Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas
Programa de Magíster en Matemática

Biálgebras y estructuras en el complejo de cohomología singular.
(Bialgebras and structures on the singular cohomology cochain complex)

Tesis para optar al grado de magíster en matemática

DANIEL ALONSO GABRIEL LÓPEZ NEUMANN
CONCEPCIÓN-CHILE
2015

Profesor Guía: Dr. Antonio Laface
Profesor Co-Guía: Dra. María Ofelia Ronco
Departamento de Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción

Bialgebras and structures on the singular cohomology cochain complex

Daniel López Neumann

August 10, 2015

Agradecimientos

Agradezco a todas las personas que han hecho de mi experiencia universitaria una etapa maravillosa. En primer lugar quiero agradecer al personal del Departamento de Matemática. A todos los profesores que he tenido, en especial a los más exigentes, a ellos les debo toda habilidad matemática adquirida. Quiero agradecer especialmente a los profesores Michela Artebani y Antonio Laface, por haber mantenido siempre despierto mi interés por la matemática y por aceptar revisar esta tesis. Agradezco infinitamente a la Sra. Erika Torres por ser simplemente lo máximo: por todo su apoyo y preocupación por las cosas que siempre olvido, y por realmente contribuir a generar un ambiente grato para hacer matemática.

Quisiera agradecer a la profesora María Ronco por toda la atención que me ha tenido, por resolver todas mis dudas y principalmente, por tenerme mucho trabajo matemático para el futuro.

Finalmente, doy gracias a mi familia, amigos y personas que me acompañaron en la universidad. Por todo lo que he podido compartir y aprender de ellos. No hay nada más estimulante que ver tanta gente realmente apasionada por lo que hace, ya sea matemática, astronomía, física o geofísica. A ellos les debo todo mi respeto y admiración.

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0.1 Introduction

One of the most important topological invariants of a space X are its homology groups $H_p(X;K)$ and cohomology groups $H^p(X;K)$, where K is a ring. As it is well known, there is additional structure on $H^*(X;K) = \oplus_{p\geq 0} H^p(X;K)$ than in homology, and this is useful for several reasons: it makes computations easier, permits to prove deep theorems and to construct other important invariants. The simplest operation defined in cohomology is its cup product. This is an associative product defined on the cochain level

$$\smile : C^p(X) \otimes C^q(X) \to C^{p+q}(X)$$

(we omit the coefficient ring K) and it gives the structure of a differential graded algebra to $C^*(X)$, so it induces a product in cohomology. The cup product is not commutative on the cochain level, but it is commutative on $H^*(X)$. The failure of commutativity on the cochain level is measured by another operation

$$\smile_1 : C^p(X) \otimes C^q(X) \to C^{p+q-1}(X)$$

which satisfies the relation

$$\pm d(x \smile_1 y) = \pm (dx) \smile_1 y \pm x \smile_1 (dy) + x \smile y - (-1)^{pq} y \smile x.$$

for $x \in C^p, y \in C^q$. One can iterate this process to get operations $\smile_i$: $C^p(X) \otimes C^q(X) \to C^{p+q-i}(X)$ called cup-i products (see [St47]). When taking mod 2 coefficients, these products are used to construct the Steenrod powers in cohomology

$$Sq^i : H^p(X;\mathbb{Z}_2) \to H^{p+i}(X;\mathbb{Z}_2)$$

and these operations generate a whole algebra $\mathcal{A}_2$ which acts on $H^*(X;\mathbb{Z}_2)$ (see [Eps62] and [Mi58] for properties of the Steenrod algebra and applications). These operations are the most classical ones and come from products defined on the cochain complex of any space. Additional structure on the (co)chain complex exists for specific classes of spaces. The most interesting case is that of d-fold loop spaces $\Omega^d X$, which are algebras over the little d-cubes operad $\mathcal{C}_d$, so $C_*(\Omega^d X)$ is an algebra over the operad $C_*(\mathcal{C}_d)$ ([BV68], [Coh76]). In this thesis, we will be mainly concerned with a much bigger operad which contains (equivalent versions of) the operads $C_*(\mathcal{C}_d)$ as suboperads. This is the McClure-Smith operad ([MS03], [BF04]), or surjection operad, $\mathcal{S}$ which acts on $S^*(X)$, the normalized cochain complex of a space, and this action contains the aforementioned cup-i products. The operad $\mathcal{S}$ is constructed from non-degenerate surjective functions and having an operad action of $\mathcal{S}$ on $S^*(X)$ means that for every such surjection $f : \overline{m+k} \to \overline{k}$ (where $\overline{m} = \{1, \dots, m\}$) there is a mapping

$$\langle f \rangle : S^*(X)^{\otimes k} \to S^*(X)$$

lowering degrees by m and satisfying certain properties. Cup-i products are the particular case $f : \overline{i+2} \to \overline{2}, f = 1212\dots$ of this construction. Moreover, the operad $\mathcal{S}$ is filtered by suboperads $\mathcal{S}_n$, that is, $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \cdots \subset \mathcal{S}$, such that each $\mathcal{S}_n$ is quasi-isomorphic to the little n-cubes chain operad $C_*(\mathcal{C}_n)$.

The objective of this thesis is to show that some operations coming from the McClure-Smith operad $\mathcal{S}$ appear in a purely algebraic context, as operations on the primitive subspace of certain bialgebras. This idea is motivated on some versions of the Milnor-Moore theorem ([MM65]). The classical version of this theorem states that the primitive subspace of a bialgebra has a Lie structure and if this bialgebra is conilpotent and cocommutative one can recover the original algebra as an enveloping algebra of its primitive subspace. For non-cocommutative bialgebras there are similar theorems, but more structure is needed. In [Lod01] dendriform algebras are defined as associative algebras in which the product splits as a sum of two operations satisfying certain relations. There is also a notion of dendriform bialgebra and it is shown in [Ron02] that a dendriform bialgebra D has a brace algebra structure, that is, there are operations $M_{1n} : D^{\otimes n+1} \to D$ satisfying certain relations, and the primitive subspace is a sub-brace algebra. These operations appear in the operad $\mathcal{S}$, the braces M_{1n} corresponding to the surjective functions $12131 \dots 1(n+1)1$. Now, in [LR01], tridendriform algebras are defined and it is shown in [BR10] that the operad $\mathcal{S}_2$ acts on the primitive subspace. Our purpose (still incomplete) is to generalize these results, that is, we would like to construct a certain kind of algebra (depending on n) defined by a non-symmetric operad such that on the primitive subspaces of the corresponding bialgebras we have operations appearing in $\mathcal{S}_n$. Eventually we would like to recover all the McClure-Smith operad as the primitive subspace of a certain bialgebra.

To study this problem, we take the following point of view: the dendriform operad can be described in terms of planar binary trees and the dendriform structure is defined in terms of the Tamari order. Now, binary trees are in bijection with 1-Dyck paths (see 3 for definitions) and the Tamari order of 1-Dyck paths (or binary trees) can be generalized to m-Dyck paths ([BP12]). We use this order to define operations $*_0, \ldots, *_m$ on the vector space $\mathcal{D}_m$ spanned by all m-Dyck paths, which give a new kind of algebras which we call Dyck^m-algebras. This is the correct structure in the sense that $\mathcal{D}_m$ becomes the free Dyck^m-algebra on one generator. We construct a bialgebra structure on $\mathcal{D}_m$ which respects the $*_i$-operations and we study its primitive subspace. We prove a Milnor-Moore type theorem for Dyck^m-algebras and GV^m-algebras, which is the structure on the primitive subspace of a Dyck^m-algebra. However this is not going to solve our problem, it is just the first step. This is because the operads $\mathcal{S}_n$ come from a filtration, that is, $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \ldots \mathcal{S}$ and we are considering Dyck paths for m-fixed, so there is no kind of filtration. In a future work, we will consider all Dyck paths (of a certain kind) and then we will take a filtration from which we expect to recover (at least part of) the $\mathcal{S}_n$-operads.

The thesis is organized as follows. In chapter 1 we give the construction of the McClure-Smith operad, following [BF04] (but the action on $S^*(X)$ is described as in [MS03]). All the necessary preliminaires are given: the construction of the cohomology chain complex, cup-i products (although these are included in the McClure-Smith operad) and operads. In chapter 2 we define differents kinds of bialgebras and we describe the different algebraic structures appearing on their primitive subspaces. We start with classical bialgebras and Lie algebras, then we discuss dendriform bialgebras and brace algebras and finally we briefly define tridendriform algebras and Gerstenhaber-Voronov algebras. The third and last chapter contains the original part of this thesis. On the space $\mathcal{D}_m$ spanned by m-Dyck paths, we construct in 3.2 a structure which generalizes dendriform algebras (which is the case $m = 1$): there are binary operations $*_i : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_m$ for $0 \leq i \leq m$ satisfying the relations

1. $x *_i (y *_j z) = (x *_i y) *_j z$ for any $i < j$;

2. $x *_i (y *_0 z + \cdots + y *_i z) = (x *_i y + \cdots + x *_m y) *_i z$ for any $0 \leq i \leq m$,

see [LPR15]. A vector space with $m+1$ binary operations satisfying such relations is called a Dyck^m-algebra and we show in 3.3 that $\mathcal{D}_m$ is the free $\mathcal{D}_m$ algebra on one generator. In 3.4 we construct a coproduct on $\mathcal{D}_m$ which respect the $*_i$-operations and in 3.5 we study the operations on $\mathrm{Prim}(A)$ of a Dyck^m-bialgebra A. Finally, in 3.6 we show a Milnor-Moore theorem for our algebras, that is, we show that the structure we define on $\mathrm{Prim}(A)$ of a

Dyck^m-bialgebra is enough to recover the original algebra A by means of an appropiate universal enveloping algebra functor.

Chapter 1

Operations on the cohomology cochain complex

In this chapter we describe the operations arising from the singular cochain complex of a topological space. In 1.1 we review the definition of the singular cohomology cochain complex. The first non-trivial operation on this complex is the cup product, which we define in 1.2 to proceed with the definition of cup-i products. We then define operads and algebras over an operad in 1.3 in order to construct the McClure-Smith operad $\mathcal{S}$ in 1.4 using surjective functions. This operad acts on the (normalized) cochain complex of a space and gives the Steenrod $\smile_i$ products for special surjective functions. Moreover, this operad is filtered by suboperads $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \cdots \subset \mathcal{S}$, these are defined in 1.5. No proofs are given in this chapter, we refer the reader to the corresponding papers (mainly [St47], [MS03], [BF04]).

1.1 The cohomology cochain complex

We start by defining the objects to consider, the singular homology chain complex with its simplicial structure and then we dualize to get the cohomology cochain complex.

Notation 1.1.1. When considering collections of groups $\{C_p\}_{p\geq 0}$, we denote by $C_*(X)$ (or $C^*(X)$ if the groups have upper indices C^p) both the collection of groups and the graded abelian group $\bigoplus_{p\geq 0} C_p$.

Convention 1.1.2. *Throughout the text we will use the Koszul convention for graded vector spaces: when two elements x, y are permuted, the sign*

$(-1)^{|x||y|}$ *is to be introduced. This also applies to maps, so for example* $f \otimes g(x \otimes y) = (-1)^{|g||x|} f(x)g(y)$.

Let Δ^p be the standard p-dimensional simplex, that is

$$\Delta^p = \{(t_0, \dots, t_p) \in \mathbb{R}^{p+1} \mid t_i \geq 0 \text{ and } \textstyle\sum_{i=0}^{p} t_i = 1)\}.$$

For each $i = 0, \dots, p$, we define maps $\delta_i : \Delta^{p-1} \to \Delta^p$ by

$$\delta_i(t_0, \dots, t_{p-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{p-1})$$

and $\sigma_i : \Delta^{p+1} \to \Delta^p$ by

$$\sigma_i(t_0, \dots, t_{p+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_{p+1}).$$

It is straightforward to verify that these maps satisfy the following relations

1. $\delta_j \delta_i = \delta_i \delta_{j-1}$, $i < j$
2. $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$, $i \leq j$
3. $\sigma_j \delta_i = \delta_i \sigma_{j-1}$, $i < j$
4. $\sigma_j \delta_j = \sigma_j \delta_{j+1} = I$
5. $\sigma_j \delta_i = \delta_{i-1} \sigma_j$, $i > j + 1$.

Now let X be a topological space.

Definition 1.1.3. A *standard p-simplex* on X is a continuous mapping $\sigma : \Delta^p \to X$. The *p-th singular chain group* of X, denoted by $C_p(X)$, is the free abelian group generated by the standard p-simplices on X.

We define homomorphisms $d_i : C_p(X) \to C_{p-1}(X)$ for $i = 0, \dots, p$ and $p \geq 1$ on generators by

$$d_i(\sigma) = \sigma \circ \delta_i.$$

These are called *face operators*. We also define *degeneracy operators* $s_i : C_{p-1}(X) \to C_p(X)$ for $i = 0, \dots p - 1$ and $p \geq 1$ by

$$s_i(\sigma) = \sigma \circ \sigma_i.$$

Dual to the relations between the maps δ_i and σ_j previously defined we have the following relations between face and degeneracy operators (which mean we have a simplicial complex structure on $C_*(X)$):

1. $d_i d_j = d_{j-1} d_i, i < j$
2. $s_i s_j = s_{j+1} s_i, i \leq j$
3. $d_i s_j = s_{j-1} d_i, i < j$
4. $d_j s_j = d_{j+1} s_j = I$
5. $d_i s_j = s_j d_{i-1}, i > j + 1$.

Using these relations, it is easy to see that if we define $d : C_p(X) \to C_{p-1}(X)$ by

$$d = \sum_{i=0}^{p} (-1)^i d_i$$

then $d^2 = 0$ so that we get a chain complex.

Definition 1.1.4. The pair $(C_*(X), d)$ is called the *singular homology chain complex* of a space X.

Now we define the singular cohomology cochain complex. Let K be a ring and define

$$C^p(X; K) = \mathrm{Hom}(C_p(X); K)$$

the homomorphisms of abelian groups. The differential $d : C_p(X) \to C_{p-1}(X)$ dualizes to a differential (which we denote by the same letter) $d : C^{p-1}(X; K) \to C^p(X; K)$. We also denote by s^i the duals of the corresponding maps.

Definition 1.1.5. The pair $(C^*(X; K), d)$ is the *singular cohomology cochain complex with coefficients in* K of X. The *normalized cochain groups*, denoted by $S^p(X; K)$, are the quotients of the $C^p(X; K)$ by the images of the s^i. The differential d induces a differential between the normalized cochain groups and the pair $(S^*(X; K), d)$ is the *normalized singular cochain complex* of X.

Remark 1.1.6. Normalization of a simplicial complex does not change its (co)homology (cf. [EM53]).

1.2 Cup-i products

Notation 1.2.1. Let $\sigma : \Delta^p \to X$ be a singular p-simplex on a space. If $0 \leq a_0 < a_1 < \cdots < a_k \leq p$ we denote by $\sigma(a_0, \ldots, a_k)$ the k-simplex obtained by composing σ with the unique linear map $\Delta^k \to \Delta^p$ which sends vertex i to vertex a_i.

Let K be a ring (commutative with identity). The usual cup product

$$\smile: C^p(X;K) \to C^q(X;K) \to C^{p+q}(X;K)$$

is defined by

$$(x \smile y)(\sigma) = x(\sigma(0,\ldots,p))y(\sigma(p,\ldots,p+q))$$

where $\sigma : \Delta^{p+q} \to X$ and we used the product structure of K on the right.

It is easy to see that this product is associative and that it satisfies the following relation with the differential (cf. [Hat02])

$$d(x \smile y) = dx \smile y + (-1)^p x \smile dy$$

for $x \in C^p, y \in C^q$, so it induces a product in cohomology. Define

$$\smile_1: C^p(X) \otimes C^q(X) \to C^{p+q-1}(X)$$

by

$$(x \smile_1 y)(\sigma) =$$
$$\sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} x(\sigma(0,\ldots,j,j+q,\ldots,p+q-1))y(\sigma(j,\ldots,j+q))$$

where $\sigma : \Delta^{p+q-1} \to X$ (see [St47]). Then the following formula holds:

$$d(x \smile_1 y) = (-1)^{p+q-1} x \smile y + (-1)^{pq+p+q} y \smile x + dx \smile_1 y + (-1)^p x \smile_1 dy.$$

This implies that if x, y are cycles, then

$$x \smile y - (-1)^{pq} y \smile y$$

is a coboundary so the cup product is commutative on the cochain level. As we stated in the introduction, this process can be iterated. In [Bre93] there is a very nice construction of the cup-i products by using the method of acyclic models and considering the cup product as the dual of the Alexander-Whitney map $C_*(X) \to C_*(X) \otimes C_*(X)$. In this case, the cup i-product is the dual of an homotopy expresing the non-commutativity of the preceding $\smile_{i-1}$. Although this method is easy and there is almost no need to prove difficult formulas, it is non explicit so we take the approach of [MS03].

Definition 1.2.2. An *overlapping partition* $\mathcal{A}$ of $\{0, \dots, p\}$ with m pieces is a collection of m nonempty sets $\mathcal{A} = \{A_1, \dots, A_m\}$ whose union is $\{0, \dots, p\}$ such that:

1. $A_i \cap A_{i+1}$ consists of exactly one point for each i.
2. If $i < j$, each element of A_i is less than or equal to every element of A_j.

Now let x, y be cochains of degrees p, q on a space X respectively.

Definition 1.2.3. The cup-i product of two cochains, denoted by $x \smile_i y$, is the $p+q-i$ cochain defined by

$$(x \smile_i y)(\sigma) = \sum \pm x(\sigma(A_1 \sqcup A_3 \sqcup \dots))y(\sigma(A_2 \sqcup A_4 \sqcup \dots))$$

where the sum is taken over all overlapping partitions of $i+2$ pieces of $\{0, \dots, p+q-i\}$ and σ is a $p+q-i$ singular simplex on X.

Remark 1.2.4. Here $A_1 \sqcup A_3 \sqcup \dots$ means disjoint union, even if they have an intersection point. In this case the singular simplex $\sigma(A_1 \sqcup A_3 \sqcup \dots)$ is degenerate, so it is zero on normalized (co)chains. When the dimensions of the $\sigma(A_1 \sqcup A_3 \sqcup \dots), \sigma(A_2 \sqcup A_4 \sqcup \dots)$ do not match the dimensions of x, y, the corresponding term is meant to be zero. The signs in this definition are hard to describe, we refer the reader to [MS03].

The following formula appears in [St47].

Proposition 1.2.5. *The cup-i products satisfy the following formula:*

$$d(x \smile_i y) = (-1)^{p+q-i} x \smile_{i-1} y + (-1)^{pq+p+q} y \smile_{i-1} x + dx \smile_i y + (-1)^p y \smile_i dx.$$

Remark 1.2.6. Cup-i products are used to define operations in mod 2 cohomology (so in this case signs in the preceding formulas do not matter). By 1.2.5 the operation $Sq_i : H^p(X; \mathbb{Z}_2) \to H^{2p-i}(X; \mathbb{Z}_2), Sq_i(x) = x \smile_i x$ is well defined (for $i \geq 0$) on cohomology. The Steenrod squares are then defined by

$$Sq^i : H^p(X; \mathbb{Z}_2) \to H^{p+i}(X; \mathbb{Z}_2), Sq^i(x) = Sq_{p-i}(x).$$

for $p \geq i$ and are zero of $p < i$. For the properties of these operations (as the Adem relations) and applications, we refer the reader to [Eps62] and for the construction and properties of the algebra they generate, see [Mi58].

1.3 Operads

In this section we will define operads and algebras over an operad (see [MSS02]). The important example to have in mind to understand the definition, is that of functions (of any arity) on a set.

Let X be a set and $\mathcal{P}(n)$ the set of maps $X^n \to X$ for $n \geq 1$. There is a map

$$\gamma : \mathcal{P}(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

given by

$$\begin{gathered}\gamma(f; g_1, \dots, g_n)(x_1, \dots, x_{m_1+\cdots+m_n}) = \\ f(g_1(x_1, \dots, x_{m_1}), \dots, g_n(x_{m_1+\cdots+m_{n-1}+1}, \dots, x_{m_1+\cdots+m_n}))\end{gathered}$$

that is, we are replacing the functions g_i into the variables of f. These maps satisfy an obvious associativity condition: suppose we are given functions $f \in \mathcal{P}(n), g_i \in \mathcal{P}(m_i)$ for $1 \leq i \leq n$ and $h_k\mathcal{P}(j_k)$ for $1 \leq k \leq m_1 + \cdots + m_n$. Then replacing the g_i on f in order, and then the functions h_k is the same as replacing first the h_k on the g_i and then replacing these new functions on f. This is expressed as the commutativity of a certain diagram.

Proposition 1.3.1. *Define*

$$\begin{gathered}\mathcal{P}[m] = \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \\ \mathcal{P}[j] = \mathcal{P}(j_1) \times \dots \mathcal{P}(j_q) \\ \mathcal{P}[m_i] = \mathcal{P}(m_i) \times \mathcal{P}(j_{m_1+\cdots+m_{i-1}+1}) \times \cdots \times \mathcal{P}(j_{m_1+\cdots+m_i})\end{gathered}$$

for each $1 \leq i \leq n$, where $q = m_1 + \cdots + m_n$, then the following diagram commutes

$$\begin{tikzcd}
(\mathcal{P}(n) \times \mathcal{P}[m]) \times \mathcal{P}[j] \arrow[r, "="] \arrow[dd, "\gamma\times\mathcal{P}[j]"] & \mathcal{P}(n) \times \mathcal{P}[m_1] \times \cdots \times \mathcal{P}[m_n] \arrow[d, "\mathcal{P}(n)\times\gamma\dots\gamma"] \\
 & \mathcal{P}(n) \times \times_{i=1}^{n} \mathcal{P}(j_{m_1+\cdots+m_{i-1}+1} + \cdots + j_{m_1+\cdots+m_i}) \arrow[d, "\gamma"] \\
\mathcal{P}(q) \times \mathcal{P}[j] \arrow[r, "\gamma"] & \mathcal{P}(j_1 + \cdots + j_q).
\end{tikzcd}$$

If $*$ denotes a set consisting of a point, there is a map $\eta : * \to \mathcal{P}(1)$ given by $\eta(*) = 1_X$, the identity map of X. The maps γ and η satisfy the following obvious relations:

$$\begin{tikzcd} \mathcal{P}(n) \times \mathcal{P}(1)^{\times n} \arrow[r, "\gamma"] & \mathcal{P}(n) \\ \mathcal{P}(n) \times *^{\times n} \arrow[u, "\mathcal{P}(n)\times\eta^{\times n}"] \arrow[ur, "\cong"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}(1) \times \mathcal{P}(n) \arrow[r, "\gamma"] & \mathcal{P}(n) \\ * \times \mathcal{P}(n) \arrow[u, "\eta\times\mathcal{P}(n)"] \arrow[ur, "\cong"] & \end{tikzcd}$$

Observe that there is a right action of S_n on $\mathcal{P}(n)$ for each n given by

$$(f\sigma)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

Let $m_1, \ldots, m_n \geq 1$ and $\sigma \in S_n$, we define a permutation $\overline{\sigma} \in S_{m_1+\cdots+m_n}$ by

$$\overline{\sigma}(m_1 + \cdots + m_{i-1} + j) = m'_1 + \cdots + m'_{\sigma(i)-1} + j$$

for $1 \leq j \leq m_i$ and $1 \leq i \leq n$, where $m'_i = m_{\sigma^{-1}(i)}$.

Proposition 1.3.2. *The action of the symmetric groups on the sets $\mathcal{P}(n)$ satisfy*

$$\gamma(f\sigma; g_1, \ldots, g_n) = \gamma(f; g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})\overline{\sigma}$$

for any $f \in P(n), g_i \in \mathcal{P}(m_i)$ for $1 \leq i \leq n$ and $\sigma \in S_n$.

Now consider an arbitrary symmetric monoidal category $\mathcal{C}$. Roughly speaking, this is a category with an associative product (under natural isomorphism) $\otimes$, an identity element for this product and a natural isomorphism $A \otimes B \cong B \otimes A$ for any pair of objects A, B in $\mathcal{C}$. For example, the category of vector spaces over a field K is a monoidal category with the usual tensor product, the field K as identity element, and symmetry isomorphism $V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v$. The category of $\mathbb{Z}$-graded (or $\mathbb{N}$-graded) modules over a ring K with the usual tensor product

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j,$$

the identity K concentrated in degree zero, and the isomorphism

$$V \otimes W \to W \otimes V,\ v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

is also a monoidal category. In 1.4 we will be concerned with the monoidal category of differential graded modules, where the differential on a tensor product is given by

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes dw$$

and where the same identity (with zero differential) and symmetry isomorphism are considered.

Let $\mathcal{C}$ be one of the preceding categories. In the diagrams of 1.3.1 and 1.3.2, replace $\times$ by $\otimes$ and $*$ by the identity I of $\mathcal{C}$.

Definition 1.3.3. A *non-symmetric operad* $\mathcal{P}$ in $\mathcal{C}$ consists of a collection of objects $\{\mathcal{P}(n)\}_{n\geq 1}$ together with the following:

1. Maps

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

   for each n and $m_1, \ldots, m_n$ such that the first diagram of 1.3.1 is commutative.

2. A map $\eta : I \to \mathcal{P}(1)$ such that the second diagrams of 1.3.1 commute.

A *symmetric operad* is a non-symmetric operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ such that each $\mathcal{P}(n)$ has a right action of the symmetric group S_n satisfying the relations of 1.3.2. An operad (symmetric or not) is said to be *unital* if the morphism $\eta : I \to \mathcal{P}(1)$ is an isomorphism. We will reserve the term *operad* for symmetric operad.

Remark 1.3.4. Operads can also be described by means of *partial composition products*. Let $\mathcal{P}$ be an operad, for any $1 \leq i \leq n$ we define

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)$$

by the composition

$$\begin{tikzcd} \mathcal{P}(n) \otimes \mathcal{P}(m) = \mathcal{P}(n) \otimes I^{i-1} \otimes \mathcal{P}(m) \otimes I^{n-i} \arrow[d] & \\ \mathcal{P}(n) \otimes \mathcal{P}(1)^{i-1} \otimes \mathcal{P}(m) \otimes \mathcal{P}(1)^{n-i} \arrow[r, "\gamma"] & \mathcal{P}(n+m-1) \end{tikzcd}$$

where the first map is $\mathcal{P}(n)\otimes\eta^{i-1}\otimes\mathcal{P}(m)\otimes\eta^{n-i}$. The associativity condition for γ implies the following relations for the $\circ_i$-products

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+r-1} g & \text{if } j < i \\ f \circ_i (g \circ_{j-i+1} h) & \text{if } i \leq j \leq i+m-1 \\ (f \circ_{j-m+1} h) \circ_i g & \text{if } i+m-1 < j \end{cases}$$

where $f \in \mathcal{P}(n), g \in \mathcal{P}(m), h \in \mathcal{P}(r)$. Conversely, if $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ is a collection of objects in a symmetric monoidal category together with products $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ for any $n, m \geq 1$ and $1 \leq i \leq n$ satisfying the preceding relations, and a morphism $\eta : I \to \mathcal{P}(1)$ satisfying obvious unit conditions with respect to the $\circ_i$-products, then $\mathcal{P}$ is a non-symmetric operad with operations $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$ defined by

$$\gamma(f; g_1, \ldots, g_n) = (((f \circ_1 g_1) \circ_{m_1+1} g_2) \circ_{m_1+m_2+1} \cdots g_{n-1}) \circ_{m_1+\cdots+m_{n-1}+1} g_n.$$

For the corresponding symmetry conditions see [MSS02].

Example 1.3.5. 1. In the category of vector spaces, let $\mathcal{E}(n) = \mathrm{Hom}(V^{\otimes n}, V)$ and let the operations γ be defined exactly as in the case of $\mathrm{Maps}(X^n, X)$ with the same S_n-action. Then $\{\mathcal{E}(n)\}$ is an operad, which we call the *endomorphism operad* and denote it by $\mathcal{E}(V)$. We can make the same definition in the category of differential graded vector spaces (with differential of degree +1). Here, each $\mathcal{E}(n)$ is graded, where a map has degree m if it lowers degrees by m and the differential (of degree -1) in $\mathcal{E}(n)$ is given by

$$\partial f = d_V f - (-1)^{|f|} f d_{V^{\otimes n}}$$

where the d_V is the differential of V.

2. Let $\mathcal{C}_d(n)$ be the collection of n-tuples $(C_1, \ldots, C_n)$ where the C_i are disjoint d-cubes linearly imbedded in the standard cube $[0,1]^d$ (and with sides parallel to the axis). Since such an n-tuple $(C_1, \ldots, C_n)$ is determined by the center of the C_i and the length of their sides, the set $\mathcal{C}_d(n)$ is in bijection with an open subset of $(\mathbb{R}^d)^n \times (\mathbb{R}^d)^n$. We topologize $\mathcal{C}_d(n)$ under this bijection. We define a map

$$\mathcal{C}_d(n) \times \mathcal{C}_d(m_1) \times \cdots \times \mathcal{C}_d(m_n) \to \mathcal{C}_d(m_1 + \cdots + m_n)$$

by inserting the first m_1 cubes into the first cube from $\mathcal{C}_d(n)$, the following m_2 cubes into the second cube from $\mathcal{C}_d(n)$ and so on. The identity element of $\mathcal{C}_d(1)$ is the whole cube $[0,1]^d$ and S_n acts on the right of $\mathcal{C}_d(n)$ by $(C_1,\dots,C_n)\sigma = (C_{\sigma(1)},\dots,C_{\sigma(n)})$. This gives an operad in the category **Top** which we denote by $\mathcal{C}_d$ and call it the *little d-cubes operad* (see [BV68]).

Definition 1.3.6. A $\mathcal{P}$*-algebra* over an operad $\mathcal{P}$ is an object A in $\mathscr{C}$ together with morphisms

$$\alpha_n : \mathcal{P}(n) \otimes A^{\otimes n} \to A$$

satisfying obvious associativity, identity and symmetry conditions (see [MSS02]). Forgetting the symmetry, we get a notion of $\mathcal{P}$-algebra for non-symmetric operads.

If we think of an element of $\mathscr{P}(n)$ as a function with n inputs, the map α_n is then thought as evaluating this function into n elements of A. Then the associativity condition means that replacing the n variables of a function $f \in \mathscr{P}(n)$ by functions $f_1,\dots,f_n$ and then evaluating at $a_1,\dots,a_q \in A$ where $q = m_1 + \dots + m_n$ is the same thing as evaluating first the f_i in the a_j (in order) and then evaluating f in the n elements of A thus obtained.

Example 1.3.7.

1. Any vector space V is an algebra over its endomorphism operad $\mathcal{E}(V)$.
2. For any space X, the d-fold loop space $\Omega^d X$ is an algebra over the little d-cubes operad $\mathcal{C}_d$. Similarly $C_*(\Omega^d X)$ (resp. $H_*(\Omega^d X)$) is an algebra over the operad $C_*(\mathcal{C}_d)$ (resp. the operad $H_*(\mathcal{C}_d)$).

This second example is pretty interesting. It was proved in [Coh76] that algebras over the operad $H_*(\mathcal{C}_d)$ are d-Gerstenhaber algebras, that is, algebras with a Lie bracket of degree $-(d-1)$ satisfying certain compatibility relations. For $d = 2$ we just call them Gerstenhaber algebras. Now, in [Gers63], the Hochschild cohomology of an associative algebra is endowed with a Gerstenhaber algebra structure, so it is an algebra over the operad $H_*(\mathcal{C}_2)$. It was asked by Deligne in [De93] whether this action lifts to an action of the chain operad $C_*(\mathcal{C}_2)$ on the Hochschild cochain complex $CC^*(A,A)$. This conjecture has been solved in several papers, for example, in [MS03], where the suboperad $\mathcal{S}_2$ of the surjection operad (see 1.5) is shown to act on $CC^*(A,A)$.

Remark 1.3.8. Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ be a unital operad of vector spaces, that is, $\mathcal{P}(1) = K$. The structure maps of $\mathcal{P}$ give the structure of an $\mathcal{P}$-algebra to the vector space $\mathcal{P} = \bigoplus \mathcal{P}_n$. Moreover, by the unitality condition, it is easy to see that this space is the free $\mathcal{P}$-algebra on one generator. Conversely, if we have a certain kind of algebras which is codified by a non-symmetric operad (that is, the variables of the defining relations stay in the same order) and $F = \bigoplus_{n\geq 1} F_n$ is the free algebra on one generator, then F has an operad structure which codifies this kind of algebras. Moreover, in this case the free $\mathcal{P}$-algebra on a vector space V is given by

$$\bigoplus_{n\geq 1} F_n \otimes V^{\otimes n}.$$

1.4 The surjection operad

Consider the category of differential graded modules over a ring K, say, indexed by the nonnegative integers. We construct an operad $\mathcal{S}$ which acts on the normalized cochain complex $S^*(X)$ of any space X. This operad is called the *surjection operad* since it will be defined in terms of surjective maps $f : \overline{m} \to \overline{k}$, where we denote by $\overline{m}$ the set $\{1, \ldots, m\}$. The following construction is taken from [MS03].

Let $f : \overline{m} \to \overline{k}$ be a surjection and $\sigma : \Delta^p \to X$ be a standard p-simplex. We define an element $\sigma[f] \in (S_* X)^{\otimes k}$ by the formula

$$\sigma[f] = \sum_{\mathcal{A}} \pm \bigotimes_{i=1}^{k} \sigma\left(\amalg_{f(j)=i} A_j\right)$$

where the sum runs over all overlapping partitions of m pieces of the set $\{0, \ldots, p\}$ (see 1.2.2). Now for every surjection $f : \overline{m} \to \overline{k}$ we define a natural transformation $\langle f \rangle : (S^* X)^{\otimes k} \to S^* X$ by

$$\langle f \rangle (x_1 \otimes \cdots \otimes x_k)(\sigma) = (-1)^{m-k} x_1 \otimes \cdots \otimes x_k(\sigma[f]).$$

Remark 1.4.1. When $f : \overline{i+2} \to \overline{2}$ is the map 1212... then the natural transformation $\langle f \rangle$ is just the usual Steenrod $\smile_i$ product (except for a sign).

Definition 1.4.2. A map $f : \overline{m} \to \overline{k}$ is said to be *degenerate* if it is non-surjective or if $f(j) = f(j+1)$ for some j.

Let $\mathcal{S}(k)$ be the graded abelian group freely generated by the maps $f : \overline{m} \to \overline{k}$ modulo de degenerate maps where such a map f has degree $m-k$. Let $\mathcal{N}(k)$ be the graded abelian group of natural transformations $(S^*X)^{\otimes k} \to S^*X$. The collection $\mathcal{N} = \{\mathcal{N}(k)\}_{k\geq 0}$ has the structure of an operad as in 1.3.5. For each k the correspondence $f \mapsto \langle f \rangle$ defines an homomorphism $\mathcal{S}(k) \to \mathcal{N}(k)$ which is easily seen to be a monomorphism. We have the following:

Theorem 1.4.3. *([MS03]) The monomorphism $\mathcal{S} \to \mathcal{N}$ embeds each $\mathcal{S}(k)$ as a subcomplex of $\mathcal{N}(k)$ and $\mathcal{S}$ as a suboperad of $\mathcal{N}$.*

We give now a more explicit description of the operadic structure of $\mathcal{S}$ as in [BF04]. This is not exactly the same operad since signs differ, but it is equivalent to it. We will use the same notation $\mathcal{S}$, although Berger and Fresse denote it by χ.

If $f : \overline{m} \to \overline{k}$ is a surjection, we denote it by $f = (f(1), \ldots, f(m))$. The signs in the differential are determined according to the following definition.

Definition 1.4.4. The *caesuras* of the surjection $(f(1), \ldots, f(m))$ are the $f(i)'s$ which are not the last occurrence of a value, that is, $f(i) = k$ is a caesura if there exists $j > i$ such that $f(i) = k$.

The differential $d : \mathcal{S}(k)_e \to \mathcal{S}(k)_{e-1}$ is defined by

$$d(f(1), \ldots, f(k+e)) = \sum \pm (f(1), \ldots, \widehat{f(i)}, \ldots, f(k+e))$$

where $\widehat{f(i)}$ means omission of that value. We will give a sign to each value $f(j)$ which appear more than once and the sign of the term

$$(f(1), \ldots, \widehat{f(i)}, \ldots, f(k+e))$$

is defined as the product of the corresponding signs. Let $f(i_1), \ldots, f(i_r)$ be the caesuras of the surjection, we give them alternate signs, starting with $+$ on the first caesura. Suppose a term $f(j)$ is the last occurrence of a value which appear more than once. The preceding occurrence of that value (a caesura) has a sign, we give $f(j)$ the opposite sign. Observe that deleting a value which appears only once gives a degenerate function, so it is zero.

Example 1.4.5. Let $f = 1243242$, the caesuras are $f(2) = 2, f(3) = 4, f(5) = 2$ so they have the signs $+, -, +$ respectively. The values $f(6) = 4, f(7) = 2$ then have signs $+, -$ so the differential is

$$d(1243242) = +143242 - 123242 + 124342 - 124324.$$

We describe the partial composition product $\circ_i : \mathcal{S}(k) \otimes \mathcal{S}(l) \to \mathcal{S}(k + l - 1)$. Let $f \in \mathcal{S}(k)_d$ and $g \in \mathcal{S}(l)_e$ and suppose there are n ocurrences of i in $(f(1), \dots, f(k+d))$. Divide $(g(1), \dots, g(l+e))$ in n blocks so that each block overlap the next one in its last element:

$$(g(0), \dots, g(j_1)), (g(j_1), \dots, g(j_2)), \dots, (g(j_{n-1}), \dots, g(j_n)).$$

We delete the ocurrences of i from f and insert the n blocks of g in order. In order to obtain a surjection, we add $i-1$ to the $g(t)$'s, add $l-1$ to the $f(t) > i$ and the $f(t) < i$ are left untouched. Then, the composition $f \circ_i g$ is defined as the sum (with signs) of all the possible ways of dividing g and inserting it on the ocurrences of i.

Example 1.4.6. Let $f = 1212$ and $g = 121$, then

$$f \circ_2 g = 123212 \pm 123132 \pm 121232.$$

The signs in the $\circ_i$-products are obtained in the following way. Let $f = (f(1), \dots, f(k+d))$ be a surjection and suppose the caesuras are $f(i_1), \dots, f(i_m)$. The *table arrangement* of f is the array

$$f = \left\{ \begin{array}{r} f(1), \dots, f(i_1) \\ f(i_1+1), \dots, f(i_2) \\ \vdots \\ f(i_m+1), \dots, f(k+d) \end{array} \right.$$

where the last terms are the caesuras of f (except on the last line). Let $(f(t_1), \dots, f(t_2))$ be a subsequence of f. We say it has *degree* p if it intersect $p+1$ lines in the table arrangement of f. When computing the product $f \circ_i g$ one has to decompose f in blocks as

$$(f(1), \dots, f(t_1)), (f(t_1), \dots, f(t_2)), \dots, (f(t_n), \dots, f(k+d))$$

(where the $t_1, \dots, t_n$ are the ocurrences of i) and also decompose g in blocks as

$$(g(0), \dots, g(j_1)), (g(j_1), \dots, g(j_2)), \dots, (g(j_{n-1}), \dots, g(j_n)).$$

Each of these blocks has a degree, and the terms of $f \circ_i g$ are obtained by permuting the blocks of g along the blocks f from right to left. We give the Koszul sign 1.1.2 to each of these terms.

Example 1.4.7. Let's see how to get the sign of the term 123212 of the preceding example. The blocks of f are (12)(212)(2) of degrees $1, 1, 0$ respectively, and the blocks of g for this term are (121)(1), of degrees 1 and 0 so the second block of g does not contribute any sign. The first block of g is permuted with (2) and (212), so the sign is -123212.

1.5 Filtration of $\mathcal{S}$

The operad $\mathcal{S}$ is filtered, that is, there are suboperads $\mathcal{S}_2 \subseteq \mathcal{S}_3 \subseteq \dots$ whose union is all of S. The operad $\mathcal{S}_n$ is homotopy equivalent to the little n-cubes chain operad. This is used in [MS03] to prove Deligne's conjecture.

Definition 1.5.1. Let $f : \overline{m} \to \overline{k}$ be a surjection. The *complexity* of f is the maximum number of jumps on $f|_{f^{-1}(A)}$, where A ranges over two element subsets of $\overline{k}$.

This definition is easier to understand with an example.

Example 1.5.2. Let $f = 12213213$, then 122121 has 4 jumps, 11313 has 3 jumps and 22323 has 3 jumps, so f has complexity 4.

We denote by $\mathcal{S}_n$ the subspace generated by all non-degenerate functions of complexity $\leq n$.

Theorem 1.5.3. *([MS03]) The subspace $\mathcal{S}_n$ is a suboperad of $\mathcal{S}$ which is quasi-isomorphic to the little n-cubes chain operad $C_*(\mathcal{C}_n)$.*

Chapter 2

Operations on primitive elements

In this chapter we turn to a more algebraic framework. We will study bialgebras (of a certain kind) and the additional structure on their primitive subspace. For each of these algebras there is a Milnor-Moore type theorem, that is, the original bialgebra can be recovered from its primitive subspace by taking an appropiate universal enveloping algebra. We only state this theorem for classical bialgebras and dendriform bialgebras. We will see that some operations appearing in the surjection operad $\mathcal{S}$ arise. Section 2.1 starts with classical bialgebras and the classical Milnor-Moore theorem ([MM65]). We then study dendriform bialgebras and brace algebras in 2.2 and 2.3. We describe in detail the free dendriform algebra on one generator and its bialgebra structure, since this will be generalized in chapter 3. In 2.4 we define the eulerian idempotent of a dendriform bialgebra and state its main properties. This is the main tool to prove a Milnor-Moore type theorem for a subcategory of dendriform bialgebras ([Ron02]). We finally define (briefly) tridendriform algebras and Gerstenhaber-Voronov algebras ([BR10]).

2.1 Bialgebras

We will start with classical bialgebras and the classical Milnor-Moore theorem (see [MM65]).

Definition 2.1.1. Let A be a vector space over a field K. A *coalgebra structure* on A consists of linear maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to K$, called *coproduct* and *counit* respectively, such that the following diagrams are commutative (the first is *coassociativity* of Δ):

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "A\otimes\Delta"] \\
A \otimes A \arrow[r, "\Delta\otimes A"] & A \otimes A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes K \arrow[dr, "\cong"'] & A \otimes A \arrow[l, "A\otimes\epsilon"'] \arrow[r, "\epsilon\otimes A"] & K \otimes A \arrow[dl, "\cong"] \\
 & A \arrow[u, "\Delta"] &
\end{tikzcd} .$$

The coalgebra is *cocommutative* if the diagonal satisfies $\tau\Delta = \Delta$ where $\tau : A \otimes A \to A \otimes A$ is the switching morphism $\tau(a \otimes b) = b \otimes a$.

We denote by $\overline{\Delta}$ the reduced diagonal

$$\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x.$$

Observe that coassociativity of Δ is equivalent to coassociativity of $\overline{\Delta}$. We define $\Delta^n : A \to A^{\otimes n}$ by $\Delta^1 = Id$ and $\Delta^n = Id \otimes \Delta^{n-1} \circ \Delta$ and similarly for $\overline{\Delta}$.

Definition 2.1.2. Let A be a coalgebra with coproduct Δ. We say that A is *conilpotent* if for each $x \in A$ there is an $n \geq 1$ such that $\overline{\Delta}^n(x) = 0$.

Definition 2.1.3. A *primitive element* on a coalgebra A is an element satisfying $\overline{\Delta}(x) = 0$. The subspace of primitive elements of A is denoted by $\mathrm{Prim}(A)$.

Definition 2.1.4. Let A be an associative algebra with unit with a coalgebra structure (Δ, ϵ). If both Δ and ϵ are algebra morphisms, we say that A is a *bialgebra*.

Example 2.1.5. Let V be a vector space, consider the tensor module $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, where $V^{\otimes 0} = K$. We define a coalgebra structure by

$$\Delta[v_1, \ldots, v_n] = \sum_{i=0}^{n} [v_1, \ldots, v_i] \otimes [v_{i+1}, \ldots, v_n]$$

where the denote $[v_1, \ldots, v_n] = v_1 \otimes \ldots \otimes v_n$, and we let $\epsilon : T(V) \to K$ be the identity on $V^{\otimes 0} = K$. Then $(T(V), \Delta, \epsilon)$ is a coalgebra which we call the *tensor coalgebra on V*. The concatenation product

$$[v_1, \ldots, v_i] \cdot [v_{i+1}, \ldots, v_n] = [v_1, \ldots, v_n]$$

gives an associative algebra structure to $T(V)$ (with this product we call $T(V)$ the *tensor algebra* over V) but it is not a bialgebra in this way (it is a unital infinitesimal bialgebra as in [LR06]) together with the preceding coproduct. We denote by $\overline{T}(V)$ the reduced tensor coalgebra $\oplus_{n\geq 1} V^{\otimes n}$.

Remark 2.1.6. When considering bialgebra structures on graded vector spaces, we require that all morphisms respect the grading. If $A = \bigoplus_{n\geq 0} A_n$ is graded and $A_0 = K$, then commutativity of the second diagram of 2.1.1 means that

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x_i \otimes x_i'$$

where $0 < |x_i|, |x_i'| < |x|$ for every i. Observe that graded coalgebras are always conilpotent.

The following proposition is obvious from the definitions.

Proposition 2.1.7. *The bracket*

$$[x, y] = xy - yx$$

defines a Lie algebra structure on an algebra A. If A is a bialgebra, then its primitive subspace is a Lie subalgebra.

We denote by A^{Lie} the space A with the preceding bracket. We now construct a functor from Lie algebras to algebras so as to recover the original algebra structure from the Lie structure of its primitive subspace.

Definition 2.1.8. Let $\mathfrak{g}$ be a Lie algebra. The *universal enveloping algebra* of $\mathfrak{g}$ is the quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ by the ideal generated by the elements of the form

$$[x, y] - x \otimes y - y \otimes x$$

for $x, y \in \mathfrak{g}$.

Let $\iota : \mathfrak{g} \to U(\mathfrak{g})$ be the obvious inclusion. The universal enveloping algebra is constructed so as to have the following (obvious) universal property.

Proposition 2.1.9. *Let $\mathfrak{g}$ be a Lie algebra and let A be an associative algebra with unit. For any Lie algebra morphism $f : \mathfrak{g} \to A^{Lie}$ there is a unique morphism of algebras $\overline{f} : U(\mathfrak{g}) \to A$ such that $\overline{f} \circ \iota = f$.*

The universal enveloping algebra has a natural bialgebra structure. Indeed, the map $\mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ given by $x \to \iota(x) \otimes 1 + 1 \otimes \iota(x)$ extends to a unique algebra map $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ which obviously satisfies the coassociativity and counit requirements of 2.1.4. Moreover, $U(\mathfrak{g})$ is always conilpotent. Observe that if A is a bialgebra and $f : \mathfrak{g} \to A$ is a Lie morphism such that $f(\mathfrak{g}) \subseteq \mathrm{Prim}(A)$, then the extension $\overline{f} : U(\mathfrak{g}) \to A$ is a bialgebra morphism.

Theorem 2.1.10. *([MM65]) Let K be a characteristic zero field. For any Lie algebra $\mathfrak{g}$ over K, the map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ gives an isomorphism $\mathfrak{g} \cong Prim(U(\mathfrak{g}))$. Let A be a conilpotent cocommutative bialgebra over K. The inclusion map $Prim(A) \to A$ extends to an isomorphism of bialgebras $U(Prim(A)) \cong A$.*

2.2 Dendriform bialgebras

The following definition was introduced by Loday in [Lod01].

Definition 2.2.1. A *dendriform algebra* is a vector space A together with two binary products $\succ, \prec : A \otimes A \to A$ satisfying

1. $(x \prec y) \prec z = x \prec (y * z)$
2. $(x \succ y) \prec z = x \succ (y \prec z)$
3. $(x * y) \succ z = x \succ (y \succ z)$

where $* = \succ + \prec$.

Observe that these relations imply that the product $* = \succ + \prec$ is associative.

Example 2.2.2. The relations of a dendriform algebra already appear in [EM53] (where the operation $\prec$ is called the *half product* and it is denoted by $\downarrow$). Let (X, μ) be an associative H-space and γ a (p, q)-shuffle of $\{0, \ldots, p+q-1\}$, that is, $\gamma(0) < \cdots < \gamma(p-1)$ and $\gamma(p) < \cdots < \gamma(p+q-1)$. Define a map $\sigma_\gamma : \Delta^{p+q} \to \Delta^p \times \Delta^q$ by

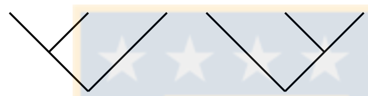
$$\sigma_\gamma = \sigma_{\gamma(p)} \circ \cdots \circ \sigma_{\gamma(p+q-1)} \times \sigma_{\gamma(0)} \circ \cdots \circ \sigma_{\gamma(p-1)}$$

where the σ_i's are the maps defined in 1.1. Then there is a dendriform structure on $C_{\geq 1}(X)$ defined by

$$x \succ y = \sum_{\gamma(p)=0} (-1)^{sgn(\gamma)} \mu \circ x \times y \circ \sigma_\gamma$$

where $x \in C_p, y \in C_q$, $sgn(\gamma)$ is the sign of a permutation and the sum is over all (p,q)-shuffles such that $\gamma(p) = 0$. The operation $\prec$ is defined similarly, summing over shuffles γ such that $\gamma(0) = 0$.

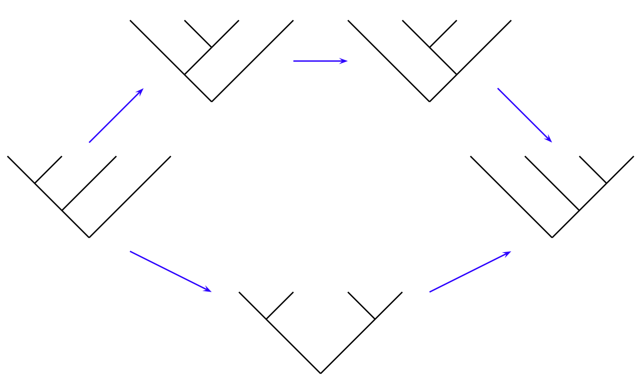
For our purposes, the most important example of a dendriform algebra is the free algebra on one generator which we now describe (see [Lod01]). For $n \geq 1$, let Y_n be the set of planar binary trees with n internal vertices. We also define $Y_0 = \{|\}$, the set consisting of the leaf with no vertex. For example, the elements of Y_2 are:



Let $\vee : Y_n \times Y_m \to Y_{n+m+1}$ be the grafting operation, joining the roots of two trees to a new root. Every binary tree can be uniquely written as $t = t_1 \vee t_2$. A *subtree* of a tree t is any binary tree obtained from a given vertex of t and considering all edges up from that vertex. There is a partial order in Y_n which turns this set into a lattice.

Definition 2.2.3. Let s be a tree and s' be a subtree of the form $s' = (s_1 \vee s_2) \vee s_3$. Let t be the tree obtained from s by substituting s' by $s_1 \vee (s_2 \vee s_3)$. The *Tamari order* on binary trees is defined by the covering relation $s < t$.

Example 2.2.4. The following is the Tamari order for $n = 3$ (increasing from left to right):



We denote by $K[Y_\infty]$ the vector space spanned by all binary trees. This space is graded by $K[Y_\infty] = \bigoplus_{n\geq 1} K[Y_n]$. We define two operations $\succ, \prec$ on $K[Y_\infty]$ inductively by

$$s \succ t = s * t_1 \vee t_2 \text{ and } s \prec t = s_1 \vee s_2 * t$$

where $s = s_1 \vee s_2, t = t_1 \vee t_2$ and where $| * t = t * | = t$, where $|$ is the leaf with no vertex. Let s, t be binary trees, we denote by s/t the binary tree obtained by gluing the root vertex of s to the end of the first leaf of t, and we let $s\backslash t$ be the binary tree obtained by gluing the root vertex of t to the end of the last leaf of s.

Theorem 2.2.5. *The space $K[Y_\infty]$ is the free dendriform algebra on one generator. The operations $\succ, \prec$ can be written as*

$$s \succ t = \sum u$$

where the sum is taken over all u such that $s/t \leq u \leq (s\backslash t_1) \vee t_2$ and

$$s \prec t = \sum u$$

where the sum is over all u such that $s_1/(s_2\backslash t) \leq z \leq s\backslash t$.

See [Lod01] for the first assertion and [LR02] for the second. This theorem will be generalized in chapter 3.

Remark 2.2.6. By 1.3.8, the free dendriform algebra on a vector space V is given by

$$\bigoplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}.$$

Let A be a dendriform algebra and let $A^+ = K \oplus A$. Let $x \in A$, we make the following definitions:

1. $x \succ 1 = 0$,
2. $1 \succ x = x$,
3. $x \prec 1 = x$,
4. $1 \prec x = 0$.

Observe that $1 \succ 1$ and $1 \prec 1$ are not defined. When we refer to a dendriform structure on A^+ we will always refer to the structure defined on $A = A^+/K$. For a linear map $\Delta : A \to A \otimes A$ we denote by Δ^+ its extension to A^+ defined by $\Delta^+(x) = x \otimes 1 + 1 \otimes x + \Delta(x)$ for $x \in A$ and $\Delta^+(1) = 1 \otimes 1$. We define a dendriform algebra structure on $A^+ \otimes A^+$ by

1. $x_1 \otimes x_2 \succ y_1 \otimes y_2 = x_1 * y_1 \otimes x_2 \succ y_2$,

2. $x_1 \otimes x_2 \prec y_1 \otimes y_2 = x_1 * y_1 \otimes x_2 \prec y_2$

whenever $x_2 \neq 1$ or $y_2 \neq 1$, otherwise we define

1. $x \otimes 1 \succ y \otimes 1 = x \succ y \otimes 1$,

2. $x \otimes 1 \prec y \otimes 1 = x \prec y \otimes 1$.

Definition 2.2.7. A *dendriform bialgebra* is a dendriform algebra A together with a coassociative coproduct $\Delta : A \to A \otimes A$ such that its extension $\Delta^+ : A^+ \to A^+ \otimes A^+$ is a dendriform algebra morphism.

Remark 2.2.8. Usually we consider non unital dendriform bialgebras, so primitive elements are those verifying $\Delta(x) = 0$.

The free dendriform algebra on one generator has a dendriform bialgebra structure (see [Ron00]). We need the following definition. Observe that the edges of a tree t are partially ordered: if e_2 is an edge inmeadiately above an edge e_1, then we define $e_1 < e_2$ and extend by transitivity.

Definition 2.2.9. An *admissible set of edges* on a tree t is a set $\Gamma = \{e_1, \dots, e_k\}$ of different edges of t such that no e_i is $<$ an e_j.

Let $\Gamma = \{e_1, \dots, e_k\}$ be an admissible set of edges of a tree t. We denote by $t_{(i)}$ the subtree of t starting from the endpoint of e_i, $i = 1, \dots, k$, and we let $t_{(k+1)}$ be the complement of the $t_{(i)}$ in t (so it has the root vertex). We define a coproduct $\Delta : K[Y_\infty] \to K[Y_\infty] \otimes K[Y_\infty]$ by

$$\Delta(t) = \sum_{\Gamma} t_{(1)} * \cdots * t_{(k)} \otimes t_{(k+1)}$$

where the sum is over all admissible set of edges Γ of t.

Theorem 2.2.10. *The pair $(K[Y_\infty], \Delta)$ is a dendriform bialgebra.*

2.3 Brace algebras

There is a version of the Milnor-Moore theorem for dendriform bialgebras. However, there is much more structure on the primitive subspace than just a Lie structure. We give now the corresponding definitions (see [Ron02]).

Definition 2.3.1. A *brace algebra* ([Gers63], [Kad88]) is a vector space B together with operations $B^{\otimes(n+1)} \to B, x \to x\{x_1, \dots, x_n\}$ (we also use the notation $M_{1n}(x; y_1, \dots, y_n)$ for these braces) satisfying

$$x\{y_1, \dots, y_n\}\{z_1, \dots, z_m\} =$$
$$\sum x\{z_1, \dots, z_{j_1}, y_1\{z_{j_1+1}, \dots, z_{k_2}\}z_{k_2+1}, \dots, z_{j_2}, \dots, z_{j_n}, y_n\{z_{j_n+1}, \dots, z_{k_n}\}, z_{k_n+1}, \dots, z_m\}$$

where the sum is taken over all possible ways of bracketing the $z_1, \dots, z_m$ in order with the $y_1, \dots, y_n$. We also admit empty braces, this is just $y_i\{\} = y_i$.

Example 2.3.2. When $n = m = 1$ we have

$$x\{y\}\{z\} = x\{y, z\} + x\{y\{z\}\} + x\{z, y\}.$$

Remark 2.3.3. Any brace algebra is a Lie algebra with

$$[x, y] = x\{y\} - y\{x\}.$$

Let $(A, \succ, \prec)$ be a dendriform algebra. We follow the notations of [Ron02] and write

1. $w_{\succ}(x_1, \dots, x_n) = (((x_1 \succ x_2) \succ x_3) \succ \dots) \succ x_n$,
2. $w_{\prec}(x_1, \dots, x_n) = x_1 \prec (\dots \prec (x_{n-2} \prec (x_{n-1} \prec x_n)))$.

We define brace operations on a dendriform algebra by

$$x\{x_1, \dots, x_n\} = \sum_{i=0}^{n} (-1)^{n-i} w_{\prec}(x_1, \dots, x_i) \succ x \prec w_{\succ}(x_{i+1}, \dots, x_n).$$

Theorem 2.3.4. *([Ron02]) The preceding operations define a brace algebra structure on a dendriform algebra A. Moreover, if A is a dendriform bialgebra, then its primitive subspace is a brace subalgebra.*

Definition 2.3.5. Let (B, M_{1n}) be a brace algebra. Let $Dend(B)$ be the free dendriform algebra over the vector space B (see 2.2.6) and let M'_{1n} be the induced brace operations on $Dend(B)$. The *universal enveloping dendriform algebra* of B is the quotient of $Dend(B)$ by the dendriform ideal generated by the elements of the form $M_{1n}(x; x_1, \dots, x_n) - M'_{1n}(x; x_1, \dots, x_n)$, where $x, x_1, \dots, x_n \in B$, and we denote it by $\mathcal{U}_{dend}(B)$.

We denote by ι the canonical map $\iota : B \to \mathcal{U}_{dend}(B)$.

Remark 2.3.6. As in 2.1.9, the universal enveloping dendriform algebra has an obvious universal property. One can check that the map

$$B \to \mathcal{U}_{dend}(B)^+ \otimes \mathcal{U}_{dend}(B)^+,\ x \mapsto \iota(x) \otimes 1 + 1 \otimes \iota(x)$$

is a brace morphism so it extends to a dendriform morphism defined on all of $\mathcal{U}_{dend}(B)$. This gives a dendriform bialgebra structure to $\mathcal{U}_{dend}(B)$.

For any dendriform bialgebra A, there is a canonical map $\mathcal{U}_{dend}(P(A)) \to A$, induced by the inclusion $\mathrm{Prim}(A) \to A$.

Theorem 2.3.7. *([Ron02]) For any brace algebra B, the canonical map $B \to \mathcal{U}_{dend}(B)$ gives an isomorphism $B \cong Prim(\mathcal{U}_{dend}(B))$. For any conilpotent dendriform bialgebra, the canonical map $\mathcal{U}_{dend}(P(A)) \to A$ is an isomorphism. In other words, the functors Prim and $\mathcal{U}_{dend}$ give an equivalence of categories between the category of conilpotent dendriform bialgebras and that of brace algebras.*

2.4 Eulerian idempotents

In this section we sketch the main ideas of [Ron02] behind the proof of 2.3.7.

Let $(A, \succ, \prec, \Delta)$ be a *conilpotent* dendriform bialgebra. Define $\succ^n : A^{\otimes n} \to A$ inductively by $\succ^1 = Id$ and $\succ^n = \succ \circ Id \otimes \succ^{n-1}$. Define a map $e : A \to A$ by

$$e = \sum_{n \geq 1} (-1)^{n+1} \succ^n \circ \Delta^n.$$

This is well-defined by conilpotency. This map is called the *eulerian idempotent* of A. Clearly, e satisfies a recursion formula

$$e(x) = x - x_{(1)} \succ e(x_{(2)})$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Proposition 2.4.1. *The eulerian idempotent e has the following properties:*

1. *$e(x)$ is primitive for all $x \in A$,*
2. *$e(x \succ y) = 0$ for any $x, y \in A$,*

3. *For any $x \in A$ the following formula holds*

$$x = e(x) + \sum e(x_{(1)}) \succ e(x_{(2)}) + \cdots + \sum w_{\succ}(e(x_{(1)}), \ldots, e(x_{(n)})) + \ldots$$

where $\Delta^n(x) = \sum x_{(1)} \otimes \ldots \otimes x_{(n)}$.

4. *$e(x \prec w_{\succ}(y_1, \ldots, y_n)) = (-1)^n M_{1n}(x; y_1, \ldots, y_n)$ for $x, y_1, \ldots, y_n \in Prim(A)$.*

Corollary 2.4.2. *On a conilpotent dendriform bialgebra A, any element can be written as a sum of elements of the form $w_{\succ}(y_1, \ldots, y_n)$ with $y_1, \ldots, y_n \in Prim(A)$.*

Proof. This is by properties 1 and 3 of 2.4.1. □

One can show easily by induction that

$$\Delta(w_{\succ}(y_1, \ldots, y_n)) = \sum_{i=1}^{n-1} w_{\succ}(y_1, \ldots, y_i) \otimes w_{\succ}(y_{i+1}, \ldots, y_n).$$

whenever $y_1, \ldots, y_n \in \mathrm{Prim}(A)$. Now, one extends the eulerian idempotent $e : A \to \mathrm{Prim}(A)$ to a map $\epsilon : A \to \overline{T}(\mathrm{Prim}(A))$ by

$$\epsilon(x) = \textstyle\sum_{n \geq 1} e(x_{(1)}) \otimes \ldots \otimes e(x_n)$$

where $\Delta(x) = \sum x_{(1)} \otimes \ldots \otimes x_{(n)}$. It is easy to see that $\epsilon(w_{\succ}(y_1, \ldots, y_n)) = y_1 \otimes \ldots \otimes y_n$ for primitive $y_1, \ldots, y_n$ by using the preceding formula for $\Delta(w_{\succ}(y_1, \ldots, y_n))$ and property 2 of 2.4.1. This implies the following:

Corollary 2.4.3. *The map $\epsilon : A \to \overline{T}(Prim(A))$ is a coalgebra isomorphism with inverse $y_1 \otimes \ldots \otimes y_n \mapsto w_{\succ}(y_1, \ldots, y_n)$.*

With all these results, it is rather easy to prove a Milnor-Moore theorem for conilpotent dendriform bialgebras. A similar proof works also for tridendriform algebras and the Dyck^m-algebras we define in chapter 3. We will give a complete proof for Dyck^m-algebras in 3.6.

2.5 Tridendriform algebras

We introduce q-tridendriform algebras by two reasons. First, to illustrate our initial objective of relating operations on the primitive subspace of bialgebras to the McClure-Smith operad. In this case, it is the operad $\mathcal{S}_2$ (without differential) which acts on $\mathrm{Prim}(A)$ of a 0-tridendriform bialgebra A. For another hand, some of the formulas between the operations on the primitive subspace of a Dyck^m-algebra are similar of those holding in a 1-Gerstenhaber-Voronov algebra. Indeed, Dyck^2-algebras are in a certain sense a non-associative version of 1-tridendriform algebras.

The material of this section is taken from [BR10]. Let K be a field, $q \in K$ and A a vector space over K.

Definition 2.5.1. A q-tridendriform algebra is a vector space A together with three binary operations $\succ, \cdot, \prec$: $A \otimes A \to A$ satisfying the following relations:

1. $(x * y) \succ z = x \succ (y \succ z)$
2. $(x \succ y) \prec z = x \succ (y \prec z)$
3. $(x \prec y) \prec z = x \prec (y * z)$
4. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
5. $(x \succ y) \cdot z = x \succ (y \cdot z)$
6. $(x \prec y) \cdot z = x \cdot (y \succ z)$
7. $(x \cdot y) \prec z = x \cdot (y \prec z)$

where $* = \succ + q \cdot + \prec$ and $x, y, z \in A$.

Remark 2.5.2. Let $\overline{\succ} = q \cdot + \succ$, then $\overline{\succ}, \prec$ defines a dendriform algebra structure on the tridendriform algebra A.

As in the case of dendriform algebras, the free q-tridendriform algebra on one generator can be described in terms of planar trees (not necessarily binary). Let T_n be the set of planar rooted trees with $n+1$ leaves and let $K[T_\infty] = \oplus_{n\geq 1} K[T_n]$. Observe that any tree t can be written in a unique way as a grafting of trees $t_1, \ldots, t_r$. We denote it by $t = (t_1, \ldots, t_r)$.

Theorem 2.5.3. *On the space $K[T_\infty]$ the operations defined inductively by*

$$t \succ u = (t * u_1, \ldots, u_s)$$
$$t \cdot u = (t_1, \ldots, t_r * u_1, \ldots, u_s)$$
$$t \prec u = (t_1, \ldots, t_r * u)$$

where $t = (t_1, \ldots, t_r), u = (u_1, \ldots, u_s)$ *and* $* = \succ + q \cdot + \prec$ *define a* q*-tridendriform algebra structure (as in the dendriform case we define* $t * | = | * t = t$*, where* $|$ *is the leaf with no vertex). Moreover,* $K[T_\infty]$ *is the free* q*-tridendriform algebra on one generator.*

Definition 2.5.4. A q-Gerstenhaber Voronov algebra is a brace algebra (A, M_{1n}) together with an associative product $\cdot$ satisfying

$$M_{1n}(x \cdot y; z_1, \ldots, z_n) = \sum_{0 \le i \le j \le n} q^{j-i} M_{1i}(x; z_1, \ldots, z_i) z_{i+1} \ldots z_j M_{1(n-j)}(y; z_{j+1}, \ldots, z_n)$$

for any $x, y, z_1, \ldots, z_n \in A$.

Remark 2.5.5. Define operations in the McClure-Smith operad by $M_{1n} = 12131 \ldots 1(n+1)1$ and $\cdot = 12$. By using the description of the McClure-Smith operad of [BF04] given in 1.4 one can see that these operations satisfy the brace relations of 2.3.1 and the relations of a 0-Gerstenhaber-Voronov algebra:

$$M_{1n}(x \cdot y; z_1, \ldots, z_n) = \sum_{i=0}^{n} M_{1i}(x; z_1, \ldots, z_i) \cdot M_{1(n-i)}(y; z_{i+1}, \ldots, z_n).$$

No signs appear on this formula, this is because the term on the right comes from expanding $121 \ldots 1(n+1)1 \circ_1 12$ and each subsequence of 12 has degree zero (there are no caesuras) so all Koszul signs are $+1$. In this way, 0-Gerstenhaber-Voronov algebras are codified by the operad $\mathcal{S}_2$ of 1.5 (expect for some signs), where the operad $\mathcal{S}_2$ is considered without differential.

Theorem 2.5.6. *Let* $(A, \succ, \cdot, \prec)$ *be a* q*-tridendriform algebra and let* $\overline{M}_{1n}$ *be the braces coming from the dendriform structure* $\overline{\succ} = q \cdot + \succ, \prec$ *on* A*. Then* $(A, \overline{M}_{1n}, \cdot)$ *is a* q*-Gerstenhaber-Voronov algebra.*

There is a Milnor-Moore theorem for q-tridendriform bialgebras and q-Gerstenhaber-Voronov algebras which we do not state here (see [BR10]).

Chapter 3

Structure on m-Dyck paths

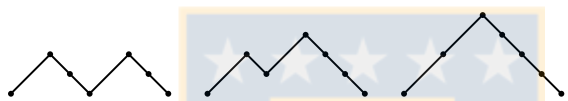
This chapter contains the original part of this thesis. We define m-Dyck paths in 3.1 and we use the m-Tamari order on these paths defined in [BP12] to construct in 3.2 $m+1$ binary operations $*_0, \ldots, *_m$ on the vector space $\mathcal{D}_m$ generated by m-Dyck paths. These operations satisfy certain relations which generalize those of dendriform algebras. We define a Dyck^m-algebra as a space with $m+1$ binary operations satisfying the same relations for $\mathcal{D}_m$. This is the correct structure on m-Dyck paths: we show in 3.3 that the space $\mathcal{D}_m$ becomes the free Dyck^m-algebra on one generator. We also show that $\mathcal{D}_m$ is Dyck^k-free for any $0 \leq k \leq m$. In 3.4 we construct a coproduct on $\mathcal{D}_m$ which respect the $*_i$-operations. We do not prove the relations for the $*_i$-operations and neither the formulas for the coproduct since these proofs are very technical (see [LPR15] for the proofs). The difficulty lies in the fact that there is not an (easy) inductive definition of the products $*_0, \ldots, *_m$ for $m > 1$ (such formulas exists for $m = 1$, see 2.2). We finally study the operations arising on the primitive subspace of a Dyck^m-bialgebra A. We show that there are brace operations M_{1n} coming from a dendriform structure on A, and together with the products $*_1, \ldots, *_{m-1}$, they generate the subspace of primitive elements. The most difficult part is to find the relations for $M_{1n}(x *_i y; -)$, this is done in 3.5. The chapter ends with the statement and proof of a Milnor-Moore type theorem for Dyck^m-bialgebras. We have not yet identified operations on the primitive subspace of a Dyck^m-bialgebra as operations of the McClure-Smith operad. This is to be done in a future work.

3.1 m-Dyck paths

Definition 3.1.1. For $m, n \geq 1$, an *m-Dyck path of size n* is a path on the real plane $\mathbb{R}^2$, starting at $(0,0)$ and ending at $(2nm, 0)$, consisting on *up steps* (m, m) and *down steps* $(1, -1)$, which never goes below the x-axis. Note that the initial and terminal points of each step lean on $\mathbb{Z}_+^2$.

We denote by Dyck_n^m the set of all m-Dyck paths of size n. We also denote by $\mathcal{D}_{m,n}$ the vector space (over a given field K) spanned by Dyck_n^m and by $\mathcal{D}_m$ the direct sum $\bigoplus_{n\geq 1} \mathcal{D}_{m,n}$. We denote by $d_{m,n} = \dim(\mathcal{D}_{m,n})$. The set of down steps of an m-Dyck path P is denoted by $\mathcal{DW}(P)$. We also denote by ρ_m the unique m-Dyck path of size 1.

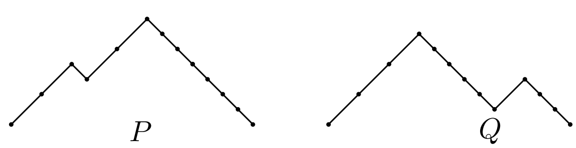
Example 3.1.2. The elements of Dyck_2^2 are



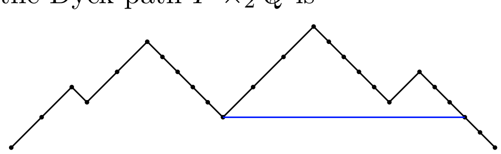
Definition 3.1.3. Let $P \in \mathrm{Dyck}_n^m$ and let $u_1, \ldots, u_n$ be the up steps of P ordered from left to right. We say that u_k has *rank* k and if d is a down step in between u_k and u_{k+1}, we say that d has *level* k. We denote by $L(P)$ the number of down steps of maximal level of P.

Definition 3.1.4. Let P, Q be two m-Dyck paths, of sizes n_1, n_2 respectively. Let $d_1, \ldots, d_{L(P)}$ be the maximal level down steps of P from left to right. For each $0 \leq j \leq L(P)$ we define $P \times_j Q$ as the m-Dyck path of size n_1+n_2 obtained by glueing the initial point of Q to the end point of $d_{L(P)-j}$ and glueing the down steps $d_{L(P)-j+1}, \ldots, d_{L(P)}$ to the end of Q.

Example 3.1.5. For the following 2-Dyck paths P, Q



the Dyck path $P \times_2 Q$ is



Definition 3.1.6. An m-Dyck path P is said to be *irreducible* (or prime) if it cannot be written as $P = Q \times_0 R$ for lower size Dyck paths Q, R. We denote by $Irr(\mathcal{D}_m)$ the set of irreducible m-Dyck paths.

Remark 3.1.7. It is clear that $\times_0$ defines an associative product on $\mathcal{D}_m$ and that any m-Dyck path P can be written uniquely as $P = P_1 \times_0 \cdots \times_0 P_r$ for irreducible Dyck paths $P_1, \ldots, P_r$. This means that $(\mathcal{D}_m, \times_0) = \overline{T}(K[Irr(\mathcal{D}_m)])$ as associative algebras.

In what follows we will also consider the point $\bullet$ as an m-Dyck path (of size 0) and $L(\bullet) = 0$, so $P \times_j \bullet = P$ for $0 \leq j \leq L(P)$ and $\bullet \times_0 P = P$ for any m-Dyck path P.

Notation 3.1.8. Let $P_0, \ldots, P_m$ be m-Dyck paths of size $n_0, \ldots, n_m \geq 0$ respectively. We denote by $(P_0, \ldots, P_m)$ the m-Dyck path

$$P_0 \times_0 (((\rho_m \times_m P_1) \times_{m-1} P_2) \times_{m-2} \ldots P_{m-1}) \times_1 P_m.$$

Proposition 3.1.9. *Any m-Dyck path P of size $n \geq 1$ can be written uniquely as $P = (P_0, \ldots, P_m)$ where the sum of the sizes of the P_i is $n - 1$.*

Proof. Write $P = P_0 \times_0 Q$ with Q irreducible of size ≥ 1. Draw an horizontal line starting from the end point of the first up step of Q. Let P_1 be the maximal sub Dyck path of P above this line. Now draw an horizontal line from the end point of the first down step after P_1 and let P_2 be the maximal sub Dyck path above this line. Repeating this process until we reach the x-axis, we get m-Dyck paths $P_0, \ldots, P_m$ which clearly satisfy $P = (P_0, \ldots, P_m)$. □

Remark 3.1.10. In a very similar way, one can prove that any m-Dyck path P can be uniquely written as

$$P = (((\rho_m \times_m P_0) \times_{m-1} P_1) \times_{m-2} \ldots) \times_0 P_m.$$

We write this by $P = \bigvee(P_0, \ldots, P_m)$.

Remark 3.1.11. Let $d_m(x)$ be the generating series of $\mathcal{D}_m^+$, that is, $d_m(x) = \sum_{n \geq 0} d_{m,n} x^n$ (where $d_{m,0} = 1$). Then the preceding proposition implies that $d_m(x)$ satisfies the following formula:

$$d_m(x) - 1 = x d_m(x)^{m+1}.$$

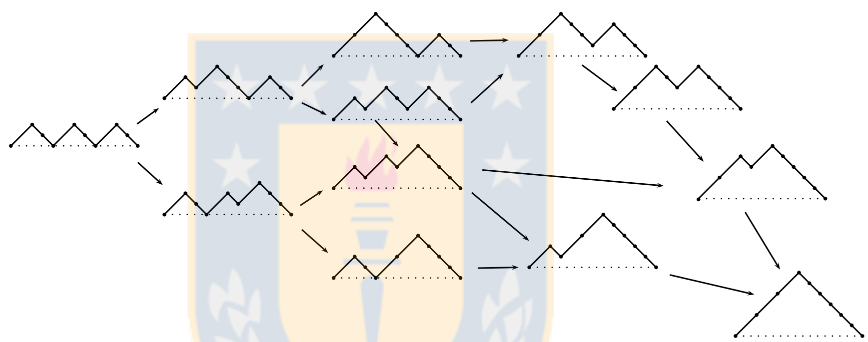
When $m = 1$ the preceding formula is also satisfied by the generating series of binary trees, so there is a bijection between 1-Dyck paths and binary trees.

3.2 Operations on m-Dyck paths

We now define the m-Tamari order of [BP12] on Dyck_n^m and we use it to define $m+1$ binary operations $*_i : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_m$ for $0 \leq i \leq m$.

Definition 3.2.1. Let $P \in \mathrm{Dyck}_n^m$ and let d be a down step of P followed by an up step u. Consider the shortest sub Dyck path P' of P starting from u. Define a new m-Dyck path $P_{(d)}$ by exchanging d with P'. The m-Tamari order on Dyck_n^m is defined by setting $P < P_{(d)}$ and extending by transitivity.

Example 3.2.2. The following is the Tamari order on Dyck_3^2:



Remark 3.2.3. A bijection $\phi : Y_n \to \mathrm{Dyck}_n^1$ can be constructed inductively by setting $\phi(t \vee s) = (\phi(t), \phi(s))$. Under this bijection, one can see that the Tamari order on binary trees correspond to the 1-Tamari order defined on 1-Dyck paths.

Definition 3.2.4. Let P be a m-Dyck path of size n. The *standard coloring* of P is a map α_P from the set of down steps $\mathcal{DW}(P)$ to the set $\{1, \ldots, n\}$, defined recursively as follows:

1. For $P = \rho_m \in \mathrm{Dyck}_1^m$, α_{ρ_m} is the constant function 1.

2. For $P = \bigvee(P_0, \ldots, P_m)$, with $P_j \in \mathrm{Dyck}_{n_j}^m$, the set of down steps of P is the disjoint union

$$\mathcal{DW}(P) = \{1, \ldots, m\} \coprod \mathcal{DW}(P_0) \coprod \cdots \coprod \mathcal{DW}(P_m),$$

where the first subset $\{1,\ldots,m\}$ corresponds to the steps of ρ_m. The map α_P is defined by:

$$\alpha_P(d) = \begin{cases} 1, & \text{for } d \in \{1,\ldots,m\}, \\ \alpha_{P_j}(d) + n_0 + \cdots + n_{j-1} + 1, & \text{for } d \in \mathcal{DW}(P_j), \end{cases}$$

where $0 \leq j \leq m$.

Let P be a m-Dyck path of size n and let $d_1, \ldots, d_{L(P)}$ be the maximal level down steps of P from left to right. For any $0 \leq i \leq m$, let

1. $c_i(P)$ be the minimal number of elements such that the word

$$\alpha_P(d_{L(P)-c_i(P)+1}) \ldots \alpha_P(d_{L(P)})$$

contains i times an integer in $\{1,\ldots,n\}$ and no integer more than i times,

2. $C_i(P)$ be the maximal integer such that the word

$$\alpha_P(d_{L(P)-C_i(P)+1}) \ldots \alpha_P(d_{L(P)})$$

contains at least one integer repeated i times and no integer repeated $i+1$ times.

Let P, Q be two m-Dyck paths of sizes n_1, n_2 respectively. For any $0 \leq i \leq m$, let $P/_iQ$ and $P\backslash_iQ$ be the Dyck paths (of size n_1+n_2) defined as follows:

1. $P/_iQ := P \times_{c_i(P)} Q$,
2. $P\backslash_iQ := (P \times_{L(P)} (Q_1 \times_0 \ldots \times_0 Q_r)) \times_{C_i(P)} Q_r$,

where $Q = Q_1 \times_0 \ldots \times_0 Q_r$, with Q_i prime for $1 \leq i \leq r$. We define a binary product $*_i : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_m$ in terms of the m-Tamari order for any $0 \leq i \leq m$ by the following formula:

$$P *_i Q = \sum_{P/_iQ \leq Z \leq P\backslash_iQ} Z.$$

For the proof of the following theorem, see [LPR15].

Theorem 3.2.5. *The products $*_i : \mathcal{D}_m \otimes \mathcal{D}_m \to \mathcal{D}_m$ satisfy the following relations:*

1. $x *_i (y *_j z) = (x *_i y) *_j z$ *for any* $i < j$;
2. $x *_i (y *_0 z + \cdots + y *_i z) = (x *_i y + \cdots + x *_m y) *_i z$ *for any* $0 \leq i \leq m$.

Definition 3.2.6. A Dyck^m-algebra is a vector space A together with $m+1$ binary operations $*_i$ for $0 \leq i \leq m$ satisfying the relations of 3.2.5.

3.3 Freedom of $\mathcal{D}_m$

We now turn to prove that $\mathcal{D}_m$ is the free Dyck^m-algebra on one generator.

Proposition 3.3.1. *Each element of $\mathcal{D}_m$ is a linear combination of elements of the form $P_1 *_i P_2$ where $0 \leq i \leq m$ and P_1, P_2 have strictly lower size.*

Proof. Let $P \in \mathcal{D}_m$ of size n and suppose the proposition is true for elements of size $< n$. The proposition is obviously true for the maximal element P_{max} of size n, so suppose it is also true for elements Q of size n such that $P < Q < P_{max}$ in the m-Tamari order. Write $P = (P_0, \ldots, P_m)$, and let $i \geq 0$ be the last index such that $P_i \neq \bullet$. If $i = 0$ then P is a concatenation of two nontrivial elements, $P = P_0 *_0 R$ and we are done. If $i > 0$ and we let P' be the Dyck path obtained from P by collapsing P_i to a point, then both P' and P_i have lower size than P and

$$P' *_{m-i+1} P_i = P + \sum Q_k$$

where the Q_k are $> P$. By our induction assumptions, this proves the proposition.

□

Theorem 3.3.2. *The free Dyck^m-algebra on one generator is isomorphic to $(\mathcal{D}_m, *_0, \ldots, *_m)$.*

Proof. We will prove this theorem following the argument of [LR06] (where a similar theorem is proved for 2-associative algebras). Let Dyck^m be the free Dyck^m-algebra on one generator, say x. Since $\mathcal{D}_m$ is a Dyck^m-algebra, sending x to the size 1 element of $\mathcal{D}_m$ defines an homomorphism of Dyck^m-algebras $\phi : \text{Dyck}^m \to \mathcal{D}_m$. By 3.3.1 this homomorphism is surjective. Let Dyck^m_n be the degree n part of Dyck^m. In order to prove that ϕ is injective, we only need to show that

$$\dim(\mathrm{Dyck}_n^m) \leq \dim(\mathcal{D}_{m,n})$$

for any $n \geq 1$. A spanning set of Dyck_n^m consists of words of the form $x *_{j_1} x *_{j_2} \dots x *_{j_{n-1}} x$ with a certain way of parenthezing. This can be identified with a binary tree with $n-1$ vertices with each vertex colored with a number between 0 and m (corresponding to the operations $*_i$, $0 \leq i \leq m$). Now, by the relations in a Dyck^m-algebra, any product $(x *_i y) *_j z$ with $i \leq j$ can be expressed in terms of products $(x *_k y) *_l z$ where $k > l$ or $x *_k (y *_l z)$ for any k, l. In this way, the dimension of Dyck_n^m is $\leq b_{n-1}$, where b_n is the number of binary trees with n vertices with the following two conditions:

1. Each vertex is colored with a color between 0 and m;
2. Whenever there is a vertex colored with k and a vertex inmediately to the right of k with colour j, then $k > j$.

Hence we only need to show that $b_n = d_{m,n+1}$. Let $b_{-1} = 1$ and $f(x) = \sum_{n \geq -1} b_n x^{n+1}$, we will show that the numbers b_n satisfy the same recursion formula as the $d_{m,n+1}$ (see 3.1.11), that is,

$$x f(x)^{m+1} = f(x) - 1 \tag{3.1}$$

and since $b_{-1} = d_{m,0} = 1$, this will prove our theorem.

Let b_n^i be the number of such trees with root vertex colored with i and let $f_i(x) = \sum_{n \geq 0} b_n^i x^{n+1}$ where we define $b_0^m = 1$ and $b_0^k = 0$ for $k < m$ so that we have

$$f - 1 = \sum_{i=0}^{m} f_i. \tag{3.2}$$

It is easy to see that $f_m = xf$ and that for $k < m$ and $n \geq 1$

$$b_n^k = \sum b_i^l b_j$$

where the sum is taken over the i, j, l such that $l > k, i + j = n - 1$ and $i, j \geq 0$. This gives the formula

$$f_k = \sum_{l > k} f_l f$$

for $k < m$ and solving we get

$$f_{m-i}(x) = x f^i (f - 1) \tag{3.3}$$

for $i = 1, \dots, m$. Putting together 3.2 and 3.3, equation 3.1 follows inmediately:

$$f-1=\sum_{i=0}^{m} f_{m-i}=\sum_{i=1}^{m} xf^i(f-1)+xf=x(f-1)\frac{f^{m+1}-f}{f-1}+xf=xf^{m+1}.$$

□

Let V be a vector space. Since $\mathcal{D}_m$ is the free Dyck^m-algebra on one generator and the variables in the relations of an Dyck^m-algebra stay in the same order, the free Dyck^m-algebra on V is

$$\mathrm{Dyck}^m(V)=\bigoplus_{n\geq 1}\mathcal{D}_{m,n}\otimes V^{\otimes n}.$$

Suppose V is graded, $V=\bigoplus_{n\geq 1} V_n$ where each V_n is finite dimensional, and let $v(x)$ be the generating series of V corresponding to this grading. If the generating series of $\mathcal{D}_m$ is $d_m(x)$, then the series of Dyck$^m(V)$ is $d(v(x))$. Using these facts, we show that $\mathcal{D}_m$ is Dyck^k-free for any $0\leq k<m$.

Lemma 3.3.3. *Let $d_m(x)$ be the generating series of $\mathcal{D}_m$. The following formula holds*

$$d_m(x)=d_k(xd_m(x)^{m-k})$$

for all $0\leq k\leq m$.

Proof. Clearly, it is enough to prove this for $k=m-1$. Let $d'_m(x)=d_m(x)-1$ and let $g_m(x)$ be the inverted series of $d'_m(x)$, that is, a series such that $d'_m(g_m(x))=g_m(d'_m(x))=x$ (such a g exists since $d'(0)=0$). Since $x(1+d'_m(x))^{m+1}=d'_m(x)$, replacing x by $g_m(x)$ we obtain the following formula for $g_m(x)$:

$$g_m(x)=\frac{x}{(1+x)^{m+1}}.$$

Since clearly $(1+x)g_m(x)=g_{m-1}(x)$, replacing x by $d'_m(x)$ and applying $d'_{m-1}(x)$ to both sides we get the desired formula $d'_{m-1}(x(1+d'_m(x)))=d'_m(x)$.

□

Observe that the operations $*_0,\dots,*_{k-1},*_k+\dots+*_m$ define a Dyck^k-algebra structure on $\mathcal{D}_m$.

Theorem 3.3.4. *As a Dyck^k-algebra, $(\mathcal{D}_m,*_0,\dots,*_{k-1},*_k+\dots+*_m)$ is free on the set W of elements of the form $(P_0,\dots,P_m)$ where $P_0=\bullet$ and $P_{m-k+1}=\dots=P_m=\bullet$.*

Proof. We show first that W has the correct dimensions, that is, that the free Dyck^k-algebra on W, which is

$$\mathrm{Dyck}^k(W) = \bigoplus_{n\geq 1} \mathcal{D}_{k,n} \otimes K[W]^{\otimes n}$$

is isomorphic to $\mathcal{D}_m$ as a graded vector space. Indeed, it is easy to see that the generating series $w(x)$ of W satisfies $w(x) = xd(x)^{m-k}$ so the series of $\mathrm{Dyck}^k(W)$ is $d_k(xd(x)^{m-k})$ and this is $d(x)$ by the preceding lemma, as we wanted. We show now that the Dyck^k-subalgebra A generated by W and the operations $*_0, \ldots, *_{k-1}, *_k + \cdots + *_m$ which we denote respectively by $*'_0, \ldots, *'_k$ coincides with $\mathcal{D}_m$. The idea is the same as in 3.3.1. Let $P = (P_0, \ldots, P_m) \in \mathcal{D}_m$ of size n and suppose that all elements of size $< n$ and all elements $Q > P$ (in the m-Tamari order) of size n are contained in A. If $P_0 \neq \bullet$ then

$$P = P_0 *_0 (\bullet, P_1, \ldots, P_m)$$

so $P \in A$ by induction. Now suppose $P_0 = \bullet$ and $P \notin W$, so there is an $i > m - k$ (and this implies $m + 1 - i \leq k$) such that $P_i \neq \bullet$ and $P_j = \bullet$ for $j > i$. Let P' be the m-Dyck path obtained from P by replacing P_i by $\bullet$. Then we have

$$P = P' *'_{m+1-i} P_i + \sum Q_k$$

where the Q_k are $> P$. By the induction assumptions, this implies that $P \in A$.

$\square$

3.4 A diagonal for m-Dyck paths

In this section we define a coproduct $\Delta : \mathcal{D}_m \to \mathcal{D}_m \otimes \mathcal{D}_m$ which respects the $*_i$-operations. This generalizes the coproduct on 1-Dyck paths (or binary trees) defined in [Ron00]. First, we extend the $*_i$-operations to $\mathcal{D}_m^+$: for $P \in \mathcal{D}_m$ (so $P \neq 1$) define

1. $P *_0 1 = 0$ and $1 *_0 P = P$,
2. $P *_i 1 = 1 *_i P = 0$ for $0 < i < m$;
3. $P *_m 1 = P$ and $1 *_m P = 0$.

As usual, $1 *_i 1$ is undefined for any i. Let $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$ be the positive degree part of $\mathcal{D}_m^+ \otimes \mathcal{D}_m^+$, that is, $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+} = \mathcal{D}_m^+ \otimes \mathcal{D}_m \oplus \mathcal{D}_m \otimes \mathcal{D}_m^+$. We define a Dyck^m-algebra structure on $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$ by

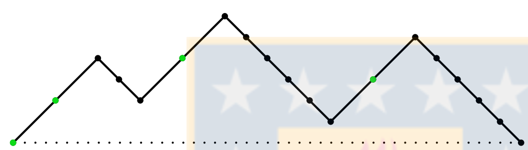
1. $(x_1 \otimes x_2) *_i (y_1 \otimes y_2) = (x_1 * y_1) \otimes (x_2 *_i y_2)$ if $x_2 \neq 1$ or $y_2 \neq 1$;
2. $(x \otimes 1) *_i (y \otimes 1) = (x *_i y) \otimes 1$.

It is easy to see that this indeed defines a Dyck^m-algebra structure.

Definition 3.4.1. Let P be an m-Dyck path. A *central vertex* of P is a vertex of an up step of P which is not a common vertex to a down step of P.

Observe that the initial vertex of P is a central vertex.

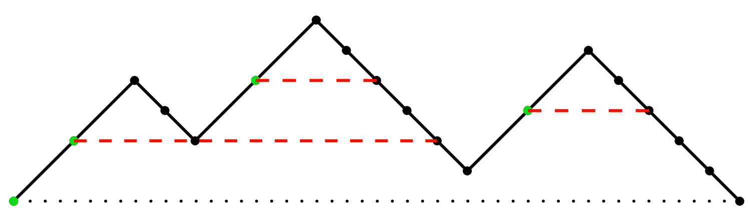
Example 3.4.2. Consider the following 2-Dyck path:



The central vertices are marked in green.

Definition 3.4.3. For each central vertex v of P consider the horizontal line through v and consider the maximal path above this line starting from v. The vertices of down steps of this path which lie on this line are called the *admissible vertices of v*. An *admissible cut* of P is any path starting from a central vertex v of P and ending on an admissible vertex of v. The whole path P is not considered as an admissible cut. The *level* of a cut P' is the level of the down steps of P of maximal level which belong to P'.

Example 3.4.4. Consider the Dyck path of the preceding example. The admissible cuts are the paths above the dotted red lines.



Observe that the central vertex (2,2) has as admissible vertices the points (4,2) and (12,2), so there are two admissible cuts corresponding to the lowest red dotted line.

Let $Ad(P)$ be the set of admissible cuts of P. This set is partially ordered in the following way: if $P_1, P_2 \in Ad(P)$, then $P_1 \leq P_2$ if P_1 is contained in P_2. We call this the *cut order* of $Ad(P)$.

Definition 3.4.5. Let P be an m-Dyck path. The (reduced) coproduct $\Delta : \mathcal{D}_m \to \mathcal{D}_m \otimes \mathcal{D}_m$ is defined by

$$\Delta(P) = \sum P_1 * \cdots * P_k \otimes P/\{P_1, \ldots, P_k\}$$

where the sum ranges over all $P_1, \ldots, P_k \in Ad(P)$ which are not comparable under the preceding partial order, ordered by increasing level, and $P/\{P_1, \ldots, P_k\}$ is the m-Dyck path obtained by collapsing all the P_i, $1 \leq i \leq k$, to a point.

The reduced coproduct extends to a coproduct $\Delta^+ : \mathcal{D}_m \to \overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$ as

$$\Delta^+(P) = \Delta(P) + P \otimes 1 + 1 \otimes P.$$

The main result is the following:

Theorem 3.4.6. *The coproduct Δ satisfies*

$$\Delta^+(P *_i Q) = \Delta^+(P) *_i \Delta^+(Q)$$

for any $0 \leq i \leq m$ and $P, Q \in \mathcal{D}_m$, that is, Δ^+ is a morphism of Dyck^m-algebras.

See [LPR15] for a proof.

Corollary 3.4.7. *The coproduct Δ^+ (hence also Δ) is coassociative.*

Proof. We need to show that the composition

$$\mathcal{D}_m \xrightarrow{\Delta^+} \overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+} \xrightarrow{\Delta^+ \otimes 1 - 1 \otimes \Delta^+} \overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$$

is zero. There is a Dyck^m-algebra structure on $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$ given by:

$$(x_1 \otimes x_2 \otimes x_3) *_i (y_1 \otimes y_2 \otimes y_3) = (x_1 * y_1) \otimes (x_2 * y_2) \otimes (x_3 *_i y_3)$$

and we make similar considerations as in the case of $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$ when $x_3 = y_3 = 1$. Since Δ^+ is a Dyck^m-morphism, it is easy to see that both $\Delta^+ \otimes 1, 1 \otimes \Delta^+$ are so. By 3.3.2, coassociativity of Δ^+ follows from the fact that

$$(\Delta^+ \otimes 1 - 1 \otimes \Delta^+)(\Delta^+ \rho_m) = 0$$

for the generator ρ_m of $\mathcal{D}_m$.

$\square$

Definition 3.4.8. A *Dyck^m-bialgebra* is a Dyck^m-algebra A together with a coassociative coproduct $\Delta : A \to A \otimes A$ such that its extension $\Delta^+ : A \to \overline{A^+ \otimes A^+}$ is a morphism of Dyck^m-algebras, where $\overline{A^+ \otimes A^+}$ has a Dyck^m-algebra structure defined in the same way as for $\overline{\mathcal{D}_m^+ \otimes \mathcal{D}_m^+}$.

In summary, $\mathcal{D}_m$ is a Dyck^m-bialgebra with the coproduct of 3.4.5.

3.5 Operations on the space of primitive elements

In this section we introduce the algebras arising on the primitive subspace of a Dyck^m-bialgebra, which we call GV^m-algebras. These are brace algebras together with $m-1$ binary operations $*_1, \dots, *_{m-1}$ satisfying certain relations. The most difficult task of the present section is to establish the relationship between braces and the operations $*_k$, which satisfy a generalisation of M. Gerstenhaber and A. Voronov formula for $M_{1n}(x \cdot y; z_1, \dots, z_n)$ given in 2.5.

Notation 3.5.1. For any binary tree $t \in Y_{n-1}$ with vertices colored with binary operations, and elements $y_1, \dots, y_n$ in a Dyck^m-algebra, we denote by $t[y_1, \dots, y_n]$ the element $y_1 \alpha_1 \dots \alpha_{n-1} y_n$ with parenthesis given by t, where the α_i are the operations in the corresponding vertices of t. If all the α_i represent the same operation α, we write $t^\alpha[y_1, \dots, y_n]$. The tree $t = |$ represents the identity operation. We denote by $r(t)$ the number of right leaves of a binary tree and by t_n the maximal element (in the Tamari order) of Y_n.

Recall that on any dendriform algebra $(A, \succ, \prec)$ there are brace operations defined by

$$M_{1n}(x; y_1, \dots, y_n) = \sum_{i=0}^{n} (-1)^{n-i} w_{\prec}(y_1, \dots, y_i) \succ x \prec w_{\succ}(y_{i+1}, \dots, y_n).$$

Consider the dendriform structure $\succ = *_0$, $\prec = *_1 + \cdots + *_m$ on a Dyck^m-algebra A. We denote by M_{1n} the corresponding braces. Define new braces, which we call *reduced braces*, $\widetilde{M}_{1n}$ by

$$\widetilde{M}_{1n}(x; y_1, \ldots, y_n) = \sum_{i=0}^{n} (-1)^{n-i} w_{\prec}(y_1, \ldots, y_i) \succ x *_m w_{\succ}(y_{i+1}, \ldots, y_n).$$

A sequence of linear operators $S_n : A^{\otimes n} \to A$ defined for $n \geq 1$ is said to satisfy the *canonical recursion formula* if

$$z_1 \succ S_n(z_2, z_3, \ldots, z_{n+1}) - S_n(z_1 \succ z_2, z_3, \ldots, z_{n+1}) = S_{n+1}(z_1, \ldots, z_{n+1})$$

for all $z_i \in A$ and $n \geq 1$. Observe that if two sequences of operators defined for $n \geq 1$ satisfy this formula and they coincide when $n = 1$, then they are equal for all n.

Lemma 3.5.2. *Let A be a Dyck^m-algebra and $*$ a binary product on A satisfying $(x \succ y) * z = x \succ (y * z)$. Consider two families of linear operators $S_n, T_n : A^{\otimes n} \to A$ for $n \geq 1$ and suppose that the family S_n satisfy the canonical recursion formula. Define S_0 and T_0 as empty or as a specified element of A (different elements could be taken). Then the operators*

$$B_n(z_1, \ldots, z_n) = \sum_{i=0}^{n} S_i(z_1, \ldots, z_i) * T_{n-i}(z_{i+1}, \ldots, z_n)$$

satisfy the canonical recursion formula provided that

$$\begin{gathered} z_1 \succ (S_0 * T_n(z_2, \ldots, z_{n+1})) - S_0 * T_n(z_1 \succ z_2, \ldots, z_{n+1}) = \\ S_0 * T_{n+1}(z_1, \ldots, z_n) + S_1(z_1) * T_n(z_2, \ldots, z_{n+1}) \end{gathered}$$

for all $z_1, \ldots, z_{n+1} \in A$ and $n \geq 1$.

Proof. This is trivial. □

Proposition 3.5.3. *The usual braces M_{1n} and the reduced braces $\widetilde{M}_{1n}$ satisfy the canonical recursion formula.*

Proof. Define $S_n(z_1, \ldots, z_n) = w_{\prec}(z_1, \ldots, z_n) \succ x$ for $n \geq 1$ and $S_0 = x$, and let $T_n(z_1, \ldots, z_n) = (-1)^n w_{\succ}(z_1, \ldots, z_n)$ (with T_0 empty). It is easy to check that these operators together with the product $* = \prec$ satisfy the conditions of the lemma and that $M_{1n}(x; z_1, \ldots, z_n) = B_n(z_1, \ldots, z_n)$ proving what we wanted. The assertion for the reduced braces is obtained by taking $* = *_m$. □

Lemma 3.5.4. *If $S_n(z_1,\dots,z_n) = M_{1n}(x; z_1,\dots,z_n)$ for all $n \geq 0$ (x fixed) and the product $*$ satisfies further that $(x \prec y) * z = x * (y \succ z) + x * (y * z)$, then the operators B_n satisfy the canonical recursion formula provided*

$$z_1 \succ T_n(z_2,\dots,z_{n+1}) - T_n(z_1 \succ z_2,\dots,z_{n+1}) + z_1 * T_n(z_2,\dots,z_{n+1}) = T_{n+1}(z_1,\dots,z_{n+1}).$$

Proof. This follows easily using the given properties of $*$, the fact that the M_{1n} satisfy the canonical recursion formula, and the recursion formula for T_n. □

The braces $\widetilde{M}_{1n}$ respect the operation $*_1$ in the sense that they satisfy a Gerstenhaber-Voronov type formula.

Proposition 3.5.5. *Given a Dyck^m algebra A, the operations M_{1n} and $\widetilde{M}_{1n}$ defined previously satisfy that:*

$$\widetilde{M}_{1n}(x *_1 y; z_1,\dots,z_n) = \sum_{0\leq i\leq j\leq n} M_{1i}(x; z_1,\dots,z_i) *_1 (z_{i+1} *_1 \cdots *_1 (z_j *_1 \widetilde{M}_{1(n-j)}(y; z_{j+1},\dots,z_n))\dots),$$

for any $x, y, z_1,\dots,z_n \in A$.

Proof. Define $T_0 = y$ and

$$T_n(z_1,\dots,z_n) = \sum_{i=0}^{n} t_i^{*_1}[z_1,\dots,z_i, \widetilde{M}_{1(n-i)}(z_{i+1},\dots,z_n)].$$

It is easy to check that these operators and the operation $*_1$ satisfy the conditions of 3.5.4. The operator $B_n(z_1,\dots,z_n)$ given by these operators and the product $*_1$ is exactly the right hand side of our equation. Since clearly $B_1(z) = x *_1 (\widetilde{M}_{11}(y; z) + z *_1 y) + M_{11}(x; z) * y$ equals $\widetilde{M}_{11}(x *_1 y; z)$ and the $\widetilde{M}_{1n}$ also satisfy the canonical recursion formula, then $\widetilde{M}_{1n}(x *_1 y; z_1,\dots,z_n) = B_n(z_1,\dots,z_n)$ for all n.

□

Proposition 3.5.6. *Given a Dyck^m algebra, for any collection of elements $x, z_1,\dots,z_n \in A$ we get that:*

$$\widetilde{M}_{1n}(x; z_1,\dots,z_n) = \sum_{i=0}^{n} t_{n-i}^{*_1'}[M_{1i}(x; z_1,\dots,z_i), z_{i+1},\dots,z_n]$$

where $*_1' = *_1 + \cdots + *_{m-1}$.

Proof. Define $T_n(z_1,\dots,z_n) = t_{n-1}^{*'_1}[z_1,\dots,z_n]$ for $n \geq 1$, and let T_0 be empty. These operators together with the product $*'_1$ satisfy the conditions of 3.5.4, and the corresponding operator $B_n(z_1,\dots,z_n)$ is the right hand side of our formula. Since clearly $\widetilde{M}_{11} = B_1$, we have $\widetilde{M}_{1n} = B_n$ for all n. □

We denote by T_n the set of all trees (not necessarily binary) with $n+1$ leaves as in 2.5 and by c_n the n-th *corolla*, that is, the tree with one vertex and $n+1$ leaves. In order to write down a nice formula for $M_{1n}(*_k;-)$ we introduce the following definitions.

Definition 3.5.7. A *right comb tree* is a binary tree obtained as a concatenation of maximal binary trees in the Tamari order. The set of right comb trees of n vertices will be denoted by RT_n. The leaf with no vertex is also considered as a right comb tree, and $r(|) = 1$.

There are exactly 2^{n-1} right comb trees of n vertices (for $n \geq 1$).

Definition 3.5.8. An *M-tree* is a tree obtained from a maximal binary tree t_n by grafting two corollas c_i, c_j $(i, j \geq 0)$ one into the first leave of t_n and the other in any other leave. We denote by MT_n the set of M-trees contained in T_n. A *right comb M-tree* is a tree of the form u/v, where u is an M-tree and v is a right comb tree. We denote by RMT_n the set of right comb M-trees of T_n. Let t be a right comb M-tree, we denote by $\widetilde{t}$ the right comb binary tree obtained from t be deleting the two corollas.

For a right comb M-tree $t \in RMT_{n+1}$ we will assume the vertex of the first corolla represents M_{1i} and the vertex of the second one represents M_{1j}. We will put a variable x into the first leave of c_i and a variable y into the first one of c_j. We put variables $z_1,\dots,z_n$ in order into the other leaves of t. All other vertices of our tree t represents a binary operation to be specified. Using these conventions, for any $t \in RMT_{n+1}$, we denote the element $t[x, z_1, \dots, z_i, \dots, y, z_{p+1}, \dots, z_{p+j}, \dots, z_n]$ just by t. Inserting 3.5.6 into 3.5.5 and using our convention, we rewrite our formula for $\widetilde{M}_{1n}(x *_1 y; -)$.

Lemma 3.5.9. *We have*

$$\widetilde{M}_{1n}(x *_1 y; z_1, \dots, z_n) = \sum_{t \in MT_{n+1}} t.$$

The vertices of $\widetilde{t}$ are colored in the following way: write $\widetilde{t} = u_1 \setminus u_2$ where the second corolla of t is inserted in the first leave of u_2. Then we color the vertices of u_1 with $*_1$ and those of u_2 with $*'_1 = *_1 + \cdots + *_{m-1}$.

Proposition 3.5.10. *The following formula holds:*

$$M_{1n}(x *_1 y; z_1, \dots, z_n) = \sum_{t \in RMT_{n+1}} (-1)^{r(\widetilde{t})-1} t.$$

The vertices of $\widetilde{t}$ are colored in the following way: write $t = u/v$ there u is an M-tree and v is a right comb tree. The vertices of u are colored as in the preceding lemma, and those of v are colored with $*'_1$.

Proof. We use the following obvious equality:

$$RMT_{n+1} = MT_{n+1} \cup \bigcup_{i=0}^{n-1} \{u/t_{n-i} \mid u \in RMT_{i+1}\}.$$

Now, by 3.5.6 we have

$$\begin{aligned} M_{1n}(*_1; -) &= \widetilde{M}_{1n}(*_1; -) - \sum_{i=0}^{n-1} t_{n-i}[M_{1i}(*_1; -), \dots] \\ &= \sum_{t \in MT_{n+1}} t - \sum_{i=0}^{n-1} \sum_{u \in RMT_{i+1}} (-1)^{r(\widetilde{u})-1} t_{n-i}[u, \dots] \end{aligned}$$

by the preceding lemma and induction. Now observe that $t_{n-i}[u, \dots] = u/t_{n-i}$ and that $r(\widetilde{u}) = r(\widetilde{u}/t_{n-i}) - 1$ so by using the equality for RMT_{n+1}, we get our theorem. The coloring of the vertices is obvious from the induction. □

Using the preceding proposition, we can find a formula for $M_{1n}(x *_k y; -)$ for any $1 \leq k \leq m-1$.

Proposition 3.5.11. *Using the same notation as before, we have*

$$M_{1n}(x *_k y; z_1, \dots, z_n) = \sum_{t \in RMT_{n+1}} (-1)^{r(\widetilde{t})-1} t$$

where the vertices of u_1 are colored in all possible ways with the operations $*_1, \dots, *_k$ with at least one $*_k$ used, and all other vertices of $\widetilde{t}$ are colored with $*'_1 = *_1 + \cdots + *_{m-1}$.

Proof. For any k let $*'_k = *_1 + \cdots + *_k$, then $*_0, *'_k, *_{k+1} + \cdots + *_m$ defines a Dyck^2-algebra structure on A, where $*'_k$ now plays the role of $*_1$. In this way, we can apply the preceding proposition to get a formula for $M_{1n}(x *'_k y; -)$. Substracting $M_{1n}(x *'_k y; -) - M_{1n}(x *'_{k-1} y; -)$ we get the desired result. □

Let us provide a formal definition for the algebraic structures which appear naturally on the subspace of primitive elements of any Dyck^m-bialgebra.

Definition 3.5.12. A GV^m-algebra W is a brace algebra (W, M_{1n}) together with binary operations $*_1, \ldots, *_{m-1}$ such that

1. $(x *_i y) *_j z = x *_i (y *_j z)$ for any $x, y, z \in W$ and $i < j$;
2. $M_{1n}(x *_k y; -)$ satisfies the formula of 3.5.11 for any $1 \leq k \leq m-1$.

Theorem 3.5.13. *Let $(A, *_0, \ldots, *_m)$ be a Dyck^m-algebra. Let M_{1n} be the brace operations coming from the dendriform structure $\succ^0 = *_0, \prec^0 = *_1 + \cdots + *_m$. Then $(A, M_{1n}, *_1, \ldots, *_{m-1})$ is a GV^m-algebra. Moreover, if A is a Dyck^m-bialgebra, then Prim(A) is a GV^m-subalgebra of A.*

3.6 Milnor-Moore theorem for Dyck^m-bialgebras

We now turn to prove that the brace operations M_{1n} together with the binary products $*_1, \ldots, *_{m-1}$ generate all the primitive elements of $\mathcal{D}_m$. We will use the eulerian idempotent of dendriform algebras defined in 2.4. Using the standard properties of this operator, we show the following two lemmas.

Lemma 3.6.1. *For any $x \in \mathcal{D}_m, y \in$ Prim$(\mathcal{D}_m)$ and $0 < i < m$ the following formula holds:*

$$e(x *_i y) = e(x) *_i y.$$

Proof. Since y is primitive and $0 < i < m$, we have

$$\overline{\Delta}(x *_i y) = x_{(1)} \otimes x_{(2)} *_i y$$

so

$$\begin{aligned} e(x *_i y) &= x *_i y - x_{(1)} \succ e(x_{(2)} *_i y) \\ &= x *_i y - x_{(1)} \succ (e(x_{(2)}) *_i y) \text{ by induction} \\ &= (x - x_{(1)} \succ e(x_{(2)})) *_i y \\ &= e(x) *_i y \end{aligned}$$

therefore proving the lemma. $\square$

Lemma 3.6.2. *For $x, y_1, \ldots, y_n \in Prim(\mathcal{D}_m)$, we have*

$$e(x *_m w_{\succ}(y_1, \ldots, y_n)) = (-1)^n \widetilde{M}_{1n}(x; y_1, \ldots, y_n).$$

Proof. By using 2.4.1 and the formulas of 3.5.6 relating M_{1n} and $\widetilde{M}_{1n}$, we have

$$\begin{aligned}\widetilde{M}_{1n}(x; y_1, \ldots, y_n) &= \sum_{i=0}^{n} M_{1i}(x; y_1, \ldots, y_i) *' t^{*'}_{n-i-1}(y_{i+1}, \ldots, y_n) \\ &= \sum_{i=0}^{n} e((-1)^i x \prec w_{\succ}(y_1, \ldots, y_i)) *' t^{*'}_{n-i-1}(y_{i+1}, \ldots, y_n) \\ &= \sum_{i=0}^{n} e((-1)^i (x \prec w_{\succ}(y_1, \ldots, y_i)) *' t^{*'}_{n-i-1}(y_{i+1}, \ldots, y_n))\end{aligned}$$

where $*' = *_1 + \cdots + *_{m-1}$ and we have used lemma 3.6.1. In this way, it is enough to prove the following formula:

$$\begin{aligned}t^{*'}_n(x, y_1, \ldots, y_n) &= \sum_{i=1}^{n} (-1)^{i-1} (x \prec w_{\succ}(y_1, \ldots, y_i)) *' t^{*'}_{n-i-1}(y_{i+1}, \ldots, y_n) \\ &+ (-1)^n x *_m w_{\succ}(y_1, \ldots, y_n).\end{aligned}$$

This is done easily by induction using the formula

$$x *' (y *' z) = (x \prec y) *' z - x *' (y \succ z).$$

$\square$

Let $Irr(\mathcal{D}_m)$ be the set of prime (or irreducible) elements of $\mathcal{D}_m$. Clearly $T(Irr(\mathcal{D}_m)) \cong \mathcal{D}_m$ and since $\mathcal{D}_m$ is a conilpotent dendriform bialgebra, by 2.4.3 we have $T(\mathrm{Prim}(\mathcal{D}_m)) \cong \mathcal{D}_m$. This implies that the number of irreducibles of degree n is equal to the dimension of the degree n part of $\mathrm{Prim}(\mathcal{D}_m)$. Using these facts, we prove:

Proposition 3.6.3. *The elements $e(x)$, where x is a prime element of $\mathcal{D}_m$, form a basis of $Prim(\mathcal{D}_m)$.*

Proof. It is enough to prove that the $e(x)$, x prime, are linearly independent. But by definition of e, we have

$$e(x) = x + \text{ reducible elements}$$

and this implies linear independence immediately. □

Theorem 3.6.4. *The subspace A of $\mathcal{D}_m$ generated by the degree 1 element ρ_m and the operations M_{1n} and $*_1, \dots, *_{m-1}$ coincides with Prim$(\mathcal{D}_m)$.*

Proof. Assume inductively that $\mathrm{Prim}(\mathcal{D}_m)_k = A_k$ for $k < n$ and let $x \in Irr(\mathcal{D}_m)_n$, we will prove that $e(x) \in A$. This will imply that $\mathrm{Prim}(\mathcal{D}_m)_n = A_n$ by 3.6.3. Suppose that $e(y) \in A$ for any $y > x$ in the m-Tamari order. Write $x = (\bullet, x_1, \dots, x_m)$ and suppose first that $x_2 = \cdots = x_m = \bullet$, that is, $x = \rho_m *_m x_1$. By 2.4.1, x_1 is a sum of elements of the form $w_\succ(y_1, \dots, y_k)$ with $y_1, \dots, y_k \in \mathrm{Prim}(\mathcal{D}_m)$ (so the $y_i \in A$ by induction) and

$$e(\rho_m *_m w_\succ(y_1, \dots, y_k)) = (-1)^k \widetilde{M}_{1k}(\rho_m; y_1, \dots y_k)$$

which implies that $e(x) \in A$ (observe that the subspace generated by the $\widetilde{M}_{1n}$ and the $*_1, \dots, *_{m-1}$ is the same as A by 3.5.6). Observe that this proves that $e(x_{max}) \in A$, where x_{max} is the maximal element in the Tamari order of $\mathcal{D}_{m,n}$. Now suppose $x_i \neq \bullet$ for $i > 1$ and $x_{i+1} = \cdots = x_m = \bullet$. Then if x' is the Dyck path obtained from x by collapsing x_i to a point (as in 3.3.1), we have

$$x' *_{m+1-i} x_i = x + \sum y_k$$

where the y_k are $> x$ in the Tamari order. By our assumptions, we have

$$e(x' *_{m+1-i} x_i) \equiv e(x) \pmod{A}$$

so the theorem will proved by the following claim.

Claim: $e(y *_i z) \in A$ for any y, z of degree $< n$ such that $|y| + |z| = n$ and $0 < i < m$.

Recall that we are assuming $e(Irr(\mathcal{D}_m)_k) = A_k$ for $k < n$. As before, it is enough to prove the claim when $z = w_\succ(y_1, \dots, y_k)$ with y_i primitive and $|z| < n$ so we perform induction on k. When $k = 1$, we have by 3.6.1 that $e(y *_i z) = e(y) *_i z$ and by induction both $e(y), z \in A$ so also $e(y *_i z) \in A$. To simplify notation, write $w^l_\succ(\overline{y})$ for $w^l_\succ(y_1, \dots, y_l)$ and note that $w^l_\succ(\overline{y}) \succ y_{l+1} = w^{l+1}_\succ(\overline{y})$ for any l. We have

$$\begin{aligned} e(y *_i w^k_\succ(\overline{y})) &= e(y *_i (w^{k-1}_\succ(\overline{y}) *_0 y_k)) \\ &= \sum_{j=1}^{i} -e(y *_i (w^{k-1}_\succ(\overline{y}) *_j y_k)) + \sum_{j=i}^{m} e((y *_j w^{k-1}_\succ(\overline{y})) *_i y_k). \end{aligned} \tag{3.4}$$

By induction, the sum on the right hand side belongs to A. Now, observe that

$$\begin{aligned} w_{\succ}^{k-1}(\overline{y}) *_j y_k &= (w_{\succ}^{k-2}(\overline{y}) \succ y_{k-1}) *_j y_k = w_{\succ}^{k-2}(\overline{y}) \succ (y_{k-1} *_j y_k) \\ &= w_{\succ}^{k-1}(y_1, \ldots, y_{k-2}, y_{k-1} *_j y_k) \end{aligned}$$

and since $0 < j < m$, $y_{k-1} *_j y_k$ is primitive. By induction again, the sum on the left hand side of (11) belongs to A, proving the claim.

□

Remark 3.6.5. Proposition 3.6.3 can be generalized easily to $\mathrm{Dyck}^m(V)$, where V is a (finite dimensional) vector space, in this case we have

$$\mathrm{Prim}(\mathrm{Dyck}^m(V))_n = e(Irr(\mathcal{D}_m)_n \otimes V^{\otimes n}).$$

It is easy to see that this coincides with $e(Irr(\mathcal{D}_m)_n) \otimes V^{\otimes n}$, so we get

$$\mathrm{Prim}(\mathrm{Dyck}^m(V))_n = \mathrm{Prim}(\mathcal{D}_m)_n \otimes V^{\otimes n}.$$

The following corollary is now obvious.

Corollary 3.6.6. *For any vector space V, the primitive subspace $Prim(Dyck^m(V))$ is generated by V, the operations M_{1n} and $*_1, \ldots, *_{m-1}$.*

Theorem 3.6.7. *For any vector space V, $Prim(Dyck^m(V))$ is the free GV^m algebra on V.*

Proof. Suppose that $\dim(V) = k$ and let $GV^m(V)$ be the free m-GV algebra on V. Then $\dim(GV^m(V)_n) = \dim(GV^m(K)_n)k^n$ and also $\dim \mathrm{Prim}(\mathrm{Dyck}^m(V))_n = \dim \mathrm{Prim}(\mathcal{D}_m)_n k^n$ so by 3.6.6, it suffices to prove the theorem for $V = K$. By 3.6.6 and 3.6.3 it suffices to prove that

$$\dim(GV^m(K)_n) = \dim K[Irr(\mathcal{D}_m)_n].$$

A basis $\mathcal{B}$ of $GV^m(K)$ is constructed inductively as follows: let $\mathcal{B}_1 = \{x\}$ and suppose that $\mathcal{B}_k$ is defined for $k < n$ and each such set is partitioned in $m-1$ subsets $\mathcal{B}_k = \mathcal{B}_{k,1} \sqcup \cdots \sqcup \mathcal{B}_{k,m-1}$. For any $1 \leq i \leq m-1$, we define

$$\mathcal{B}_{n,i} = \{y *_i z \mid y \in \mathcal{B}_{k,j}, z \in \mathcal{B}_l \text{ such that } j \geq i, k+l = n\}$$

and we let $\mathcal{B}_n = \mathcal{B}_{n,1} \sqcup \cdots \sqcup \mathcal{B}_{n,m-1}$. It is easy to see this defines a basis of $GV^m(K)_n$. We define a map $\varphi : \mathcal{B}_n \to Irr(\mathcal{D}_m)_n$ inductively as follows:

1. $\varphi(x) = \rho_m$;
2. $\varphi(M_{1i}(x; y_1, \ldots, y_i)) = \rho_m *_m w^i_{\succ}(\varphi(y_1), \ldots, \varphi(y_i))$;
3. $\varphi(x *_j y) = \varphi(x) \times_j \varphi(y)$.

It is easy to see that this map is injective, and since there is a surjection $GV^m(K) \to \mathrm{Prim}(\mathcal{D}_m)$, this is a bijection, therefore proving the theorem.

□

Definition 3.6.8. Let $(W, M_{1n}, *_1, \ldots, *_{m-1})$ be a GV^m algebra. The *universal enveloping Dyck^m-algebra* on W, denoted by $\mathcal{U}_{\mathrm{Dyck}^m}(W)$, is the quotient of $\mathrm{Dyck}^m(W)$ by the Dyck^m-ideal generated by the elements of the form $M_{1n}(x; y_1, \ldots, y_n) - M'_{1n}(x; y_1, \ldots, y_n)$ and $x *_i y - x *'_i y$ for $x, y, y_1, \ldots, y_n \in W$, $1 \leq i \leq m-1$ and where $M'_{1n}, *'_1, \ldots, *'_{m-1}$ is the induced GV^m-algebra structure on $\mathrm{Dyck}^m(W)$.

Remark 3.6.9. The universal enveloping Dyck^m-algebra has an obvious adjointness property. Using this, it is easy to see that it also has a canonical Dyck^m-bialgebra structure.

As usual, there is a canonical GV^m-algebra morphism $W \to \mathcal{U}_{\mathrm{Dyck}^m}(W)$ for any GV^m-algebra W, and a canonical Dyck^m-morphism $\mathcal{U}_{\mathrm{Dyck}^m}(\mathrm{Prim}(A)) \to A$ for any Dyck^m-algebra A.

Theorem 3.6.10. *For any GV^m-algebra W, the canonical morphism gives an isomorphism $W \cong Prim(\mathcal{U}_{Dyck^m}(W))$. For any conilpotent Dyck^m-bialgebra A, the canonical morphism $\mathcal{U}_{Dyck^m}(Prim(A)) \to A$ is an isomorphism. In other words, the functors Prim and $\mathcal{U}_{Dyck^m}$ give an equivalence between the category of conilpotent Dyck^m-bialgebras and GV^m-algebras.*

Proof. The subspace $\mathrm{Prim}(\mathcal{U}_{\mathrm{Dyck}^m}(W))$ is the image of $\mathrm{Prim}(\mathrm{Dyck}^m(W)) = GV^m(W)$ under the projection, and the image of this is exactly W. For the second assertion, observe that conilpotent Dyck^m-bialgebras are conilpotent dendriform bialgebras (for example, with $\succ^0, \prec^0$). Consider the composition

$$A \xrightarrow{\epsilon} \overline{T}(\mathrm{Prim}(A)) \xrightarrow{\varphi} \mathcal{U}_{\mathrm{Dyck}^m}(\mathrm{Prim}(A)),$$

where ϵ is map of 2.4.3 and $\varphi(y_1 \otimes \ldots \otimes y_n) = w_{\succ}(y_1, \ldots, y_n)$. Since $\mathrm{Prim}(\mathcal{U}_{\mathrm{Dyck}^m}(\mathrm{Prim}(A))) = \mathrm{Prim}(A)$ by the first part, any element of $\mathcal{U}_{\mathrm{Dyck}^m}(\mathrm{Prim}(A))$ can be written as a sum of elements of the form $w_{\succ}(y_1, \ldots, y_n)$ for $y_1, \ldots, y_n \in \mathrm{Prim}(A)$ and this is also true for A. It is now easy to see that the composition $\varphi \circ \epsilon$ is an inverse to the canonical map. $\square$

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