left((t,s),(u,v)\right) = \frac{1}{2}\left(f(t,s,u,v) + f(u,v,t,s)\right).$$

We have

$$D_{t,s}\frac{1}{\sigma_{T,S}}F_{T,S} = \frac{1}{\sigma_{T,S}}D_{t,s}I_2\left(f\right) = \frac{1}{\sigma_{T,S}}D_{t,s}I_2\left(\tilde{f}\right) = \frac{1}{\sigma_{T,S}}\left(I_1(f(\cdot,\cdot,t,s)) + I_1(t,s,\cdot,\cdot)\right).$$

Here "$\cdot,\cdot$" represents the variable with respect to which the integral I_1 acts.

We obtain

$$\begin{aligned}
\|D\frac{1}{\sigma_{T,S}}F_{T,S}\|^2_{\mathcal{H}^{\alpha,\beta}} &= \frac{1}{\sigma^2_{T,S}}c(\alpha)c(\beta)\int_0^T dt \int_0^S ds \int_0^T dt_0 \int_0^S ds_0 \\
&\quad \times\left[I_1\left(f(\cdot,\cdot,t,s)\right) + I_1\left(f(t,s,\cdot,\cdot)\right)\right]\left[I_1\left(f(\cdot,\cdot,t_0,s_0)\right) + I_1\left(f(t_0,s_0,\cdot,\cdot)\right)\right] \\
&\quad \times|t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2} \\
&= \frac{1}{\sigma^2_{T,S}}c(\alpha)c(\beta)\int_0^T dt \int_0^S ds \int_0^T dt_0 \int_0^S ds_0 \\
&\quad \times\left[I_2\left(f(\cdot,\cdot,t,s)\otimes f(\cdot,\cdot,t_0,s_0)\right) + I_2\left(f(\cdot,\cdot,t,s)\otimes f(t_0,s_0,\cdot,\cdot)\right)\right. \\
&\quad \left.+I_2\left(f(t,s,\cdot,\cdot)\otimes f(\cdot,\cdot,t_0,s_0)\right) + I_2\left(f(t,s,\cdot,\cdot)\otimes f(t_0,s_0,\cdot,\cdot)\right)\right] \\
&\quad \times|t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2} + \mathbf{E}\|D\frac{1}{\sigma_{T,S}}F_{T,S}\|^2_{\mathcal{H}^{\alpha,\beta}} \\
&:= A^{(1)}_{T,S} + A^{(2)}_{T,S} + A^{(3)}_{T,S} + A^{(4)}_{T,S} + \mathbf{E}\|D\frac{1}{\sigma_{T,S}}F_{T,S}\|^2_{\mathcal{H}^{\alpha,\beta}}
\end{aligned}$$

Let us note that that $\mathbf{E}\|D\frac{1}{\sigma_{T,S}}F_{T,S}\|^2_{\mathcal{H}^{\alpha,\beta}}$ is equal to 2. This follows from the fact that for any multiple integral of order n we have

$$\mathbf{E}\|DI_n(f)\|^2_{\mathcal{H}} = n\mathbf{E}I_n(f)^2.$$

It remains to show that the terms containing multiple integrals of order 2 does not converges to zero in $L^2(\Omega)$ as $T,S\to\infty$. The first and the fourth summand are similar, as they are the second one and the third one. We will handle only the first summand denoted by $A^{(1)}_{T,S}$ (because the other three terms can be studied analogously). We can write, using the definition of the scalar product in the Hilbert space $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes 2}$ and the expression (2.10) of the kernel f in terms of the Bessel function J_0

$$\begin{aligned}
\mathbf{E}\left|A^{(1)}_{T,S}\right|^2 &= \frac{1}{\sigma^4_{T,S}}\mathbf{E}\Bigg(\int_0^T dt\int_0^S ds\int_0^T dt_0\int_0^S ds_0 |t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}\\
&\quad I_2\left(f(\cdot,\cdot,t,s)\otimes f(\cdot,\cdot,t_0,s_0)\right)\Bigg)^2\\
&\sim 2\int_0^T dt\int_0^S ds\int_0^T dt_0\int_0^S ds_0\int_0^T du\int_0^S dv\int_0^T du_0\int_0^S dv_0\\
&\quad |t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\\
&\quad \langle f(\cdot,\cdot,t,s)\otimes f(\cdot,\cdot,t_0,s_0), f(\cdot,\cdot,u,v)\otimes f(\cdot,\cdot,u_0,v_0)\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes 2}}\\
&= C\frac{1}{\sigma^4_{T,S}}\int_0^T dt\int_0^S ds\int_0^T dt_0\int_0^S ds_0\int_0^T du\int_0^S dv\int_0^T du_0\int_0^S dv_0\\
&\quad \int_0^T dx\int_0^S dy\int_0^T dx_0\int_0^S dy_0\int_0^T da\int_0^S db\int_0^T da_0\int_0^S db_0\\
&\quad f(x,y,t,s)f(x_0,y_0,t_0,s_0)f(a,b,u,v)f(a_0,b_0,t_0,s_0)\\
&\quad |t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\\
&\quad |x-a|^{2\alpha-2}|y-b|^{2\beta-2}|x_0-a_0|^{2\alpha-2}|y_0-b_0|^{2\beta-2}\\
&= C\frac{1}{\sigma^4_{T,S}}\int_0^T dx\int_0^S dy\int_0^T dx_0\int_0^S dy_0\int_0^T da\int_0^S db\int_0^T da_0\int_0^S db_0\\
&\quad \int_0^x dt\int_0^y ds\int_0^{x_0} dt_0\int_0^{y_0} ds_0\int_0^a du\int_0^b dv\int_0^{a_0} du_0\int_0^{b_0} dv_0\\
&\quad J_0(2\sqrt{\theta(x-t)(y-s)})J_0(2\sqrt{\theta(x_0-t_0)(y_0-s_0)})\\
&\quad J_0(2\sqrt{\theta(a-u)(b-v)})J_0(2\sqrt{\theta(a_0-u_0)(b_0-v_0)})\\
&\quad |t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\\
&\quad |x-a|^{2\alpha-2}|y-b|^{2\beta-2}|x_0-a_0|^{2\alpha-2}|y_0-b_0|^{2\beta-2}
\end{aligned}$$

making $\tilde{t} = \frac{t}{T}, \tilde{s} = \frac{s}{S}$ and similar for the other variables we get

$$\begin{aligned}
\mathbf{E}\left|A^{(1)}_{T,S}\right|^2 &= C\frac{1}{\sigma^4_{T,S}}T^{8\alpha}S^{8\beta}\int_0^1 dx\int_0^1 dy\int_0^1 dx_0\int_0^1 dy_0\int_0^1 da\int_0^1 db\int_0^1 da_0\int_0^1 db_0\\
&\int_0^x dt\int_0^y ds\int_0^{x_0} dt_0\int_0^{y_0} ds_0\int_0^a du\int_0^b dv\int_0^{a_0} du_0\int_0^{b_0} dv_0\\
&J_0(2\sqrt{\theta(x-t)(y-s)TS})J_0(2\sqrt{\theta(x_0-t_0)(y_0-s_0)TS})\\
&J_0(2\sqrt{\theta(a-u)(b-v)TS})J_0(2\sqrt{\theta(a_0-u_0)(b_0-v_0)TS})\\
&|t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\\
&|x-a|^{2\alpha-2}|y-b|^{2\beta-2}|x_0-a_0|^{2\alpha-2}|y_0-b_0|^{2\beta-2}
\end{aligned}$$

.

Using the asymptotic behavior of the Bessel function when its variable is close to infinity (see (2.11)) we see that

$$\begin{aligned}
\mathbf{E}\left|A^{(1)}_{T,S}\right|^2 &\approx C\frac{1}{\sigma^4_{T,S}}T^{8\alpha}S^{8\beta}\int_0^1 dx\int_0^1 dy\int_0^1 dx_0\int_0^1 dy_0\int_0^1 da\int_0^1 db\int_0^1 da_0\int_0^1 db_0\\
&\int_0^x dt\int_0^y ds\int_0^{x_0} dt_0\int_0^{y_0} ds_0\int_0^a du\int_0^b dv\int_0^{a_0} du_0\int_0^{b_0} dv_0\\
&((x-t)(y-s)(x_0-t_0)(y_0-s_0)(a-u)(b-v)(a_0-u_0)(b_0-v_0))^{\frac{-1}{4}}\\
&|t-t_0|^{2\alpha-2}|s-s_0|^{2\beta-2}|u-u_0|^{2\alpha-2}|v-v_0|^{2\beta-2}\\
&|x-a|^{2\alpha-2}|y-b|^{2\beta-2}|x_0-a_0|^{2\alpha-2}|y_0-b_0|^{2\beta-2}.
\end{aligned}$$

and considering the fact that the last integral is finite for $\frac{1}{2} < \alpha, \beta < \frac{5}{8}$ (the proof of this fact is similar to the proof of Lemma 2) it is straightforward to see that, for T, S close to infinity, the quantity $\mathbf{E}\left|A^{(1)}_{T,S}\right|^2$ does not converges to zero.

■

Chapter 3

CAPÍTULO III: Hitting times for the stochastic wave equation with fractional-colored noise

3.1 Introduction

The recent development of the stochastic calculus with respect to the fractional Brownian motion (fBm) naturally led to the study of stochastic partial differential equations (SPDEs) driven by this Gaussian process. The motivation comes from the wide area of applications of the fBm. We refer, among others, to [50], [69], [80], [86] and [98] for theoretical studies of SPDEs driven by fBm. To list only a few examples of the appearance of fractional noises in practical situations, we mention [61] for biophysics, [11] for financial time series, [40] for electrical engineering, and [26] for physics.

The purpose of our paper is to study the stochastic wave equation driven by fractional-colored Gaussian noise. Our work continues, in part, the line of research which concerns SPDEs driven by the fBm but at the same time it follows the research line initiated by Dalang in [30] which treats equations with white noise in time and correlated in space. More precisely, we consider a system of k stochastic wave equations

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) = \Delta u_i(t,x) + \dot{W}_i(t,x), \qquad t \in [0,T], x \in \mathbb{R}^d \tag{3.1}$$

with initial condition $u_i(t,x) = 0$ and $\frac{\partial u_i}{\partial t}(0,x) = 0$ for every $x \in \mathbb{R}^d$ and for every $i = 1, ..., k$. The driving Gaussian process behaves as a fractional Brownian motion in time and has spatial covariance given by the Riesz kernel. More precisely

$$\mathbf{E}(W_i(t,A)W_j(s,B)) = \delta_{i,j} R_H(t,s) \int_A \int_B f(x-y)dxdy$$

for every $t,s \in [0,T]$ and A, B Borel sets in $\mathbb{R}^d$ where R_H is the covariance of the fractional Brownian motion (3.5), $f : \mathbb{R}^d \to \mathbb{R}_+$ is the Fourier transform of a non-negative tempered measure

μ on $\mathbb{R}^d$ whose density with respect to the Lebesgue measure is $|\xi|^{-(d-\beta)}$, $0 < \beta < d$. Above $\delta_{i,j}$ denotes the Kronecker symbol.

The equation (3.1) has been recently studied in [7] for $H > \frac{1}{2}$. It is has been proven that (3.1) admits a unique mild solution if and only if $\beta < 2H + 1$ which extends the result obtained in [30] in the case $H = \frac{1}{2}$. The purpose of this work is to analyze further the solution of (3.1). We will actually give sharp results for the regularity of it, in time and in space, and we apply these regularity results to study the hitting probabilities for the solution u to (3.1). More precisely, given a Borel set $A \subset \mathbb{R}^k$ we want to determine whether the process $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ hits the set A with positive probability. Recently, there has been several papers on hitting probabilities, and more generally speaking, on potential theory for systems of SPDEs. We refer, among others, to [31], [32], [33], [35] or [72]. The study of hitting probabilities for stochastic partial differential equations with fractional noise in time is new. As far as we know, only the paper [81] treated this problem. Actually, in this reference the authors give upper and lower bounds for the hitting times of solution to a system of stochastic heat equations on the circle with fractional noise in time.

Our aim is to make a new step in this research direction. As we mentioned before, we make a potential analysis of the solution to the stochastic wave equation with fractional-colored noise. That means the noise behaves as the fractional Brownian motion with respect to the time variable and it is a "colored" non-white spatial covariance. In our work this spatial covariance will be described by the Riesz kernel. It is now widely accepted the fact that in order to obtain results on the hitting times of a stochastic process, a detailed analysis of the behavior of the increments of the process is needed. We address this question in our paper and we find the following: the solution $u(t,x), t \in [0,T], x \in \mathbb{R}^d$ to (3.1) is Hölder continuous of order $\frac{1}{2}(2H+1-\beta)$, $\beta \in (2H-1, d \wedge 2H+1)$ in time as well as with respect to the space variable. This generalizes the result obtained in [35] and [36] for the wave equation with white noise in time and Riesz covariance in space. Although the main lines of our work follow the approach of [36], we stress that, as per usual, the fractional case involves more complex calculation and the techniques used in the standard white noise case need to be substantially adapted, this is mainly due to the nature of the noise and to the structure of the Gaussian space associated to the noise. We will point out later in our paper, how the fractional noise involves more complexity than in e.g. [32] or [36]. Moreover, the study of the solution to the wave equation is generally recognized to be more difficult than, for example, the solution to the heat equation, due to the appearance of the trigonometric functions, and this is also the case in our work.

We mention that there are more or less general criteria to determine the hitting times for a stochastic process. Such criteria have been given in [17], [32], [33] or [36] among others. We will use the approach in [17] because it concerns Gaussian processes and fits well with our context (note that the solution to (3.1) is Gaussian).

Our paper is structured as follows. Section 2 contains some preliminaries, we briefly describe the basic properties of the Gaussian noise and its associated Hilbert space, we list the elements of the potential theory that we will use in our paper and we will recall some facts related to the solution to the stochastic wave equation with fractional-colored noise. In Section 3 we analyze the Hölder regularity of the solution with respect to its time and space variables. Section 4 is devoted to the study of the hitting probabilities for this solution, based on a criterium in [17].

3.2 Preliminaries

This section is devoted to introduce the basic notion that we will need throughout the paper. We first introduce the canonical Hilbert space associated to the fractional-colored Gaussian noise. In the second part we present the basic elements related to the potential theory that it is involved in the last section.

3.2.1 The canonical Hilbert space

We denote by $C_0^\infty(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$ with compact support, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^d$. For $\varphi \in L^1(\mathbb{R}^d)$, we let $\mathcal{F}\varphi$ be the Fourier transform of φ:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)dx.$$

We begin by introducing the framework of [30]. Let μ be a non-negative tempered measure on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty, \quad \text{for some } l > 0. \tag{3.2}$$

Since the integrand is non-increasing in l, we may assume that $l \geq 1$ is an integer. Note that $1+|\xi|^2$ behaves as a constant around 0, and as $|\xi|^2$ at ∞, and hence (3.2) is equivalent to:

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty, \quad \text{and} \quad \int_{|\xi|\geq 1} \mu(d\xi)\frac{1}{|\xi|^{2l}} < \infty, \quad \text{for some integer } l \geq 1. \tag{3.3}$$

Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be the Fourier transform of μ in $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Simple properties of the Fourier transform show that for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi).$$

An approximation argument shows that the previous equality also holds for indicator functions $\varphi = 1_A, \psi = 1_B$, with $A, B \in \mathcal{B}_b(\mathbb{R}^d)$, where $\mathcal{B}_b(\mathbb{R}^d)$ is the class of bounded Borel sets of $\mathbb{R}^d$:

$$\int_A\int_B f(x-y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}\mu(d\xi). \tag{3.4}$$

Now we introduce the *fractional Brownian motion* (fBm) with Hurst index $H \in (0,1)$. This is a zero-mean Gaussian process $(B_t^H)_{t\in[0,T]}$ with covariance

$$R_H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,T]. \tag{3.5}$$

Let us denote by $\mathcal{H}$ the canonical Hilbert space associated with this Gaussian process. This canonical Hilbert space is defined as the closure of the linear space generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

It is well known that for $H > 1/2$ we have the expression

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv \tag{3.6}$$

for every $s, t \in [0,T]$ with $\alpha_H := H(2H-1)$. More generally, for $H > 1/2$ and every $\psi, \phi \in \mathcal{H} = \mathcal{H}([0,T])$ we have

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \psi(u)\phi(v)|u-v|^{2H-2} du dv \tag{3.7}$$

As in [6], on a complete probability space $(\Omega, \mathcal{F}, P)$, we consider a zero-mean Gaussian process $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W_t(A)W_s(B)) = R_H(t,s) \int_A \int_B f(x-y) dx dy =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{HP}}. \tag{3.8}$$

Let $\mathcal{E}$ be the set of linear combinations of elementary functions $1_{[0,t]\times A}$, $t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{HP}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$. (Alternatively, $\mathcal{HP}$ can be defined as the completion of $C_0^\infty(\mathbb{R}^{d+1})$, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$; see [6].)

The map $1_{[0,t]\times A} \mapsto W_t(A)$ is an isometry between $\mathcal{E}$ and the Gaussian space H^W of W, which can be extended to $\mathcal{HP}$. We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) W(dt,dx).$$

In the present work, we assume that $H > 1/2$. Hence, (3.6) holds. From (3.4) and (3.6), it follows that for any $\varphi, \psi \in \mathcal{E}$,

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{HP}} &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(u,x)\psi(v,y) f(x-y)|u-v|^{2H-2} dx\, dy\, du\, dv \\ &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(u,\cdot)(\xi)\overline{\mathcal{F}\psi(v,\cdot)(\xi)}|u-v|^{2H-2} \mu(d\xi)\, du\, dv. \end{aligned}$$

Moreover, we can interchange the order of the integrals $dudv$ and $\mu(d\xi)$, since for indicator functions φ and ψ, the integrand is a product of a function of (u,v) and a function of ξ. Hence, for $\varphi, \psi \in \mathcal{E}$, we have:

$$\langle \varphi, \psi \rangle_{\mathcal{HP}} = \alpha_H \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \mathcal{F}\varphi(u,\cdot)(\xi)\overline{\mathcal{F}\psi(v,\cdot)(\xi)}|u-v|^{2H-2} du\, dv\, \mu(d\xi). \tag{3.9}$$

The space $\mathcal{HP}$ may contain distributions, but contains the space $|\mathcal{HP}|$ of measurable functions $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\varphi\|^2_{|\mathcal{HP}|} := \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(u,x)||\varphi(v,y)| f(x-y)|u-v|^{2H-2} dx\, dy\, du\, dv < \infty.$$

3.2.2 Elements of the potential theory

Our aim is to analyze the probability

$$P\left(u(I) \cap A\right) \neq \emptyset$$

where u is the solution to (3.1), I is a Borel set included in $[0,T] \times \mathbb{R}^d$ and A is a Borel set in $\mathbb{R}^k$. Here $u(I)$ means the image of I under the random map $(t,x) \to u(t,x)$.

We will briefly present the notion of the potential theory that we will need in our paper. For all Borel sets $F \subset \mathbb{R}^d$ we define $\mathcal{P}(F)$ to be the set of all probability measures with compact support included in F. For all $\mu \in \mathcal{P}(\mathbb{R}^d)$, let us denote by $I_\beta(\mu)$ the so-called β-energy of the measure μ defined by

$$I_\beta(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\beta(\|x-y\|)\mu(dx)\mu(dy) \tag{3.10}$$

where

$$K_\beta(r) = \begin{cases} r^{-\beta} & \text{if } \beta > 0; \\ log\left(\frac{N_0}{r}\right) & \text{if } \beta = 0; \\ 1 & \text{if } \beta < 0. \end{cases} \tag{3.11}$$

Here N_0 is a constant.

For all $\beta \in \mathbb{R}$ and $F \in \mathcal{B}(\mathbb{R}^d)$ we define the β-dimensional capacity of F by

$$\mathrm{Cap}_\beta(F) = \left[\inf_{\mu \in \mathcal{P}(F)} I_\beta(\mu)\right]^{-1} \tag{3.12}$$

with the convention $1/\infty := 0$. The β-dimensional Hausdorff measure of the set $F \in \mathcal{B}(\mathbb{R}^d)$ is given by

$$\mathcal{H}_\beta(F) = \lim_{\varepsilon \to 0^+} \inf \left[\sum_{i=1}^{\infty}(2r_i)^\beta;\ F \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ \sup_{i \geq 1} r_i \leq \varepsilon\right] \tag{3.13}$$

where $B(x,r)$ denotes the Euclidean ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. When $\beta < 0$, the β-dimensional Hausdorff measure of F is infinite by definition.

3.2.3 The stochastic wave equation with linear fractional-colored noise

Consider the linear stochastic wave equation driven by an infinite-dimensional fractional Brownian motion W with Hurst parameter $H \in (0,1)$. That is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.14}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}(W_t(A)W_s(B)) = R_H(t,s)\int_A\int_B f(x-y)dxdy,$$

where R_H is the covariance of the fractional Brownian motion and f is the Riesz kernel.

Let G_1 be the fundamental solution of $u_{tt} - \Delta u = 0$. It is known that $G_1(t, \cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease, and

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{3.15}$$

for any $\xi \in \mathbb{R}^d, t > 0, d \geq 1$ (see e.g. [99]). In particular,

$$\begin{aligned} G_1(t,x) &= \frac{1}{2} 1_{\{|x|<t\}}, \quad \text{if } d = 1 \\ G_1(t,x) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \quad \text{if } d = 2 \\ G_1(t,x) &= c_d \frac{1}{t} \sigma_t, \quad \text{if } d = 3, \end{aligned}$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t.

The solution of (3.14) is a square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds, dy). \tag{3.16}$$

By definition, $u(t,x)$ exists if and only if the stochastic integral above is well-defined, i.e. $g_{tx} := G_1(t - \cdot, x - \cdot) \in \mathcal{HP}$. In this case, $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|^2_{\mathcal{HP}}$.

The following result has been proved in [7].

Theorem 3 *The stochastic wave equation (3.14) admits an unique mild solution $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ if and only if*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{H+\frac{1}{2}} \mu(d\xi) < \infty. \tag{3.17}$$

Remark 12 *Note that (3.17) is equivalent to:*

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty, \quad \textit{and} \quad \int_{|\xi|\geq 1} \mu(d\xi) \frac{1}{|\xi|^{2H+1}} < \infty. \tag{3.18}$$

As mentioned in the Introduction, we will consider throughout the paper that the spatial covariance of the noise W is given by the Riezs kernel. That means the measure μ is

$$d\mu(\xi) = |\xi|^{-d+\beta} d\xi \text{ with } \beta \in (0,d).$$

In this case the kernel f is given by

$$f(\xi) = |\xi|^{-\beta} \text{ with } \beta \in (0,d).$$

Note that in the case of the Riesz kernel, condition (3.17) is equivalent to

$$\beta \in (0, d \wedge (2H+1)). \tag{3.19}$$

Remark 13 *Since $H > \frac{1}{2}$ and so $2H+1 \in (2,3)$, for dimension $d = 1,2$ we have $\beta \in (0,d)$ while for $d \geq 3$ we have $\beta \in (0, 2H+1)$.*

3.3 Regularity of the solution

3.3.1 Time regularity

In this part we will focus our attention on the behavior of the increments of the solution $u(t,x)$ with respect to the variable t. We will give upper and lower bounds for the L^2-norm of this increment. Usually, obtaining upper bounds is recognized to be easier than obtaining lower bounds, this is also the case in our work. Actually, to get the sharpness of the regularity of u with respect to the time variable, we need to impose a stronger assumption than (3.19) on the parameters β and H (condition (3.20) below). This is due to the characteristics of the scalar product in the $\mathcal{HP}$.

We will start with the following useful lemma that gives an explicit expression for the $\mathcal{H}$ norm of the cosinus and sinus functions. These norms will widely appear further in our computations as well as the trigonometric identities that follow, which will be used several times in the demostrations.

$$\begin{aligned}
\sin(u \pm v) &= \sin u \cos v \pm \cos u \sin v \\
sin(x) - sin(y) &= 2cos(\frac{x+y}{2})sin(\frac{x-y}{2}) \\
\cos(u \pm v) &= \cos u \cos v \mp \sin u \sin v \\
cos(x) - cos(y) &= -2sin(\frac{x+y}{2})sin(\frac{x-y}{2})
\end{aligned}$$

Lemma 4 *Let $f(x) = \cos(x)$ and $g(x) = \sin x$ for $x \in \mathbb{R}$. Then for every $a, b \in \mathbb{R}$, $a < b$*

$$\|f1_{(a,b)}\|_{\mathcal{H}}^2 = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) + \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).$$

and

$$\|g1_{(a,b)}\|_{\mathcal{H}}^2 = \alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v) - \alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).$$

Proof: Using the expression of the scalar product in the Hilbert space $\mathcal{H}$ and the trigonometric identities we can write

$$\begin{aligned}
\|f1_{(a,b)}\|_{\mathcal{H}}^2 + \|g1_{(a,b)}\|_{\mathcal{H}}^2 &= \alpha_H \int_a^b \int_a^b |u-v|^{2H-2} (\cos u \cos v + \sin u \sin v)\, dudv \\
&= \alpha_H \int_a^b du \int_a^b dv |u-v|^{2H-2} \cos(u-v) \\
&= 2\alpha_H \int_a^b du \int_0^{u-a} dv \cos(v) v^{2H-2} \\
&= 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v)
\end{aligned}$$

where we made the change of variables $\tilde{v} = u - v$ in the integral dv above and we computed the integral du. Similarly

$$\begin{aligned}
\|f1_{(a,b)}\|_{\mathcal{H}}^2 - \|g1_{(a,b)}\|_{\mathcal{H}}^2 &= \alpha_H \int_a^b du \int_a^b dv |u-v|^{2H-2} (\cos u \cos v - \sin u \sin v) \\
&= \alpha_H \int_a^b \int_a^b |u-v|^{2H-2} \cos(u+v) dudv
\end{aligned}$$

and by the change of variable $\tilde{v} = u - v$ in the integral dv,

$$\begin{aligned}\|f1_{(a,b)}\|^2_{\mathcal{H}} - \|g1_{(a,b)}\|^2_{\mathcal{H}} &= 2\alpha_H \int_a^b du \int_0^{u-a} dv \cos(2u-v) v^{2H-2} \\ &= 2\alpha_H \int_0^{b-a} dv v^{2H-2} \int_{v+a}^b du \cos(2u-v) \\ &= \alpha_H \int_0^{b-a} dv v^{2H-2} \left(\sin(2b-v) - \sin(2a+v)\right) \\ &= 2\alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).\end{aligned}$$

■

Remark 14 *As a consequence of the Lemma 4 we deduce the following*

i. For every $a, b \in \mathbb{R}$, $a < b$

$$\|f1_{(a,b)}\|^2_{\mathcal{H}} \leq 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2} (b-a-v)$$

ii. For any $x > 0$ the quantity $\int_0^x v^{2H-2} \cos(v)(x-v)dv$ is positive (it is the sum of two norms).

iii. For every $a, b \in \mathbb{R}$, $a < b$

$$\|f1_{(a,b)}\|^2_{\mathcal{H}} \geq 2\alpha_H \cos(a+b) \int_0^{b-a} dv v^{2H-2} \sin(b-a-v).$$

Later, we use also the following lemma.

Lemma 5 *For every $a, b \in \mathbb{R}$ with $a < b$,*

$$\int_a^b \int_a^b dudv \sin(u-v)|u-v|^{2H-2} = 0.$$

Proof: This follows from the trivial equality

$$\int_a^b \int_a^b \sin(u)\cos(v)|u-v|^{2H-2} dudv = \int_a^b \int_a^b \sin(v)\cos(u)|u-v|^{2H-2} dudv.$$

■

Concretely, we will prove the following result concerning the time regularity of the solution to (3.14). We mention that, in the rest of our paper, $c, C...$ will denote generic positive constants that may change from line to line.

Proposition 13 *Assume that*

$$\beta \in (2H-1, d \wedge (2H+1)). \tag{3.20}$$

Let $t_0, M > 0$ and fix $x \in [-M, M]^d$. Then there exists a positive constants c_1, c_2 such that for every $s, t \in [t_0, T]$

$$c_1 |t-s|^{2H+1-\beta} \leq \mathbf{E}\, |u(t,x) - u(s,x)|^2 \leq c_2 |t-s|^{2H+1-\beta}.$$

Proof: Let $h>0$ and let us estimate the $L^2(\Omega)$-norm of the increment $u(t+h,x)-u(t,x)$. Splitting the interval $[0,t+h]$ into the intervals $[0,t]$ and $[t,t+h]$, and using the inequality $|a+b|^2\le 2(a^2+b^2)$, we obtain:

$$\begin{aligned}\mathbf{E}|u(t+h,x)-u(t,x)|^2 &\le 2\{\|(g_{t+h,x}-g_{t,x})1_{[0,t]}\|^2_{\mathcal{HP}}+\|g_{t+h,x}1_{[t,t+h]}\|^2_{\mathcal{HP}}\}\\ &=: 2[E_{1,t}(h)+E_2(h)]. \end{aligned}\tag{3.21}$$

The first summand can be handled in the following way.

$$\begin{aligned}E_{1,t}(h) &= \alpha_H\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t dvdv|u-v|^{2H-2}\mathcal{F}(g_{t+h,x}-g_{tx})(u,\cdot)(\xi)\\ &\quad\times\overline{\mathcal{F}(g_{t+h,x}-g_{tx})(v,\cdot)(\xi)}\\ &= \alpha_H\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t dudv|u-v|^{2H-2}[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)]\\ &\quad\times\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)}\\ &= \alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}I_h,\end{aligned}$$

where

$$\begin{aligned}I_h &= \int_{\mathbb{R}^d}\mu(d\xi)[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)][\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)}]\\ &= \int_{\mathbb{R}^d}\mu(d\xi)\frac{(\sin((u+h)|\xi|)-\sin(u|\xi|))}{|\xi|}\frac{(\sin((v+h)|\xi|)-\sin(v|\xi|))}{|\xi|}.\end{aligned}$$

Using the trigonometric identities we obtain

$$\begin{aligned}E_{1,t}(h) &= \alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\mu(d\xi)\frac{\sin(\frac{h|\xi|}{2})^2}{|\xi|^2}\cos(\frac{(2u+h)|\xi|}{2})\cos(\frac{(2v+h)|\xi|}{2})\\ &= c\cdot\alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^{d-\beta+2}}\sin(h|\xi|)^2\cos((2u+h)|\xi|)\cos((2v+h)|\xi|),\end{aligned}$$

and by making the change of variables $\tilde{u}=(2u+h)|\xi|,\tilde{v}=(2v+h)|\xi|$,

$$\begin{aligned}E_{1,t}(h) &= c\cdot\alpha_H\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^{d-\beta+2H+2}}\sin(h|\xi|)^2\int_{h|\xi|}^{(2t+h)|\xi|}\int_{h|\xi|}^{(2t+h)|\xi|}dudv|u-v|^{2H-2}\cos u\cos v\\ &= c\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^{d-\beta+2H+2}}\sin(h|\xi|)^2\|\cos(\cdot)1_{(h|\xi|,(2t+h)|\xi|)}(\cdot)\|^2_{\mathcal{H}},\end{aligned}\tag{3.22}$$

and using Lemma 4,

$$
\begin{aligned}
E_{1,t}(h) &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \times \left[\int_0^{2t|\xi|} \cos(v) v^{2H-2}(2t|\xi|-v)dv \right.\\
&\quad \left. + \cos(2t|\xi|+2h|\xi|) \int_0^{2t|\xi|} v^{2H-2}(\sin(2t|\xi|-v))\right]\\
&= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \times \left[2t|\xi| \int_0^{2t|\xi|} \cos(v) v^{2H-2}dv \right.\\
&\quad - \sin(2t|\xi|)(2t|\xi|)^{2H-1} + (2H-1)\int_0^{2t|\xi|} \sin(v)v^{2H-2}dv\\
&\quad \left. + \cos(2t|\xi|+2h|\xi|) \int_0^{2t|\xi|} v^{2H-2}(\sin(2t|\xi|-v))\right] \qquad (3.23)
\end{aligned}
$$

where we use integration by parts. By Remark 14, point i. we have the upper bound

$$
\begin{aligned}
E_{1,t}(h) &\leq c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2\\
&\quad \times \left[2t|\xi| \int_0^{2t|\xi|} \cos(v)v^{2H-2}dv - \sin(2t|\xi|)(2t|\xi|)^{2H-1} \right.\\
&\quad \left. + (2H-1)\int_0^{2t|\xi|} \sin(v)v^{2H-2}dv\right].
\end{aligned}
$$

We will treat separately the three summands above. Concerning the first one,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 2t|\xi| \int_0^{2t|\xi|} \cos(v)v^{2H-2}dv\\
&= c_{t,H} h^{2H+1-\beta} \left| \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \cos(v)v^{2H-2}dv \right|\\
&\leq c_{t,H} h^{2H+1-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin(|\xi|)^2 \left| \int_0^{\frac{2t|\xi|}{h}} \cos(v)v^{2H-2}dv \right|\\
&\leq c_{t,H} h^{2H+1-\beta}
\end{aligned}
$$

using condition (3.19) and the fact that the integral $\int_0^\infty \cos(v)v^{2H-2}dv$ is convergent (this implies that the function $x\in[0,\infty)\to\int_0^x \cos(v)v^{2H-2}dv$ admits a limit at infinity and it is therefore bounded). On the other hand

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \sin(2t|\xi|)(2t|\xi|)^{2H-1}\\
&= c_t h^{3-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right)\\
&= c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right)\\
&\quad + c_t h^{3-\beta} \int_{|\xi|>1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right).
\end{aligned}
$$

The second part over the region $|\xi| \geq 1$ is bounded by $ch^{3-\beta}$ simply by majorizing sinus by one. The second integral has a singularity for $|\xi|$ close to zero. Using that $sin(x) \leq x$ for all $x \geq 0$, we will bound it above by

$$\begin{aligned} & h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin(|\xi|)^2 \sin\left(\frac{2t|\xi|}{h}\right) \\ & \leq c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} |\xi|^2 \left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2-2H} \left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2H-1} \\ & \leq c_t h^{2H+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}} \end{aligned}$$

where we bounded $\left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2-2H}$ by $c_t(|\xi|h^{-1})^{2-2H}$ and $\left|\sin\left(\frac{2t|\xi|}{h}\right)\right|^{2H-1}$ by 1. The last integral is finite since $\beta > 2H-1$ (assumption (3.20)).
Finally

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \int_0^{2t|\xi|} \sin(v) v^{2H-2} dv \\ & = h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv \\ & = h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv \\ & + h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(|\xi|)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv \\ & \leq h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} |\xi|^2 \int_0^{\frac{2t|\xi|}{h}} |\sin v| v^{2H-2} dv \\ & + h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv. \end{aligned} \tag{3.24}$$

Using again the fact that $\int_0^\infty \sin(v) v^{2H-2} dv$ is convergent is easy to see that the integral over the region $|\xi| \geq 1$ is bounded by $c_t h^{2H+2-\beta}$. For the intgral over $|\xi| \leq 1$ we make the change of variables $\tilde{v} = \frac{vh}{\xi}$ and we get

$$\begin{aligned} & h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} |\sin\left(\frac{v|\xi|}{h}\right)| \, v^{2H-2} dv \\ & = h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2-2H} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2H-1} v^{2H-2} dv \\ & \leq c_t h^{2h+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}}, \end{aligned}$$

where we have made the same considerations as for the second summand in the decomposition of $E_{1,t}(h)$. In this way, we obtained the upper bound for the summand $E_{1,t}(h)$ in (3.21)

$$E_{1,t}(h) \leq Ch^{2H+1-\beta}. \tag{3.25}$$

We study now the term $E_2(h)$ in (3.21) (its notation $E_2(h)$ instead of $E_{2,t}(h)$ is due to the fact that it does not depend on t, see below). Using successively the change of variables $\tilde{u} = \frac{u}{h}, \tilde{v} = \frac{v}{h}$ in the integral $dudv$ and $\tilde{\xi} = h\xi$ in the integral $d\xi$, the summand $E_2(h)$ can be written as

$$\begin{aligned} E_2(h) &= \alpha_H \int_{\mathbb{R}^d} \int_t^{t+h} \int_t^{t+h} \mathcal{F}G_1(t+h-u,\cdot)(\xi)\overline{\mathcal{F}G_1(t+h-v,\cdot)(\xi)}|u-v|^{2H-2} du\ dv\ \mu(d\xi) \\ &= \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^h \int_0^h \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} dudv \\ &= \alpha_H h^{2H} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin(u|\xi|h)\sin(v|\xi|h)\,|u-v|^{2H-2} dudv \\ &= \alpha_H h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} dudv. \end{aligned}$$

Let us use the following notation:

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} dudv, \quad t \in [0,T], \xi \in \mathbb{R}^d. \tag{3.26}$$

By Proposition 3.7 in [7] the term

$$N_1(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^1 \int_0^1 \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} dudv$$

satisfies the inequality

$$N_1(\xi) \quad \leq \quad C_H \left(\frac{1}{1+|\xi|^2}\right)^{H+1/2},$$

with C_H a positive constant not depending on h. Consequently the term $E_2(h)$ is bounded by

$$E_2(h) \leq Ch^{2H+2-\beta} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \mu(d\xi) \tag{3.27}$$

and this is clearly finite due to (3.17). Relations (3.25) and (3.27) give the first part of the conclusion.

Let us analyze now the lower bound of the increments of $u(t,x)$ with respect to the variable t. Let $h > 0, x \in [-M,M]^d$ and $t \in [t_0,T]$ such that $t+h \in [t_0,T]$. From the decomposition

$$\begin{aligned} \mathbf{E}\,|u(t+h,x)-u(t,x)|^2 &= \|\,(g_{t+h,x}-g_{t,x})\,1_{[0,t]}\|^2_{\mathcal{HP}} + \|g_{t+h,x}1_{[t,t+h]}\|^2_{\mathcal{HP}} \\ &\quad +2\langle (g_{t+h,x}-g_{t,x})\,1_{[0,t]}, g_{t+h,x}1_{[t,t+h]}\rangle_{\mathcal{HP}} \end{aligned}$$

we immediately obtain, since the second summand in the right-hand side is positive,

$$\begin{aligned} \mathbf{E}\,|u(t+h,x)-u(t,x)|^2 &\geq \|\,(g_{t+h,x}-g_{t,x})\,1_{[0,t]}\|^2_{\mathcal{HP}} + 2\langle (g_{t+h,x}-g_{t,x})\,1_{[0,t]}, g_{t+h,x}1_{[t,t+h]}\rangle_{\mathcal{HP}} \\ &:= E_{1,t}(h) + E_{3,t}(h). \end{aligned}$$

We can assume, without any loss of the generality, that $t = \frac{1}{2}$. Denote $E_{1,\frac{1}{2}}(h) := E_1(h)$. We first prove that

$$E_1(h) \geq ch^{2H+1-\beta} - c'h^{2H+2-\beta}. \tag{3.28}$$

for h small enough. Recall that we have an exact expression for $E_1(h)$ (see (3.23)). Actually,

$$\begin{aligned}
E_1(h) &= \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \| \cos(\cdot) 1_{(h|\xi|, h|\xi|+|\xi|)} \|_{\mathcal{H}}^2 \\
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \int_{h|\xi|}^{(1+h)|\xi|} \int_{h|\xi|}^{(1+h)|\xi|} dudv |u-v|^{2H-2} \cos u \cos v \\
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \int_0^{|\xi|} \int_0^{|\xi|} dudv \cos(u + h|\xi|) \cos(v + h|\xi|) |u-v|^{2H-2}.
\end{aligned}$$

By the trigonomtric formula $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ we can write

$$\begin{aligned}
E_1(h) &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 \Big[\cos(h|\xi|)^2 \int_0^{|\xi|} \int_0^{|\xi|} dudv \cos u \cos v |u-v|^{2H-2} \\
&\quad -2 \sin(h|\xi|) \cos(h|\xi|) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \cos v |u-v|^{2H-2} \\
&\quad + \sin(h|\xi|)^2 \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \sin v |u-v|^{2H-2} \Big] \\
&:= A + B + C.
\end{aligned}$$

We will neglect the first term since it is positive. We will bound the second one above by $ch^{2H+2-\beta}$. Using again the trigonometric identities, Lemma 5 (used at the third line below), and the change of variables $\tilde{v} = u - v$ we have

$$\begin{aligned}
& -2 \sin(h|\xi|) \cos(h|\xi|) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin u \cos v |u-v|^{2H-2} \\
&= -\sin(h|\xi|) \cos(h|\xi|) \int_0^{|\xi|} \int_0^{|\xi|} dudv \left(\sin(u+v) + \sin(u-v) \right) |u-v|^{2H-2} \\
&= -\sin(h|\xi|) \cos(h|\xi|) \int_0^{|\xi|} \int_0^{|\xi|} dudv \sin(u+v) |u-v|^{2H-2} \\
&= c \cdot sin(h|\xi|) \cos(h|\xi|) \int_0^{|\xi|} v^{2H-2} (\cos(2|\xi| - v) - \cos(v)) dv
\end{aligned}$$

and thus

$$
\begin{aligned}
B &= c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^2 sin(h|\xi|)\cos(h|\xi|) \int_0^{|\xi|} v^{2H-2}(\cos(2|\xi|-v)-\cos(v))dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^3 \cos(h|\xi|)\sin(|\xi|) \int_0^{|\xi|} v^{2H-2}\sin(|\xi|-v)dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^3 \cos(h|\xi|)\sin(|\xi|) \int_0^{|\xi|} v^{2H-2}\left(\sin(|\xi|)\cos(v)-\cos(|\xi|)\sin(v)\right)dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^3 \cos(h|\xi|)\sin(|\xi|) \\
&\quad \times \left(\sin(|\xi|)\int_0^{|\xi|} v^{2H-2}\cos(v)dv - \cos(|\xi|)\int_0^{|\xi|} v^{2H-2}\sin(v)dv\right) \\
&= -c \cdot \alpha_H \int_{|\xi|\le 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^3 \cos(h|\xi|)\sin(|\xi|) \\
&\quad \times \left(\sin(|\xi|)\int_0^{|\xi|} v^{2H-2}\cos(v)dv - \cos(|\xi|)\int_0^{|\xi|} v^{2H-2}\sin(v)dv\right) \\
&- c \cdot \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin(h|\xi|)^3 \cos(h|\xi|)\sin(|\xi|) \\
&\quad \times \left(\sin(|\xi|)\int_0^{|\xi|} v^{2H-2}\cos(v)dv - \cos(|\xi|)\int_0^{|\xi|} v^{2H-2}\sin(v)dv\right)
\end{aligned}
$$

Taking absolute value we see that the part over the set $|\xi| \le 1$ is bounded by ch^3 by simply majorizing $\sin(h|\xi|)$ by $h|\xi|$, $\cos(h|\xi|)\sin(|\xi|)$ by one, and

$$
\left|\sin(|\xi|)\int_0^{|\xi|} v^{2H-2}\cos(v)dv - \cos(|\xi|)\int_0^{|\xi|} v^{2H-2}\sin(v)dv\right|
$$

by a constant. Concerning the part over the region $|\xi| \ge 1$ we bound again the last expression by a constant and we use the change of variables $\tilde{\xi} = h\xi$. This part will by bounded by

$$
\begin{aligned}
& h^{2H+2-\beta} \int_{|\xi|\ge h} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \left|\sin(|\xi|)^3 \cos(|\xi|)\sin(|\xi|/h)\right| \\
& \le h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \left|\sin(|\xi|)^3\right| \\
& \le ch^{2H+2-\beta}
\end{aligned}
$$

since the last integral is convergent at infinity by bounded sinus by one and at zero by bounding $\sin(x)$ by x and using the assumption $\beta > 2H-1$. Therefore

$$
B \le ch^{2H+2-\beta}. \tag{3.29}
$$

We bound now the summand C below. In this summand the $\mathcal{H}$ norm of the sinus function appear and this has been analyzed in [7]. We have, after the change of variables $\tilde{u} = \frac{u}{|\xi|}, \tilde{v} = \frac{v}{|\xi|}$,

$$\begin{aligned} C &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin(h|\xi|)^4 \int_0^1 \int_0^1 \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} du dv \\ &\geq \alpha_H \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin(h|\xi|)^4 \int_0^1 \int_0^1 \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} du dv. \end{aligned}$$

We will use Proposition 3.8 in [7] (more precisely, we will use the inequality (34) in that paper with $k = 0$; we notice that the term $\sin(h|\xi|)^4$ does not appear in this proof but analyzing step by step the proof we can see that this term can be added without problems). For h small, we will have that,

$$\begin{aligned} C &\geq \alpha_H \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin(h|\xi|)^4 \frac{1}{|\xi|^2} \int_0^1 \int_0^1 \sin(u|\xi|)\sin(v|\xi|)|u-v|^{2H-2} du dv \\ &\geq \alpha_H \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin(h|\xi|)^4 \frac{1}{|\xi|^{2H+1}} \\ &= \alpha_H h^{2H+1-\beta} \int_{|\xi|\geq h} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin(|\xi|)^4 \\ &\geq \alpha_H h^{2H+1-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin(|\xi|)^4 \\ &= c \cdot \alpha_H h^{2H+1-\beta}. \end{aligned} \tag{3.30}$$

Relations (3.29) and (3.30) imply (3.28). Now, from relation (3.28), for every $t_0 \leq s < t < T$ with s, t close enough

$$E_1(t-s) \geq c(t-s)^{2H+1-\beta} - c'(t-s)^{2H+2-\beta} \geq \frac{c}{2}(t-s)^{2H+1-\beta}$$

if $|t-s| \leq \frac{c}{2c'}$. To extend the above inequality to arbitrary values of $|t-s|$, we proceed as in [36], proof of Proposition 4.1. Notice that the function $g(t,s,x,y) := \mathbf{E}\,|u(t,x)-u(s,x)|^2$ is positive and continuous with respect to all its arguments and therefore it is bounded below on the set $\{(t,s,x,y) \in [t_0,T]^2 \times [-M,M]^{2d}; |t-s| \geq \varepsilon\}$ by a constant depending on $\varepsilon > 0$. Hence for $|t-s| \geq \frac{c}{2c'}$ it also holds that

$$E_1(t-s) \geq c_1|t-s|^{2H+1-\beta}.$$

On the other side, from (3.22) and (3.27) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} E_{3,t}(h) &= \langle (g_{t+h,x} - g_{t,x})\, 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \rangle_{\mathcal{HP}} \\ &\leq \|\, (g_{t+h,x} - g_{t,x})\, 1_{[0,t]} \|_{\mathcal{HP}} \| g_{t+h,x} 1_{[t,t+h]} \|_{\mathcal{HP}} \\ &\leq c h^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}. \end{aligned}$$

Consequently,

$$\mathbf{E}\,|u(t+h,x) - u(t,x)|^2 \geq C h^{2H+1-\beta} - C' h^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}$$

and this implies that for every $s,t \in [t_0,T]$ and $x \in [-M,M]^d$

$$\mathbf{E}\,|u(t,x)-u(s,x)|^2 \;\geq\; \frac{C}{2}|t-s|^{2H+1-\beta} \quad \text{if} \quad |t-s| \leq \left(\frac{C}{2C'}\right)^{\frac{1}{2}}.$$

Similarly as above, the previous inequality can be extended to arbitrary values of $s,t \in [t_0,T]$. ■

Proposition (13) implies the following Hölder property for the solution to (3.14).

Corollary 2 *Assume (3.20). Then for every $x \in \mathbb{R}^d$ the application*

$$t \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in \left(0, \frac{2H+1-\beta}{2}\right)$.

Proof: This is consequence of the relations (3.22) and (3.27) in the proof of Proposition 13 and of the fact that u is Gaussian. ■

Let us make some comments on the result in Proposition 13.

Remark 15

- *Following the proof of Theorem 5.1 in [35] we can show that the mapping $t \to u(t,x)$ is not Hölder continuous of order $\frac{2H+1-\beta}{2}$.*
- *When $H=\frac{1}{2}$, we recover the results in [36], [35] in the linear case.*

3.3.2 Space regularity

Let us discuss the behavior of the solution u to the equation (3.14) with respect to the spatial variable. We have

Proposition 14 *Assume (3.20), fix $M>0$ and $t \in [t_0,T]$. Then there exist positive constants c_3, c_4 such that for any $x,y \in [-M,M]^d$*

$$c_3|x-y|^{2H+1-\beta} \leq \mathbf{E}\,|u(t,x)-u(t,y)|^2 \leq c_4|x-y|^{2H+1-\beta}.$$

Proof: Let $z \in \mathbb{R}^d$. We compute

$$\begin{aligned}
&\mathbf{E}|u(t,x+z)-u(t,x)|^2 = \|g_{t,x+z}-g_{t,x}\|^2_{\mathcal{HP}} \\
&= \alpha_H \int_{\mathbb{R}^d}\int_0^t\int_0^t \mathcal{F}(g_{t,x+z}-g_{t,x})(u,\cdot)(\xi)\overline{\mathcal{F}(g_{t,x+z}-g_{t,x})(v,\cdot)(\xi)}|u-v|^{2H-2}du\,dv\;\mu(d\xi) \\
&= \alpha_H \int_0^t\int_0^t |u-v|^{2H-2}du\,dv \int_{\mathbb{R}^d} |e^{-i\xi\cdot(x+z)}-e^{-i\xi\cdot x}|^2 \mathcal{F}G_1(u,\cdot)(\xi)\overline{\mathcal{F}G_1(v,\cdot)(\xi)}\;\mu(d\xi) \\
&= \alpha_H \int_0^t\int_0^t |u-v|^{2H-2}du\,dv \int_{\mathbb{R}^d} |e^{-i\xi\cdot z}-1|^2 \frac{\sin(u|\xi|)}{|\xi|}\cdot\frac{\sin(v|\xi|)}{|\xi|}\mu(d\xi) \\
&=: E_{1,x}(z)+E_{2,x}(z),
\end{aligned}$$

where $E_{1,x}(z)$ and $E_{2,x}(z)$ are the integrals over the regions $|\xi| < 1$ and $|\xi| \geq 1$ respectively.
For the first expresion is easy to see that, using the inequality $|1 - e^{-i\xi z}|^2 \leq |\xi|^2|z|^2$, we get the bound

$$E_{1,x}(z) \leq C|z|^2 \int_{|\xi|\leq 1} \mu(d\xi).$$

Developing the second expresion we get

$$\begin{aligned} E_{2,x}(z) &= \alpha_H \int_0^t \int_0^t |u-v|^{2H-2} du\, dv \int_{|\xi|\geq 1} |e^{-i\xi\cdot z} - 1|^2 \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|} \mu(d\xi) \\ &= 2\alpha_H \int_0^t \int_0^t |u-v|^{2H-2} du\, dv \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta}} (1 - \cos(z\cdot\xi)) \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|}, \end{aligned}$$

where $z \cdot \xi$ means the scalar product in $\mathbb{R}^d$. Again from Proposition 3.7 in [7] we have that

$$N_t(\xi) \leq c_{t,H} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}$$

for any $t > 0$, $|\xi| \geq 1$, where $N_t(\xi)$ is given by (3.26). Hence, denoting by $e = \frac{z}{|z|}$

$$\begin{aligned} E_{2,x}(z) &\leq C \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta}} (1 - \cos(z\cdot\xi)) \left(\frac{1}{1+|\xi|^2} \right)^{H+\frac{1}{2}} \\ &= C z^{2H+1-\beta} \int_{\mathbb{R}^d} \frac{dw}{|w|^{d-\beta}} (1 - \cos(w\cdot e)) \left(\frac{1}{|w|^2+|z|^2} \right)^{H+\frac{1}{2}} \\ &\leq C|z|^{2H+1-\beta}, \end{aligned}$$

where we used the change of variables $w = \xi|z|$. This proves the upper bound.

Let us prove the sharpness of this bound (i.e. the lower bound). We can assume, without losing the generality, that $t = 1$. We note that

$$\begin{aligned} & \mathbf{E}\, |u(1, x+z) - u(1,x)|^2 \geq F_2(z) \\ := \quad & 2\alpha_H \int_0^1 \int_0^1 dudv |u-v|^{2H-2} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta}} (1 - \cos(\xi\cdot z)) \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|}. \end{aligned}$$

Condition (3.18) implies that

$$\int_{|\xi|\geq 1} \frac{\mu(d\xi)}{|\xi|^3} \leq \int_{|\xi|\geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+1}} < \infty.$$

We apply Proposition 3.8 in [7] (more precisely, the inequality (34) in [7] with $k = 0$) and we get (note that the result in [7] is stated without the factor $(1 - \cos(\xi \cdot z))$ but by analyzing the steps of the proof we can see that this factor may be added without problems)

$$F_2(z) \geq C \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} (1 - \cos(\xi\cdot z)) - C' \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} (1 - \cos(\xi\cdot z))$$

(here $C > C'$) and by the change of variables $\xi|z| = w$ in the integral $d\xi$

$$\begin{aligned} F_2(z) &\geq Cz^{2H+1-\beta}\int_{|w|\geq|z|}\frac{dw}{|w|^{d-\beta+2H+1}}(1-\cos(w\cdot e)) \\ &\quad -C'z^{2H+2-\beta}\int_{|w|\geq|z|}\frac{dw}{|w|^{d-\beta+2H+2}}(1-\cos(w\cdot e)) \\ &\geq Cz^{2H+1-\beta}. \end{aligned} \tag{3.31}$$

As in the proof of Theorem 5.1 in [35], we obtain that the integral $\int_{|w|\geq|z|}\frac{dw}{|w|^{d-\beta+2H+1}}(1-\cos(w\cdot e))$ is bounded below by a constant. (Notice that $\beta > 2H-1$, implies that the first integral above is convergent when z is zero, because $1-\cos(x) \approx x^2$ around zero). Thus, it is immediate that

$$\mathbf{E}\,|u(1,x+z)-u(1,x)|^2 \geq Cz^{2H+1-\beta}.$$

■

We have the following result concerning the Hölder continuity in space. We mention that it is a little bit more than an extension of Proposition 14.

Proposition 15 *Assume $\beta \in (0, d\wedge(2H+1))$. Then for any $t \in [t_0, T]$ the application*

$$x \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in \left(0, \left(\frac{2H+1-\beta}{2}\right)\wedge 1\right)$.

Proof: We claim that

$$\mathbf{E}\,|u(t,x)-u(t,y)|^2 \leq c|x-y|^{(2H+1-\beta)\wedge 2} \tag{3.32}$$

whenever $|x-y|$ is sufficiently small. From Proposition 14, (3.32) is true when $\beta > 2H-1$. When $\beta \in (0, 2H-1]$ then it suffices to regards the part of the quantity $\mathbf{E}\,|u(t,x+z)-u(t,x)|^2$ over the region $|\xi| \leq 1$ (the part over the region $|\xi| > 1$ is, as in the proof of Proposition 14, bounded by $cz^{2H+1-\beta}$ so by cz^2 for z small). It is immediate to see that, using the inequality $|1-e^{-i\xi z}|^2 \leq |\xi|^2|z|^2$ the considered part is less than $C|z|^2\int_{|\xi|\leq 1}\mu(d\xi)$. This concludes the proof of (3.32).

The conclusion is a consequence of Proposition 14, the Gaussianity of u and the Kolmogorov continuity theorem.

■

Remark 16

- *When $H = \frac{1}{2}$, the above result coincides with the findings in [36], [35]).*
- *We distinguish in Proposition 15 two cases: if $\beta \in (0, 2H-1)$ then the solution to (3.14) has spatial Hölder continuity of order α for every $\alpha \in (0,1)$ while if $\beta \in (2H-1, d\wedge(2H+1))$ the Hölder exponent is $\delta \in (0, \frac{2H+1-\beta}{2}) < 1$.*

- *There is another way to see why the cases $\beta \in (0, 2H-1]$ and $\beta \in (2H-1, d \wedge (2H+1))$ need to be separated. Denote by*

$$\begin{aligned} g_t(z) &:= \mathbf{E}\left|u(t,x+z)-u(t,x)\right|^2 \\ &= 2\alpha_H \int_0^t \int_0^t |u-v|^{2H-2} du\, dv \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta}} (1-\cos(z\cdot\xi)) \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|} \end{aligned}$$

and let us study the behavior of g_t around $z=0$. Let us also assume that $d=1$. Notice first that $g_t(0)=0$ and

$$g_t'(z) = 2\alpha_H \int_0^t \int_0^t |u-v|^{2H-2} du\, dv \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta-1}} \sin(z\cdot\xi) \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|}$$

and thus $g_t'(0)=0$ provided that $\beta < 2H$. Moreover

$$g_t''(z) = 2\alpha_H \int_0^t \int_0^t |u-v|^{2H-2} dudv \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta-2}} \cos(z\cdot\xi) \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|}$$

and

$$\begin{aligned} g_t''(0) &= 2\alpha_H \int_0^t \int_0^t |u-v|^{2H-2} dudv \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta-2}} \frac{\sin(u|\xi|)}{|\xi|} \cdot \frac{\sin(v|\xi|)}{|\xi|} \\ &\leq C_t 2\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta-2}} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \end{aligned}$$

which is a finite constant for $\beta < 2H-1$. Therefore $g_t(z)$ behaves as Cz^2 for z close to zero.

3.3.3 Joint regularity

Let us denote by Δ the following metric on $[0,T]\times\mathbb{R}^d$

$$\Delta\left((t,x);(s,y)\right) = |t-s|^{2H+1-\beta} + |x-y|^{2H+1-\beta}. \tag{3.33}$$

From Propositions 13 and 14, we obtain the following result:

Theorem 4 *Fix $M>0$ and assume (3.20). For every $t,s \in [t_0,T]$ and $x,y \in [-M,M]^d$ there exist positive constants C_1, C_2 such that*

$$C_1 \Delta\left((t,x);(s,y)\right) \leq \mathbf{E}\left|u(t,x)-u(s,y)\right|^2 \leq C_2 \Delta\left((t,x);(s,y)\right).$$

Proof: The upper bound can be easily obtained by using the upper bound in Propositions 13 and 14 since

$$\begin{aligned} \mathbf{E}\left|u(t,x)-u(s,y)\right|^2 &\leq 2\mathbf{E}\left|u(t,x)-u(s,x)\right|^2 + 2\mathbf{E}\left|u(s,x)-u(s,y)\right|^2 \\ &\leq C_2\left(|t-s|^{2H+1-\beta} + |x-y|^{2H+1-\beta}\right). \end{aligned}$$

Concerning the lower bound, it suffices follow the lines of the proof of Lemma 2.1 in [81] (see also Steps 3 and 4 in the proof of Proposition 4.1 in [36]). We will briefly explain the main lines of the proof. The demostration needs to be divided upon three cases: $|t-s|^{2H+1-\beta} \leq \frac{c_3}{4c_2}|x-y|^{2H+1-\beta}$, $|t-s|^{2H+1-\beta} \geq \frac{4c_4}{c_1}|x-y|^{2H+1-\beta}$ and $\frac{4c_4}{c_1}|x-y|^{2H+1-\beta} \geq |t-s|^{2H+1-\beta} \geq \frac{c_3}{4c_2}|x-y|^{2H+1-\beta}$ with the constants c_1, c_2, c_3, c_4 appearing in the statements of Propositions 13 and 14. The first case can be handled as follows

$$\begin{aligned}
\mathbf{E}\left|u(t,x)-u(s,y)\right|^2 &\geq \frac{1}{2}\mathbf{E}\left|u(t,x)-u(t,y)\right|^2 - \mathbf{E}\left|u(t,y)-u(s,y)\right|^2 \\
&\geq \frac{1}{2}c_3|x-y|^{2H+1-\beta} - c_2|t-s|^{2H+1-\beta} \\
&\geq \frac{1}{2}c_3|x-y|^{2H+1-\beta} - \frac{1}{4}c_3|x-y|^{2H+1-\beta} \\
&= \frac{1}{4}c_3|x-y|^{2H+1-\beta} \\
&\geq \frac{c_3}{8}|x-y|^{2H+1-\beta} + \frac{c_3}{8}\frac{4c_2}{c_3}|t-s|^{2H+1-\beta} \\
&\geq C_1\Delta\left((t,x);(s,y)\right).
\end{aligned}$$

The other cases follows similarly from Lemma 3.1 in [81], by replacing their exponents with our exponents.

■

Remark 17 *The result of Theorem 4 can be stated also in the following form: Fix $M > 0$ and assume (3.20). For every $t, s \in [t_0, T]$ and $x, y \in [-M, M]^d$ with (t,x) close enough to (s,y), there exist positive constants C_1, C_2 such that*

$$C_1\left(|t-s|+|x-y|\right)^{2H+1-\beta} \leq \mathbf{E}\left|u(t,x)-u(s,y)\right|^2 \leq C_2\left(|t-s|+|x-y|\right)^{2H+1-\beta}.$$

3.4 Hitting times

Let us discuss the upper and lower bounds for the hitting probabilities of the solution u to equation (3.14). These bounds will be given in terms of the Newtonian capacity and the Hausdorff measure of the hit set (see Section 2 for the definition). Let us recall the notation: if $V = (V(x), x \in \mathbb{R}^m)$ is a $\mathbb{R}^k$ valued stochastic process then $V(S)$ denote the range of the Borel set S under the random mapping $x \to V(x)$.

Our result is based on the following criteria for the hitting probabilities proven in [17], Theorem 2.1.

Theorem 5 *Let $X = X(t), t \in \mathbb{R}^N$ be a $\mathbb{R}^k$-valued centered Gaussian process and fix $I \subset \mathbb{R}^N$. Assume that there exist positive constants a_1, a_2, a_3, a_4 such that*

i. For every $t \in I$, $\mathbf{E}\left[X(t)^2\right] \geq a_1 > 0$.

ii. There exists $\alpha_1, ..., \alpha_N \in (0,1)$ such that for every $t = (t_1, ..., t_N), s = (s_1, ..., s_N) \in I$ it holds that

$$a_2\sum_{j=1}^{N}|t_j-s_j|^{2\alpha_j} \leq \mathbf{E}\left|X(t)-X(s)\right|^2 \leq a_3\sum_{j=1}^{N}|t_j-s_j|^{2\alpha_j}.$$

iii. For every $t = (t_1, ..., t_N), s = (s_1, ..., s_N) \in I$

$$Var(X(t)|X(s)) \geq a_4 \sum_{j=1}^{N} |t_j - s_j|^{2\alpha_j}.$$

Then there exist positive constants a_5, a_6 *such that for every Borel set* A *in* $\mathbb{R}^k$

$$a_5 Cap_{k-Q}(A) \leq P(X(I) \cap A \neq \emptyset) \leq a_6 \mathcal{H}_{k-Q}(A)$$

where $Q = \sum_{j=1}^{N} \frac{1}{\alpha_j}$.

Next, we will show that the solution to (3.14) satisfies the assumptions of the previous result. This will be done via several lemmas.

Lemma 6 *Assume (3.20) and let* u *be the solution to (3.14). Then for every* $t \in [t_0, T]$ *and* $x \in \mathbb{R}^d$

$$\mathbf{E}u(t,x)^2 \geq C.$$

Proof: *Let* $\sigma^2_{t,x}$ *be the variance of* $u(t,x)$*. We need to give a lower bound for this variance. Assume for simplicity* $t = 1$*. Then*

$$\begin{aligned}
\sigma^2_{1,x} &= \mathbf{E}\,|u(1,x)|^2 \\
&= \alpha_H \int_0^1 \int_0^1 dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin(u|\xi|)\sin(v|\xi|) \\
&\geq \alpha_H \int_0^1 \int_0^1 dudv|u-v|^{2H-2} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin(u|\xi|)\sin(v|\xi|) \\
&\geq \alpha_H \sin^2 1 \int_0^1 \int_0^1 dudv|u-v|^{2H-2} uv = C > 0
\end{aligned}$$

where we used the bound $\sin x \geq x \sin 1$ *for every* $x \in [0,1]$*. The general case* $t \in [t_0, T]$ *follows in the same way by doing the change of variables* $\tilde{u} = \frac{u}{t}, \tilde{v} = \frac{v}{t}$ *and then working on the domain* $D = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{1}{ut}\}$. ∎

Now, we bound the conditional variance (condition iii. in Theorem 5).

Lemma 7 *Assume (3.20) and fix* $t_0, M > 0$*. Then for every* $s, t \in [t_0, T]$ *and* $x, y \in [-M, M]^d$

$$Var(u(t,x)|u(s,y)) \geq C\Delta((t,x);(s,y))$$

where Δ *is the metric given by (3.33).*

Proof: We will use the following formula: if (U, V) is a centered Gaussian vector, then

$$Var(U,V) = \frac{(\rho^2_{U,V} - (\sigma_U - \sigma_V)^2)((\sigma_U + \sigma_V)^2 - \rho^2_{U,V})}{4\sigma^2_V} \tag{3.34}$$

where $\rho^2_{U,V} = \mathbf{E}(U-V)^2, \sigma^2_U = \mathbf{E}U^2, \sigma^2_V = \mathbf{E}V^2$. Denote by

$$\rho^2_{t,x,s,y} = \mathbf{E}\left|u(t,x)-u(s,y)\right|^2, \;\; \sigma^2(t,x) = \mathbf{E}u(t,x)^2, \;\; \sigma^2_{s,y} = \mathbf{E}u(s,y)^2.$$

It suffices to show that

$$(\rho^2_{t,x,s,y} - (\sigma_{t,x}-\sigma_{s,y})^2)((\sigma_{t,x}+\sigma_{s,y})^2 - \rho^2_{t,x,s,y}) \geq c\Delta((t,x);(s,y))$$

for every $s,t \in [t_0,T]$ and $x,y \in [-M,M]^d$. By Theorem 4 the second factor in the left-hand side above is bounded below by a constant. So it remains to check that

$$(\rho^2_{t,x,s,y} - (\sigma_{t,x}-\sigma_{s,y})^2) \geq c\Delta((t,x);(s,y))$$

but this has been done in [81], proof of Proposition 3.2. (see also [32], proof of Lemma 4.3).

■

Remark 18 *Using the previous result we can give a bound on the joint density $p_{t,x,s,y}$ of the vector $(u(t,x),u(s,y))$. Actually, one can show that for every $t \in [t_0,T]$ and $x,y \in [-M,M]^d$ we have the inequality*

$$p_{t,s,x,y}(z_1,z_2) \leq C_1\Delta\left((t,x);(s,y)\right)^{-\frac{k}{2}} \exp\left(-\frac{C_2|z_1-z_2|^2}{\Delta\left((t,x);(s,y)\right)}\right)$$

for every $z_1,z_2 \in [-N,N]^k$, where Δ is the metric defined by (3.33). It suffices to follow the lines of Proposition 3.2 in [81].

We can state now the main result of this section.

Theorem 6 *Assume (3.20) and let us consider I,J non-trivial compacts sets in $[t_0,T]$ and $[-M,M]^d$ respectively. Fix $N>0$ and let u be the solution to the system (3.1). Then for every Borel set A contained in $[-N,N]^k$ it holds that*

$$C^{-1}Cap_{k-\gamma}(A) \leq P\left(u(I\times J)\cap A \neq \emptyset\right) \leq C\mathcal{H}_{k-\gamma}(A)$$

with

$$\gamma = k - \frac{2(d+1)}{2H+1-\beta}.$$

Proof: The proof is a consequence of Theorem 5 and of the preceding two lemmas.

■

Remark 19

- *Of course, for $H=\frac{1}{2}$, our result recovers the findings in [36] in the linear case.*
- *it is also possible to give some results concerning the probability that, for fixed t,x, the sets $u(\{t\}\times J)$ and $u(I\times\{x\})$ (as before I,J non-trivial compacts sets in $[t_0,T]$ and in $[-M,M]^d$ respectively) to hit a given Borel set A contained in $[-N,N]^k$. Actually, by routine arguments we will have*

  $$C^{-1}Cap_{k-\frac{2d}{2H+1-\beta}}(A) \leq P\left(u(\{t\}\times J)\cap A \neq \emptyset\right) \leq C\mathcal{H}_{k-\frac{2d}{2H+1-\beta}}(A)$$

  and

  $$C^{-1}Cap_{k-\frac{2}{2H+1-\beta}}(A) \leq P\left(u(I\times\{x\})\cap A \neq \emptyset\right) \leq C\mathcal{H}_{\frac{2}{2H+1-\beta}}(A).$$

Chapter 4

CAPÍTULO IV: Wiener integrals with respect to the Hermite random field and a wave equation

Introduction

The random fields or multiparameter stochastic processes have focused a significant amount of attention among scientists due to the wide range of applications that they have. Particularly, self-similar random fields find some of their applications in various kind of phenomena, going from hydrology and surface modeling to network traffic analysis and mathematical finance, to name a few. From other side, this type of processes are also quite interesting when they appear as solutions to Stochastic Parial Differential Equations (SPDE's) in several dimensions, such as the wave or heat equations.

A class of processes that lies in the family described above are the Hermite random fields or Hermite sheets (from now on). Inside this class we can find the well-known and studied fractional Brownian sheet and the Rosenblatt processes, among others.

The Hermite processes of order $q \geq 1$ are self-similar with stationary increments and live in the qth Wiener chaos, that is, it can be expressed as a q times iterated integral with respect to the Wiener process. The class of Hermite process includes the fractional Brownian motion which is the only Gaussian process in this family. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes. Since they are non-Gaussian and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian.

The Hermite sheet of order q it is only known in his representation as a non-central limit of a particularly normalized Hermite variation of the fractional Brownian sheet, see [87] for the two-parameter case and [19] for the general d-parametric case. In both cases the authors also prove self-similarity, stationary increments and Hölder continuity.

In the present work we deal directly with the multi-parametric case building the Hermite sheet as a natural extension of the expression for the Hermite process studied as a non-central limit in [41] and [96].

Fix $d \in \mathbb{N} \setminus \{0\}$ and let $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$ a multi-Hurst index

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{t_1} \cdots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} \cdots (s_d - y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} \right) \\ & \quad ds_d \ldots ds_1 \quad dW(y_{1,1}, ..., y_{d,1}) \ldots dW(y_{1,q}, ..., y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q). \end{aligned} \tag{4.1}$$

The above integrals are Wiener-Itô multiple integrals of order q with respect to the d-parametric standard Brownian sheet $(W(\mathbf{y}))_{\mathbf{y}\in\mathbb{R}^d}$ (see [77] for the definition) and $c(\mathbf{H}, q)$ is a positive normalization constant depending only on $\mathbf{H}$ and q. We designate the process $Z^q_{\mathbf{H}}(\mathbf{t})$ as the *Hermite sheet* or *Hermite random field*.

From expression (4.1) is possible to note that for $d = 1$ we recover the *Hermite process* which represent a family that has been recently studied in [29], [64] and [85]. As a particular case ($q = 1$) we recover the most known element of this family, the *fractional Brownian motion*, which has been largely studied due to its various applications. Recently, a rich theory of stochastic integration with respect to this process has been introduced and stochastic differential equations driven by the fractional Brownian motion have been considered for several purposes. The process obtained in (4.1) for $d = 1, q = 2$ is known as the *Rosenblatt process*, it was introduced by Rosenblatt in [88] and it has been called in this way by Taqqu in [95]. Lately, this process have been increasingly studied by his different interesting aspects like wavelet type expansion or extremal properties, parameter estimations, discrete approximations and others potential applications (see [1], [2], [9], [28], [100]).

As far as we know, the only well-known multiparameter process that can be obtained from (4.1) is the *fractional Brownian sheet* ($d > 1$ and $q = 1$). This processes has been recently studied as a driving noise for stochastic differential equations and stochastic calculus with respect to it have been developed. We refer to [5], [56], [101] for only a few works on various aspects of the fractional Brownian sheet.

In one hand the purpose of this article is to study the basic properties of the multiparamarmeter Hermite process and then to introduce Wiener integrals with respect to the Hermite sheet in order to generalize and continue the line introduced in [64] putting a new brick in the construction of stochastic calculus driven by this class of processes in several dimensions. As in [19] the covariance structure of the Hermite sheet is like the one of the fractional Brownian sheet, enabling the use of the same classes of deterministic integrands as in the fractional Brownian sheet profiting its well-known properties.

Also in the aim of this work lives the idea of making an approach to the study of stochastic partial differential equations in several dimensions driven by non-Gaussian noises, giving a specific expression for the driving noise allowing to use in a better way the properties of the equations by taking advantage of the results already existent in the literature. Is in this sense that, inspired by the works [7], [24] or [36] and exploiting these, we present a stochastic wave equation with respect to the Hermite sheet in spatial dimension $d \geq 1$ and we study the existence, regularity, and other properties of the solution, including the existence of local times and of the joint density.

We organize our paper as follows. Section 2 present the necessary notations and prove several properties of the Hermite sheet. In Section 3, we construct Wiener integrals with respect to this process. Section 4 is devoted to present the wave equation and discuss the existence and regularity of the solution and other properties.

4.1 Notation and the Hermite sheet

Throughout the work we use the notation introduced in [19]. Fix $d \in \mathbb{N} \setminus \{0\}$ and consider multi-parametric processes indexed in $\mathbb{R}^d$. We shall use bold notation for multi-indexed quantities, i.e., $\mathbf{a} = (a_1, a_2, ..., a_d)$, $\mathbf{ab} = (a_1 b_1, a_2 b_2, ..., a_d b_d)$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, ..., a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_i^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_i^d (a_i, b_i)$, $\sum_{\mathbf{i} \in [\mathbf{0}, \mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \dots \sum_{i_d}^{N_d} a_{i_1, i_2, ..., i_d}$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, and $\mathbf{a} < \mathbf{b}$ iff $a_1 < b_1, a_2 < b_2, ..., a_d < b_d$ (analogously for the other inequalities).

Before introducing the *Hermite sheet* we briefly recall the *fractional Brownian sheet* and the *standard Brownian sheet*.

The *d-parametric anisotropic fractional Brownian sheet* is the centered Gaussian process $\{B_{\mathbf{t}}^{\mathbf{H}} : \mathbf{t} = (t_1, ..., t_d) \in \mathbb{R}^d\}$ with Hurst multi-index $\mathbf{H} = (H_1, ..., H_d) \in (0, 1)^d$. It is equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, and its covariance function is given by

$$\begin{aligned} R_{\mathbf{H}}(\mathbf{s}, \mathbf{t}) &= \mathbb{E}[B_{\mathbf{s}}^{\mathbf{H}} B_{\mathbf{t}}^{\mathbf{H}}] \\ &= \prod_i^d R_{H_i}(s_i, t_i) = \prod_i^d \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2}. \end{aligned} \tag{4.2}$$

The *d-parametric standard Brownian sheet* is the Gaussian process $\{W_{\mathbf{t}} : \mathbf{t} = (t_1, ..., t_d) \in \mathbb{R}^d\}$ equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, and covariance function given by

$$R(\mathbf{s}, \mathbf{t}) = \mathbb{E}[W_{\mathbf{s}}, W_{\mathbf{t}}] = \prod_i^d R(s_i, t_i) = \prod_i^d s_i \wedge t_i. \tag{4.3}$$

Let $q \geq 1, q \in \mathbb{Z}$ and the Hurst multi-index $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order q* is given by

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H},q)\int_{\mathbb{R}^{d\cdot q}}\int_0^{t_1}\cdots\int_0^{t_d}\left(\prod_{j=1}^{q}(s_1-y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}\cdots(s_d-y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)}\right)\\ &\quad ds_d\ldots ds_1 \quad dW(y_{1,1},...,y_{d,1})\ldots dW(y_{1,q},...,y_{d,q})\\ &= c(\mathbf{H},q)\int_{\mathbb{R}^{d\cdot q}}\int_0^{\mathbf{t}}\prod_{j=1}^{q}(\mathbf{s}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}d\mathbf{s}\quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q). \end{aligned} \tag{4.4}$$

where $x_+ = max(x,0)$. For a better understanding about multiple stochastic integrals we refer to [77]. As pointed out before, when $q=1$, (4.4) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H}=(H_1,H_2,...,H_d)\in(\frac{1}{2},1)^d$. For $q\geq 2$ the process $Z^q_{\mathbf{H}}(\mathbf{t})$ is not Gaussian and for $q=2$ we denominate it as the *Rosenblatt sheet*.

Now let's calculate the covariance $R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t})$ of the Hermite sheet. Using the isometry of multiple Wiener-Itô integrals and Fubini one get

$$\begin{aligned} R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= \mathbb{E}[Z^q_{\mathbf{H}}(\mathbf{s})Z^q_{\mathbf{H}}(\mathbf{t})]\\ &= \mathbb{E}\left\{c(\mathbf{H},q)^2\int_{\mathbb{R}^{d\cdot q}}\int_0^{\mathbf{s}}\prod_{j=1}^{q}(\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}d\mathbf{u}\quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q)\right.\\ &\quad\left.\cdot\int_{\mathbb{R}^{d\cdot q}}\int_0^{\mathbf{t}}\prod_{j=1}^{q}(\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{\mathbf{q}}\right)}d\mathbf{v}\quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q)\right\}\\ &= c(\mathbf{H},q)^2\int_{\mathbb{R}^{d\cdot q}}\left\{\int_0^{s_1}\cdots\int_0^{s_d}\prod_{j=1}^{q}\prod_{i=1}^{d}(u_i-y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)}du_d\ldots du_1\right.\\ &\quad\left.\cdot\int_0^{t_1}\cdots\int_0^{t_d}\prod_{j=1}^{q}\prod_{i=1}^{d}(v_i-y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)}dv_d\ldots dv_1\right\}dy_{1,1}\ldots dy_{d,1}\ldots dy_{1,q}\ldots dy_{d,q}\\ &= c(\mathbf{H},q)^2\int_0^{t_1}\int_0^{s_1}\int_{\mathbb{R}^q}\prod_{j=1}^{q}(u_1-y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}(v_1-y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}dy_{1,1}\ldots dy_{1,q}du_1dv_1\\ &\quad\vdots\\ &\quad\int_0^{t_d}\int_0^{s_d}\int_{\mathbb{R}^q}\prod_{j=1}^{q}(u_d-y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)}(v_d-y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)}dy_{d,1}\ldots dy_{d,q}du_ddv_d \end{aligned}$$

but

$$\begin{aligned} &\int_{\mathbb{R}^q}\prod_{j=1}^{q}(u-x_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}(v-x_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}dx_1\ldots dx_q\\ &= \left[\int_{\mathbb{R}}(u-x)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}(v-x)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\right]^q, \end{aligned} \tag{4.5}$$

so

$$\begin{aligned} R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= c(\mathbf{H},q)^2\int_0^{t_1}\int_0^{s_1}\left[\int_{\mathbb{R}}(u_1-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}(v_1-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}\right]^q du_1dv_1 \\ &\vdots \\ &\int_0^{t_d}\int_0^{s_d}\left[\int_{\mathbb{R}}(u_d-y_d)_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)}(v_d-y_d)_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)}\right]^q du_ddv_d. \end{aligned}$$

Recalling that the Beta function $\beta(p,q)=\int_0^1 z^{p-1}(1-z)^{p-1}dz, p,q>0$, satisfies the following identity

$$\int_{\mathbb{R}}(u-y)_+^{a-1}(v-y)_+^{a-1}dy = \beta(a,2a-1)|u-v|^{2a-1} \tag{4.6}$$

we see that

$$\begin{aligned} R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= c(\mathbf{H},q)^2\int_0^{t_1}\int_0^{s_1}\beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q\cdot|u_1-v_1|^{2(H_1-1)}du_1dv_1 \\ &\quad\ldots\int_0^{t_d}\int_0^{s_d}\beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q\cdot|u_d-v_d|^{2(H_d-1)}du_ddv_d \\ &= c(\mathbf{H},q)^2\beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q\frac{1}{2H_1(2H_1-1)}\left(s_1^{2H_1}+t_1^{2H_1}-|t_1-s_1|^{2H_1}\right) \\ &\quad\ldots\beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q\frac{1}{2H_d(2H_d-1)}\left(s_d^{2H_d}+t_d^{2H_d}-|t_d-s_d|^{2H_d}\right) \end{aligned}$$

So now we choose

$$c(\mathbf{H},q)^2=\left(\frac{\beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q}{H_1(2H_1-1)}\right)^{-1}\cdots\left(\frac{\beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q}{H_d(2H_d-1)}\right)^{-1} \tag{4.7}$$

in this way we get $\mathbb{E}\left(Z^q_{\mathbf{H}}(\mathbf{t})^2\right)=\mathbf{t}^{2\mathbf{H}}=t_1^{2H_1}\ldots t_d^{2H_d}$, and finally

$$\begin{aligned} R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= \frac{1}{2}\left(s_1^{2H_1}+t_1^{2H_1}-|t_1-s_1|^{2H_1}\right)\ldots\left(s_d^{2H_d}+t_d^{2H_d}-|t_d-s_d|^{2H_d}\right) \\ &= \prod_i^d\frac{s_i^{2H_i}+t_i^{2H_i}-|t_i-s_i|^{2H_i}}{2} \\ &= \prod_i^d R_{H_i}(s_i,t_i)=R_{\mathbf{H}}(\mathbf{s},\mathbf{t}) \end{aligned} \tag{4.8}$$

Remark 20 *As mentioned at the beginning, from the previous development we see that the covariance structure is the same for all $q\geq 1$, so it coincides with the covariance of the fractional Brownian sheet.*

We will next prove the basic properties of the Hermite sheet: self-similarity, stationarity of the increments and Hölder continuity.

Let us first recall the concept of self-similarity for multiparameter stochastic processes.

Definition 3 *A stochastic process $(X_{\mathbf{t}})_{\mathbf{t}\in T}$, where $T \subset \mathbb{R}^d$ is called self-similar with self-similarity order $\alpha = (\alpha_1, ..., \alpha_d) > 0$ if for any $\mathbf{h} = (h_1, .., h_d) > 0$ the stochastic process $(\hat{X}_{\mathbf{t}})_{\mathbf{t}\in T}$ given by*

$$\hat{X}_{\mathbf{t}} = \mathbf{h}^{\alpha} X_{\frac{\mathbf{t}}{\mathbf{h}}} = h_1^{\alpha_1} ... h_d^{\alpha_d} X_{\frac{t_1}{h_1}, ..., \frac{t_d}{h_d}}$$

has the same law as the process X.

Proposition 16 *The Hermite sheet is self-similar of order $\mathbf{H} = (H_1, ..., H_d)$.*

Proof: The scaling property of the Wiener sheet implies that for every $0 < \mathbf{c} = (c_1, .., c_d) \in \mathbb{R}^d$ the processes $(W(\mathbf{ct})_{\mathbf{t}\geq 0})$ and $\left(\sqrt{\mathbf{c}}W(\mathbf{t})\right)_{\mathbf{t}\geq 0}$ have the same finite dimensional distributions. Therefore, if $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^d$, using obvious changes of variables in the integrals $d\mathbf{s}$ and dW,

$$\begin{aligned}
\hat{Z}^q_{\mathbf{H}}(t) &= \mathbf{h}^{\mathbf{H}} Z^q_{\frac{\mathbf{t}}{\mathbf{h}}} \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\frac{\mathbf{t}}{\mathbf{h}}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\frac{\mathbf{s}}{\mathbf{h}} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\frac{\mathbf{s}}{\mathbf{h}} - \frac{\mathbf{y}_j}{\mathbf{h}})_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{h}^{-1}\mathbf{y}_1) \ldots dW(\mathbf{h}^{-1}\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}}\mathbf{h}^{q\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{h}^{-1}\mathbf{y}_1) \ldots dW(\mathbf{h}^{-1}\mathbf{y}_q) \\
&=^{(d)} c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-\mathbf{1}}\mathbf{h}^{q\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} \mathbf{h}^{-\frac{q}{2}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\
&= Z^q_{\mathbf{H}}(t)
\end{aligned}$$

where $=^{(d)}$ means equivalence of finite dimensional distributions.

■

Let us recall the notion of the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, ..., s_d), \mathbf{t} = (t_1, ..., t_d)$, with $\mathbf{s} \leq \mathbf{t}$. This increment is denoted by $\Delta X_{[\mathbf{s},\mathbf{t}]}$ and it is given by

$$\Delta X_{[\mathbf{s},\mathbf{t}]} = \sum_{r\in\{0,1\}^d} (-1)^{d-\sum_i r_i} X_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}. \tag{4.9}$$

When $d = 1$ one obtains $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_t - X_s$ while for $d = 2$ one gets $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}$.

Definition 4 *A process $(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^d)$ has stationary increments if for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$ the stochastic processes $(\Delta X_{[0,\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ and $(\Delta X_{[\mathbf{h},\mathbf{h}+\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ have the same finite dimensional distributions.*

Proposition 17 *The Hermite sheet $(Z^q(\mathbf{t}))_{\mathbf{t}\geq 0}$ has stationary increments.*

Proof: Developing the increments of the process using the definition of the Hermite sheet and proceding as in the proof of Proposition 1 using the change of variables $\mathbf{s}' = \mathbf{s} - \mathbf{h}$, it is immediate to see that for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$,

$$\Delta Z^q_{[\mathbf{h},\mathbf{h}+\mathbf{t}]} =^d \Delta Z^q_{[0,\mathbf{t}]}$$

for every $\mathbf{t}$. ■

Proposition 18 *The trajectories of the Hermite sheet $(Z^q(\mathbf{t}), \mathbf{t} \geq 0)$ are Hölder continuous of any order $\delta = (\delta_1, .., \delta_d) \in [0, \mathbf{H}]$ in the following sense: for every $\omega \in \Omega$, there exists a constant $C_\omega > 0$ such that for every $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d, \mathbf{s}, \mathbf{t} \geq 0$,*

$$|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}| \leq C_\omega |t_1 - s_1|^{\delta_1} ....|t_d - s_d|^{\delta_d} = C_\omega |\mathbf{t} - \mathbf{s}|^{\delta}.$$

Proof: Using the Cencov's criteria (see [22]) and the fact that the process Z^q is almost sure equal to 0 when $t_i = 0$, it suffices to check that

$$\mathbb{E}\left|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\right|^p \leq C\left(|t_1 - s_1|....|t_d - s_d|\right)^{1+\gamma} \tag{4.10}$$

for some $p \geq 2$ and $\gamma > 0$. From the self-similarity and the stationarity of the increments of the process Z^q, we have for every $p \geq 2$

$$\mathbb{E}\left|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\right|^p = \mathbb{E}\left|Z_{\mathbf{1}}\right|^p \left(|t_1 - s_1|....|t_d - s_d|\right)^{p\mathbf{H}}$$

and this obviously implies (4.10). ■

4.2 Wiener integrals with respect to the Hermite sheet

Now we are well positioned to present Wiener integrals with respect to the d-parametric Hermite sheet. Let us consider a Hermite sheet $\left(Z^q_{\mathbf{H}}(\mathbf{t})\right)_{t\in\mathbb{R}^d}$. Denote $\mathscr{E}$ the family of elementary functions on $\mathbb{R}^d$ of the form

$$\begin{aligned} f(\mathbf{u}) &= \sum_{l=1}^{n} a_l 1_{(\mathbf{t}_l,\mathbf{t}_{l+1}]}(\mathbf{u}) \qquad (4.11)\\ &= \sum_{l=1}^{n} a_l 1_{(t_{1,l},t_{1,l+1}]\times\ldots\times(t_{d,l},t_{d,l+1}]}(u_1, \ldots, u_d), \quad \mathbf{t}_l < \mathbf{t}_{l+1},\ a_l \in \mathbb{R},\ l = 1, \ldots, n. \end{aligned}$$

For functions like f above we can naturally define it's Wiener integral with respect to the Hermite sheet $Z^q_{\mathbf{H}}$ as

$$\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) = \sum_{l=1}^{n} a_l \Delta(Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]} \tag{4.12}$$

where $(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]}$ (see (4.9)) stands for the generalized increments of $Z^q_{\mathbf{H}}$ on the rectangle

$$\Delta_{\mathbf{t}_l} := [\mathbf{t}_l, \mathbf{t}_{l+1}] = \prod_{i=1}^{d} [t_{i,l}, t_{i,l+1}]$$

given by

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]} = \sum_{\xi\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} Z^q_{\mathbf{H}}(t_{1,l+\xi_d}, \dots, t_{d,l+\xi_1}). \tag{4.13}$$

In the case $d = 1$, we simply have

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]} = Z^q_{\mathbf{H}}(t_{1,l+1} - t_{1,l})$$

while for $d = 2$

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]} = Z^q_{\mathbf{H}}(t_{1,l+1}, t_{2,l+1}) - Z^q_{\mathbf{H}}(t_{1,l}, t_{2,l+1}) - Z^q_{\mathbf{H}}(t_{1,l+1}, t_{2,l}) + Z^q_{\mathbf{H}}(t_{1,l}, t_{2,l}).$$

With the purpose of extend the definition (4.12) to a larger family of integrands, we will point out some observations before. Let's consider the mapping J on the set of functions $f : \mathbf{R}^d \to \mathbf{R}$ to the set of functions $f : \mathbf{R}^{d\cdot q} \to \mathbf{R}$ such that

$$\begin{aligned} J(f)(\mathbf{y}_1, \dots, \mathbf{y}_q) &= c(\mathbf{H}, q) \int_{\mathbb{R}^d} f(\mathbf{u}) \prod_{j=1}^{q} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{u} \qquad (4.14) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^d} f(u_1, \dots, u_d) \prod_{j=1}^{q}\prod_{i=1}^{d} (u_i - u_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} du_1, \dots du_d. \end{aligned}$$

Using the mapping J we see that definition (4.4) can be re-expressed as follows

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q) \\ &= \int_{\mathbb{R}^{d\cdot q}} J(1_{[0,t_1]\times\dots\times[0,t_d]})(\mathbf{y}_1, \dots, \mathbf{y}_q) dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q). \qquad (4.15) \end{aligned}$$

As J is clearly linear, definition (4.12) can be tailored to

$$\begin{aligned} \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) &= \sum_{l=1}^{n} a_l \Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right) \\ &= \sum_{l=1}^{n} a_l \left(\sum_{\xi\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} Z^q_{\mathbf{H}}(t_{1,l+\xi_1}, \dots, t_{d,l+\xi_d}) \right) \\ &= \sum_{l=1}^{n} a_l \sum_{\xi\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} \int_{\mathbb{R}^{d\cdot q}} J(1_{[0,t_{1,l+\xi_1}]\times\dots\times[0,t_{d,l+\xi_d}]})(\mathbf{y}_1, \dots, \mathbf{y}_q) dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q) \\ &= \sum_{l=1}^{n} a_l \int_{\mathbb{R}^{d\cdot q}} J(1_{[t_{1,l},t_{1,l+1}]\times\dots\times[t_{d,l},t_{d,l+1}]})(\mathbf{y}_1, \dots, \mathbf{y}_q) dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q) \\ &= \int_{\mathbb{R}^{d\cdot q}} J(f)(\mathbf{y}_1, \dots, \mathbf{y}_q) dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q). \qquad (4.16) \end{aligned}$$

In this way we introduce the space

$$\mathcal{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^{d\cdot q}} (J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q))^2 \, d\mathbf{y}_1, \ldots, d\mathbf{y}_q < \infty \right\} \tag{4.17}$$

equipped with the norm

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^{d\cdot q}} (J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q))^2 \, d\mathbf{y}_1, \ldots, d\mathbf{y}_q. \tag{4.18}$$

Working the expression for the norm we see that

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= c(\mathbf{H}, q)^2 \int_{\mathbb{R}^{d\cdot q}} \left\{ \left(\int_{\mathbb{R}^d} f(\mathbf{u}) \prod_{j=1}^{q} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{u} \right) \right. \\
&\quad \left. \cdot \left(\int_{\mathbb{R}^d} f(\mathbf{v}) \prod_{j=1}^{q} (\mathbf{v} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{v} \right) \right\} d\mathbf{y}_1, \ldots, d\mathbf{y}_q \\
&= c(\mathbf{H}, q)^2 \int_{\mathbb{R}^{d\cdot q}} \left\{ \left(\int_{\mathbb{R}^d} f(u_1, \ldots, u_d) \prod_{j=1}^{q} \prod_{i=1}^{d} (u_i - y_{i,j})_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} du_1, \ldots du_d \right) \right. \\
&\quad \left. \cdot \left(\int_{\mathbb{R}^d} f(v_1, \ldots, v_d) \prod_{j=1}^{q} \prod_{i=1}^{d} (v_i - y_{i,j})_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} dv_1, \ldots dv_d \right) \right\} d\mathbf{y}_1, \ldots, d\mathbf{y}_q
\end{aligned}$$

Using (4.5), (4.6) and (4.7) we get that

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= c(\mathbf{H}, q)^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u_1, \ldots, u_d) f(v_1, \ldots, v_d) \\
&\quad \left\{ \prod_{i=1}^{d} \int_{\mathbb{R}^q} \prod_{j=1}^{q} (u_i - y_{i,j})_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} (v_i - y_{i,j})_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} dy_{i,1}, \ldots dy_{i,q} \right\} du_1, \ldots du_d dv_1, \ldots dv_d \\
&= c(\mathbf{H}, q)^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u_1, \ldots, u_d) f(v_1, \ldots, v_d) \\
&\quad \cdot \prod_{i=1}^{d} \left(\int_{\mathbb{R}} (u_i - y)_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} (v_i - y)_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} dy \right)^q du_1, \ldots du_d dv_1, \ldots dv_d \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u_1, \ldots, u_d) f(v_1, \ldots, v_d) \prod_{i=1}^{d} H_i(2H_i - 1)|u - v|^{2H_i - 2} du_1, \ldots du_d dv_1, \ldots dv_d \\
&= \mathbf{H}(\mathbf{2H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H}-\mathbf{2}} d\mathbf{u} d\mathbf{v},
\end{aligned} \tag{4.19}$$

hence

$$\mathcal{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H}-\mathbf{2}} d\mathbf{u} d\mathbf{v} < +\infty \right\} \tag{4.20}$$

and

$$\|f\|_{\mathcal{H}}^2 = \mathbf{H}(\mathbf{2H} - \mathbf{1}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u} - \mathbf{v}|^{\mathbf{2H}-\mathbf{2}} d\mathbf{u} d\mathbf{v}.$$

The mapping

$$f \to \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) \tag{4.21}$$

provides an isometry from $\mathscr{E}$ to $L^2(\Omega)$. Indeed, for f like (4.11) it holds that

$$\begin{aligned}
&\mathbb{E}\left\{\left(\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u})\right)^2\right\} \\
=& \sum_{k,l=0}^{n-1} a_k a_l \mathbb{E}\left(\Delta_{\mathbf{t}_k}\left(Z^q_{\mathbf{H}}(\mathbf{t}_k)\right) \cdot \Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right)\right) \qquad (4.22)\\
=& \sum_{k,l=0}^{n-1} a_k a_l \sum_{\xi \in \{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} \sum_{\rho \in \{0,1\}^d} (-1)^{d-\sum_{j=1}^d \rho_j} \mathbb{E}\left\{Z^q_{\mathbf{H}}(\mathbf{t}_{k+\xi}) Z^q_{\mathbf{H}}(\mathbf{t}_{l+\rho})\right\} \\
=& \sum_{k,l=0}^{n-1} a_k a_l \sum_{\xi \in \{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} \sum_{\rho \in \{0,1\}^d} (-1)^{d-\sum_{j=1}^d \rho_j} R_{\mathbf{H}}(\mathbf{t}_{k+\xi}, \mathbf{t}_{l+\rho}) \\
=& \sum_{k,l=0}^{n-1} a_k a_l H_1(2H_1-1) \dots H_d(2H_d-1) \int_{t_{1,k}}^{t_{1,k+1}} \dots \int_{t_{d,k}}^{t_{d,k+1}} \cdot \int_{t_{1,l}}^{t_{1,l+1}} \dots \int_{t_{d,l}}^{t_{d,l+1}} \\
& |u_1 - v_1|^{2H_1-2} \dots |u_d - v_d|^{2H_d-2} du_1 \dots du_d dv_1 \dots dv_d \\
=& \sum_{k,l=0}^{n-1} a_k a_l < 1_{[t_{1,k}, t_{1,k+1}] \times [t_{d,k}, t_{d,k+1}]}, 1_{[t_{1,l}, t_{1,l+1}] \times [t_{d,l}, t_{d,l+1}]} >_{\mathcal{H}} \\
=& < f, f >_{\mathcal{H}}, \qquad (4.23)
\end{aligned}$$

where we have made a slight abuse of notation, $\mathbf{t}_{k+\xi} = (t_{1,k+\xi_1}, \dots t_{d,k+\xi_d})$.

On the other hand, from what shown in [84] it follows that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (4.14) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ and relation (4.15) still holds.

Remark 21 *The elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is*

$$|\mathcal{H}| \;=\; \left\{ f : \mathbb{R}^d \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\mathbf{u})||f(\mathbf{v})||\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{v} d\mathbf{u} < \infty \right\}.$$

Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^{\mathbf{2}}(\mathbb{R}^d) \cap L^{\mathbf{1}}(\mathbb{R}^d) \subset L^{\frac{1}{\mathbf{H}}}(\mathbb{R}^d) \subset |\mathcal{H}| \subset \mathcal{H}, \tag{4.24}$$

where $L^{\mathbf{p}}$ denotes $L^{p_1} \otimes \dots \otimes L^{p_d}$.
The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} \;=\; \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\mathbf{u})||f(\mathbf{v})||\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{v} d\mathbf{u}$$

Remark 22 *Expression (4.16) present a useful interpretation for the Wiener integrals with respect to the Hermite sheet; as elements in the q-th Wiener chaos generated by the d-parametric standard Brownian field.*

4.3 Application: The Hermite stochastic wave equation

In this section we present the linear stochastic wave equation as an example of equations driven by a Hermite sheet. We show the existence of the solution and study some properties of it thanks to the definition of the Wiener integrals with respect to the Hermite sheet.

Consider the linear stochastic wave equation driven by an infinite-dimensional Hermite sheet $Z^q_{\mathbf{H}}$ with Hurst multi-index $\mathbf{H} \in (1/2, 1)^{(d+1)}$. That is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{Z}^q_{\mathbf{H}}(t,\mathbf{x}), \quad t>0, \mathbf{x}\in\mathbb{R}^d \\ u(0,x) &= 0, \quad \mathbf{x}\in\mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad \mathbf{x}\in\mathbb{R}^d. \end{aligned} \tag{4.25}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $Z^q_{\mathbf{H}} = \{Z^q_{\mathbf{H}}(t,\mathbf{x}); t\geq 0, \mathbf{x}\in\mathbb{R}^d\}$ is the $(d+1)$-parametric Hermite sheet whose covariance is given by

$$\mathbb{E}\left\{Z^q_{\mathbf{H}}(s,\mathbf{x})Z^q_{\mathbf{H}}(t,\mathbf{y})\right\} = R_H(t,s)R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y})$$

if $\mathbf{H} = (H, H_1, .., H_d)$ and we denoted by $\mathbf{H}_0 = (H_1, ..., H_d)$. Equivalently we can write

$$\mathbb{E}\left\{\dot{Z}^q_{\mathbf{H}}(s,\mathbf{x})\dot{Z}^q_{\mathbf{H}}(t,\mathbf{y})\right\} = H(2H-1)|t-s|^{2H-2}\prod_{i=1}^d (H_i(2H_i-1)\cdot|x_i-y_i|^{2H_i-2}) \tag{4.26}$$

Let G_1 be the fundamental solution of $u_{tt} - \Delta u = 0$. It is known that $G_1(t,\cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease, and

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{4.27}$$

for any $\xi\in\mathbb{R}^d, t>0, d\geq 1$ (see e.g. [99]). In particular,

$$\begin{aligned} G_1(t,\mathbf{x}) &= \frac{1}{2}1_{\{|x|<t\}}, \quad \text{if } d=1 \\ G_1(t,\mathbf{x}) &= \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}}, \quad \text{if } d=2 \\ G_1(t,\mathbf{x}) &= c_d\frac{1}{t}\sigma_t, \quad \text{if } d=3, \end{aligned}$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t.

The *mild* solution of (4.25) is a square-integrable process $u = \{u(t,\mathbf{x}); t\geq 0, \mathbf{x}\in\mathbb{R}^d\}$ defined by:

$$u(t,\mathbf{x}) = \int_0^t\int_{\mathbb{R}^d} G_1(t-s,\mathbf{x}-\mathbf{y})Z^q_{\mathbf{H}}(ds,d\mathbf{y}). \tag{4.28}$$

The above integral is a Wiener integral with respect to the Hermite sheet, as introduced in Section 2.

4.3.1 Existence and regularity of the solution

By definition, $u(t,\mathbf{x})$ exists if and only if the stochastic integral above is well-defined, i.e. $g_{t,\mathbf{x}} := G_1(t-\cdot,\mathbf{x}-\cdot) \in \mathcal{H}$. In this case, $\mathbb{E}|u(t,x)|^2 = \|g_{t,\mathbf{x}}\|^2_{\mathcal{H}}$.

We state the result on the existence and the regularity of the solution to (4.25).

Proposition 19 *Let $Z^q_{\mathbf{H}}(t,\mathbf{x})$ be the $(d+1)$-parametric Hermite sheet of order q. Denote by*

$$\beta = d - \sum_{i=1}^{d}(2H_i - 1). \tag{4.29}$$

Then the following statements are true

a.- *The stochastic wave equation (4.25) admits an unique mild solution $(u(t,\mathbf{x}))_{t\in[0,1],\mathbf{x}\in\mathbb{R}^d}$ if and only if*

$$\sum_{i=1}^{d}(2H_i - 1) > d - 2H - 1. \tag{4.30}$$

b.- *Assume $\beta > 2H-1$ and let t_0 and $\mathbf{x} \in \mathbb{R}^d$ fixed. Then there exists positive constants c_1, c_2 such that for every $s,t \in [t_0,1]$*

$$c_1|t-s|^{2H+1-\beta} \leq \mathbb{E}\left|u(t,\mathbf{x}) - u(s,\mathbf{x})\right|^2 \leq c_2|t-s|^{2H+1-\beta}.$$

Also for every fixed $\mathbf{x} \in \mathbb{R}^d$ the application

$$t \to u(t,\mathbf{x})$$

is almost surely Hölder continuous of order $\delta \in \left(0, \frac{2H+1-\beta}{2}\right)$.

c.- *Fix $t \in [t_0,T]$. Then there exist positive constants c_3, c_4 such that for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$*

$$c_3|\mathbf{x}-\mathbf{y}|^{2H+1-\beta} \leq \mathbb{E}\left|u(t,\mathbf{x}) - u(t,\mathbf{y})\right|^2 \leq c_4|\mathbf{x}-\mathbf{y}|^{2H+1-\beta}.$$

Also, for any $t \in [t_0,1]$ the application

$$\mathbf{x} \to u(t,\mathbf{x})$$

is almost surely Hölder continuous of order $\delta \in \left(0, \left(\frac{2H+1-\beta}{2}\right) \wedge 1\right)$.

d.- *Denote by Δ the following metric on $[0,T]\times\mathbb{R}^d$*

$$\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right) = |t-s|^{2H+1-\beta} + |\mathbf{x}-\mathbf{y}|^{2H+1-\beta}. \tag{4.31}$$

Fix $M > 0$ and assume (4.30). For every $t,s \in [t_0,1]$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$ there exist positive constants C_1, C_2 such that

$$C_1\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right) \leq \mathbb{E}\left|u(t,\mathbf{x}) - u(s,\mathbf{y})\right|^2 \leq C_2\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right). \tag{4.32}$$

Proof: By the isometry of the Wiener integral with respect to the Hermite sheet, the L^2 norm will be

$$\begin{aligned}\mathbb{E}u(t,\mathbf{x})^2 &= \alpha_H \int_0^t du \int_0^t dv |u-v|^{2H-2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{y}d\mathbf{z} G_1(t-u,\mathbf{x}-\mathbf{y})G_1(t-v,\mathbf{x}-\mathbf{z}) \\ &\times \prod_{i=1}^d (H_i(2H_i-1))|x_i-y_i|^{2H_i-2} \\ &= \alpha_H \int_0^t du \int_0^t dv |u-v|^{2H-2} \int_{\mathbb{R}^d} \frac{\sin(u|\xi|)\sin(v|\xi|)}{|\xi|^2}\mu(d\xi)\end{aligned}$$

where

$$\mu(d\xi) = c_{\mathbf{H}} \prod_{i=1}^d |\xi_i|^{-(2H_i-1)} \tag{4.33}$$

with $\xi = (\xi_1, ..., \xi_d)$. This is, $u(t,\mathbf{x})$ has the same L^2 norm as in the case $q = 1$, that means, when the noise of the equation is a fractional Brownian sheet. It therefore follows from [7], Theorem 3.1 that the above integral is finite if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \mu(d\xi) < \infty$$

with μ given by (4.33). The above condition is equivalent to $\sum_{i=1}^d (2H_i - 1) > d - 2H - 1$, see Example 3.4 in [7].

The proof of the other two items items is strongly held in the covariance structure of the Hermite sheet, which is the same as for the fractional Brownian sheet. By a carefully revision of the proofs of Theorem 3.1 in [7], Propositions 1, 2, 3 and Corollary 1 in [24], is possible to appreciate that the computations are also valid for any process with a covariance structure like the one presented in these articles, in particular our case.

- The bounds for the increments are consequence of Proposition 1 in [24], and the Hölder regularity comes from Corollary 1 in [24].
- The bounds are deduced from Proposition 2 in [24], and the space Hölder regularity is direct from Proposition 3 in [24].
- Point **d** follows from **b** and **c** by following the lines of the proof of Theorem 2 in [24].

■

4.3.2 Existence of local times

We will show that the solution to (4.25), viewed as a process in (t,x), admits a square integrable local time.

Let us define the local time of a stochastic process $(X_t)_{t\in T}$. Here T denotes a subset of $\mathbb{R}^d$. For any Borel set $I \subset T$ the occupation measure of X on I is defined as

$$\mu_I(A) = \lambda\left(t \in I, X_t \in A\right), \qquad A \in \mathcal{B}(\mathbb{R})$$

where λ denotes the Lebesque measure. If μ_I is absolutely continuous with respect to the Lebesque measure, we say that X has local time on I. The local time is defined as the Radon-Nykodim derivative of μ_I

$$L(I,x) = \frac{d\mu_I}{d\lambda}(x), \qquad x \in \mathbb{R}.$$

We will use the notation

$$L(\mathbf{t},x) := L([0,\mathbf{t}],x), \qquad \mathbf{t} \in \mathbb{R}^d_+, x \in \mathbb{R}.$$

The local time satisfies the occupation time formula

$$\int_I f(X_\mathbf{t})d\mathbf{t} = \int_\mathbb{R} f(y)L(I,y)dy \tag{4.34}$$

for any Borel set I in T and for any measurable function $f: \mathbb{R}^d \to \mathbb{R}$.

Proposition 20 *Let $u(t,\mathbf{x}), t \geq 0, \mathbf{x} \in \mathbb{R}^d$ be the solution to (4.25) and assume $\beta > 2H-1$ where β is given by (4.29). Then on each set $[a,b]\times[A,B] \subset [0,\infty)\times\mathbb{R}^d$ the process $\left(u(t,x), t \geq 0, \mathbf{x} \in \mathbb{R}^d\right)$ admits a local time $(L([a,b]\times[A,B],y), y \in \mathbb{R})$ which is square integrable with respect to y*

$$\mathbb{E}\int_\mathbb{R} L([a,b]\times[A,B],y)^2 dy < \infty \ \ a.s.\ .$$

Proof: It is well known from [13] (see also Lemma 8.1 in [104]) that, for a jointly measurable zero-mean stochastic process $X = (X(\mathbf{t}), t \in [0,\mathbf{T}])$ ($\mathbf{T}$ belongs to $\mathbb{R}^d$) with bounded variance, the condition

$$\int_{[0,\mathbf{T}]}\int_{[0,\mathbf{T}]} (\mathbb{E}[X(\mathbf{t}) - X(\mathbf{s})]^2)^{-1/2} d\mathbf{s}d\mathbf{t} < \infty$$

is sufficient for the local time of X to exist on $[0,\mathbf{T}]$ almost surely and to be square integrable as a function of u.
According to the inequality (4.32), for all $I = [a,b]\times[A,B]$ interval included in $[0,\infty)\times\mathbb{R}^d$ we have,

$$\int_I\int_I (\mathbb{E}(u(t,\mathbf{x}) - u(s,\mathbf{y}))^{-1/2} dtd\mathbf{x}dsd\mathbf{y} < C\int_I\int_I \left(|t-s|^{2H+1-\beta} + |\mathbf{x}-\mathbf{y}|^{2H+1-\beta}\right)^{-\frac{1}{2}} dtd\mathbf{x}dsd\mathbf{y}$$

and this is finite for $\beta > 2H-1$. Thus almost surely the local time of u exists and is square integrable. ■

Remark 23 *It follows as a consequence of Lemma 8.1 in [104] that the local time of the solution u admits the following L^2 representation*

$$L([a,b]\times[A,B],x) = \frac{1}{2\pi}\int_\mathbb{R} dze^{-izx}\int_{[a,b]\times[A,B]} dsd\mathbf{y}e^{iu(s,\mathbf{y})z}$$

for every $x \in \mathbb{R}$.

4.3.3 Existence of the joint density for the solution in the Rosenblatt case

It is possible to obtain the existence of the joint density of the random vector $(u(t,x), u(s,y))$ with $s \neq t$ or $\mathbf{x} \neq \mathbf{y}$ in the case when the wave equation (4.25) is driven by a Hermite sheet of order $q = 2$ (the Rosenblatt sheet). The result is based on a criterium for the existence of densities for vectors of multiple integrals which has recently been proven in [75].

Let us state our result.

Proposition 21 *Let $u(t,x), t \geq 0, \mathbf{x} \in \mathbb{R}^d$ be the mild solution to (4.25). Then for every $(t,\mathbf{x}) \neq (s,\mathbf{y})$, $(t,\mathbf{x}), (s,\mathbf{y}) \in (0,\infty) \times \mathbb{R}^d$, the random vector*

$$(u(t,\mathbf{x}), u(s,\mathbf{y}))$$

admits a density.

Proof: Note that for every $t \geq 0$ and $x \in \mathbb{R}^d$, the random variable $u(t,\mathbf{x})$ is a multiple integral of order 2 with respect to the d-parametric Brownian sheet. A result present in [75] states that a two-dimensional vector of multiple integrals of order 2 admits a density if and only if the determinant of the covariance matrix is strictly positive. Denote by $C(t,s,\mathbf{x},\mathbf{y})$ the covariance matrix of $(u(t,\mathbf{x}), u(s,\mathbf{y}))$. The determinant of this matrix is the same for every $q \geq 1$, from the covariance structure of the Hermite sheet. It is clear that for $q = 1$ obviously $\det C(t,s,\mathbf{x},\mathbf{y})$ is strictly positive, since the vector $(u(t,\mathbf{x}), u(s,\mathbf{y}))$ is a Gaussian vector and hence admits a density when $(t,\mathbf{x}) \neq (s,\mathbf{y})$. This implies that $\det C(t,s,\mathbf{x},\mathbf{y})$ is also strictly positive for $q = 2$ and so the vector $(u(t,\mathbf{x}), u(s,\mathbf{y}))$ admits a density also for $q = 2$. ■

Chapter 5

Conclusiones y trabajo futuro

En este capítulo se presenta un resumen de los principales aportes de esta tesis y una descripción del trabajo futuro a desarrollar.

5.1 Conclusiones

La tesis abordó el análisis matemático de diferentes ecuaciones diferenciales estocásticas (EDE's) dirigidas por procesos autosimilares. El estudio de este tipo de ecuaciones es motivado por las diversas aplicaciones de los procesos autosimilares. Modelos basados en este tipo de procesos resultan idóneos para lograr una mejor comprensión de fenómenos presentes en diversas áreas del conocimiento como finanzas, tráfico de redes o procesamiento de imagenes. Además de ello, el estudio de estos procesos es en si una motivación. Por tales razónes y con el objetivo de obtener un conocimiento más acabado de las propiedades y comportamiento de este tipo de procesos, en la tesis se estudiaron EDE's dirigidas por estos, desde diversos puntos de vista.

A continuación se enumeran los resultados más importantes

- Se demostró un resultado de convergencia fuerte de un esquema numérico para la solución de una EDE's con restardo dirigida por un movimiento Browniano fraccionario (mBf) con parámetro de autosimilaridad $H > 1/2$, se establece también la tasa de convergencia. Este tipo de resultados ya ha sido considerado y estudiado para el caso de EDE's dirigidas por un movimiento Brownianos standard (mBs), el caso con delay ha sido menos considerado y, hasta donde sabemos, la variante analizada en esta tesis no ha sido considerada antes. El resultado ilustra la relación existente entre el parámetro de autosimilaridad y la tasa de convergencia del esquema numérico. Utilizando resultados de aproximaciones para el mBf se demuestra también la convergencia en ley de una familia de procesos discretos hacia la solución de la EDE con delay.

- Se presentó un Estimador de Mínimos Cuadrados (EMC) para la sábana fraccionaria de Ornstein-Uhlenbeck (sfOU) y se estudió su comportamiento asintótico basado en observaciones a tiempo continuo. Se demostró la consistencia fuerte y la no-normalidad aisntótica del

EMC. Las herramientas utilizadas para demostrar los resultados difieren considerablemente de lo requerido para el caso uniparamétrico (proceso de Ornstein-Uhlenbeck fraccionario), esto es un aspecto relevante del trabajo presentado y resulta ser teoricamente atractivo.

- Se analizó un sistema lineal de k EDE's de la onda dirigídas por un campo Gaussiano, fraccionario en la variable temporal y coloreado en la variable espacial, esto último se representa a través de una función de covarianza homogenea dada por un kernel de Riesz de índice $d-\beta$ donde d es la dimensión de la variable espacial y β es un parámetro relacionado con la existencia de solución del sistema a través del parámetro de autosimilaridad de la parte fraccionaria del campo Gaussiano. De este análisis se desprenden los siguientes resultados:

  - Se establecen cotas óptimas para el modulo de continuidad de la métrica canónica del proceso.
  - Se establece que la solución del sistema de ecuaciones es fuertemente localmente no-determinista.
  - Se establecen cotas superiores e inferiores para las probabilidades de arrivo de la solución en términos de la Capacidad de Riesz y la medida de Hausdorff del conjunto involucrado.

  Los métodos necesarios para probar el último punto han sido establecidos recientemente en la literatura.

- Se definió el campo aleatorio de Hermite (caH) como una integral múltiple con respecto al campo Browniano standard (cBs). Se definieron también integrales múltiples con respecto al caH y esto fue utilizado para estudiar un sistema lineal de EDE's de la onda dirigidas por un caH, de donde se obtuvieron los siguientes resultados:

  - Se demostró la existencia de solución al sistema de ecuaciones.
  - Se demostraron cotas óptimas para la regularidad de las trayectorias de la solución.
  - Se probo la existencia de una densidad para la solución en un caso particular.
  - Se estableció la existencia de tiempos locales para la solución.

  El caH es un proceso no-Gassiano, siendo así una herramienta interesante al momento de cosiderar aplicaciones o modelar fenómenos donde la hipotesis de Gaussianeidad de los datos no es siempre satisfecha. La definición presentada involucra una fórmula explícita para el proceso, esto es de gran útilidad para trabajar con integrales y ecuaciones con respecto al caH pués permite una manipulación directa de las expresiones involucradas.

  Se demostró también que el caH es un proceso auto-similar, posee incrementos estacionarios y sus trayectorias son Hölder continuas.

5.2 Trabajo futuro

El trabajo desarrollado en la tesis otorga una variada gama de posibilidades que dan continuidad a lo ya hecho. En términos generales se puede considerar extender los resultados obtenidos a otros procesos autosimilares como el movimiento Browniano sub-fraccionario, multi-fraccionario, o considerar procesos autosimilares con incrementos no estacionarios. Estudiar otras EDE's dirigidas por los mismo procesos, la ecuación del calor, Schrödinger ó Helmhöltz, incluso es posible considerar operadores diferenciales más generales.

Específicamente las siguientes variantes representan interesantes lineas de continuación para el trabajo realizado:

1. El resultado de convergencia en la EDE's con retardo considera que el coeficiente de difusión solo depende del proceso. Ferrante y Rovira probaron en [46] la existencia y unicidad de solución para una ecuación con difusión dependiendo del tiempo y del proceso. Extender los resultados numéricos a este caso más general y/o estudiar este tipo de ecuaciones utilizando otros esquemas es una opción a considerar.

2. La convergencia fuerte del EMC probada en el capítulo 2 consideró una normalización particular del estimador lo cual acarreó como consecuencia la restricción de los parámetros de autosimilaridad de la sábana al intervalo $(1/2, 5/8)$ y la no-normalidad del estimador. Es posible considerar una normalización más general y estudiar bajo que condiciones específicas el estimador será asintóticamente normal, buscar debilitar la restricción a los parámetros de autosimilaridad, y considerar otros estimadores (máximo verosimil, por ejemplo).

3. El método para obtener las cotas para las probabilidades de arrivo requiere, entre otras cosas, que la densidad y la densidad bi-variante del proceso en cuestión satisfagan ciertas propiedades. De esta forma resulta bastante interesante caracterizar una familia de EDE's dirigidas por procesos fraccionarios-coloreados cuyas densidades satisfagan las hipótesis de probabilidades de arrivo. Analizar también ecuaciones no-lineales o con ruído multiplicativo es otra variante bastante interesante.

4. La forma en que se presentan las integrales múltiples con respecto al campo aleatorio de Hermite (caH) permite interpretar dichas integrales como elementos en los caos de Wiener generados por el campo Browniano standard. Como consecuencia una familia de EDE's dirigidas por el caH, en caso de tener solución, puede ser representada como un elemento en un caos de Wiener particular. De esta forma, estudiar la existencia de densidades para elementos en diferentes caos de Wiener y que propiedades satisfacen dichas densidades, permitiría caracterizar familias de EDE's dirigidas por el caH que satisfagan las hipotesis necesarias para probar las cotas de las probabilidades de arrivo.

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