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Copias de Gribov y Ecuación de Gap en Presencia de Potencial Químico para la Teoría de Yang-Mills $SU(2)$

Gribov Copies and Gap Equation in Presence of Chemical Potential for $SU(2)$ Yang-Mills Theory

Tesis para optar al grado de Magíster en Ciencias con Mención en Física por:

DIEGO ROBERTO HIDALGO TECAY

Profesor Guía: Dr. Fabrizio Canfora Tartaglia
Laboratorio de Física Teórica
Centro de Estudios Científicos (CECs)

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Director de Tesis	:	Dr. Fabrizio Canfora Tartaglia
Codirector de Tesis	:	Dr. Fernando Izaurieta Aranda
Comisión	:	Dr. Julio Oliva Zapata

A mi familia y Cristina...

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Resumen

En esta tesis hacemos uso del Método de Campo de Fondo (MCF) para encontrar la ecuación de las copias de Gribov en el caso de $SU(2)$, en la presencia de potencial químico y temperatura. También es analizada la ecuación de gap para el parámetro de Gribov para la teoría de Yang-Mills $SU(2)$ en presencia de potencial químico. Las soluciones numéricas de esta ecuación determinan cómo el parámetro de Gribov depende del potencial químico, y por lo tanto también la estructura no-perturbativa del propagador del gluón en la aproximación semi-clásica.

Abstract

In this thesis, we make use of the Background Field Method (BFM) to find the equation for Gribov's copies in the $SU(2)$ case, in the presence of chemical potential and temperature. Also the gap equation for the Gribov's parameter for $SU(2)$ Yang-Mills theory in presence of chemical potential and temperature is analyzed. The numerical solutions of this equation determine how the Gribov's parameter depends on the chemical potential, and therefore also the non-perturbative structure of the gluon propagator in the semi-classical approximation.

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Chapter 1

Introduction

From earliest times, the physicists have had the tendency to unify the theories that explain the workings of nature. Apparently, this natural inclination is generated from events that have occurred along the history of Physics, just as it was for example, the unification of terrestrial and celestial gravity, the unification of electricity and magnetism, or more recently the unification of electromagnetism and weak nuclear interactions. It is well known nowadays that there are four fundamental forces that govern the Universe: gravity, electromagnetism and, strong and weak interactions. Unfortunately, there is no concrete way to unify gravity with the other three forces. Despite this, one of the things that these forces have in common is the existence of symmetries. The symmetries provide, to some extent, the structure and coherence to the laws of nature just as the laws of nature do on a set of events. They play an important role in Physics for the understanding in depth the essence of the problem. Therefore, from this common quality that the four fundamental forces have, it is believed that the possible unification towards a full theory would be through a single principle, *the gauge principle*.

The symmetries and gauge principle are very well represented by the known gauge theories. These theories are invariant under certain local symmetry transformations, which maintain unaltered the physics of the problem. The gauge theories gave a description to all the fundamental forces along with their interactions. For example has been found that gravity has as a mediator particle, the graviton of spin 2, for the electromagnetism the photon and, the gluon and W and Z bosons for the strong and electroweak interactions, respectively. It is important to remark that the gauge principle introduces all the radiation fields, which are responsible to explain the interactions of the particles in a natural way.

The appearance of the gauge principle arises basically of Hermann Weyl's work which had as main consequence to elevate the grade of the gauge invariance, from a symmetry to a fundamental principle of Physics. This idea took about sixty years to consolidate, beginning with the Einstein's theory of gravitation, and ending with the discovery of the strong interactions described by the Yang-Mills gauge theory. In the electroweak case, it was established by the Glashow-Salam-Weinberg model with a gauge group $G = SU(2) \times U(1)$. On the other hand, for the strong interaction, the necessity to understand it like a gauge theory, gave the remarkable result of a new phenomenon: the asymptotic freedom. This phenomenon pointed out that at short distances quarks and gluons behave as free particles.
From the experiments of 60s, and the discovery of asymptotic freedom in the quarks, the theoretical physicists could develop strong interactions as a gauge theory invariant under a non-abelian group, $G = SU(3)$. This gauge theory for strong interactions is known as Quantum Chromodynamics or

QCD.

Quantum Chromodynamics is a theory that describes strong interactions by field called quarks, with spin 1/2, and with the mediators known as gluons, with spin 1. In particular, in this thesis we only worry about the part that describes the Yang-Mills field of QCD, i.e., the gluonic part.
The QCD gluonic part is very similar to the photonic part in Quantum Electrodynamics, since both describe (in principle) massless particles. However, the big difference that exists between them is on the QCD gluonic part, because unlike photons (observable particles), the gluons (also the quarks) have never been observed in an isolated form. This phenomenon is known as *confinement*. It will, in the second part of this thesis, be the central topic.
Although Yang-Mills theory managed to explain certain things about strong interactions, it left many open problems: in particular confinement. Many of the works involved in the understanding of the confinement problem, elucidate the solutions from different perspectives, but with certain things in common. All of these approaches try to capture, in theoretical way, the possible solutions in relation to the observed world in the experiments. In this sense, this thesis will be based on how the insertion of the Yang-Mills part in the QCD action can determines possible signals about the confinement. It leads us directly to the quantization of the Yang-Mills theory. In this framework, we will try to understand if the residual gauge degrees of freedom left after the gauge fixing (when we perform the quantization), could play a important role for the understanding of the infrared behaviour in Yang-Mills theory. It is worth to mention, that the first one to realize the importance of this issue was V. Gribov [1] (his approach was refined later by Zwanziger [2]).

On the other hand, it is important remark that the thermodynamics of QCD nowadays is an active area of research. The nature of the phase transitions caused by strong quantum corrections, finite temperature and density effects, have direct implications in the physical world, from high energy physics to the dynamics of how the universe is cooled and expanded after the Big Bang. In particular, gauge theories with non-zero temperature and chemical potential are of interest in plasma physics, for quark-gluon plasma [3, 4], the color superconductivity in QCD [5], astrophysics [6], and cosmology [7, 8]. In this direction, the first person that considered different thermal relativistic non-abelian field theories in presence of chemical potentials was Kapusta in Ref.[9]. In this work, the chemical potentials are related to the (Noether) conserved charges, which (after an integration over momenta) can be interpreted as background fields for the temporal components of the gauge fields. Although the relation between the temporal-components of the gauge fields and chemical potentials can be obtained by a straightforward calculation, there is an argument purely thermodynamic, which suggests that the chemical potentials can be associated to the expectation values of the temporal components of the fields. For these reasons, in order to devise an approach where the chemical potential is able to distinguish between confinement and deconfinement phase in the Gribov-Zwanziger approach, we make use of the Background Field Method (BFM). This method is inserted in the Chapter 5. The consequences of BFM in the first part of this thesis on the finite gauge transformations for the GZ action throw a key vision about how the degrees of freedom are affected. On the other hand, the dynamical part of the thesis will show the effects of chemical potential in the gap equation of Gribov-Zwanziger approach, and subsequently to the gluon propagator.
Thus, a kinematic and dynamic part will be the basic structure of this thesis. The first part (the kinematic part) corresponds to the study of the gauge symmetry with non-zero chemical potential of the Yang-Mills theory. In the second part we will focus to the determination of the Gribov parameter, and how it is affected by the presence of chemical potential.
More precisely, the thesis is structured in the following way: the first four chapters correspond to introductory chapters about the theme what we will build; in particular, for the second chapter we

will develop the conventions and properties of the Yang-Mills theory, next we develop the quantization of it. We will analyse briefly the infrarred behaviors of the Yang-Mills theory, and some possible confinement scenarios for the gluon as well as the gauge fixation. We will utilize the usual quantization method for non-abelian gauge theories: the so-known Faddeev-Popov procedure. By a particular gauge choice, and in order to perform the quantization of the theory, non-physical extra fields are necessary.

The Gribov's view on the Faddeev-Popov procedure involved new changes in the theory, in order to avoid the overcounting of the degrees of freedom. The Gribov's solution, and subsequently all the experimental data in accordance to it, will be analyzed. In particular, the infrarred behaviors of the Yang-Mills theory together with the Gribov's ideas will give a description of the confinement problem. It will be seen in Chapter 3.

Later on, Zwanziger [2] by a thorough study about to the Gribov's idea managed to put it within a framework of a renormalizable theory. Due to this last, a new symmetry and extra fields are necessary, which will be explicated in Chapter 4.

The second part of the thesis is dedicated to the dynamic study of the Gribov-Zwanziger theory, in particular to the analysis of the gluon propagator. The standard way on how one must insert chemical potentials in abelian and non-abelian gauge theories will be seen in Chapter 5.

Finally, I will present an appropriate procedure to insert these chemical potentials like background fields. The Gribov mass parameter that appears in the gluon propagator of the Gribov-Zwanziger model, will be determined in presence of this chemical potential by means original regularization methods, explicated in Chapter 6. We will obtain the gap equation that give us as solution the Gribov mass parameter.

Chapter 2

Yang-Mills Theory and its Infrared Behaviour

QCD is the theory of the strong interactions, and it possesses two parts: the spinorial part, describing particles with half-integer spin, and the gauge term, describing the gluons with 1-spin. The Yang-Mills theory describes the second part, i.e., the gluonic part.
Due to the existence of a gauge symmetry in Yang-Mills theory, we should select only one representative among all the physically equivalent gauge field configurations to avoid any overcounting of degrees of freedom. In Sect. 2.3, we will describe the standard procedure for gauge fixing in perturbation theory as well as in non-perturbative regime.
The infrared behaviour (IR) of the gluons correlation functions are investigated in Sect. 2.3 by means the usual relativistic invariant quantization method elaborated by Faddeev and Popov [10]. Next, we will present the Gribov's idea about the gauge fixing in the Faddeev-Popov quantization, and how he showed that for a non-abelian gauge theory (for example $SU(2)$) there are distinct equivalent transverse configurations satisfying the same gauge condition. In the final section of this chapter we will shortly review the relation between IR behaviours of propagators and confinement, and some confinement scenarios. In particular, one of them latter will be developed in Chapter 4.

2.1 The action of Yang-Mills theory

The Maxwell's electromagnetism theory was the first gauge theory invariant under an abelian group, the known $U(1)$ group. Some years later, physicists as Feynman, Dyson, Schwinger and Tomonaga developed the quantum version of this theory. This gave birth to the Quantum Electrodynamics or QED, which gives a correctly accurate account between electromagnetic fields and forces. This theory predicted new splitting of energy levels, and scattering process between photons and electrons. In 1954, in order to have a better understanding of the strong interactions in the atomic nucleus, the physicists Chen Yang and Robert Mills extended the concept on a gauge theory invariant under an abelian group to the case of a gauge theory invariant under a non-abelian group [11]. Nowadays, this models are known as *non-abelian gauge theories*.

The fundamental property in common between Maxwell and Yang-Mills equations is that both provide a classical description of massless waves travelling at the speed of light. In the 50s, it was hard understand whether a non-abelian gauge theory could describe other forces in the nature. The QCD theory emerged by a series of experimental and theoretical discoveries made in the 1960s and

1970s, involving exclusively the strong interactions. We will see how the non-abelian Yang-Mills action works in the description of QCD theory.

The Lagrangian density of QCD is

$$\mathcal{L}_{QCD} = \bar{\psi}(-\not{D} + m)\psi + \mathcal{L}_{YM}, \tag{2.1}$$

where $\not{D} = \gamma^\rho D_\rho$, with γ^ρ the four-dimensional Pauli matrices. Here, the quarks are in the fundamental representation of the gauge group while the gauge potential in the adjoint representation (see Appendix A for more details about these representations). The second term corresponds to the Yang-Mills Lagrangian defined by

$$\mathcal{L}_{YM} = \frac{1}{2g^2}\mathrm{Tr}F \wedge *F = \frac{1}{4g^2}\delta_{ab}F^a_{\rho\sigma}F^{\rho\sigma b} \tag{2.2}$$

where δ_{ab} corresponds to the Killing metric on the $SU(N)$ algebra, such that

$$\mathrm{Tr}\,(T_a T_b) = \frac{\delta_{ab}}{2}, \tag{2.3}$$

and with g a coupling constant. The first term in $\mathcal{L}_{QCD}$ describes the propagation of the massive quarks ψ and its interactions with gluons by means the covariant derivative D_ρ,

$$D_\rho = \partial_\rho(\cdot) + ig[A_\rho, \cdot]. \tag{2.4}$$

The term $\mathcal{L}_{YM}$ constitutes the gluonic part of QCD which contains the propagation of the gluons and its self-interactions. In some sense, QCD appears as an expanded version of QED, but with some differences (for more details see Appendix C).
Moreover, continuing with the study of Yang-Mills action, the symbol "Tr" in Eq. (2.2) denotes an invariant quadratic form on the Lie algebra of $SU(N)$, "$\wedge$" denotes the product of differential forms, and "*" represents to the Hodge operator. [1]

The two-form F in Eq.(2.2) can be defined considering to T_a as the hermitian generators which form a Lie algebra of the form

$$[T_a, T_b] = f^c_{ab}T_c \tag{2.6}$$

where f^c_{ab} are the structure constants of $SU(N)$ group. Therefore, the components of the curvature F, to say $F^a_{\rho\sigma}$, are defined as

$$F^a_{\rho\sigma} = \partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + ig[A^a_\rho, A^b_\sigma]. \tag{2.7}$$

The gauge field A_ρ lives in an algebra defined by the hermitian generators T_a. Therefore, the decomposition of the gauge field is

$$A_\rho = A^a_\rho T_a, \tag{2.8}$$

and similarly for the field strength tensor:

$$F^a_{\rho\sigma} = \partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + igf^{abc}A^b_\rho A^c_\sigma, \tag{2.9}$$

[1] The Hodge operator is an invertible map of type

$$* : \Omega^p(M^{(d)} \to \Omega^{d-p}(M^{(d)}) \tag{2.5}$$

where p is the range of the differential form and d is the dimension of the manifold Ω.

where we have used the relation (2.6). The finite transformation of the gluon field under which the YM action (2.2) is invariant, is given by

$$A_\rho(x) \to A^U_\rho(x) \equiv U(x)A_\rho(x)U^{-1}(x) + (\partial_\rho U(x))U^{-1}(x), \tag{2.10}$$

where $U(x)$ is

$$U(x) = e^{ig\omega(x)}, \tag{2.11}$$

and $\omega(x)$ a Lie algebra valued gauge parameter: $\omega(x) = \omega(x)^a T_a$. This kind of transformations are known as *gauge transformations*, and their infinitesimal version reads

$$\delta A^a_\rho = -\partial_\rho \omega^a_\rho - g f^{abc}\omega^b A^c_\rho = -D^{ab}_\rho \omega^a \tag{2.12}$$

where the covariant derivative D^{ab}_ρ, in the adjoint representation, is defined as

$$D^{ab}_\rho = \delta^{ab}\partial_\rho + g f^{abc} A^c_\rho. \tag{2.13}$$

As we mentioned in the introduction, the strong interaction is governed by the compact simple Lie group $SU(3)$, but during this thesis only the $SU(2)$ case will be considered to explain key features of our problem.

If we vary the action of the theory associated to $\mathcal{L}_{YM}$, we obtain the following field equations:

$$(D_\rho)^a_b F^{\rho\sigma b} = 0. \tag{2.14}$$

On the other hand, the two-form $F^a_{\rho\sigma}$ also satisfies a Bianchi identity

$$(D_\rho F_{\sigma\delta})^a + (D_\delta F_{\rho\sigma})^a + (D_\sigma F_{\delta\rho})^a = 0. \tag{2.15}$$

which is equivalent to the Jacobi identity

$$[D_\rho, [D_\sigma, D_\delta]] + [D_\delta, [D_\rho, D_\sigma]] + [D_\sigma, [D_\delta, D_\sigma]] = 0 \tag{2.16}$$

since $[D_\rho, F^a_{\rho\delta}] = D_\rho F^a_{\sigma\delta}$. Also very useful is to define the strength tensor $\tilde{F}^{\rho\sigma} = \frac{1}{2}\epsilon^{\rho\sigma\delta\gamma}F_{\delta\gamma}$, then the Bianchi identity (2.15) can be rewritten as

$$D_\rho \tilde{F}^{\rho\sigma} = 0. \tag{2.17}$$

Is noteworthy that the field equations in Eq.(2.14) are non-linear- in contrast to the Maxwell equations, but have certain properties in common with them. Like the Eintein equations, few exact solutions are known. (See for example Ref. [12]).

2.2 Path integral quantization of the Yang-Mills theory

The path integrals provide an alternative to the canonical quantization of complicated gauge theories, and it also gives a direct route to the study regimes where perturbation theory is either inadequate or is not well-defined.
A brief topic and systematic details on the functional integral can be seen in Appendix B.

The perturbative quantization problem, was solved locally by L. Faddeev and A. Popov [10]. Their method of quantization starts from the functional integral [2]

$$Z = \int \mathcal{D}A_\rho e^{-\int \mathcal{L}_{YM} d^4x} \tag{2.18}$$

in Euclidean space-time and the integral run over all gauge potential A_ρ including those which are related by the gauge transformation (2.10). Due to this last, the integral diverges, and hence some sort of restriction is necessary. This restriction exists and is known as the gauge fixing condition.
In the following section, we will describe the Faddeev-Popov technique for the particular Landau gauge condition.

2.3 Fixing the gauge

The source of problem in quantizing the Yang-Mills action is due to that one integrates, in the usual Feynman path integral, over equivalent gauge field configurations. But obviously we only want to take into account gauge fields physically different. Due to the symmetry transformation in Eq. (2.10), there are many redundant degrees of freedom, which prevent sensible computations. In particular, the configuration space $\mathbb{A}$ for a gauge theory is made of different equivalence classes which each class correspond to a different physical state. Thus, the "physical" configuration space, $\mathbb{A}_{phys}$, is given by

$$\mathbb{A}_{phys} = \mathbb{A}/G, \tag{2.19}$$

where G is the gauge group (in our case $SU(2)$ group). $\mathbb{A}/G$ means that gauge fields which are related by a gauge transformation must be identified. The process to obtain the physical configuration space from $\mathbb{A}$ is called "gauge fixing" and consist in imposing some constraints or "gauge conditions" over the gauge fields, or sometimes only over certain components of it.
It is useful to introduce here the notion of a *gauge orbit*, which is the set of all the field configurations connected by gauge transformations

$$O[A] := \{A'_\rho | \quad A'_\rho = A^U_\rho\} \tag{2.20}$$

and where A^U_ρ is given in Eq. (2.10). Thus, if one choose a gauge condition $G^a[A_\rho] \equiv 0$, it should fix completely the gauge. In the present thesis, we will only consider the covariant Landau gauge condition, i.e., $G^a[A_\rho] = \partial^\rho A^a_\rho = 0$. Even though the gauge fixing condition guarantees that the gauge is locally fixed, it does not imply that it is globally fixed. The gauge fixing condition works as follows. The gauge orbits can be thought a foliation of the functional space of the gauge field A_ρ. We want to select one and only one representative for each gauge orbit. Thus, the gauge fixing condition, defined by $G^a[A_\rho] = \partial_\rho A^a_\rho = 0$, should intersect every gauge orbit once and only once (see Figure 2.1).

[2] Here we will use the natural units which $c = \hbar = 1$. For this reason, the fields in $d = 4$ has mass units for that exponential function it is well defined.

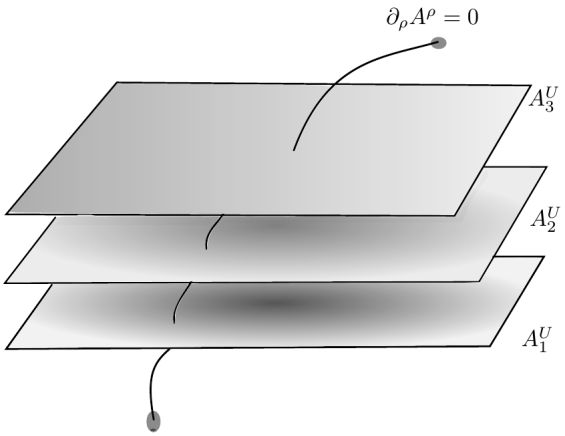


Figure 2.1: Ideal situation: Landau gauge condition intersecting once and only once the gauge orbits in the functional space.

Apparently, up to here the gauge fixing process is quite right. In particular, in the usual commutative QED the gauge fixing condition $\partial^\rho A_\rho = 0$ fixes the gauge completely. In other words, if A_ρ satisfies the Landau gauge fixing condition, the transformed field under the transformation (2.10) does not

$$\partial^\rho \bar{A}_\rho[\omega] \neq 0, \quad \omega \neq 0, \tag{2.21}$$

where ω is some function of x considered as a parameter of the transformation. One can conclude that the equation (2.21) has only the trivial solution $\omega = 0$. A recent work in non-commutative QED [13] explains that the zero mode equation (2.21) can exhibit non-trivial solutions.
However, it may happens that gauge fixing conditions intersect more than once the gauge orbits, and consequently the gauge fixing is unable to completely eliminate the overcounting of the degrees of freedom; this is a typical phenomenon in non-abelian gauge theories, and is known as *Gribov ambiguity* [1]. Due to the Gribov ambiguity, gauge fixing conditions, for example the Landau gauge fixing $\partial^\rho A_\rho = 0$, needs to be improved at non-perturbative level.

In order to have a better understanding about this overcounting in the degrees of freedom, let us start with a simple case of a Klein-Gordon field $\phi(x)$ with mass m, which its dynamics is described by the action functional

$$\begin{aligned} S_{KG} &= \int d^4x \left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{m^2}{2}\phi^2 \right) \\ &= \int d^4x \left(-\frac{1}{2}\phi(\Box + m^2)\phi \right) \end{aligned} \tag{2.22}$$

where we have integrated by parts in the last step and neglected boundary terms. Our goal is try to perform the path integral for the partition function associated to S_{KG} in presence of an external source $J(x)$:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\left(i \int_{-\infty}^{+\infty} d^4x \left[\mathcal{L}_{KG} + J(x)\phi(x) + \frac{i}{2}\epsilon\phi^2(x) \right] \right) \\ &= \int \mathcal{D}\phi \exp\left(-i \int d^4x d^4y \left[\frac{1}{2}\phi(x)(\Box(x) + m^2 - i\epsilon)\delta(x-y)\phi(y) + i\int d^4x J(x)\phi(x) \right] \right) \end{aligned}$$

where we have introduced a damping term proportional to $\epsilon > 0$. Realizing a Gaussian integration, we obtain

$$Z[J] = \frac{1}{\sqrt{i\det(\Box + m^2 - i\epsilon)}} \exp\left(\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4xd^4y\right). \tag{2.23}$$

Here Δ_F denotes the Feynman propagator and corresponds to the inverse of the operator $(\Box+m^2-i\epsilon)$ that satisfies the Green's equation:

$$(\Box(x) + m^2 - i\epsilon)\Delta_F(x-y) = \delta(x-y). \tag{2.24}$$

It is direct show that the operator $(\Box(x)+m^2-i\epsilon)$ has inverse, and therefore it is possible determine the Feynman propagator Δ_F.

Now, let's see what happens in the Yang-Mills case. In fact, the quadratic part of Yang-Mills action is give by

$$S_{YM} = \int d^4x \left(\frac{1}{2}A^\rho_a \left(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma\right) A^\sigma_a\right). \tag{2.25}$$

Here, we need the inverse of the operator $(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)$, and so be able to find the Feynman propagator which satisfies

$$(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)D^{\rho\sigma}_{ab}(x-y) = \delta_{ab}\delta^{(4)}(x-y). \tag{2.26}$$

Unfortunately this operator has no inverse. If the operator $(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)$ would be invertible, then the equation below

$$(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)X^\sigma = 0, \tag{2.27}$$

would only have the trivial solution $X^\sigma \equiv 0$. But the simple choose $X^\sigma = \partial^\sigma\Lambda$ also satisfies the equation (2.27). This proves that $(g_{\rho\sigma}\Box - \partial_\rho\partial_\sigma)$ is not invertible, and therefore we cannot define the Feynman propagator.

This zero mode arises due to the gauge symmetry. Thus, we must require the insertion of a gauge fixing condition through the path integration such as proposed by Faddeev and Popov.

In order to have a deep understanding about this problem, we start showing a simple example of the ordinary integration. Let us consider the integral

$$I := \int\int dxdye^{-(x^2+y^2)} \tag{2.28}$$

which is invariant under the usual two-dimensional rotation (the $SO(2)$ symmetry). Using polar coordinates, we find

$$I = \boxed{\int d\theta} \int drre^{-r^2} \tag{2.29}$$

where the redundant factor in the box due to the rotational symmetry is factored out. In the Yang-Mills path integral there is a similar redundancy which must be eliminated.

We note that (2.29) can be written as,

$$I = \int d\theta \int drr \int d\theta e^{-r^2}\delta(\theta), \tag{2.30}$$

where we have choose the path $\theta = 0$. Instead of doing that, we consider the generalized non-zero θ path, for example

$$y\cos\theta = x\sin\theta. \tag{2.31}$$

To apply it to the integration (2.30), the path function is chosen as

$$f(\theta) = y\cos\theta - x\sin\theta = 0. \tag{2.32}$$

Now, from the property of the Dirac-delta one has

$$\delta(f(\theta)) = \sum_j \frac{1}{\left|\frac{df(\theta_j)}{d\theta}\right|}\delta(\theta - \theta_j), \quad f(\theta_j) = 0. \tag{2.33}$$

In our case, we have

$$\int d\theta\delta(f(\theta)) = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2}}. \tag{2.34}$$

To compensate the additional factor, we define the new function $\Delta(r)$ such that

$$\Delta(r)\int \delta(f(\theta))d\theta = 1, \quad \Delta(\sqrt{x^2+y^2}) = \frac{\sqrt{x^2+y^2}}{2}. \tag{2.35}$$

Then, the function f is the simple rotation operation,

$$y' = y\cos\theta - x\sin\theta, \quad x' = x\cos\theta + y\sin\theta. \tag{2.36}$$

From this last, the equality (2.35) is rewritten as

$$\Delta(\sqrt{x'^2+y'^2})\int \delta(f(y'))d\theta = 1. \tag{2.37}$$

This equation seem to be the identity. Hence, (2.30) becomes

$$I = \boxed{\int d\theta}\int dx'dy'e^{-(x'^2+y'^2)}\Delta(\sqrt{x'^2+y'^2})\delta(y'). \tag{2.38}$$

We observe that Δ plays a role to isolate the volume factor of each gauge orbit. We can calculate this factor explicitly as

$$(\Delta(r))^{-1} = \int \delta(f(\theta))d\theta = \int \delta(f(\theta))\left|\det\left(\frac{d\theta}{df}\right)\right|df = \left|\det\left(\frac{d\theta}{df}\right)\right|\Big|_{f=0}. \tag{2.39}$$

Thus,

$$\Delta(r) = \left|\det\left(\frac{df}{d\theta}\right)\right|\Big|_{f=0}. \tag{2.40}$$

Now, our goal is use a similar method to remove the redundant gauge symmetry over the Yang-Mills action, applying the same previous computations for gauge fields.

2.3.1 The Faddeev-Popov operator

The Faddeev and Popov idea is similar to the previous computation. In order to avoid the over-counting of the degrees of freedom, we consider the gauge condition $f[A] = 0$ (here the fields A are functions of spacetime coordinates), which can be inserted in the path integral by a delta functional $\delta(f[A])$. Thus, we need that the functional Jacobian of the gauge variation of $f[A]$ to be non-degenerate.

Let U be the (finite) gauge parameter, and A^U be the transformed field configuration associated to A. One can interpret this transformation as two points of one equivalence class, which are always connected by U. The Jacobian factor of this gauge transformation U is

$$\mathcal{J}[A,U] = \left|\det \frac{\delta f[A^U]}{\delta U}\right| \tag{2.41}$$

where $|\cdot|$ denotes the absolute value. Formally, we may insert

$$1 = \int \mathcal{D}U \delta(f[A^U]) \left|\det \frac{\delta f[A^U]}{\delta U}\right| \tag{2.42}$$

into (2.18), and then write the path integral as

$$Z = \int \mathcal{D}A\mathcal{D}U\delta(f[A^U]) \left|\det \frac{\delta f[A^U]}{\delta U}\right| e^{-\int \mathcal{L}_{YM} d^4x}.$$

We now make the change of variables

$$A^U_\rho \to A^{U'}_\rho \tag{2.43}$$

where $A^{U'}_\rho$ is other arbitrary gauge transformation, and then the partition function reads:

$$\begin{aligned} Z &= \int \mathcal{D}U\mathcal{D}A^{U'} \delta(f[A^{U\circ U'}]) \left|\det \frac{\delta f[A^{U\circ U'}]}{\delta U}\right| e^{-\int \mathcal{L}_{YM}[A^{U'}] d^4x} \\ &= \int \mathcal{D}U\mathcal{D}A^{U'} \delta(f[A^{U\circ U'}]) \left|\det \frac{\delta f[A^{U\circ U'}]}{\delta (U\circ U')}\right| \left|\det \frac{\delta (U\circ U')}{\delta U}\right| e^{-\int \mathcal{L}_{YM}[A^{U'}] d^4x}, \end{aligned} \tag{2.44}$$

Here $U \circ U'$ denotes the composition of the two gauge transformations. We now choose $U' = U^{-1}$, and we use the gauge invariance of the Lagrangian $\mathcal{L}_{YM}[A]$ and of the measure $\mathcal{D}A_\rho$ to write the partition function as

$$Z = \left[\int \mathcal{D}U\right] \int \mathcal{D}A \left|\det \frac{\delta f[A^U]}{\delta U}\right|_{U=0} \left| \delta(f[A]) e^{-\int \mathcal{L}_{YM}[A] d^4x}. \right. \tag{2.45}$$

The Jacobian factor

$$\Delta_{FP} \equiv \left|\det \frac{\delta f[A^U]}{\delta U}\right|_{U=0}\Bigg| \tag{2.46}$$

is known as the Faddeev-Popov determinant. On the other hand, the factor in brackets in Eq. (2.45) is the infinite constant

$$\int \mathcal{D}U = v(G)^V \tag{2.47}$$

where $v(G)$ is the volume of the gauge group and V is the infinite volume of space-time. As well as in Eq. (2.38), this infinite constant is nothing but the result of summing over gauge-equivalent states. Therefore, we were able in find the rule to fix the gauge by dividing out the infinite factor of the volume of the group element which comes directly of the insertion of det $\frac{\delta f[A^U]}{\delta U}$.
For example, we can calculate the quantity Δ_{FP} for $U(1)$ case, i.e., Maxwell's electromagnetic theory. In fact, if we choose the gauge $f[A_\rho] = \partial_\rho A^\rho$ and the gauge transformation $U = U(x) = e^{i\Lambda(x)}$, we get

$$f[A^U_\rho] = \partial_\rho(A^\rho + \partial^\rho\Lambda) = \partial_\rho A^\rho + \partial^2\Lambda. \tag{2.48}$$

Therefore,

$$\frac{\delta f(x)}{\delta\Lambda(y)} = \partial^2\delta(x-y). \tag{2.49}$$

Thus, for an abelian theory under the particular Landau gauge condition, the Faddeev-Popov determinant acquires the following form:

$$\Delta_{FP}[A_\rho] = \det\partial^2. \tag{2.50}$$

The peculiarity of be an independent non-vanishing constant of the fields A_ρ, it comes from the abelianility of the Maxwell's theory. It also occur in the case of axial gauge for non-abelian gauge theories. However, we see will that it be a non-trivial functional with the covariant Landau gauge fixing condition in non-abelian gauge theories.

For our case of a non-abelian gauge theory, in particular for the gauge transformation (2.12), our gauge condition it yields

$$f[(A^a)^U] = \partial_\rho(A^a_\rho)^U = \partial^\rho A^a_\rho - \partial_\rho D^{ab}_\rho\omega \tag{2.51}$$

where ω is defined by $U(x) = 1 + \omega(x)$. Thus, the Faddeev-Popov determinant reads

$$|\Delta_{FP}(x,y)| = \left|\det\frac{\delta f[A^U(x)]}{\delta\omega(y)}\right| = \left|\det\Big(\partial_\rho D^\rho\delta^{(4)}(x-y)\Big)\right|, \tag{2.52}$$

where D^ρ is the usual covariant derivative defined in Eq. (2.13), and $\delta^{(4)}(x-y)$ is the 4-dimensional Dirac-delta. Moreover, notice that it is an explicit function of the gauge field. To write Eq.(2.52) $\mathcal{D}U = \prod_a \mathcal{D}\omega^a$ have been assumed in expression (2.42). This can be done when dealing with infinitesimal gauge transformations, i.e., $U = 1 + \omega$. Thus, in the case $f_a[A^U] = 0$ there is a solution for U, and therefore the integration in (2.42) runs over an infinitesimal region around that solution.

A simple covariant Landau gauge choice for non-abelian gauge theory is

$$f_a[A] = \partial_\rho A^\rho_a. \tag{2.53}$$

Following 't Hooft, we now can make the following (also known as the 't Hooft trick). We note that physical quantities are independent of $f_a[A]$. Hence we can multiply the generating functional by a weight factor and integrate all f. Thus, the Gaussian form necessary for that be inserted in the partition function, it is defined as

$$F[f_a] := \int \mathcal{D}f\delta(f_a - g_a)e^{-\frac{i}{2\xi}\int d^4x g_a g_a} = e^{-\frac{i}{2\xi}(\partial_\rho A_\rho)^2}. \tag{2.54}$$

Therefore, the gauge fixed Lagrangian is

$$\mathcal{L} = \mathcal{L}_{YM} - \frac{1}{2\xi}(\partial_\rho A^\rho_a)^2, \tag{2.55}$$

where ξ is known as the gauge fixing parameter. Also one can see that the gauge fixing term only modifies the kinetic term of the Lagrangian, because is quadratic in the gauge fields.

On the other hand, since the quantity $\Delta_{FP}[A_\rho]$ is a determinant, it can be written as a path integral over a set of fermionic *ghost* fields, denoted by $c_\alpha(x)$ and $\bar{c}_\alpha(x)$, one per generator:

$$\Delta_{FP} = \det \frac{\delta f^\alpha[A^U]}{\delta U^\beta} = \int \prod_\alpha d\bar{c}_\alpha \prod_\beta dc^\beta \exp\left(i\bar{c}_\alpha \frac{\delta f^\alpha[A^U]}{\delta U^\beta}\bigg|_{U=0} c^\beta \right). \tag{2.56}$$

Here α, β are indices labelling the gauge fixing condition and the gauge parameter at all spacetime points (appropriately discrete) or in momenta space, all momenta. We must note that the absolute value is neglected in order to introduce the $c, \bar{c}$-variables. Thus, one is implicitly assuming that the Faddeev-Popov determinant does not changes sign.

Therefore, the continuum version of the Grassmannian integral on the RHS of the partition function is

$$\Delta Z = \int \mathcal{D}\bar{c}\mathcal{D}c e^{-S_{gh}[A,\bar{c},c]}. \tag{2.57}$$

where the "ghost" action S_{gh} is obtained by linearising $f[A^U]$ with respect to the gauge parameter U, replace the infinitesimal gauge parameter U_a by c_a, then multiply by $\bar{c}$ and integrate over the space-time. c and $\bar{c}$ are Grassmannian scalar fields, and are referred as *Faddeev-Popov ghosts*. They do not appear in asymptotic states, their role only is played through coupling to the gauge fields (and matter fields if they enter the gauge fixing condition), and enter loop diagrams.

In particular, the ghost action associated with the Landau covariant gauge fixing function $\partial_\rho A^\rho_a = 0$ is

$$\begin{aligned} S_{gh} &= \int d^4x \bar{c}_a \partial_\rho (\partial^\rho c_a + f_{abc} A^\rho_b c_c) \\ &= -\int d^4x (\partial_\rho \bar{c}_a \partial^\rho c_a + f_{abc} \partial_\rho \bar{c}_a A^\rho_b c_c). \end{aligned} \tag{2.58}$$

We see that c, $\bar{c}$ has the same propagator as massless scalar fields (up to the overall sign which is a matter of convention), but couple to A^a_ρ through a cubic vertex only (unlike the minimally coupled scalar). Furthermore, since c, $\bar{c}$ are Grassmannian ghost fields, each ghost loop comes with a minus sign, as in the Feynman rules for fermions. These "particles" do not satisfy the general conditions for causality and unitarity, therefore these cannot create physical states.

In summary, the full form of the Yang-Mills partition function in a Feynman-'t Hooft covariant gauge, is given by

$$Z = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c} e^{i\int d^D x \mathcal{L}_{YM}[A,c,\bar{c}]} \tag{2.59}$$

where the new effective Lagrangian density is

$$\mathcal{L}_{YM}[A,c,\bar{c}] = -\frac{1}{4g^2} F^a_{\rho\sigma} F^{\rho\sigma a} - \frac{1}{2\xi}(\partial_\rho A^\rho_a)^2 + \bar{c}^a \partial_\rho D^{ab}_\rho c^b. \tag{2.60}$$

As wee this pure gauge theory is non-linear. It will have relevancy in the perturbative and non-perturbative aspects of Yang-Mills gauge theories.

In summary, the prescription for the functional integral in a gauge theory consist in to choose a gauge fixing, and then construct $\mathcal{L}_{gf}$ by adding the f^2-term to the classical Lagrangian density and also adding the Faddeev-Popov ghost term.[3] Now, a "good" gauge fixing must produce a hyperplane which intersects each gauge orbit transversally once and only once. When the gauge fixing surface intersect more than once, then the integration over all points on the chosen surface leads to overcounting of the degrees of freedom. However, the overcounting of the intersections in the gauge orbits is unavoidable for all smooth gauge-fixings in a non-abelian gauge theory or restrictions on the integration in the path integral, as will be discussed in more detail in the following section.

2.4 Replacing gauge symmetry with the BRST symmetry

The full Lagrangian density defined in Eq. (2.60), was fixed to a hyperplane in the field configuration space. Due to the insertion of the ghost fields, a new symmetry emerges. This new Lagrangian density, with gauge fixing and ghost terms, develops a global fermionic symmetry which, in some sense, remembers the gauge invariance of the original theory. It is known as *BRST symmetry*, and in this section we will see how works.

2.4.1 BRST symmetry

The BRST (Becchi-Rouet-Stora-Tyutin) symmetry [14, 15, 16], arises when the gauge fixing and the ghost Lagrangian densities have been added to the original gauge invariant Yang-Mills Lagrangian density. It is very useful to prove the renormalizability and unitarity of a theory, see for example Refs. [17], [22].
The corresponding transformations of BRST symmetry can be derived from the standard gauge transformations by replacing the gauge parameter ω by a ghost field c. It is a convenient way to introduce the gauge fixing condition via an auxiliary Lie-algebra valued field b as

$$\int \mathcal{D}b^a e^{-\int dx(ib^a f^a[A])} = N\delta(f^a[A]), \tag{2.61}$$

where we used the integral representation for the Dirac delta $\delta(f^a[A])$. The field b^a is known as "Hubbard-Stratonovich field" and, it works as an auxiliary field which has no dynamics itself, and transforming as a vector in the adjoint representation of some G group. Furthermore, it will be useful to express the Faddeev-Popov determinant as an integral over Grassmann variables, i.e.,

$$\det(i\mathcal{M}^a_b) = \det(i\mathbf{1})\det(\mathcal{M}^a_b) = \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(-i\int d^4x d^4y \bar{c}_a(x)\mathcal{M}^{ab}(x,y)c_b(y)\right).$$

$$\Rightarrow \quad \det(\mathcal{M}^a_b) \sim \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(i\int d^4x \bar{c}_a \partial^\rho D^{ab}_\rho c_b\right), \tag{2.62}$$

where $\bar{c}_a$ and c_b are the Faddeev-Popov ghosts. The factor $\det(i\mathbf{1})$ can be absorbed in a normalization factor N of Z, which do not affects the normalized correlation functions of the theory. It is important to emphasize that this procedure is valid when the Faddeev-Popov determinant is

[3] We note that f^2-term cannot be gauge invariant because in such case $\mathcal{M}^{ab}(x,y)$ will has zero modes, i.e., $\det\mathcal{M}=0$, and then $\mathcal{M}^{-1}$ does not exist.

not zero.
Thus, the generating functional in the Landau gauge can be written as

$$\begin{aligned} Z &= \int \mathcal{D}A_\rho \det(\mathcal{M}^a_b)\delta(f[A_\rho])e^{i\int d^4x\mathcal{L}} \\ &= \int \mathcal{D}A_\rho \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}b \exp\left(i\int d^4x(\mathcal{L}_{YM} - b^a\partial^\rho A^a_\rho\right) \\ &= \int \mathcal{D}A_\rho \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}b \exp\left(i\int d^4x\mathcal{L}_{eff}\right). \end{aligned} \tag{2.63}$$

where we have considered the QCD Lagrangian $\mathcal{L}$ in the Feynman-'tHooft covariant gauge. The effective Lagrangian density is defined by

$$\mathcal{L}_{eff} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma}_a - \frac{1}{2}b_a b_a + b_a\partial^\rho A^a_\rho + \bar{c}^a\partial^\rho D^{ab}_\rho c^b. \tag{2.64}$$

Due to that $\mathcal{L}_{eff}$ has been gauge fixed, then it does not posses the gauge invariance of the original theory. However, there is a global fermionic residual symmetry, reads

$$\delta A^a_\rho = -\epsilon(D_\rho)^a_b\eta^b, \tag{2.65}$$
$$\delta\psi = ig\epsilon\eta^a t^a\psi, \tag{2.66}$$
$$\delta\eta^a = -\frac{\epsilon}{2}f^{abc}\eta^b\eta^c, \tag{2.67}$$
$$\delta\bar{\eta}^a = \epsilon b^a, \tag{2.68}$$
$$\delta b^a = 0. \tag{2.69}$$

where ϵ is an anticommuting constant parameter. The transformations (2.65) and (2.66), are local gauge transformations as such leave invariant the first two terms of the effective Lagrangian of Eq. (2.64). The invariance of the fourth and fifth terms holds because the change of δA in the fourth term cancels against the change of $\bar{c}$ in the fifth term. Also the changes of the fields A_ρ and c in the fifth term of Eq.(2.64) cancel out. The third term of Eq. (2.64) is trivial.
A substantial property of the BRST symmetry is its nilpotency, $\delta^2 = 0$, off-shell. For this reason the gauge parameter ϵ is an anticommuting variable. The nilpotency property allows an easy way to fix the gauge without the need of a path integral [23]. This statement is based on the fact that it is possible add a quantity to Lagrangian that comes from a BRST transformation. It quantity usually is known as exact BRST quantity.
In addition, the total Lagrangian density $\mathcal{L}_{eff}$ is also invariant under the infinitesimal bosonic symmetry transformations

$$\delta'\eta^a = \epsilon\eta^a, \tag{2.70}$$
$$\delta'\bar{c}^a = -\epsilon\bar{\eta}^a \tag{2.71}$$

with all others fields remaining inert. Here ϵ represents a commuting infinitesimal parameter, and the generator of this symmetry transformation corresponds to the operator that counts the number of ghost fields. This is known as the ghost scaling symmetry of the theory. [4]
Furthermore, in the case of the Landau gauge, where $f[A] = \partial_\rho A_\rho$, this prescription to fix the gauge directly leads to the known gauge fixing terms:

$$\begin{aligned} \mathcal{L}_{gf} &= \delta(\bar{\eta}^a f^a[A]) = \delta(\bar{\eta}^a\partial_\rho A^a_\rho) = \\ &= ib^a(\partial_\rho A^a_\rho) - \bar{\eta}^a\partial_\rho(-D^{ab}_\rho\eta^b) = ib^a(\partial_\rho A^a_\rho) - \bar{\eta}^a\mathcal{M}^{ab}\eta^b. \end{aligned} \tag{2.72}$$

[4] The fact that these transformations are like scale transformations are in accordance with the particular hermiticity properties that the ghost and anti-ghost fields satisfy for a consistent covariant quantization of the theory.

The minus sign stems from the anti-commutativity property of ϵ and $\bar{\eta}$. This method can be used also for other gauges and allows the use of gauge fixing conditions depending on ghost fields [23].

2.4.2 The propagator

When the gauge fixing condition breaks the gauge invariance, it renders the theory non-singular make it possible to define the propagator of the theory and derive the Feynman rules for the non-abelian gauge theory. In fact, we return to the full Lagrangian density obtained after of the gauge fixing

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma a} - \frac{1}{2\xi}(\partial^\rho A^a_\rho)^2 + \bar{c}^a\partial^\rho D_\rho c^a. \tag{2.73}$$

In order to derive the propagators, we only take care of the quadratic part of $\mathcal{L}_{YM}$, i.e.,

$$\begin{aligned}
\mathcal{L}^{quad}_{YM} &= -\frac{1}{4}(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho)(\partial^\rho A^{\sigma a} - \partial^\sigma A^{\rho a}) - \frac{1}{2\xi}(\partial_\rho A^a_\rho)^2 + \partial^\rho\bar{c}^a\partial_\rho c^a, \\
&= \frac{1}{2}A^a_\rho\left(g^{\rho\sigma}\Box - \left(1-\frac{1}{\xi}\right)\partial^\rho\partial^\sigma\right)A^a_\sigma - \bar{c}^a\Box c^a, \\
&= \frac{1}{2}A^a_\rho(\mathbb{O}^{\rho\sigma})^{ab}A^b_\sigma + \bar{c}^a\mathbb{M}^{ab}c^b,
\end{aligned} \tag{2.74}$$

where in the step last we used integration by parts, neglected boundary terms. The operators $\mathbb{O}$ and $\mathbb{M}$ are defined as

$$(\mathbb{O}^{\rho\sigma})^{ab} = \delta^{ab}\left(g^{\rho\sigma}\Box - \left(1-\frac{1}{\xi}\right)\partial^\rho\partial^\sigma\right), \tag{2.75}$$

$$\mathbb{M}^{ab} = -\delta^{ab}\Box. \tag{2.76}$$

These operators in the momentum space take the forms

$$(\mathbb{O}^{\rho\sigma})^{ab}(p) = -\delta^{ab}\left(g^{\rho\sigma}p^2 - \left(1-\frac{1}{\xi}\right)p^\rho p^\sigma\right), \tag{2.77}$$

$$\mathbb{M}^{ab}(p) = \delta^{ab}p^2. \tag{2.78}$$

The reason to go to momentum space is because the propagators for the gauge and ghost fields are the inverses of $\mathbb{O}$ and $\mathbb{M}$, respectively, and therefore are easily calculates in this space.

In fact, the inverse of $\mathbb{M}^{ab}(p)$ is (analogous to the propagator of a massless scalar field)

$$(\mathbb{M}^{ab})^{-1}(p) = \frac{\delta^{ab}}{p^2}. \tag{2.79}$$

On the other hand, the inverse of $\mathbb{O}$ can be calculated by the following parametrization:

$$(\mathbb{O}^{ab}_{\rho\sigma})^{-1}(p) = \delta^{ab}\left(a_0 g_{\rho\sigma} + a_1\frac{p_\rho p_\sigma}{p^2}\right) \tag{2.80}$$

where a_0, a_1 are arbitrary parameters must be determined. In order to determine these parameters, we make use of the identity that must satisfy $\mathbb{O}$ with its inverse $\mathbb{O}^{-1}$,

$$\mathbb{O}^{\rho\sigma ab}(p)(\mathbb{O}^{bc}_{\sigma\lambda})^{-1}(p) = \delta^{ac}\delta^\rho_\lambda \tag{2.81}$$

which later of a small algebra, we obtain

$$a_0 = -\frac{1}{p^2}, \quad a_1 = \frac{(1-\xi)}{p^2}. \tag{2.82}$$

So that we can write the inverse in (2.80) as

$$(\mathbb{O}^{ab}_{\rho\sigma})^{-1}(p) = -\frac{\delta_{ab}}{p^2}\left(g_{\rho\sigma} + (\xi-1)\frac{p_\rho p_\sigma}{p^2}\right) \tag{2.83}$$

Now, the generating functional Z associated to $\mathcal{L}^{quad}_{YM}$ can be written as

$$Z_{quad} = N\exp\left(-\frac{i}{2}\left[J^{\rho a}(-p),(\mathbb{O}^{ab}_{\rho\sigma})^{-1}(p)J^{\sigma b}(p)\right] + i\left[\bar{c}^a(-p),(\mathbb{M}^{ab})^{-1}(p)c^b(p)\right]\right) \tag{2.84}$$

where $(,)$ denotes the integration over p, and N a normalization constant. Therefore, one can deduces the propagators of the free theory as

$$iG^{ab}_{\rho\sigma}(p) = \frac{(-i)^2}{Z_{quad}}\cdot\frac{\delta^2 Z_{quad}}{\delta J^{\rho a}(-p)\delta J^{\sigma b}(p)}\bigg|_{J^{\rho a}=c^a=\bar{c}^a=0} = i(\mathbb{O}^{ab}_{\rho\sigma})^{-1}(p). \tag{2.85}$$

From definition of the gluon propagator in (2.83), one can observe that when $\xi = 0$, i.e., in Landau gauge, this propagator is transverse. Mathematically, transverse reads

$$p_\rho\left(g^{\rho\sigma} - \frac{p^\rho p^\sigma}{p^2}\right) = 0. \tag{2.86}$$

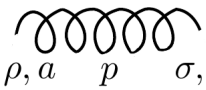
This follows from the gauge condition $\partial_\rho A^\rho = 0$. Since there is a pole at $p^2 = 0$, it means that $p_0 = \sqrt{\vec{p}^2} = |\vec{p}|$, then it tells us that this theory has a massless particle, the photon.

While for $\xi = 1$ (Feynman gauge) the propagator has a much simpler form which is more suitable for perturbative calculations.

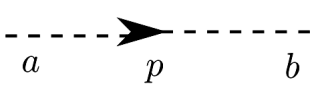
Similarly, the ghost propagator can be deduce as,

$$iG^{ab}(p) = \frac{(-i)^2}{Z_{quad}}\cdot\frac{\delta^2 Z_{quad}}{\delta\bar{c}^a(-p)\delta c^b(p)}\bigg|_{J^{\rho a}=c^a=\bar{c}^a=0} = \frac{i\delta^{ab}}{p^2}. \tag{2.87}$$

The gluon and ghost propagators can be diagrammatically represented, respectively, as



$$= iG^{ab}_{\rho\sigma}(p)$$



$$= iG^{ab}(p)$$

.

We can now write the complete generating functional as

$$Z[J] = \exp\left(i\int d^4x\mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right)Z_{quad}. \tag{2.88}$$

Here $\mathcal{L}_{int}$ indicates some interaction as a monomial in the basic field variables and derivatives, and with factors depending on the coupling constant g,

$$\mathcal{L}_{int} = \mathcal{L}_{YM} - \mathcal{L}_{YM}^{quad} = g f^{abc} \partial_\rho A^a_\sigma A^{\rho b} A^{\sigma c} - \frac{g^2}{4} f^{abp} f^{cdp} A^a_\rho A^b_\sigma A^{\rho c} A^{\sigma d} - g f^{abc} A^a_\rho \partial^\rho \bar{c}^c c^b. \quad (2.89)$$

From this interaction Lagrangian density, all the Feynman rules can be determined, as for example the interactions vertices, and others perturbative calculations useful to understand the renormalization process. For more details about the Feynman rules in Yang-Mills theories, see Ref. [24].

2.5 Some confinement scenarios

During the past forty years, much attention has been devoted to the study of the gluon and ghost propagator, including their low energy behavior where Yang-Mills gauge theories are confining. Due to the confined property, the gluon cannot be considered as a free particle.
An intuitive view of the confinement mechanism is that if one tries to separate a quark-antiquark pair, the energy necessary to perform this action grows linearly with distance between the quark and antiquark. This should happen because a chromo-electric flux appears in the non-perturbative vacuum between two heavy test quarks (see Fig. 2.2).

Figure 2.2: Simulation of a quark-antiquark pair (labeled q and $\bar{q}$), which a little "colour electric flux tube" forms between them, when one tries to separate.

Thereby it is impossible to observed isolated quarks. By the 70s the hypothesis that the quarks are confined, was tested in computer simulations as well as by ordinary experiments. However, the full understanding of this problem is still a lacking.
In this section we will give a short overview about some confinement scenarios directly related to propagator behaviours. In particular, one of them will be analysed in details more later.

2.5.1 The Gribov-Zwanziger confinement scenario

As we mentioned in Sec.2.3, a Lorentz invariant gauge fixing condition which would select one and only one representative for each gauge orbit, is not possible [25]. The problem was first noticed by Gribov [1] and nowadays, it is solved by the so-called Gribov-Zwanziger action [2]. Its derivation and IR analysis are described in Chapter 4. For the moment we only will mention the qualitative consequences of this improved gauge fixing for the theory.
One of the main features of the Gribov-Zwanziger confinement scenario in the Landau gauge is that the gluon propagator vanishes at zero momentum. In fact, the three-level gluon propagator derived from this approach is

$$D^{ab}_{\rho\sigma}(p^2) = \delta^{ab} \left(g_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \frac{p^2}{p^4 + \lambda^4}, \quad \lambda^4 = 2 g^2 N \gamma^4, \quad (2.90)$$

where N is the number of colors and γ a mass parameter, which is not free but determined by a horizon condition (or called gap equation). This condition has to be enforced in order to make the

theory well-defined. The expression (2.90) does not allow us to attach the usual particle meaning to the gluon propagator. Then gluons cannot be considered as part of the physical spectrum. In other words, the gluons are confined by the Gribov horizon, whose presence is encoded in the explicit dependence of expression (2.90) on the Gribov parameter γ.

2.5.2 Others confinement scenarios

In addition to the Gribov-Zwanziger confinement scenario, there are different approaches to the confinement problem. For example, the mechanism proposed by Kugo and Ojima [17] has the same qualitative predictions in the IR behavior on the gluon and ghost propagators, such as the Gribov-Zwanziger scenario, but the fundamental difference is that here the BRST symmetry is a global and non-perturbatively symmetry. Another idea is to try solve non-perturbatively for quark and gluon propagators and vertex functions, by means an infrared expansion of the complete set of Schwinger-Dyson equations [18, 19, 20]. However, the most popular scenario is described in terms of Wilson loop [26] with the requirement to fall off exponentially with the minimal area of the loop. Here the QCD functional integral is dominated by certain class of field configurations, such as magnetic monopoles and center vortices, and sometimes merons [27] and calorons [28, 29].

Chapter 3

The Gribov Problem

The usual method of quantization of Faddeev and Popov in the Landau gauge for non-abelian gauge theories, needs improvements to fix the gauge at non-perturbative level since in 1978, Gribov showed that the gauge-fixing condition doesn't fix completely the gauge. In the same year, I. Singer [25] demonstrated that Gribov copies appear in all gauge fixings with derivatives acting on the gauge fields. In this chapter, we will analyse the Gribov's solution by means the limitation on the integration range in the path integral which suggest a possible case for eliminating equivalent fields.

3.1 Equivalent field configurations: Gribov copies

As was explained in Section (2.3), abelian gauge theories as QED, the Gribov problem does not apply, because the zero mode equation (2.21) has only trivial solutions. Now, let us consider the non-abelian case with two field configurations A_ρ and $\bar{A}_\rho$, which are connected by the following gauge transformation

$$\bar{A}_\rho = U A_\rho U^{-1} - ig(\partial_\rho U)U^{-1} \tag{3.1}$$

with U a $SU(N)$ unitary matrix. When it happens that both field configurations meet the same gauge fixing condition (in this case the Landau gauge), one call to $\bar{A}_\rho$ a *copy* of A_ρ. Nowadays it is known as a *Gribov copy*.

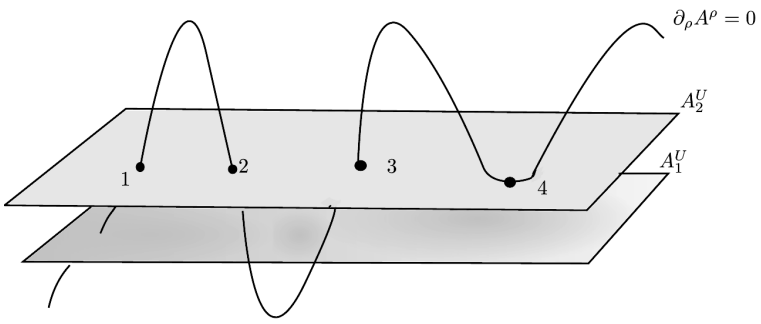


Figure 3.1: Equivalent field configurations satisfying (or cutting of in points 1,2,3,4) the Landau gauge condition over the same gauge orbit.

The Gribov copy equation reads

$$\partial_\rho A_\rho = 0 = \partial_\rho \bar{A}_\rho, \tag{3.2}$$

$$\Leftrightarrow \quad \partial_\rho(U) A_\rho U^{-1} + U A_\rho \partial_\rho U^{-1} - ig\partial_\rho^2(U)U^\dagger - ig\partial_\rho(U)\partial_\rho(U^{-1}) = 0. \tag{3.3}$$

If we consider an infinitesimal transformation $U^{\pm 1} \approx 1 \pm \alpha$, with $\alpha = \alpha_a T^a$, and T_a the generators of the $SU(N)$ group, this last expression can be expand to first order

$$-\partial_\rho^2\alpha - ig\partial_\rho[A_\rho, \alpha] = 0 \quad \Rightarrow \quad -\partial_\rho(\partial_\rho\alpha - ig[\alpha, A_\rho]) = 0. \tag{3.4}$$

Using the definition of covariant derivative give in Eq. (2.13), we obtain

$$-\partial_\rho D_\rho \alpha = 0. \tag{3.5}$$

From this equation one can see that the Gribov copies connected with the identity are nothing more than the zero modes of the Faddeev-Popov operator $\mathcal{M}^{ab} = -(\partial_\rho D_\rho)^{ab}$. Thus, the presence of "small" copies affect the Faddeev-Popov procedure of quantization. Some explicit examples of zero modes of it operator can be seen in [30, 31].

Let us remark that when $A_\rho \to 0$, the equation (3.4) reduces to

$$-\partial_\rho\partial_\rho\alpha = 0. \tag{3.6}$$

But, since one needs to have smooth solutions for the parameters α, then the unique smooth solution will be $\alpha = 0$. Therefore, it turns out that the gauge transformation, which relates the trivial perturbative vacuum $A_\rho = 0$ and its Gribov copies, is far apart within the functional space. Thus, the copies are relevant for the non-perturbative dynamic regions, but irrelevant as far as perturbative expansions are concerned. Due to this, the semi-classical expansion around the perturbative vacuum makes sense.

3.1.1 Gribov region and gauge fixing for Z

The structure of the Faddeev-Popov determinant to the covariant Landau gauge fixing condition is:

$$\Delta_{FP} \equiv \mathcal{M} = \det(\partial^\rho D_\rho[A]). \tag{3.7}$$

We see that the derivation of the Faddeev-Popov determinant is valid under the requirement that the gauge-fixing condition intersects only once each gauge orbit. Therefore, the equation for $\partial_\rho A_\rho^U = 0$ should only have the trivial solution. However, for the Landau gauge, non-trivial solutions for the equation $\partial_\rho A_\rho^U = 0$ can be found, and so there are gauge equivalent fields satisfying the same gauge fixing condition. This implies that the Landau gauge fixing is not complete. However, the fact that gauge is not fixed completely by a local gauge fixing condition is not a specific property of the Landau gauge, but valid for all local gauge fixing conditions. The Singer's theorem formalizes this statement, by establishing that a choice of unique representative field, on each gauge orbit by a linear gauge condition, such as $\partial_\rho A^\rho = 0$, is impossible in a non-abelian gauge theory [25].

Starting with a gauge configuration fulfilling the Landau gauge condition, $\partial_\rho A_\rho = 0$, we can perform an infinitesimal gauge transformation like Eq.(2.10) demanding that the results again fulfils the Landau gauge condition:

$$\partial_\rho A_\rho^U = 0 \to \partial_\rho A_\rho - \partial_\rho D_\rho \omega \stackrel{!}{=} 0 \Rightarrow \mathcal{M}\omega \stackrel{!}{=} 0. \tag{3.8}$$

It indicates that if the Faddeev-Popov operator has zero modes, then there are still gauge equivalent configurations left.
In order to differentiate perturbative and non-perturbative analysis in the functional space, firstly one encounters that there are no zero modes of the Faddeev-Popov operator in the case for $A_\rho \approx 0$, i.e., the equation

$$-\partial_\rho \partial_\rho \psi = \epsilon(A)\psi \tag{3.9}$$

is solvable only for positive ϵ. Thus, for a greater value of the functional norm of the field A, zero modes can appear, and thus the Faddeev-Popov will have zero eigenvalues. This implies the existence of Gribov copies, in other words the positivity of the Faddeev-Popov operator is no longer ensured.

Following to Gribov [1], one can imagine the functional space divided into regions C_n, whose boundaries are determined by the vanishing of one or more eigenvalues of Faddeev-Popov operator.
In particular, Gribov proposed to restrict the functional space in the path integral to the first region $C_1 \equiv \Omega$, where all the eigenvalues of the Faddeev-Popov operator $\mathcal{M}(A)$ are strictly positive:

$$\Omega := \{A; \quad \partial_\rho A^\rho = 0; \quad \mathcal{M}(A) > 0\}. \tag{3.10}$$

Inside Ω, the operator $\mathcal{M}$ is invertible and then the propagators of the Faddeev-Popov ghost exist. The Ω region is known as *Gribov region* and has the following properties at the Landau gauge:

- Perturbation theory, i.e., $A = 0$ lies within the Gribov region because $\mathcal{M}(0) = -\Box$ is a positive operator, i.e., quantum fluctuations around the trivial vacuum does not feel the presence of Gribov copies.

- The Gribov region is bounded in all the directions [32]. This is not valid in any gauge [33].

- It is a convex region. Let us consider two arbitrary configurations A_1 and A_2 within the Gribov region. These configurations can be combined by a new configuration A_3 that lies also within the Gribov region as follows [32] :

$$A_3 = \alpha A_1 + (1-\alpha)A_2, \quad 0 \leq \alpha \leq 1. \tag{3.11}$$

- Every gauge orbit passes at least once through the Gribov region [34, 1]. This property permits the Gribov restriction without missing any gauge orbits. In other words, all the physical field configurations of the theory are within the Gribov region.

- However, there are still gauge copies in the Gribov region (however a region completely free of copies, called modular region can be defined) so that the Gribov approach is not complete [31], but some arguments suggest that expectation values are not influenced by these additional copies [32]. Therefore, it is common to employ the Gribov region restriction as we will do in this thesis.

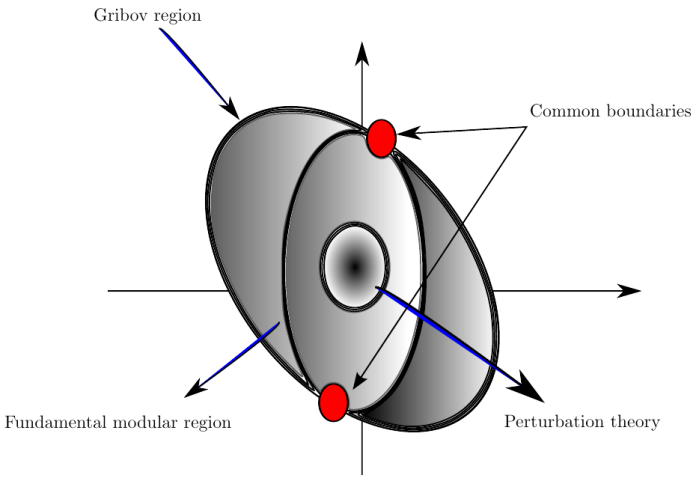


Figure 3.2: Intersection between the Gribov region and the fundamental modular region.

3.2 Gribov region implemented in generating functional

As we saw in the previous section, Gribov proposed to restrict the path integral to the so-called Gribov region Ω defined in Eq.(3.10). Let's see how to insert this statement in the Landau gauge [1] on the path integral. In fact, let us consider the following partition function

$$Z_G = \mathcal{N}\int [\mathcal{D}A][\mathcal{D}\bar{c}][\mathcal{D}c]\delta(\partial^\rho A_\rho)\det(\mathcal{M})e^{-S_{EYM}}\nu(\Omega). \tag{3.12}$$

Here the factor $\nu(\Omega)$ ensures that the integration is performed only over Ω. From relationship between the ghost sector and the Faddeev-Popov determinant, one can see clearly how is the factor $\nu(\Omega)$. Then, we start calculating the ghost propagator given by

$$\begin{aligned}\langle \bar{c}_a(x)c_b(y)\rangle_c &= \frac{\delta}{\delta J^b_c(y)}\frac{\delta}{\delta J^a_{\bar{c}}(x)}Z_G \\ &= \mathcal{N}\int [\mathcal{D}A_\rho][\mathcal{D}c][\mathcal{D}\bar{c}]\delta(\partial A)\bar{c}^a(x)c^b(y)e^{-(S_{EYM}+\int d^4x\bar{c}^a\partial_\rho D^{ab}_\rho c^b)}\nu(\Omega) \\ &= \mathcal{N}\int [\mathcal{D}A_\rho]\delta(\partial A)e^{-S_{EYM}}\det(-\partial_\rho D_\rho)[\partial_\rho D^{ab}_\rho\delta(x-y)]^{-1}.\end{aligned} \tag{3.13}$$

Applying Fourier transform to the last expression, we obtain

$$\langle \bar{c}_a(p)c_b(-p)\rangle_c = \int [\mathcal{D}A]\delta(\partial_\rho A^a_\rho)\det(-\partial_\rho D^{ab}_\rho)\int d(x-y)e^{ip(x-y)}\mathcal{M}^{-1}_{ab}(x,y)e^{-S_{EYM}}. \tag{3.14}$$

To bring about the integral on the momentum space in the expression for the ghost propagator, one make use of the Wick theorem (for most details about the derivation of it propagator, the reader can see Ref. [35]), then we obtain the following expression

$$\langle \bar{c}_a(p)c_b(k)\rangle_c = \delta(p+k)\delta_{ab}G(k^2), \tag{3.15}$$

[1] From now on, we shall work in the Landau gauge, unless explicitly mentioned

with

$$G(k^2) = \frac{1}{k^2}\frac{1}{\left(1-\frac{11g^2N}{48\pi^2}\ln\frac{\Lambda^2}{k^2}\right)^{\frac{9}{44}}} \tag{3.16}$$

where Λ is the ultraviolet cutoff, N is the color indices and g the coupling constant of S_{EYM}. From the expression for $G(k^2)$ one obtains the following conclusions:

- The pole $k^2 = 0$ tells us that we are approaching to the horizon, and when $1/k^2 > 0$, then we stay inside the Gribov region.
- The pole $k^2 = \Lambda^2 \exp\left(-48\pi^2/11g^2N\right)$ can be complex or switch his sign, indicating that we have left the Gribov region.
- On the other hand, we can only approach the Gribov horizon when $k = 0$. So, this pole must be eliminated.

In order to stay inside the Gribov region, we must to impose some condition, where can be well-defined the function $G(k^2)$. In fact, this relation exists and is known as the *no-pole condition*.

3.2.1 No-pole condition

Let us evaluate the two-point ghost function, given by

$$\begin{aligned}\delta^{ab}\langle\bar{c}^a(x)c^b(y)\rangle &= -\langle x,a|1/\mathcal{M}|y,a\rangle\delta^{ab} = \mathcal{N}\int[\mathcal{D}A][\mathcal{D}c][\mathcal{D}\bar{c}]\delta(\partial A)\bar{c}^a c^a e^{-S_{EYM}-\int d^4x\bar{c}^a\mathcal{M}^{ab}c^b}\\ &= \mathcal{N}\int[\mathcal{D}A]\delta(\partial A)e^{-S_{EYM}}G(x,y;A).\end{aligned} \tag{3.17}$$

Now we will try to write $G(x,y;A)$ in perturbation theory up to second order and considering the gluonic field A_ρ as a external field [35]. Thus, $G(k,A)$ function at space momentum, reads

$$G(k,A) = \frac{V}{k^2}\left(1+\sigma(k,A)\right) \approx \frac{1}{k^2}\frac{V}{1-\sigma(k,A)} = (\mathcal{M}^{-1})^{ab}(k,A), \tag{3.18}$$

with

$$\sigma(k,A) = \frac{4}{k^2}\int\frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(q)A^a_\sigma(-q)k_\rho(k_\sigma-q_\sigma)}{(k-q)^2}. \tag{3.19}$$

Here $A^a_\rho(p)$ is the Fourier component of the field A_ρ, V is the volume of the system, and $\sigma(k,A)$ defines positions of the poles of $G(k,A)$.

Due to that in the Landau gauge the fields $A_\rho(p)$ are transverse, i.e., $p_\rho A_\rho = 0$, thus overaring over the gluon polarization directions λ, we have

$$\sigma(k,A) = 4\int\frac{d^4p}{(2\pi)^4}\frac{|A^{a,\lambda}(p)|^2}{(k-p)^2}\left(1-\frac{(kp)^2}{k^2p^2}\right). \tag{3.20}$$

Assuming that the quantity $|A^{a,\lambda}(p)|^2$ over the main range of integration turns out to decrease with p^2, so that $\sigma(k,A)$ decreases as k^2 increases, hence one can imposes

$$1-\sigma(0,A) > 0. \tag{3.21}$$

This condition implies that the ghost propagator $G^{ab}(k,A)$ has no poles at finite non-vanishing k, and that the Faddeev-Popov operator has no zero modes. Moreover, positivity of $G^{ab}(k,A)$ ensures that the Gribov horizon $\delta\Omega$ is not crossed. Later on, Zwanziger [2] was able to implement the no-pole condition to all orders, relying on the equivalence between the microcanonical and the canonical Boltzmann ensemble [36].

Finally, taking Eq.(3.21) as a condition for $\nu(\Omega)$, we obtain a functional integral which is easy to calculate, if $\nu(1-\sigma(k,A))$ is written in the form [2]

$$\begin{aligned}\nu(\Omega) &= \nu(1-\sigma(0,A)) = \theta(1-\sigma(0,A)) \\ &= \int \frac{d\beta}{2i\pi\beta} e^{\beta(1-\sigma(0,A))}. \end{aligned} \tag{3.22}$$

Now, we have the expression for the factor $\nu(\Omega)$ we were looking, then the generating functional (3.12) yields,

$$Z = \mathcal{N}\int \frac{d\beta}{2\pi i\beta}\int [\mathcal{D}A_\rho]\delta(\partial A) e^{\beta(1-\sigma(0,A))} e^{-S_{EYM}} \det(-\partial_\rho D_\rho). \tag{3.23}$$

3.3 Gluon propagator and Gribov parameter

Due to the insertion of the Gribov restriction in the generating functional, the Gribov propagator differs substantially from the perturbative propagator in the infrared region.
Let us consider the path integral (3.23) under the influence of an external source $J(x)$ as

$$Z = \mathcal{N}\int \frac{d\beta}{2\pi i\beta}\int \mathcal{D}A e^{\beta(1-\sigma(0,A))} e^{-S_{YM}+\int d^dx \frac{1}{2\xi}(\partial_\rho A_\rho)^2 + \int d^dx A^a_\rho(x) J^a_\rho(x)}, \tag{3.24}$$

where $\mathcal{N}$ is the normalization factor. At the same way that we calculated the gluon propagator of the Yang-Mills theory in Sec. 2.4.2 is only necessary the quadratic (or free) part in the Yang-Mills Lagrangian. Therefore, by a integration by parts, we have

$$\begin{aligned} S^{\text{quad}}_{YM} &+ \int d^dx \frac{1}{2\xi}(\partial_\rho A^\rho)^2 + \int d^dx A^a_\rho(x) J^a_\rho(x) \\ &= \int d^dx \left(\frac{1}{2}A_\rho\left(g^{\rho\sigma}\partial^2 - \frac{\xi-1}{\xi}\right)A_\sigma - J_\rho A^\rho\right). \end{aligned} \tag{3.25}$$

This last result can be expressed in Fourier space as follows

$$\begin{aligned} S^{\text{quad}}_{YM} &+ \int d^dx \frac{1}{2\xi}(\partial_\rho A^\rho)^2 + \int d^dx A^a_\rho(x) J^a_\rho(x) \\ &= \int \frac{d^dp}{(2\pi)^d}\left[\frac{1}{2}A^a_\rho(p)\left(\delta_{\rho\sigma}p^2 - \frac{\xi-1}{\xi}p_\rho p_\sigma\right)A^a_\sigma(-p) - A^a_\rho(p)J^a_\rho(-p)\right], \end{aligned} \tag{3.26}$$

[2] Here, we will use the step function defined as $\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$, and moreover his useful integral representation: $\theta(x) = \frac{1}{2i\pi\beta}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\beta e^{\beta x}$, with $\epsilon \to 0^+$.

then the gluon propagator takes the following form

$$\begin{aligned}\langle A^a_\rho(k)A^a_\rho(p)\rangle &= \frac{\delta^2}{\delta J^a_\rho(-p)\delta J^b_\sigma(-p)}\int\frac{de^\beta}{2i\pi\beta}\int\mathcal{D}A \\ &\times \exp\left[-\int\frac{d^dp}{(2\pi)^d}\left\{A^a_\rho(p)\frac{\beta Ng^2}{Vd(N^2-1)}\frac{\delta_{\rho\sigma}}{p^2}A^a_\sigma(-p)\right.\right. \\ &+ \left.\left.\frac{1}{2}A^a_\rho(p)\left(\delta_{\rho\sigma}p^2-\frac{\xi-1}{\xi}p_\rho p_\sigma\right)A^a_\sigma(-p)\right\}\right]\Big|_{J=0} \\ &= \frac{\delta^2}{\delta J^a_\rho(-p)\delta J^b_\sigma(-p)}\int\frac{de^\beta}{2i\pi\beta}\int\mathcal{D}A\exp\left[-\int\frac{d^dp}{(2\pi)^d}\left\{\frac{1}{2}A^a_\rho(p)\mathbb{P}^{ab}_{\rho\sigma}A^b_\sigma(-p)\right.\right. \\ &+ \left.\left.A^a_\rho(p)J^a_\rho(-p)\right\}\right]\Big|_{J=0}\end{aligned}$$

where,

$$\mathbb{P}^{ab}_{\rho\sigma}=\delta^{ab}\left(\frac{2\beta Ng^2}{Vd(N^2-1)}\frac{\delta_{\rho\sigma}}{p^2}+\delta_{\rho\sigma}p^2-\frac{\xi-1}{\xi}p_\rho p_\sigma\right). \tag{3.27}$$

Evaluating the Gaussian integral for the field A under Fourier transform [35], we arrived to

$$\langle A^a_\rho(k)A^b_\rho(p)\rangle=\mathcal{N}'\delta(k+p)\int\frac{d\beta e^\beta}{2\pi i\beta}\frac{1}{\sqrt{\det(\mathbb{P}^{ab}_{\rho\sigma})}}(\mathbb{P}^{ab}_{\rho\sigma})^{-1} \tag{3.28}$$

with $\mathcal{N}'$ being other normalization constant obtained after of the Gaussian integration.

Following to [37], the determinant of the operator $\mathbb{P}$ is possible evaluated, resulting a factor independent of p, thus the gluon propagator reads

$$\langle A^a_\rho(k)A^a_\rho(p)\rangle=\tilde{\mathcal{N}}\delta(k+p)\int\frac{d\beta e^\beta}{2i\pi\beta}e^{f(\beta)}(\mathbb{P}^{ab}_{\rho\sigma})^{-1} \tag{3.29}$$

where,

$$f(\beta)=\beta-\ln\beta-\frac{d-1}{2}(N^2-1)\int\frac{d^dp}{(2\pi)^d}\ln\left(p^2+\frac{\beta Ng^2}{d(N^2-1)}\frac{2}{Vp^2}\right). \tag{3.30}$$

The integration over β can be evaluated by applying the saddle point method, which yields the following minimum condition

$$f'(\beta_0)=0, \tag{3.31}$$

which yields

$$1=\frac{1}{\beta_0}+\frac{d-1}{d}Ng^2\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4+\gamma^4},\quad \gamma^4=\frac{2g^2\beta_0 N}{d(N^2-1)V}. \tag{3.32}$$

Since, in the thermodynamic limit V is infinity (Euclidean volume), then in order to have a finite γ , $\beta_0\sim V$. Thus, Eq. (3.32) becomes the gap equation defined as:

$$1=\frac{d-1}{d}Ng^2\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4+\gamma^4},\quad \gamma^4=\frac{2g^2N}{d(N^2-1)}. \tag{3.33}$$

Thus, it is important to emphasize that the Gribov parameter is not a free parameter of the theory but is self-consistently determined by the gap equation (3.33) as a function of some cut-off (the QCD

scale).

The only thing we have to do is to determine the inverse of the operator $\mathbb{P}$ in Eq.(3.29), but this kind of computations was realized in Eq. (2.80), then we can find the following propagator:

$$D^{ab}_{\rho\sigma}(p) \equiv \langle A^a_\rho(k) A^b_\sigma(p)\rangle = \delta(k+p)\delta^{ab}\frac{p^2}{p^4+\gamma^4}\left[\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right], \tag{3.34}$$

where we have evaluated in the Landau gauge, i.e., $\xi = 0$. This result is very important to analyse the physic behaviours of the gluon, since in contrast to the result found in Eq. (2.83), now the propagator depends of a parameter γ, known as *Gribov mass parameter*, it turn depends of the gap equation (3.33). This propagator violates positivity of the spectral density function of the Källén–Lehmann representation [38], then this can be interpreted as a manifestation of the gluon confinement.
It is straightforward to verify that the propagator (3.34) has two complex conjugated poles

$$\frac{p^2}{p^4+\gamma^4} = \frac{1}{2}\left(\frac{1}{p^2+i\gamma^2} + \frac{1}{p^2-i\gamma^2}\right). \tag{3.35}$$

The unphysical excitations associated to the complex masses $p^2 = \pm i\gamma^2$ have been called $i-$particles [39]. Although, the $i-$particles do not correspond to observable particles, they provide a useful set up in order to extract the analytic properties of correlation functions of gauge invariant composite operators [40].

On the other hand, it is important to remark that this same expression for the gluon propagator has been determined in $d = 4$ and 3 dimensions, and in all this cases the gluon propagator turns out to be suppressed in the infrared region. Unfortunately, this suppression are only observed for lattice data in 2 dimensions, and for recent lattice data in $d = 4, 3$ dimensions this computations are not agreement [41, 42, 43]. However, it does not happen for Refined Gribov-Zwanziger formulation [44] in $d = 4, 3$ dimensions, which is in very good agreement with the numerical data. (See for example for $d = 4$ Ref. [45]).
In this thesis we will work in the Gribov-Zwanziger approach because it describes very well the phase diagrams with temperature and chemical potential.

Chapter 4

The Gribov-Zwanziger Confinement Scenario

Although Gribov suggested improved gauge fixing, he did not manage to get a local action. At the end of the 80s, Zwanziger was able to show that the restriction to the Gribov region can be done within a local and renormalizable field theory framework. The resulting action is called Gribov-Zwanziger action [1, 2] and constitutes the best option available so far to achieve a complete gauge fixing in the functional space.
This Chapter is organized as follows: firstly, from the global Gribov action, we will derive a standard local form for this action by means the insertion of auxiliary fields. Due to the implementation of this fields, a new symmetry emerges. The topic about this new symmetry will be explained briefly from the local action. Finally, we will derive the gluon propagator in the Landau gauge.

4.1 The Gribov-Zwanziger action

In this section we deliver a brief derivation of the local formulation of the Gribov-Zwanziger action. His name it must be to Gribov, who was the first to suggest a solution to overcome the problem with gauge fixing, and Zwanziger, who brought this idea more formally, in terms of a local action and several properties of the first Gribov horizon.

4.1.1 Motivation

Let us start with the Gribov's solution [1] about the problem of gauge fixing condition, by means the restriction on the integration domain in the path integral,

$$\Omega := \{A^a_\rho; \quad \partial_\rho A^a_\rho = 0, \quad \mathcal{M}^{ab} > 0, \}. \tag{4.1}$$

It is defined by these gauge field configurations for which the lowest non-trivial eigenvalue of the Faddeev-Popov operator vanishes.

Also, as we saw in the previous chapter, Gribov parametrized the ghost propagator as

$$D^{ab}_{c\bar{c}}(p) \;=\; -\frac{\delta^{ab}}{p^2}\frac{1}{(1-\sigma(p,A))}, \tag{4.2}$$

where $\sigma(p, A)$ can be calculated as a series in A , which increases when p decreasing. In order to avoid this increase, it was sufficient demand that $\sigma(0, A) < 1$. The so-called no-pole condition.

Although the Gribov's idea is correct, Zwanziger [2] derived a closed and formal expression to perform the integration within Gribov region, by means the function

$$h(x,y) := \lim_{\gamma(x)\to\gamma} \int dy (D^{ac}_{\rho}(x)\gamma^2(x))(\mathcal{M}^{-1})^{ab}(x,y)(D^{bc}_{\rho}(y)\gamma^2(y)). \tag{4.3}$$

The limit $\gamma(x) \to \gamma$ must be taken after localizing the action. The condition to be within the Gribov horizon is phrased as the horizon condition [2]:

$$\int dx h(x) < \gamma^4 d(N^2-1)V, \tag{4.4}$$

where d, N , V are the number of dimensions, the numbers of colours and the space-time volume, respectively. Notice that this condition can be enforced via a Heaviside functional as the no-pole condition.
Moreover, the interesting observation is that the resulting path integral has similarities with the partition function of a canonical ensemble which is equivalent to the microcanonical ensemble in the thermodynamic limit [46], so that the Heaviside functional transforms by a delta functional, and the horizon condition becomes

$$\int dx h(x) = \gamma^4 d(N^2-1)V. \tag{4.5}$$

The resulting action that enforces to the restriction (4.5) is known as *Gribov-Zwanziger action*, and is given by

$$S^{NL} = S_{FP} + S_h, \tag{4.6}$$

where,

$$\begin{aligned} S_{FP} &= S_{YM} + S_{gf}, & (4.7)\\ S_h &= \int dx (h(x) - \gamma^2(N^2-1)V), & (4.8)\\ S_{YM} &= \frac{1}{4}\int dx F^a_{\rho\sigma}F^{\rho\sigma a}, & (4.9)\\ S_{gf} &= \int dx (ib^a\partial_\rho A^a_\rho + \bar{c}^a D^{ad}_\rho c^d). & (4.10) \end{aligned}$$

Since S_h is non local (NL) the standard tools of quantum field theory can not be employed. Then, in order to express the action S^{NL} in a local form, auxiliary fields will are necessary.

4.2 From non-local action to a local action

In fact, in order to localize the horizon function we need two pairs of additional fields: φ^{ab}_ρ and $\bar{\varphi}^{ab}_\rho$, which are complex conjugate to each other, and ω^{ab}_ρ and $\bar{\omega}^{ab}_\rho$ Grassmann fields.

These new fields are BRST doublets, i.e.,

$$\begin{aligned} s\varphi^{ab}_\rho &= \omega^{ab}_\rho, \quad s\omega^{ab}_\rho = 0,\\ s\bar{\omega}^{ab}_\rho &= \bar{\varphi}^{ab}_\rho, \quad s\bar{\varphi}^{ab}_\rho = 0. \end{aligned} \tag{4.11}$$

Here the Greek indices run as $\rho = 1, ..., d$ and the Latin indices as $a, b = 1, ..., (N^2 - 1)$. From the Gaussian integrations, one can see

$$\begin{aligned}\int \mathcal{D}[\varphi\bar{\varphi}]e^{-\int dx\left(\bar{\varphi}^{ac}_{\rho}\mathcal{M}^{ab}\varphi^{bc}_{\rho}+\gamma^2 g f^{abc}A^{a}_{\rho}(\varphi^{bc}_{\rho}-\bar{\varphi}^{bc}_{\rho})\right)} &= (\det\mathcal{M})^{-d(N^2-1)}e^{-\gamma^4 g^2 f^{ace}A^{a}_{\rho}(\mathcal{M}^{-1})^{cd}f^{bde}A^{b}_{\rho}},\\ \int \mathcal{D}[\omega\bar{\omega}]e^{\int dx\bar{\omega}^{ac}_{\rho}\mathcal{M}^{ab}\omega^{bc}_{\rho}} &= (\det\mathcal{M})^{d(N^2-1)}.\end{aligned}$$

Then, the bosonic felds are used for localizing the non-local term S_h, and the fermionic fields for canceling of determinant. From this, one can show

$$e^{-S_h} = \int \mathcal{D}[\varphi\bar{\varphi}\omega\bar{\omega}]e^{-\int dx\left(\bar{\varphi}^{ac}_{\rho}\mathcal{M}^{ab}\varphi^{bc}_{\rho}+\gamma^2 g f^{abc}A^{a}_{\rho}(\varphi^{bc}_{\rho}-\bar{\varphi}^{bc}_{\rho})-\bar{\omega}^{ac}_{\rho}\mathcal{M}\omega^{bc}_{\rho}-d\gamma^4(N^2-1)\right)}. \tag{4.12}$$

Therefore, the local Gribov-Zwanziger action turns out to be

$$\begin{aligned}S^{local} &= S_{FP} + S_{GZ}, \qquad (4.13)\\ S_{GZ} &= \int dx\left(\bar{\varphi}^{ac}_{\rho}\mathcal{M}^{ab}\varphi^{bc}_{\rho} + \gamma^2 g f^{abc}A^{a}_{\rho}(\varphi^{bc}_{\rho}-\bar{\varphi}^{bc}_{\rho}) - \bar{\omega}^{ac}_{\rho}\mathcal{M}^{ab}\omega^{bc}_{\rho} - d\gamma^4(N^2-1)\right).\end{aligned}$$

Let us now translate the nonlocal horizon condition (4.5) into a local version [2]. The local action that we obtained previously, $S_{YM} + S_{gf} + S_{GZ}$, can be related as follows,

$$\int [\mathcal{D}A][\mathcal{D}b][\mathcal{D}c][\mathcal{D}\bar{c}]e^{-(S_{YM}+S_{gf}+S_{GZ})} = \int [\mathcal{D}A][\mathcal{D}b][\mathcal{D}c][\mathcal{D}\bar{c}][\mathcal{D}\varphi][\mathcal{D}\bar{\varphi}][\mathcal{D}\omega][\mathcal{D}\bar{\omega}]e^{-S^{local}}. \tag{4.14}$$

Next, if we take the partial derivative in both sides with respect to γ^2 (with $\gamma \neq 0$), we obtain

$$-2\gamma^2\frac{\int dx h(x)}{\gamma^4} = \langle g f^{abc}A^{a}_{\rho}(\varphi^{bc}_{\rho}+\bar{\varphi}^{bc}_{\rho})\rangle. \tag{4.15}$$

In this last result, we used the fact that $\langle\partial_\rho\varphi^{aa}\rangle = 0$ and $\langle\partial_\rho\bar{\varphi}^{aa}\rangle = 0$, with $\langle\cdots\rangle$ stands functional integral over the fields.

Therefore, using the expression (4.5), and assuming that $\gamma \neq 0$, we can write the local version of the horizon condition (4.15) as

$$\langle g f^{abc}A^{a}_{\rho}(\varphi^{bc}_{\rho}+\bar{\varphi}^{bc}_{\rho})\rangle + 2\gamma^2 V d(N^2-1) = 0. \tag{4.16}$$

Moreover by adding the vacuum term $\int d^4x\gamma^4 d(N^2-1)$ to S_h, we can write the horizon condition as

$$\frac{\partial\Gamma}{\partial\gamma^2} = 0, \tag{4.17}$$

with Γ the quantum action defined as

$$e^{-\Gamma} = \int [d\Phi]e^{-S_{GZ}}, \tag{4.18}$$

where $\int[d\Phi]$ indicates the integration over all the fields.

For end this section, we recall that sometimes in the literature the action S^{local} contains the term

$$S_{\Delta_{BRST}} = \int dx g f^{abe} \bar{\omega}^{ac}_{\rho} \partial_{\sigma}[(D^{ed}_{\sigma} c^{d}) \varphi^{bc}_{\rho}]. \tag{4.19}$$

This term appears as a shift in the field ω and has the advantage of remains BRST symmetry most manifest. The only term in $S^{local} + S_{\Delta_{BRST}}$ invariant is the one proportional to γ^2

$$s\left(\gamma^2 g f^{abc} A^{a}_{\rho}(\varphi^{bc}_{\rho} - \varphi^{bc}_{\rho})\right) = \gamma^2 g f^{abc}\left((-D^{ad}_{\rho} c^{d})(\varphi^{bc}_{\rho} - \bar{\varphi}^{bc}_{\rho}) + A^{a}_{\rho}\omega^{bc}_{\rho}\right). \tag{4.20}$$

Moreover to make explicit the BRST symmetry, this shift over the field ω^{a}_{ρ} implies a Gribov-Zwanziger action renormalizable. This insertion of a term in S_g is far from being trivial, since no new parameter now is needed to take into account vacuum divergences [21].

4.3 The gluon and ghost propagator

In this section, our disposal is to calculate the gluon and ghost propagator from the local Gribov-Zwanziger action, at lowest order. We verify that the same result is obtained as in the semi-classical approximation obtained by Gribov.

4.3.1 The gluon propagator

In order to calculate the propagators, we again need only the free part of S_{GZ},

$$\begin{aligned} S^{quad}_{GZ} &= \int d^{d}x \left(\frac{1}{4}(\partial_{\rho} A^{a}_{\sigma} - \partial_{\sigma} A^{a}_{\rho})^2 + \frac{1}{2\xi}(\partial_{\rho} A^{a}_{\rho})^2 + \bar{\varphi}^{ab}_{\rho}\partial^2 \varphi^{ab}_{\rho} \right. \\ &- \left. \gamma^2 g(f^{abc} A^{a}_{\rho}\varphi^{bc}_{\rho} + f^{abc} A^{a}_{\rho}\bar{\varphi}^{bc}_{\rho}) + \cdots \right), \end{aligned} \tag{4.21}$$

The $\cdots$ stands for the constant term $-d(N^2-1)\gamma^4$ and other terms in the ghost and $\omega, \bar{\omega}$ fields irrelevant for the calculation of the gluon propagator. Next, we integrate out the φ- and $\bar{\varphi}$-fields. Since we are only interested in the gluon propagator, we simply use the equations of motion: $\delta S^{quad}_{GZ}/\delta\varphi^{bc}_{\rho} = 0$ and $\delta S^{quad}_{GZ}/\delta\bar{\varphi}^{bc}_{\rho} = 0$, which results

$$\partial^2 \varphi^{bc}_{\rho} = \partial^2 \bar{\varphi}^{bc}_{\rho} = \gamma^2 g f^{abc} A^{a}_{\rho}. \tag{4.22}$$

Utilizing these equations, we can rewrite S^{quad}_{GZ} as

$$\begin{aligned} S^{quad}_{GZ} &= \int d^{d}x \left(\frac{1}{4}(\partial_{\rho} A^{a}_{\sigma} - \partial_{\sigma} A^{a}_{\rho})^2 + \frac{1}{2\xi}(\partial_{\rho} A^{a}_{\rho})^2 + \gamma^4 g^2 f^{abc} A^{a}_{\rho} \frac{1}{\partial^2} f^{dbc} A^{d}_{\rho} \right. \\ &- \left. 2\gamma^4 g(f^{abc} A^{a}_{\rho} \frac{1}{\partial^2} g f^{dbc} A^{d}_{\rho}) + \cdots \right) \\ &= \int d^{d}x \left(\frac{1}{4}(\partial_{\rho} A^{a}_{\sigma} - \partial_{\sigma} A^{a}_{\rho})^2 + \frac{1}{2\xi}(\partial_{\rho} A^{a}_{\rho})^2 - N\gamma^4 g^2 A^{a}_{\rho} \frac{1}{\partial^2} A^{a}_{\rho} + \cdots \right), \end{aligned} \tag{4.23}$$

where in the last step we used the relation of $SU(N)$ group theory $f^{abc} f^{dbc} = N\delta^{ad}$. Then, from a small algebraic exercise, S^{quad}_{GZ} can be rewrite as:

$$S^{quad}_{GZ} = \int d^{d}x \left(\frac{1}{2} A^{a}_{\rho} \Delta^{ab}_{\rho\sigma} A^{b}_{\sigma} + \cdots \right) \tag{4.24}$$

with

$$\Delta^{ab}_{\rho\sigma} = \left[\left(\partial^2 + \frac{2g^2N\gamma^4}{\partial^2}\right)\delta_{\rho\sigma} - \partial_\rho\partial_\sigma\left(1-\frac{1}{\xi}\right)\right]\delta^{ab}. \tag{4.25}$$

The inverse of this operator is easily calculated in the momentum space, which takes the form

$$\Delta^{ab}_{\rho\sigma}(p) = -\left[\left(p^2 + \frac{2g^2N\gamma^4}{p^2}\right)\delta_{\rho\sigma} - p_\rho p_\sigma\left(1-\frac{1}{\xi}\right)\right]\delta^{ab}. \tag{4.26}$$

The derivation of the inverse of $\Delta^{ab}_{\rho\sigma}(p)$ is a bit involved. Firstly, we note that the Green's function $(\Delta^{ab}_{\rho\sigma})^{-1}$ is a symmetric second rank tensor and with all the Lorentz structures available $(p_\rho, \delta_{\rho\sigma})$, then we can parametrize the most general form of the inverse as

$$(\Delta^{ab}_{\rho\sigma})^{-1} = \left(\alpha\delta_{\rho\sigma} + \beta\frac{p^2p_\rho p_\sigma}{p^4+\lambda^4}\right)\delta^{ab}, \quad \lambda^4 = 2Ng^2\gamma^4, \tag{4.27}$$

where α, β are arbitrary parameters to be determined. From this, we note that the inverse is defined to satisfy

$$\begin{aligned}
\Delta^{\rho\sigma}_{ab}(p)(\Delta^{bc}_{\sigma\lambda})^{-1} &= \delta^c_a\delta^\rho_\lambda \\
\left[\left(\frac{p^4+\lambda^4}{p^2}\right)\delta^{\rho\sigma} - p^\rho p^\sigma\left(1-\frac{1}{\xi}\right)\right]\left[\alpha\delta_{\sigma\lambda} + \beta\frac{p^2p_\sigma p_\lambda}{p^4+\lambda^4}\right] &= -\delta^\rho_\lambda \\
\left[\alpha\left(\frac{p^4+\lambda^4}{p^2}\right)+1\right]\delta^\rho_\lambda + \left[\beta - \alpha\left(1-\frac{1}{\xi}\right) - \beta\left(1-\frac{1}{\xi}\right)\left(\frac{p^4+\lambda^4}{p^2}\right)\right]p^\rho p_\lambda &= 0.
\end{aligned}$$

Setting the coefficients of each distinct Lorentz structure to zero and taking the limit $\xi \to 0$, we obtain

$$\alpha = -\frac{p^2}{p^4+\lambda^4} \tag{4.28}$$

$$\beta = \frac{1}{p^2}. \tag{4.29}$$

So that we can write the inverse of the operator. Thus, the gluon propagator can be determined by taking the inverse of $\Delta^{ab}_{\rho\sigma}$, which results

$$\langle A^a_\rho(p)A^b_\sigma(k)\rangle = \delta(p+k)\frac{p^2}{p^4+\lambda^4}\left[\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right]\delta^{ab}, \tag{4.30}$$

which coincides with the expression obtained by Gribov (see Eq. (3.34)). Once more, due to the complex poles in this last expression, it does not allow us to attach the usual particle meaning to the gluon propagator, which implies that gluons cannot be considered as part of the physical spectrum; they are confined by the Gribov horizon, whose presence is encoded in the explicit dependence the propagator on the Gribov parameter γ.

4.4 Why a breaking in BRST symmetry?

Due to the existence of Gribov copies, an intuitive argument suggests that it could affect the BRST symmetry. In 1983, Fujikawa showed [47] (see also [48]) that the Gribov problem may then induce

dynamical instability of the BRST symmetry.
The BRST transformations of the GZ action (4.13) are

$$\begin{aligned} sA^a_\rho &= -D^{ab}_\rho c^b = -(\partial_\rho \delta^{ab} + g f^{abc} A^c_\rho) c^b, \\ sc^a &= \frac{g}{2} f^{acb} c^b c^c, \\ s\bar{c}^a &= ib^a, \\ sb^a &= 0, \\ s\varphi^{ab}_\rho &= \omega^{ab}_\rho, \quad s\omega^{ab}_\rho = 0, \\ s\bar{\omega}^{ab}_\rho &= \bar{\varphi}^{ab}_\rho, \quad s\bar{\varphi}^{ab}_\rho = 0. \end{aligned} \tag{4.31}$$

From this only some terms of the local GZ action are BRST invariant. In particular, one finds

$$s \int d^4x \left(\frac{1}{4} F^a_{\rho\sigma} F^{\rho\sigma a} + s \left[\bar{c}^a \partial_\rho A^a_\rho - \bar{\omega}^{ac}_\sigma \partial_\sigma D^{ab}_\sigma \varphi^{bc}_\rho \right] \right) = 0. \tag{4.32}$$

On the other hand, the full GZ action is not left invariant by the BRST transformations (4.31), since the term S_γ it broken, namely

$$\begin{aligned} sS_\gamma &= \int d^4x \left[\gamma^2 g f^{abc} s A^a_\rho (\varphi^{bc}_\rho - \bar{\varphi}^{bc}_\rho) + \gamma^2 g f^{abc} A^a_\rho (s\varphi^{bc}_\rho - s\bar{\varphi}^{bc}_\rho) \right] \\ &= \gamma^2 \int d^4x \left(-g f^{abc} D^{ad}_\rho c^d (\varphi^{bc}_\rho - \bar{\varphi}^{bc}_\rho) + g f^{abc} A^a_\rho \omega^{bc}_\rho \right) \neq 0. \end{aligned} \tag{4.33}$$

We note that the breaking is quadratic in the fields, so that it has to be treated as a composite field operator, a feature which requires the introduction of a suitable set of external sources in order to implement the Slavnov-Taylor identities [1]. In this direction, many works has shown that this breaking can be converted into a linear one, resulting a nilpotent linearly broken BRST symmetry [49]. As consequence, the linearly broken BRST symmetry can be thus directly converted into a set of useful Slavnov-Taylor identities. Therefore, the quantum aspects of the Gribov-Zwanziger theory can be analyzed by means of the cohomology of a local nilpotent operator. In [50, 51], it was pointed out that the softly broken BRST symmetry of the GZ action can be converted into a exact symmetry, however non-local. This non-local invariance has been localized in [52], though the resulting BRST symmetry is not nilpotent.

In summary, we can state that this breaking is clearly due to the introduction of the horizon into the Yang-Mills action. Similarly, the BRST invariance may be regarded as a consequence of the fact that the non-perturbative gauge fixing introduced above has been done in the Landau gauge, and has not been carried out in other covariant gauges. The main result obtained so far remains that of the renormalizability of the theory.

[1] The Slavnov-Taylor identities are relations between various scattering amplitudes of the non-abelian theory, namely are the analogous to Ward-Takahashi identities in a abelian gauge theory.

Chapter 5

Field Theory at Finite Temperature and Density

The study of the Feynman path integral implies a direct connection between the quantum field theories and the equilibrium statistical physics, by means the partition function. Due to this connection, many authors [53, 54] have studied what happens when a system described by a quantum field theory, is heated or exposed to certain densities.
In the present chapter, we will review the basic procedures to induce chemical potential and temperature to a field theory. In this line, many aspects about thermal field theories are presented in many textbooks. In particular, pedagogical presentations can to visit in Refs. [55, 56, 57].

5.1 Statistical mechanics

A brief summary about the principal elements of statistical mechanics are analysed in this section. We will use the notation ϕ for we refer to bosonic fields and ψ for fermionic fields. We restrict ourselves to grand canonical ensemble in order to include the effects of chemical potentials.

To obtain the equilibrium properties of a system in the canonical grand ensemble, we define the density matrix:

$$\hat{\rho} = \frac{1}{Z} \exp\left[-\beta\left(H - \sum_j \mu_j Q_j\right)\right] \tag{5.1}$$

where the dynamics of the system it is described by the Hamiltonian H, Q_j denotes all the conserved charges, μ_j the chemical potentials associated to those charges, and $\beta = 1/T$ being the inverse of the temperature. From this definition, we can determine the expectation values of some observable $\mathcal{O}$, by means:

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\hat{\rho}\mathcal{O}). \tag{5.2}$$

In general a chemical potential μ can be associated to any conserved charge of the system. Then, these conserved charges can be the number of particles, electric charge, isospin, or as we will see below conserved charges in the internal space of a gauge theory. Thus, the system can be described by:

$$Z = \mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{Q})}\right), \tag{5.3}$$

where the operators $\hat{Q}$ denote the conserved charges. In the case that exist several conserved charges simultaneously, the operators must be diagonalized in the same basis, i.e., they must commute. Therefore, a simple way to include chemical potentials in the theory is:

$$H \to H - \mu_j \cdot \hat{Q}_j, \tag{5.4}$$

such that

$$\begin{aligned} \left[H, \hat{Q}_j\right] &= 0, \\ \left[\hat{Q}_i, \hat{Q}_j\right] &= 0. \end{aligned} \tag{5.5}$$

A requirement for chemical potentials is that the corresponding charges must be additive observables, i.e., the eigenvalues of an operator for a composite system is the sum of eigenvalues for the components of the system.

It is known that the most important quantity in equilibrium statistical mechanics is the partition function. It is a function of the temperature T and the chemical potential in the grand canonical ensemble, and is enough to determine other thermodynamical properties of the system. The partition function is denoted by Z, as it is shown in Eq. (5.1).

We know that for canonical ensembles, the partition function can be written as a trace of the operator $e^{-\beta\hat{H}}$, which in the basis of eigenstates of energies $|n\rangle$, with eigenvalues of energies E_n, is given by:

$$Z = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right) = \sum_n \langle n|e^{-\beta\hat{H}}|n\rangle = \sum_n e^{-\beta E_n}. \tag{5.6}$$

Here the sum correspond to all the eigenstates of the system. In the case in which there is also a chemical potential the expression is

$$Z(T,\mu,V) = \mathrm{Tr}\exp\left[-\beta(\hat{H}-\mu\hat{N})\right] = \sum_{\phi_n}\langle\phi_n|\exp\left(-\beta(\hat{H}-\mu\hat{N})\right)|\phi_n\rangle, \tag{5.7}$$

where ϕ_n denote the eigenstates of the operator $(\hat{H}-\mu\hat{N})$. All the thermodynamical properties can be derived from Z, for example the pressure p, energy E, entropy S, and the number of particles N. Many times (by practical purposes) it is useful to define the thermodynamical potential, known as the free energy F, by $F(T,\mu,N) = -T\ln Z(T,\mu,N)$, such that

$$p = -\frac{\partial F}{\partial V}, \quad S = -\frac{\partial F}{\partial T}, \tag{5.8}$$

$$N = -\frac{\partial F}{\partial \mu}, \quad E = F + TS + \mu N. \tag{5.9}$$

Continuing with our study about the partition function in the continue case, we can consider this integral as a path integral. In fact, the matrix elements $\langle\phi_n| - \beta(\hat{H}-\mu\hat{N})|\phi_n\rangle$ it can be interpreted as a transition amplitude in the imaginary time $\tau = it$, carrying states from $\tau = 0$ to $\tau = \beta$, obviously in the case that the "temporal evolution" is governed by the operator $(\hat{H}-\mu\hat{N})$. Thus, we have

$$\begin{aligned} &\langle\phi_1|\exp\left[-\beta(\hat{H}-\mu\hat{N})\right]|\phi_0\rangle = \\ &= \int_{\phi(\vec{x},0)=\phi_0}^{\phi(\vec{x},\beta)=\phi_1}\mathcal{D}\phi\int\mathcal{D}\pi\exp\left[\int_0^\beta d\tau\int d^3\vec{x}\left(i\pi(\vec{x},\tau)\dot{\phi}(\vec{x},\tau) - \mathcal{H}(\pi,\phi) + \mu\mathcal{N}(\pi,\phi)\right)\right]. \end{aligned} \tag{5.10}$$

where π denotes the canonically conjugate fields of $\phi(\vec{x},\tau)$, such that are solved from

$$i\dot{\phi} = \frac{\partial(\mathcal{H}-\mu\mathcal{N})}{\partial\pi}, \tag{5.11}$$

On the other hand, $\mathcal{H}$ and $\mathcal{N}$ correspond to Hamiltonian and number densities, respectively. Due to our case of study, the Lagrangian density will be quadratic in π [58], so that a Gaussian integration is possible. The partition function reads

$$Z(T,\mu,V) = \int \mathcal{D}\phi \exp\left(\int_0^\beta d\tau \int d^3\vec{x}\mathcal{L}'(\phi,\dot{\phi})\right), \tag{5.12}$$

It is common to write the subscript "periodic" to indicate explicitly that the integration is performed over all the fields satisfying boundary conditions in τ. We will not follow this notation, but the periodicity of the fields will be explicated below.
The Lagrangian density $\mathcal{L}'(\phi,\dot{\phi})$ in Eq. (5.12) differs from the Lagrangian defined of the vacuum theory when the chemical potentials are non-zero. In other words, we have

$$\mathcal{L}'(\phi,\dot{\phi}) = i\pi(\phi,\dot{\phi})\dot{\phi} - \mathcal{H}(\pi(\phi,\dot{\phi}),\phi) + \mu\mathcal{N}(\pi(\phi,\dot{\phi}),\phi), \tag{5.13}$$

Due to the presence of the trace in Eq. (5.7) it follows that bosonic fields are periodic

$$\phi(\vec{x},0) = \phi(\vec{x},\beta), \tag{5.14}$$

while for the fermionic fields will be anti-periodic, $\psi(\vec{x},0) = -\psi(\vec{x},\beta)$. It is appropriate to expand the fields in Fourier series at imaginary-time. Thus, from Eq. (5.14) follows that

$$\phi(\vec{x},\tau) = T\sum_{n=-\infty}^{+\infty} e^{i\omega_n\tau}\phi_n(\vec{x}), \tag{5.15}$$

where the frequencies ω_n are given by:

$$\omega_n = \begin{cases} 2\pi nT, & \text{for bosonic fields,} \\ (2n+1)\pi T, & \text{for fermionic fields.} \end{cases} \tag{5.16}$$

known as the Matsubara frequencies. This Fourier expansion gives us a new interpretation of the finite-theory as a tridimensional theory with infinite fields $\phi_n, n\in\mathbb{N}$.

We now consider a gauge model which contains a charge density $q = eN/V$, with e the fundamental charge and N the number of external particles. It charge can be add to the partition function as:

$$Z(T,q) = \int \mathcal{D}\phi \exp\left[\int_0^\beta d\tau \int d^3\vec{x}(\mathcal{L}_E + iqA_0)\right]. \tag{5.17}$$

Therefore, the chemical potential will be the conjugate variable N:

$$\begin{aligned} \mu_q &= -T\frac{\partial}{\partial N}\ln\left(Z(T,q)\right), \\ &= -\frac{ie}{Z(T,q)}\int \mathcal{D}\phi\left(\frac{T}{V}\int_0^\beta d\tau\int d^3\vec{x}A_0\right)\exp\left[\int_0^\beta d\tau\int d^3\vec{x}(\mathcal{L}_E + ieA_0)\right], \\ &= -\frac{ieT}{V}\int_0^\beta d\tau\int d^3\vec{x}A_0 = -ie\langle A_0\rangle. \end{aligned} \tag{5.18}$$

This result indicates that, the chemical potentials can be associated with the temporal components of the fields.
On the other hand, for a gauge theory one know that the thermal equilibrium is achieved when the free energy is stationary with respect to the fluctuations around to the medium value of the gauge field $\langle A_0 \rangle$. Therefore, the thermal equilibrium is achieved only when:

$$Q \sim \frac{\partial F}{\partial \mu_Q} \sim \frac{\partial F}{\partial \langle A_0 \rangle} = 0. \tag{5.19}$$

This tells us that for the gauge charges case, the chemical potentials must be choose in such way that the system be neutral with respect to them.

Next, we will show three cases of field theories, where a relation between the temporal components of the gauge fields and the chemical potentials is realized. We start with a simple scalar field, then we analyse the Yang-Mills gauge case in $SU(2)$, and finally coupled Yang-Mills theory with scalar and fermionic fields.

5.1.1 Scalar field case

Let us consider a non-interacting charged scalar field with the Lagrangian density

$$\mathcal{L} = \partial^\rho \phi^* \partial_\rho \phi - m^2 \phi^* \phi, \quad \rho = 1, 2 \tag{5.20}$$

with

$$\phi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2), \quad \phi^* = \frac{1}{\sqrt{2}}(\phi^1 - i\phi^2). \tag{5.21}$$

There is a gauge symmetry $U(1)$, and then a conserved current given by

$$Q_\rho = i(\phi^* \partial_\rho \phi - \phi \partial_\rho \phi^*). \tag{5.22}$$

The Hamiltonian density has the form

$$\mathcal{H} = \frac{1}{2}\left[\pi_\rho \pi_\rho + (\vec{\nabla}\phi_\rho) \cdot (\vec{\nabla}\phi_\rho) + m^2 \phi_\rho \phi_\rho\right], \tag{5.23}$$

with π_ρ the momenta associated to the fields ϕ_ρ. From these expressions, we can build (in the euclidean-time) the partition function associated to (5.20) in presence of a chemical potential μ, which acquires the following form

$$\begin{aligned} Z &= N \int [d\pi_1][d\pi_2] \int [d\phi_1][d\phi_2] \exp\left[\int_0^\beta d\tau \int_V d^3x \left(i\pi_1 \frac{\partial \phi_1}{\partial \tau} + i\pi_2 \frac{\partial \phi_2}{\partial \tau} + \mu(\phi_2 \pi_1 - \phi_1 \pi_2) \right.\right. \\ &\left.\left. -\frac{1}{2}(\pi_1)^2 - \frac{1}{2}(\pi_2)^2 - \frac{1}{2}(\vec{\nabla}\phi_1)^2 - \frac{1}{2}(\vec{\nabla}\phi_2)^2 - \frac{1}{2}m^2(\phi_1)^2 - \frac{1}{2}m^2(\phi_2)^2 \right)\right]. \end{aligned} \tag{5.24}$$

Following the calculation of Bernard [59], it is convenient realize a careful evaluation of the $\int [d\pi]$, by an interchange between the integrals and sums. The full computation is realized in the Kapusta's work in [9]. It and others field theories are analyzed in details under the presence of chemical potentials.
Therefore, evaluating the Gaussian integration over the momenta, the partition function reads

$$\begin{aligned} Z &= [\tilde{N}(\beta)]^2 \int [d\phi_1][d\phi_2] \exp\left(-\frac{1}{2} \int_0^\beta d\tau \int_V d^3x \left[\left(\frac{\partial \phi_1}{\partial \tau} \right)^2 + \left(\frac{\partial \phi_2}{\partial \tau} \right)^2 + (\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2 \right.\right. \\ &\left.\left. + m^2(\phi_1)^2 + m^2(\phi_2)^2 + 2i\mu \left(\phi_1 \frac{\partial \phi_2}{\partial \tau} - \phi_2 \frac{\partial \phi_1}{\partial \tau} \right) - \mu^2[(\phi_1)^2 + (\phi_2)^2] \right] \right). \end{aligned} \tag{5.25}$$

Finally, considering the forms of the fields in (5.21) and integrating by parts, it yields

$$Z = [\tilde{N}(\beta)]^2 \int [d\phi][d\phi^*] \exp\left(\int_0^\beta d\tau \int_V d^3x \{-\phi^*[(\partial_0 + i\mu)^2 - \vec{\nabla}^2 + m^2]\phi\} \right). \tag{5.26}$$

It is the final form of the partition function in presence of a background charge (chemical potential), which is only affected in the temporal-component of the derivative by the shift $\partial_0 \to \partial_0 + i\mu$. It is direct show that over a euclidean metric the shift is $\partial_0 \to \partial_0 - \mu$.

5.1.2 Non-abelian case

Let us consider a pure Yang-Mills theory described by the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\rho\sigma} F^a_{\rho\sigma}, \tag{5.27}$$

$$F^a_{\rho\sigma} = \partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + g f^{abc} A^b_\rho A^c_\sigma, \tag{5.28}$$

where f^{abc} are the structure constants of the group. For $SU(2)$, one has $f_{abc} = \epsilon_{abc}$, then by Noether's theorem [60], the global symmetry implies that there is a conserved current Q^a_ρ given by

$$Q^a_\rho = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_n)} \frac{\delta \phi_n}{\partial \epsilon}. \tag{5.29}$$

Here ϵ^a are infinitesimal parameters of the gauge transformation and ϕ_n denotes some arbitrary fields. Since the Lagrangian (5.27) is invariant under the infinitesimal gauge transformation $\delta A^a_\rho = \frac{1}{g}\partial_\rho \epsilon^a - f^{abc} A^b_\rho \epsilon^c$, then our charge reads

$$Q^a_\rho = -\epsilon^{abc} F^b_{\rho\sigma} A^c_\sigma. \tag{5.30}$$

In this case the charge Q^a_ρ is associated to the diagonalized generator τ_3 of the $SU(2)$ group, such that commutes with the others elements of the group. In the language of group theory, τ_3 belongs to the $SU(2)$ Cartan subalgebra [61] (for more details see appendix A).

Our form to evaluate the partition function will be defining the Hamiltonian density associated to (5.27), given by

$$\mathcal{H}_{YM} = \frac{1}{2}\Pi^i_a \Pi^i_a + \frac{1}{4} F^a_{ij} F^{ija}, \quad i,j = 1,2,3. \tag{5.31}$$

Since the chemical potential is associated to Q^a_0, then the partition function takes the form

$$\begin{aligned} Z &= N(\beta) \int \prod_a [dP^a_j] \int [dA^a_j] \exp\left\{ \int_0^\beta d\tau \int d^3x [i P^a_j \dot{A}^a_j - \mathcal{H}(A^a_j, P^a_j) + \mu Q^a_0(P^a_j)] \right\} \\ &= N(\beta) \int \prod_a [dP^a_1][dP^a_2][dP^a_3] \int [dA^a_1][dA^a_2][dA^a_3] \exp\left\{ \int_0^\beta d\tau \int d^3x \left[-\frac{1}{2}\left(P^1_a - B^1_a\right)^2 \right.\right. \\ &+ \left.\left. \frac{1}{2} B^1_a B^1_a - \frac{1}{2}\left(P^2_a - B^2_a\right)^2 + \frac{1}{2} B^2_a B^2_a - \frac{1}{2}\left(P^3_a - B^3_a\right)^2 + \frac{1}{2} B^3_a B^3_a - \frac{1}{4} F^a_{ij} F^{ija} \right] \right\} \end{aligned}$$

where B_a^i is defined as $B_a^i \equiv F_a^{0i} + \mu\epsilon_{abc}A_c^i$. Realizing the integration over the momenta, we obtain

$$Z = N(\beta)(2\pi)^{3/2}\prod_a \int [dA_i^a] \exp\left(\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[(B_a^1)^2 + (B_a^2)^2 + (B_a^3)^2 - \frac{1}{2}F_{ij}^a F^{ija}\right]\right). \quad (5.32)$$

From the definition of the B_i^a fields, the compact term $F_{\rho\sigma a}F^{\rho\sigma a}$ it is produced shifting the usual field A_ρ as

$$A_a^\rho \to A_a^\rho + \frac{\mu}{g}\delta_{\rho 0}\tau^a\delta_{a3}. \quad (5.33)$$

Therefore, we arrived to

$$Z = N'(\beta)\int \mathcal{D}A\delta(\partial_\rho A_a^\rho)\exp\left(\int_0^\beta \int d^3x \mathcal{L}_{\text{eff}}(A, i\dot{A})\right) \quad (5.34)$$

where the effective Lagrangian is defined by

$$\mathcal{L}_{eff} = -\frac{1}{4}\mathcal{F}_{\rho\sigma}^a\mathcal{F}_a^{\rho\sigma} = \partial^\rho A_a^\sigma - \partial^\sigma A_a^\rho - g\epsilon_{abc}\left(A_b^\rho + \frac{\mu}{g}\delta^{\rho 0}\delta_{b3}\right)\left(A_c^\sigma + \frac{\mu}{g}\delta^{\sigma 0}\delta_{c3}\right). \quad (5.35)$$

In conclusion the non-abelian case has as usual prescription transferred a shift in the vectorial gauge field A_ρ. This also is a simple prescription to introduce chemical potentials into an abelian gauge theory.

As a last example, let us consider a $SU(2)$ non-abelian gauge field theory with a fermion and a scalar field transforming in the fundamental representation of the gauge group. The density Lagrangian is given by

$$\mathcal{L} = D_\rho\Phi^\dagger D^\rho\Phi - m^2\Phi^\dagger\Phi - \frac{1}{4}F_{\rho\sigma}^a F^{\rho\sigma a} + \bar{\psi}i\not{D}\psi. \quad (5.36)$$

This theory has two conserved charges, associated to the baryon number and the third component of isospin,

$$\begin{aligned} B &= \int d^3\vec{x}\bar{\psi}\gamma_0\psi, \\ Q_3 &= \int d^3\vec{x}\left(\frac{1}{2}\bar{\psi}\gamma_0\tau^3\psi - \frac{i}{2}\left[(D_0\Phi)^\dagger\tau^3\Phi - \Phi^\dagger\tau^3 D_0\Phi\right] - \epsilon^{3bc}A^{\sigma b}F_{\rho\sigma}^c\right), \end{aligned} \quad (5.37)$$

with γ_ρ denoting the usual three-dimensional Pauli-matrices. Thus, in order to insert chemical potential in the theory, we must insert two chemical potentials associated to each charge of Eq.(5.37). Therefore, the partition function obtained (already realized the integration over momenta), has the following form:

$$Z = \int \mathcal{D}\phi \exp\left[\int_0^\beta d\tau \int d^3\vec{x}(\mathcal{L}_E + \mu_B\bar{\psi}\gamma_0\psi)\right] \quad (5.38)$$

where μ_B corresponds to the chemical potential associated to the baryonic number, and $\mathcal{L}_E$ is the Lagrangian density obtained from the original Lagrangian (5.36) going to imaginary time and by making the change

$$A_0^3 \to A_0^3 - \frac{i\mu_Q}{g}. \quad (5.39)$$

with μ_Q the chemical potential associated to the conserved charge of the third component isospin. Once more, we thus see that the chemical potentials can be interpreted as background fields for the temporal components of the gauge fields.

In summary, the rule to insert a chemical potential in a field theory is thus to perform a shift on A_0 as a constant imaginary term $-i\mu$. As we see this term is considered not dynamical. This leads to consider the chemical potential as a background field which does not fluctuate.
In the next section, we will deepen this idea in detail, and we will explain the relation of the effects of chemical potential in the usual calculations of functional integrations [62].

5.2 The background field method

In this section we will analyze the tools of the Background Field Method (BFM) following the line of Refs. [62, 63, 64].
This method was first introduced by DeWitt [62] for one-loop process. Later on, 't Hooft in [63] reformulated this approach for multi-loops computations. Due to its effectiveness, BFM have been useful for studies of quantum theory of gravity [65], and also in QCD, e.g for the computations of β-function [64, 66, 67]. Although an adequate choice of the background field reduces considerably the computations, for example the elements of the $S-$matrix or the independence of the quantum field in the gauge fixing condition, the BFM still remains complicated to apply to the renormalization of the electroweak interactions in the Standard Model. However, it should be noted that recent works have made possible the application of BFM to QCD and electroweak interactions of Standard Model [68].

This method is useful every time the fields of the path integral can be divided into a "classical part" which does not fluctuate, and a quantum fluctuate.
In the first part, we present the generating functional for the connected and irreducible Green's functions of the conventional theory and in the BFM. In the final part is developed an equivalence between the BFM and the conventional approach.

5.2.1 Background field method for Yang-Mills theories

Let us consider the generating functional for pure Yang-Mills field, [1]

$$Z[J] = \int \mathcal{D}A \det\left[\frac{\delta G^a}{\delta \omega^b}\right] \exp\left(i \int d^4x \left[\mathcal{L}(A) - \frac{1}{2\xi} G_a G^a + J_a^\rho A_\rho^a\right]\right), \tag{5.40}$$

with the usual definitions given in (5.27). G^a is the gauge-fixing term, where we will consider the covariant Landau gauge: $G^a = \partial^\rho A_\rho^a$. Moreover, $\delta G^a/\delta\omega^b$ is the derivative of the gauge-fixing term under an infinitesimal gauge transformations

$$\delta A_\rho^a = -f^{abc}\omega^b A_\rho^c + \frac{1}{g}\partial_\rho \omega^a. \tag{5.41}$$

Due to this last transformation, the field $F^{\rho\sigma}$ becomes $F_{\rho\sigma}^a - f^{abc}\omega^b F_{\rho\sigma}^c$ and $\mathcal{L}$ remains his form, i.e., is gauge-invariant.

[1] Here fermions play no role in the background field method, they are treated as in the ordinary formalism, and therefore will be neglected.

We now define the partition function, analogous to Z, in the background field method. We denote this by $\tilde{Z}$. It is defined exactly like the conventional generating functionals except that the field in the classical Lagrangian is written not A_ρ but as $A_\rho + B_\rho$, where B_ρ is the background field, (we do not couple the background field to the source J.). Thus, our new partition function is defined by

$$\tilde{Z}[J,B] = \int \mathcal{D}A \det\left[\frac{\delta\tilde{G}^a}{\delta\omega^b}\right] \exp\left(i\int d^4x\left[\mathcal{L}(A+B) - \frac{1}{2\xi_A}\tilde{G}_a\tilde{G}^a + J^\rho_a Q^a_\rho\right]\right), \tag{5.42}$$

where $\delta\tilde{G}^a/\delta\omega^b$ is the derivative of the gauge-fixing term under the infinitesimal gauge transformation $\delta A^a_\rho = -f^{abc}\omega^b(A^c_\rho + B^c_\rho) + (1/g)\partial_\rho\omega^a$. We know that the Green's functions of the theory are generated by means the following definitions:

$$W[J] = -i\ln Z[J], \tag{5.43}$$

thus by means of a Legendre's transformation, we find that

$$\Gamma[\bar{A}] = W[J] - \int d^4x J^\rho_a \bar{A}^a_\rho, \quad \text{with} \quad \bar{A}^a_\rho = \frac{\delta W}{\delta J^\rho_a}. \tag{5.44}$$

Here the field $\bar{A}$ denotes an argument of the effective action Γ. Therefore, the derivative of the effective action with respect to $\bar{A}$ will be the irreducible one-particle Green's function of the theory [69].

Up to now, we have defined the generating functional $\tilde{Z}$ in the background field method, then we only need to determine the analogous quantities $\tilde{W}$ and $\tilde{\Gamma}$. In fact, we define

$$\tilde{W}[J,B] = -i\ln\tilde{Z}[J,B] \tag{5.45}$$

with the background effective action given by

$$\Gamma[\tilde{A},B] = \tilde{W}[J,B] - \int d^4x J^\rho_a \tilde{A}^a_\rho, \quad \text{with} \quad \tilde{A}^a_\rho = \frac{\delta\tilde{W}}{\delta J^\rho_a}. \tag{5.46}$$

On the other hand, we know that $\mathcal{L}(A)$ is invariant under the transformation: $\delta A^a_\rho = -f^{abc}\omega^b A^c_\rho + \frac{1}{g}\partial_\rho\omega^a$, then $\mathcal{L}(A+B)$ is invariant under

$$\begin{aligned} \delta A^a_\rho &= \bar{D}_\rho\omega^a - f^{acb}\omega^b A^a_\rho, & (5.47)\\ \delta B^a_\rho &= 0, & (5.48) \end{aligned}$$

where $\bar{D}_\rho = \partial_\rho + ig[B_\rho,\cdot]$ is the background covariant derivative. As we see, in the spirit of the background field, the field B is not dynamical and therefore does not fluctuate. Moreover, since we will consider situations in which the background field represents the chemical potential, B does not transforms under gauge transformation.

In order to maintain invariant the gauge term in $\tilde{Z}[J,B]$, then we choose the background field gauge condition:

$$\tilde{G}^a = \partial^\rho A^a_\rho + gf^{acb}B^b_\rho A^c_\rho. \tag{5.49}$$

Therefore, $\tilde{\Gamma}[\tilde{A},B]$ is invariant under the transformations:

$$\begin{aligned} \delta\tilde{A}^a_\rho &= \bar{D}_\rho\omega^a - f^{acb}\omega^b A^a_\rho & (5.50)\\ \delta B^a_\rho &= 0, & (5.51) \end{aligned}$$

in the background field gauge (5.49).

5.2.2 Background field method for scalar fields

Let us look how works the background field method over a simple field scalar ϕ [2]. In fact, we define the generating functional

$$\exp\left(-\frac{i}{\hbar}W[J]\right) \equiv \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}S[\phi] - \frac{i}{\hbar}\int dx J(x)\phi(x)\right), \tag{5.52}$$

where $S[\phi]$ denotes the full quantum action including the classical action. By means the Legendre transform we find that

$$\Gamma[\bar{\phi}] \equiv -W[J] - \int J\bar{\phi}, \quad \text{with} \quad \bar{\phi}(x) = \frac{\delta W[j]}{\delta J(x)}. \tag{5.53}$$

Once more, the field $\bar{\phi}$ denotes the argument of the effective action Γ. Now inserting the definition $J(x) = \delta\Gamma[\bar{\phi}]/\delta\bar{\phi}(x)$ and using Eq. (5.53), we arrived to

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{\phi}]\right) = \int \mathcal{D}\phi \exp\frac{i}{\hbar}\left(S[\phi] - \int dx(\phi - \bar{\phi})(x)\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}\right). \tag{5.54}$$

By means the use of the background field method we can shift the integration field as: $\phi \to \bar{\phi} + \sqrt{\hbar}\varphi$,

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{\phi}]\right) = \int \mathcal{D}\phi \exp\frac{i}{\hbar}\left(S[\bar{\phi} + \sqrt{\hbar}\varphi] - \sqrt{\hbar}\int dx\varphi(x)\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}\right). \tag{5.55}$$

The principal gain with this last expression is achieved by the following expansion around to $\hbar$:

$$S[\phi] = S_0[\phi] + (\hbar)^n S_n[\phi], \tag{5.56}$$

$$\Gamma[\bar{\phi}] = \Gamma_0[\bar{\phi}] + (\hbar)^n \Gamma_n[\bar{\phi}] \quad n \geq 1. \tag{5.57}$$

Obviously the terms $S_n, n \geq 1$ are clearly the counterterms to loop n, and $\Gamma_n[\bar{\phi}]$ the n-loop corrections to effective action. Finally, considering that S_0 describes a field like Eq.(2.22), and by means the expansion of perturbation theory, we obtain the 1-loop correction:

$$\Gamma_1[\bar{\phi}] = \Gamma_1[0] + \frac{i}{2}\ln\det\left(1 + V''(\bar{\phi})(\Box + m^2)^{-1}\right) + S_1[\bar{\phi}] \tag{5.58}$$

At first sight, this expression is the same standard 1-loop expansion that is obtained without the use of the background field method. In conclusion, the background field method over scalar fields is only like a minimal coupling to the usual field ϕ.

[2] Only for convenience in this section, we will use explicitly the Planck's constant $\hbar$.

Chapter 6

Background Field Method in Gribov-Zwanziger Approach

In the present chapter, we implement a chemical potential as our background field within the GZ scenario. Firstly, we discuss the Gribov copies in the Landau-DeWitt gauge. The resulting gauge-fixation for the Yang-Mills action is equivalent to consider the chemical potential as an external and non-dynamic field, proportional to the temporal-component of the gauge field. Finally, we will see that the chemical potential affects directly the solution of gap equation for the Gribov parameter, and therefore the gluon propagator.

6.1 Background field method in GZ action

We consider the thermal $SU(N)$ YM theory in $d = 4$ dimensions with Euclidean time $\tau \in [0, \beta]$, where $\beta = 1/T$ is the inverse temperature. The Euclidean action reads

$$S_{YM} = \frac{1}{4}\int_0^\beta d\tau \int d^4x F^a_{\rho\sigma} F^{\rho\sigma a} \tag{6.1}$$

where $F^a_{\rho\sigma} = \partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + g f^{abc} A^b_\rho A^c_\sigma$.

In the background field method one introduces a fixed background gauge field configuration B_ρ, through the splitting

$$A_\rho \to a_\rho \equiv A_\rho + B_\rho, \tag{6.2}$$

Here, the A and B fields has different roles in the theory. In this case, A is the quantum field, i.e., the variable of integration in the functional formalism, meanwhile B meet the role of a classical background field without dynamical (as we will see more later in our case the background field represents the chemical potential). In terms of a_ρ, our Landau gauge condition $G^a[A] = \partial_\rho A^a_\rho = 0$ for the quantum field takes the following form

$$\tilde{G}^a[B] \equiv \bar{D}_\rho A^a_\rho = 0. \tag{6.3}$$

It is known as the Landau-DeWitt (LDW) gauge fixing condition, and where $\bar{D}_\rho = \partial_\rho \delta^{ab} + g f^{acb} B^c_\rho$ is the "background" covariant derivative.

From this new assumption for the gauge fixing condition, one can rewritten the action (4.8) as

$$S_B^{gf} = \int_0^\beta d\tau \int d^{d-1}x \left(\frac{1}{4} F^a_{\rho\sigma} F^{\rho\sigma a} + \bar{c}^a \bar{D}_\rho(B) D^\rho(a) c^a - \frac{(\bar{D}_\rho(B) A^\rho)^2}{2\xi} \right) \tag{6.4}$$

with c and $\bar{c}$ denoting the ghost and antighost fields, respectively. In particular, the LDW gauge can be recovered taken the limit $\xi \to 0$ at the very end of each computation. On the other hand, we can identify the new FP operator $\bar{M} \equiv -\bar{D}_\rho D_\rho(a)$ which with our suitable choose of the background field is invertible. Analogously, the horizon condition (4.5) also is well-defined with the choose of our background field.

Since even the Landau-DeWitt gauge is plagued by Gribov copies, then the procedure of Gribov-Zwanziger applies. Therefore, the Gribov-Zwanziger local action in Eq. (4.13) acquires the following form:

$$\begin{aligned} S_{GZ} &= \int_0^\beta d\tau \int d^{d-1}x \left(\frac{1}{4} F^a_{\rho\sigma} F^{\rho\sigma a} + \bar{c}^a \bar{D}_\rho(B) D^\rho(a) c^a - \frac{(\bar{D}_\rho(B) A^\rho)^2}{2\xi} \right. \\ &\quad \left. + \bar{\omega}^{ac}_\rho \bar{D}^{ab}_\sigma(B) D^{bd}_\sigma(a) \omega^{dc}_\rho - g\gamma^2 f^{abc} A^a_\rho (\varphi^{bc}_\rho + \bar{\varphi}^{bc}_\rho) - \gamma^4 d(N^2-1) \right). \end{aligned} \tag{6.5}$$

The last action has the symmetry [9]:

$$S_{GZ}[\Phi] = S_{GZ}[U\Phi U^{-1}] \tag{6.6}$$

where Φ accounts for the quantizing fields, $A, \bar{c}, c, \bar{\omega}, \omega, \bar{\varphi}$, and φ, and U is a $SU(N)$ matrix. The gauge symmetry of (6.5) in the presence of a background field becomes

$$A_\rho + B_\rho \to A'_\rho + B'_\rho = U^{-1}\partial_\rho U + U^{-1} A_\rho U + U^{-1}(A_\rho + B_\rho)U. \tag{6.7}$$

At the infinitesimal level [70], $U \approx 1 + \omega$, $\omega \ll 1$, the above gauge symmetry reads

$$\delta A^a_\rho = -f^{abc}\omega^b (A^a_\rho + B^c_\rho) + \frac{1}{g}\partial_\rho \omega^a, \tag{6.8}$$

$$\delta B^a_\rho = 0. \tag{6.9}$$

Clearly, such transformation has no effects to B, because is just prescribed as a classical background field, i.e., it does not transform under gauge transformations. In our case, it background field will corresponds to the chemical potential which do not fluctuate like the quantum field A_ρ. These transformations induce an ordinary gauge transformation on $A + B$, i.e.,

$$\delta(A^a_\rho + B^a_\rho) = \frac{1}{g}\partial_\rho \omega^a - f^{abc}\omega^b (A^c_\rho + B^c_\rho). \tag{6.10}$$

Consequently, the symmetry transformation in Eq. (6.7) can be written as

$$\boxed{A_\rho \to A^U_\rho = U^{-1}\partial_\rho U + U^{-1} A_\rho U + U^{-1} B_\rho U - B_\rho}. \tag{6.11}$$

6.2 Chemical potential as background field

In order to describe the effects of this background field on the Gribov copies, and in particular on the solution of the gap equation, we will use the following prescription for the background field:

$$B^a_\rho = -\frac{i\mu}{g}\delta^{a3}\delta_{\rho 0}. \tag{6.12}$$

We have chosen the third component of the internal $SU(2)$ group. The i factor denotes explicitly the euclidean form as well as the apparition of the chemical potential like a charge density non-fluctuating.

From now on, we restrict ourselves only to $SU(2)$ case, thus the gauge fixing condition defined in Eq.(6.3) under the prescription (6.12) acquires the form

$$\tilde{G}_\rho[B]A^\rho = \partial_\rho A^a_\rho - i\mu\epsilon^{a3b}A^b_0 = 0. \tag{6.13}$$

Therefore, the Gribov's equation for field configurations satisfying the same LDW gauge fixing condition, i.e., $\bar{D}\cdot A = \bar{D}\cdot A^U = 0$, becomes

$$\partial^\rho A^U_\rho + g[B^\rho, A^U_\rho] = 0, \tag{6.14}$$

where A^U_ρ is defined in Eq. (6.11).
A straightforward exercise tells us that the background gauge field B_ρ satisfies also the LDW gauge fixing condition

$$\partial_\rho B^\rho + g[B^\rho, B_\rho] = 0. \tag{6.15}$$

In some sense, the last equation indicates that the background gauge field play the role of vacuum, i.e., $B_\rho = 0$, and due to this the "chemical vacuum" satisfies the chosen gauge fixing condition.

The following standard parametrization of the $SU(2)$-valued functions $U(x^i)$ will be useful

$$U(x^i) = Y^0\mathbf{1} + Y^a\tau_a, \quad U^{-1}(x^i) = Y^0\mathbf{1} - Y^a\tau_a, \tag{6.16}$$

$$Y^0 = Y^0(x^i), \quad Y^a = Y^a(x^i), \tag{6.17}$$

$$(Y^0)^2 + Y^aY_a = 1. \tag{6.18}$$

where, of course, the sum over repeated indices is understood also in the case of the group indices (in which case the indices are raised and lowered with the flat metric δ_{ab}). The $SU(2)$ generators τ_a satisfy the relation (A.20).

6.2.1 Gribov copies of the vacuum with chemical potential

From Eq. (6.11), the gauge transformation of the vacuum is given by

$$0 \to U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho. \tag{6.19}$$

Subsequently, the equation for the Gribov copies of the vacuum are

$$\partial^\rho\left(U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho\right) + g[B^\rho, U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho] = 0\ , \tag{6.20}$$

which under the prescription (6.12) and the functions (6.16) becomes

$$(\partial_\rho - i\mu\delta_{\rho 0}[\tau_3,\cdot])\times\Bigg(\epsilon^{abc}Y_a\partial^\rho Y_b\tau_c + Y^0\partial^\rho Y^c\tau_c - Y^c\partial^\rho Y^0\tau_c - \frac{2i\mu}{g}\delta^{\rho 0}\left(Y^aY^0\epsilon_{a3c} + Y^3Y^c\right)\tau_c$$
$$-\frac{2i\mu}{g}\delta^{\rho 0}\left(Y^0\right)^2\tau_3\Big) = 0. \tag{6.21}$$

As we see it corresponds to a system of coupled non-linear partial differential equations, and due to the intrinsic non-linear nature of this system, it is necessary to introduce a suitable technical tool to study and reduced the number of equations. The derivation of this equation for the vacuum is shown in Appendix C.

In particular, we will choose the geometrical interpretation of *hedgehog ansatz* as a form to reduce the system (6.21) to a single scalar non-linear differential equation. This systematic reduction is the first ansatz in general coordinates beyond spherical symmetry [72].

In fact, let us consider the standard spherically symmetric hedgehog ansatz:

$$U \;=\; \mathbf{1}\cos\alpha(x^\rho) + \hat{n}^a(x^\rho)\tau_a \sin\alpha(x^\rho), \tag{6.22}$$

where $\alpha(x^\rho)$ is a scalar function, and with the $\hat{n}_a$ normalized with respect to the internal metric δ_{ab} as

$$\delta_{ab}\hat{n}^a\hat{n}^b = 1, \tag{6.23}$$

which means that in terms of the variable Y^0 and Y^a corresponds to

$$Y^0(x^\rho) = \cos\alpha(x^\rho), \quad Y^a(x^\rho) = \hat{n}^a \sin\alpha(x^\rho). \tag{6.24}$$

6.2.2 Vacuum Gribov copies with T^3 topology

Here we will study the presence of Gribov copies with chemical potential in the space with T^3-topology. The choose of this kind of regions is due to the research in QCD theory, as for example glueballs, and also to study of the topology of the Yang-Mills configuration spaces [71], [72].

In this topology, the borderless flat spatial metric describing T^3 is

$$ds^2 = \sum_{i=1}^{3} \lambda_i^2 (d\phi_i)^2, \tag{6.25}$$

where $\lambda_i \in \mathbb{R}$ and the coordinates $\phi_i \in [0, 2\pi)$ correspond to the $i-$th factor S^1 in T^3 while λ_i represents the size of the i-th factor S^1.

Since the transformation must be *proper* [35, 72], i.e., to be everywhere smooth and vanish into infinity, then in this case U is proper if

$$U(\phi_i + 2m_i\pi) = U(\phi_i), \quad m_i \in \mathbb{Z}, \quad i = 1, 2, 3. \tag{6.26}$$

We now consider the following *ansatz* for the unitary vectors (time-independent) $\hat{n}^a$, defined by

$$\alpha = \alpha(\phi_1), \quad \hat{n}^1 = \cos(p\phi_2 + q\phi_3), \quad \hat{n}^2 = \sin(p\phi_2 + q\phi_3), \quad \hat{n}^3 = 0, \tag{6.27}$$

with p, q arbitrary integers. From this ansatz, the Eq. (C.10) reduces to the following single scalar non-linear differential equation:

$$\frac{d^2\alpha}{d\phi_1^2} = \xi \sin(2\alpha), \tag{6.28}$$

with

$$\xi = \frac{\lambda_1^2}{2}\left[\left(\frac{p}{\lambda_2}\right)^2 + \left(\frac{q}{\lambda_3}\right)^2\right] - \left(\frac{\sqrt{2}\mu\lambda_1}{\sqrt{g}}\right)^2. \tag{6.29}$$

The Eq. (6.28) can be reduced to a first order conservation law

$$\begin{aligned} V &= \frac{1}{2}\left[\left(\frac{d\alpha}{d\phi_1}\right)^2 + \xi\cos(2\alpha)\right] \\ \Rightarrow \phi_1 - \phi_0 &= \pm\int_{\alpha(\phi_0)}^{\alpha(\phi_1)} \frac{dy}{\sqrt{2V - \xi\cos(2y)}}. \end{aligned} \tag{6.30}$$

where ϕ_0, V are integration constants. Since ϕ_1 belongs to the range $[0, 2\pi)$, so let us take $\phi_1 = 2\pi$ and $\phi_0 = 0$. In turn, the condition (6.26) implies $\alpha(\phi_1) = \alpha(\phi_0) + 2\pi k$, where $k \in \mathbb{Z}$. Taking this into account, we have for (6.30) the following expression

$$2\pi = \pm\frac{1}{\xi^{1/2}}\int_{\alpha(0)}^{\alpha(0)+2\pi k} \frac{dy}{\sqrt{W - \cos(2y)}}, \quad W = \frac{2V}{\xi}$$

then the integrand must be well defined at least in the range $y \in (0, 2\pi k)$. For this, it is necessary that $W > 1$ because $\cos(2y) \in [-1, 1]$.

In order to analyse explicitly the effect of the chemical potential on the Gribov copies for this ansatz, let us consider the weight of a copy A^U as its norm in the functional space [74]

$$N[U] = \int_{T^3} d^4x\sqrt{g}\mathrm{Tr}\left[(U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho)^2\right] \tag{6.31}$$

where g corresponds to the determinant of the metric given in Eq. (6.25). Therefore, the norm of the copy in presence of chemical potential acquires the following form

$$\begin{aligned} N[U] &= \frac{(2\pi)^2\lambda_2\lambda_3}{\lambda_1}\int_0^{2\pi} d\phi_1\left[\left(\frac{d\alpha}{d\phi_1}\right)^2 + 2\xi\sin^2\alpha\right], \\ &= \frac{(2\pi)^2\lambda_2\lambda_3}{\lambda_1}\int_0^{2\pi} d\phi_1\left[2V + 3\xi\sin^2\alpha(\phi_1) - \xi\cos^2\alpha(\phi_1)\right], \end{aligned} \tag{6.32}$$

where in the last equality we used the definition (6.30) of the constant V. In the figure 6.1, we show the curve of the norm $N[U]$ for values $p = q = \lambda_i = 1$ increasing when $r_0 \equiv -i\mu$ grows both for $k = 1$ and $k = 2$. In some sense, the importance of the Gribov copies decreases when the chemical potential acquires high values. Moreover, it is possible to carry out a thorough analysis outside the range studied here, for example $|p| > 1$ and $|q| > 1$. However, just with simple calculation one can suggests that the Gribov gap equation in presence of chemical potential also should be affected.
It is worth to emphasize [74] that the larger is $N[U]$, the less relevant is the copy A^U (at least from the path integral point of view). Thus the above numerical results suggest that the bigger is the chemical potential, the less relevant is the Gribov problem.

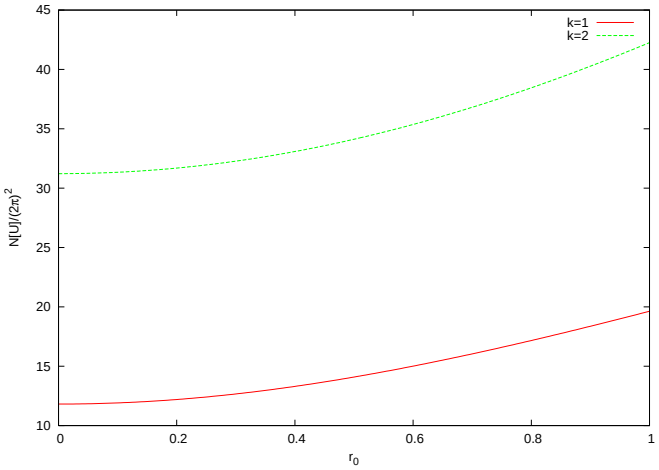


Figure 6.1: The norm of the copies (6.32) in the case $p = q = \lambda_i = 1$ versus the chemical potential $r_0 \equiv \mu$ for $k = 1$ (red line) and $k = 2$ (green line). It is worth mention that $\alpha(\phi_1)$ fulfil the condition (6.26).

6.2.3 Another example of Gribov copies in T^3

Now, let us consider a non-vacuum gauge field configuration $\bar{A}_\rho$ in the presence of chemical potential. A suitable family of gauge fields (like pure gauge) for which the generalized hedgehog ansatz (6.22) works is

$$\bar{A}_\rho = \eta(\phi_1)(U^o)^{-1}\partial_\rho U^o, \tag{6.33}$$

with

$$U^o = \hat{v}^a\tau_a, \quad (U^o)^{-1} = -U^o, \tag{6.34}$$

and

$$\hat{v}^1 = \cos(\bar{p}\phi_2 + \bar{q}\phi_3), \quad \hat{v}^2 = \sin(\bar{p}\phi_2 + \bar{q}\phi_3), \quad \hat{v}^3 = 0, \quad \bar{p}, \bar{q} \in \mathbb{Z}, \tag{6.35}$$

where $\eta(\phi_1)$ is a scalar function. In order to show that the $\bar{A}_\rho$ field also satisfies the LDW gauge fixing condition, we write it as

$$\bar{A}_\rho = \eta\hat{v}^a\partial_\rho(\hat{v}_a) + \eta\hat{v}_a\partial_\rho(\hat{v}_b)\tau_c\epsilon^{abc}. \tag{6.36}$$

Thus, the LDW gauge fixing condition becomes

$$\begin{aligned}
\partial_\rho\bar{A}^\rho + g[B_{\rho 3}, \bar{A}^\rho] &= \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{abc}\tau_c, \\
&= \sum_c \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{abc}\tau_c, \\
&= \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{ab1}\tau_1 + \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{ab2}\tau_2 + \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{ab3}\tau_3, \quad (\text{but} \quad \hat{v}^3 = 0), \\
&= 0 + 0 + \eta\hat{v}_a^{'}\hat{v}_b^{'}\epsilon^{ab3}\tau_3 \\
&= -\gamma^2(\bar{p}, \bar{q})\eta\hat{v}_{[b}\hat{v}_{a]}\epsilon^{(ab)3}\tau_3 \\
&= 0.
\end{aligned}$$

where the prime denotes derivative with respect to the ϕ_i angles. Therefore, this suitable gauge field configuration satisfies the LDW gauge fixing condition (6.3). Inserting $\bar{A}_\rho$ into the equation of Gribov's copies in Eq. (C.12), we obtain the following single scalar differential equation:

$$\alpha''(\phi_1) + L\sin(2\alpha) = 0, \tag{6.37}$$

where

$$\begin{aligned} L &= \frac{\eta(\phi_1)}{g} - \xi, \\ \xi &= \frac{\beta(p,q)}{2} - \left(\frac{\sqrt{2}\mu\lambda_1}{\sqrt{g}}\right)^2, \\ \beta(p,q) &= \lambda_1^2\left(\left(\frac{p}{\lambda_2}\right)^2 + \left(\frac{q}{\lambda_3}\right)^2\right). \end{aligned}$$

We observe that Eq.(C.12) reduces to the equation (6.37) of sine-Gordon type, with a vertical force proportional to $\eta(\phi_1)$ function, very similar to the obtained in [35]. Thus, one can constructs many explicit examples of copies using the freedom in choosing the function $\eta(\phi_1)$.
It is important remark that the equations of the Gribov's copies depend only of the gauge symmetry of the theory, and then it have no influence on some physical property of the theory. Once more, as in the previous case, the Gribov copies are affected by the chemical potential non-trivially, and as we will see in the next section the gap equation also.

6.3 Solving the GZ gap equation

6.3.1 The background field potential at one-loop

In order to determine the gap equation, we will proceed to calculate the effective potential of the Gribov-Zwanziger action at the one-loop approximation.
We know that the thermal effective action at one-loop is given by

$$\exp\left(\Gamma^{(1)}[\Phi_c]\right) = \int \mathcal{D}\Phi e^{\int_0^\beta d\tau \int d^4x \mathcal{L}_{GZ}^{quad}(\bar{x})} \tag{6.38}$$

with $\bar{x} = (i\tau, \vec{x})$, and $\mathcal{L}_{GZ}^{quad}$ corresponding to the free Gribov-Zwanziger Lagrangian density associated to (6.5). On the other hand, it is well known [69] that the effective potential corresponds to the effective action when the classical field is a constant, i.e.,

$$\Gamma^{(1)}[\Phi_c = cte = v] = \beta \int d^3x \varepsilon_v = \beta V \varepsilon_v. \tag{6.39}$$

The quadratic Gribov-Zwanziger Lagrangian is

$$\begin{aligned} \mathcal{L}_{GZ}^{quad} &= \frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma a} - \frac{(\bar{D}_\rho(B)A^\rho_a)^2}{2\xi} + \bar{c}^a\bar{D}^{ab}_\rho(B)D^{bd}_\rho(a)c^d + \bar{\varphi}^{ac}_\rho\bar{D}^{ab}_\sigma(B)D^{bd}_\sigma(a)\varphi^{dc}_\rho + \bar{\omega}^{ac}_\rho\bar{D}^{ab}_\sigma(B)D^{bd}_\sigma(a)\omega^{dc}_\rho \\ &\quad -g\gamma^2 f^{abc}A^a_\rho(\varphi^{bc}_\rho + \bar{\varphi}^{bc}_\rho) - \gamma^4 d(N^2-1). \end{aligned} \tag{6.40}$$

We use the equations of motion $\frac{\delta S_{GZ}}{\delta\varphi} = 0$ and $\frac{\delta S_{GZ}}{\delta\bar{\varphi}} = 0$, which give

$$\varphi^{bc}_\rho = \bar{\varphi}^{bc}_\rho = -\frac{1}{\bar{D}D}\gamma^2 g f^{abc}A^a_\rho. \tag{6.41}$$

Thus, if we use this result to rewrite $\mathcal{L}_{GZ}^{quad}$, we obtain

$$\mathcal{L}_{GZ}^{quad} \approx \frac{1}{2}A_{\rho}^{a}\Delta_{\rho\sigma}^{ab}A_{\sigma}^{b} + \bar{c}\bar{D}Dc + \bar{\omega}\bar{D}D\omega - \gamma^4 d(N^2-1) \tag{6.42}$$

where

$$\Delta_{\rho\sigma}^{ab} = \left[\left(-\bar{D}D - \frac{2g^2N\gamma^4}{\bar{D}D}\right)\delta_{\rho\sigma} - \bar{D}_{\rho}D_{\sigma}\left(\frac{1}{\xi}-1\right)\right]\delta^{ab}. \tag{6.43}$$

Utilizing Eqs. (6.38) and (6.39), and evaluating the limit $\xi \to 0$, then the background field potential reads

$$\varepsilon_v(T,\mu) = -\frac{d(N^2-1)}{2Ng^2}\lambda^4 + \frac{1}{2\beta V}(d-1)\mathrm{Tr}\ln\left(\frac{\bar{D}^4+\lambda^4}{-\bar{D}^2}\right) - \frac{1}{2\beta V}\mathrm{Tr}\ln(-\bar{D}^2). \tag{6.44}$$

where V is the spatial volume, $\lambda^4 = 2Ng^2\gamma^4$, with γ the Gribov parameter, and $\bar{D}$ the background covariant derivative in the adjoint representation.
After the Fourier transform, one obtains that the covariant derivative go to $\bar{D}^2 = (\omega_n - is\mu)^2 + \vec{p^2}$, where $\omega_n = 2\pi nT$ are the Matsubara frequencies, and $\vec{p}$ is the spacelike momentum component, and s is the isospin $SU(2)$ which take the values $-1, 0$ and $+1$.

6.3.2 Case for $T = 0$

We will proceed first to show the effective potential to GZ action (6.44) at zero temperature as, in this case, there is no need to introduce two different form factors in the Landau-De Witt propagator for the gluon as in [75] nor to consider the plasma approximation as in [76]. From (6.44) the effective potential without temperature acquires the form

$$\varepsilon_v(\mu) = -\frac{d(N^2-1)}{2Ng^2}\lambda^4 + \frac{1}{2V}(d-1)\mathrm{Tr}\ln\frac{\bar{D}^4+\lambda^4}{\Lambda^4} - \frac{d}{2V}(d-1)\mathrm{Tr}\ln\frac{-\bar{D}^2}{\Lambda^2}. \tag{6.45}$$

Here Λ^2 is a scale parameter in order to regularize the result.

Considering the general trace, under Matsubara formalism [55], one obtains

$$\begin{aligned}\frac{1}{\beta V}\mathrm{Tr}\ln(-\bar{D}^2+m^2) &= \sum_s\sum_{n=-\infty}^{+\infty}\int\frac{d^4p}{(2\pi)^4}\ln\left((-is\mu)^2+\vec{p}^2+m^2\right)\\ &\equiv \sum_s I(m^2,\mu,s).\end{aligned} \tag{6.46}$$

with the function I defined by

$$I(m^2,\mu,s) = \sum_{n=-\infty}^{+\infty}\int\frac{d^4p}{(2\pi)^4}\ln\left((-is\mu)^2+\vec{p}^2+m^2\right), \tag{6.47}$$

where m^2 is an arbitrary constant.

In order to calculate the momenta integral in Eq. (6.47), we make use of Zeta function regularization techniques. We can rewritten $I(m^2,\mu,s)$ as the derivative respect of some auxiliary variable ϵ and then taking the limit ϵ:

$$\ln x = -\lim_{\epsilon\to 0}\frac{d}{d\epsilon}x^{-\epsilon}. \tag{6.48}$$

From this prescription, (6.47) reads

$$\begin{aligned} I &= \sum_{n=-\infty}^{+\infty}\frac{d^4p}{(2\pi)^4}\ln\left(\frac{(-is\mu)^2+\vec{p}^2+m^2}{\Lambda^2}\right) \\ &= -\sum_{n=-\infty}^{+\infty}\int\frac{d^4p}{(2\pi)^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\left[(-is\mu)^2+\vec{p}^2+m^2\right]^{-\epsilon}\right). \end{aligned}$$

Defining a new variable t as $|\vec{q}| = t\sqrt{(-i\mu)+m^2}$ and passing to spherical coordinates, we get

$$\int d^4q = 2\pi^2\int_0^{+\infty} dtt^3((-is\mu)^2+m^2)^2, \tag{6.49}$$

which give us

$$I(\mu,m^2,s) = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(-\frac{\Lambda^{2\epsilon}}{8\pi^2}((-is\mu)^2+m^2)^{2-\epsilon}\int_0^{+\infty}dtt^3(1+t^2)^{-\epsilon}\right). \tag{6.50}$$

Carry out the last integral by means the change of variable $u = 1+t^2$, one obtains

$$\int_0^{+\infty}dtt^3(1+t^2)^{-\epsilon} = \frac{1}{2}\frac{1}{(\epsilon-2)(\epsilon-1)}. \tag{6.51}$$

So, we have

$$I(\mu,m^2,s) = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(-\frac{\Lambda^{2\epsilon}}{16\pi^2}\frac{((-i\mu)^2+m^2)^{2-\epsilon}}{(\epsilon-1)(\epsilon-2)}\right). \tag{6.52}$$

Finally, a direct computation of the derivative and the limit on ϵ allow us get

$$I(\mu,m^2) = \sum_s\frac{((-is\mu)^2+m^2)^2}{32\pi^2}\left(\ln\left(\frac{s^2(-is\mu)^2+m^2}{\Lambda^2}\right)-\frac{3}{2}\right). \tag{6.53}$$

Inserting now (6.53) in (6.57), and then in (6.45), we have

$$\begin{aligned} \varepsilon_v(\mu) &= -d\frac{(N^2-1)\lambda^4}{2Ng^2} - \frac{(d-1)\lambda^4}{32\pi^2}\left[\ln\left(\frac{\lambda^4+(i\mu)^4}{\Lambda^4}\right)+\ln\left(\frac{\lambda^4}{\Lambda^4}\right)-6\right] \\ &\quad +\frac{(d-1)(i\mu)^4}{32\pi^2}\left[\ln\left(\frac{\lambda^4+(i\mu)^4}{\Lambda^4}\right)-3\right] - \frac{d(i\mu)^4}{32\pi^2}\left[\ln\left(\frac{(i\mu)^4}{\Lambda^4}\right)-3\right] \\ &\quad -\frac{(d-1)(i\mu)^4\lambda^2}{8\pi^2}\arctan\left(\frac{\lambda^2}{(i\mu)^2}\right) \end{aligned} \tag{6.54}$$

where in the last term we took the principal branch of the ln in the complex plane [77]. In order to normalize the last equation, we shall choose Λ^2 in order that for $\mu = 0$ the solution is $\lambda_0 = 1$.

Because we are interested in solving the gap equation $\partial\varepsilon_v/\partial\lambda^2 = 0$, we can re-scale it in the following way

$$\frac{\partial\varepsilon_v(\mu,\lambda)}{\partial\lambda^2} - \frac{\lambda^2}{\lambda_0^2}\frac{\partial\varepsilon_v(\mu=0,\lambda)}{\partial\lambda^2} = -\frac{(d-1)\lambda^2}{16\pi^2}\left[\ln\left(\frac{\lambda^4+(i\mu)^4}{\lambda_0^4}\right)+\ln\left(\frac{\lambda^4}{\Lambda_0^4}\right)\right] - \frac{(d-1)(i\mu)^2}{8\pi^2}\arctan\left(\frac{\lambda^2}{(i\mu)^2}\right) = 0. \tag{6.55}$$

The equation (6.55) can be straightforwardly solved numerically for the critical chemical potential. In the figure (6.2) it is plotted the left hand side of he gap equation (6.55) as a function of λ^2 for different values of μ.

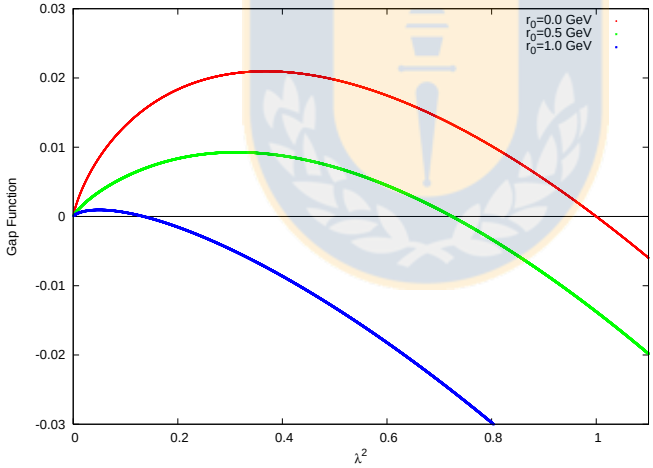


Figure 6.2: The gap equation (6.55) as a function of λ^2 for different values of $r_0 \equiv -i\mu$. The value of λ which corresponds the curve intersects the $x-$axis is the solution of (6.55).

Clearly we can see that the intersection value of the curve (which corresponds to the solution of the gap equation for a given value of μ) decrease when the chemical potential μ grows, as it is shown in Figure (6.3).

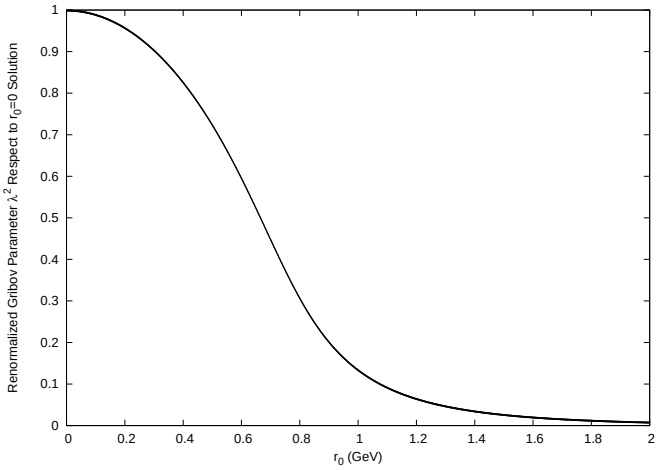


Figure 6.3: The zeros of the gap equation (6.55) decreasing as a function of the chemical potential $r_0 \equiv -i\mu$ when it last grows.

This analysis of the semi-classical Gribov gap equation in the chosen background field and of the dependence of the Gribov mass γ^2 on the chemical potential itself, confirms the results about the norm of the Gribov copy. We could interpret this as a theory which the gluons becomes less confined as the Gribov parameter γ reduces. Unfortunately, nowadays the physical interpretation of such bosonic chemical potential is rather a mistake in the case of non-perturbative gluons.

6.3.3 Case $T \neq 0$

This section only presents a brief introduction to the computational computation done for the case of finite-temperature of the effective potential to GZ action (6.44) [1]. Moreover, due to this last and the little knowledge about the chemical potential in gluons then just a possible statement is established.

Well, we consider the effective potential (6.44) to finite-temperature:

$$\varepsilon_v(T,\mu) = -\frac{d(N^2-1)}{2Ng^2}\lambda^4 + \frac{1}{2\beta V}(d-1)\mathrm{Tr}\ln\left(\frac{\bar{D}^4+\lambda^4}{-\bar{D}^2}\right) - \frac{1}{2\beta V}\mathrm{Tr}\ln(-\bar{D}^2). \tag{6.56}$$

The general trace is of the form

$$\begin{aligned}\frac{1}{\beta V}\mathrm{Tr}\ln(-\bar{D}^2+m^2) &= T\sum_s \sum_{n=-\infty}^{+\infty}\int \frac{d^3p}{(2\pi)^3}\ln\left((\omega_n - is\mu)^2 + \vec{p}^2 + m^2\right) \\ &\equiv \sum_s I_T(m^2,\mu,s,T).\end{aligned} \tag{6.57}$$

[1] It is necessary more computational power to see the full behaviour of the gap equation.

with the function I_T defined by

$$I_T(m^2,\mu,T) = T\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\ln\left((\omega_n - is\mu)^2 + \vec{p}^2 + m^2\right), \tag{6.58}$$

where m^2 is an arbitrary constant.

In order to calculate the momenta integral in Eq.(6.58), we make use of Zeta function regularization techniques. The prescription starts again rewritten the logarithm function as

$$\ln x = -\lim_{\epsilon\to 0}\frac{d}{d\epsilon}x^{-\epsilon}. \tag{6.59}$$

From this prescription, (6.58) reads

$$\begin{aligned}
I_T &= T\sum_{n=-\infty}^{+\infty}\frac{d^3p}{(2\pi)^3}\ln\left(\frac{(2\pi nT - i\mu)^2 + \vec{p}^2 + m^2}{\Lambda^2}\right)\\
&= -T\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\left[(2\pi nT - i\mu)^2 + \vec{p}^2 + m^2\right]^{-\epsilon}\right)\\
&= -T\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\sum_{n=-\infty}^{+\infty}\frac{\Lambda^{2\epsilon}}{4\pi}\int dp p^2\left[(2\pi nT - i\mu)^2 + p^2 + m^2\right]^{-\epsilon}\\
&= -T\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\sum_{n=-\infty}^{+\infty}\frac{\Gamma(\epsilon - 3/2)}{8\pi^{\frac{3}{2}}\Gamma(\epsilon)}\left[(2\pi nT - i\mu)^2 + m^2\right]^{\frac{3}{2}-\epsilon}\right)
\end{aligned}$$

where we had to introduce a renormalization scale Λ to get dimensional agreement, and $\Gamma(\epsilon)$ is the usual Gamma function defined as

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx. \tag{6.60}$$

From the integral representation of the Gamma function (6.60), we can be recast I_T to

$$I_T = -T\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\sum_{n=-\infty}^{+\infty}\frac{1}{8\pi^{\frac{3}{2}}\Gamma(\epsilon)}\int t^{\epsilon-5/2}e^{-t\left[(2\pi nT - i\mu)^2 + m^2\right]}\right).$$

This last integral can be realized by the change of variable $z = 4\pi^2T^2t \geq 0$, then I_T yields

$$\begin{aligned}
I_T &= -\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\frac{T^{2-2\epsilon}}{4^\epsilon\pi^{2\epsilon-3/2}\Gamma(\epsilon)}\int_0^\infty\sum_{n=-\infty}^{+\infty}z^{\epsilon-3/2}e^{-z\left(n^2 - \frac{i\mu n}{\pi T} + \frac{i\mu^2+m^2}{4\pi^2T^2}\right)}dz\right)\\
&= -\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left(\Lambda^{2\epsilon}\frac{T^{4-2\epsilon}}{4^\epsilon\pi^{2\epsilon-3/2}\Gamma(\epsilon)}\int_0^\infty z^{\epsilon-5/2}\sum_{n=-\infty}^{+\infty}e^{-z\left(n - \frac{i\mu}{2\pi T}\right)^2}e^{\frac{-zm^2}{4\pi^2T^2}}dz\right).
\end{aligned}$$

Using the Poisson rule for sums

$$\sum_{n=-\infty}^{+\infty}e^{-(n+x)^2\omega} = \sqrt{\frac{\pi}{\omega}}\left(1 + 2\sum_{n=1}^{+\infty}e^{-\frac{n^2\pi^2}{\omega}}\cos(2n\pi x)\right), \tag{6.61}$$

then after of a numerical integration over z, we arrived to

$$\begin{aligned} I_T &= -\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\Lambda^{2\epsilon}\left(\frac{\Gamma(\epsilon-2)T^{4-2\epsilon}}{4^{\epsilon}\pi^{2\epsilon-2}\Gamma(\epsilon)}\left(\frac{m^2}{4\pi^2T^2}\right)^{2-\epsilon}+\right. \\ &\left.\frac{T^{4-2\epsilon}}{4^{\epsilon-1}\pi^{\epsilon}\Gamma(\epsilon)}\left(\frac{m^2}{4\pi^2T^2}\right)^{1-\epsilon/2}\sum_{n=1}^{\infty}n^{\epsilon-2}\cosh\left(\frac{n\mu}{T}\right)K_{2-\epsilon}\left(\frac{nm}{T}\right)\right) \end{aligned} \tag{6.62}$$

where $K_n(x)$ is the modified Bessel function of the second kind. Taking the computations over the variable ϵ, we obtain

$$I_T[m^2,T,\mu]=\sum_{s=-1}^{+1}\left(\frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{m^2}{\Lambda^2}\right)-\frac{3}{2}\right]-\sum_{n=1}^{\infty}\frac{m^2T^2}{\pi^2n^2}\cosh\left(\frac{ns\mu}{T}\right)K_2\left(\frac{nm}{T}\right)\right). \tag{6.63}$$

From this last expression, we can see that the first term is the $T=0$ contribution, which it depends of the renormalization scale as it appears in [35]. The second term, is a new term which it indicates the finite-temperature correction. Unlike the analysis done in [78], here the full analysis of the convergence of I_T is more complicated because the argument of the Bessel function is complex, i.e., $m^2=\pm i\lambda^2$, so that oscillate and moreover it must decays faster than the hyperbolic cosine of the chemical potential in order to get an adequate convergence. This suggest that probably for $\lambda\approx 0$, it could pass through to cut the x-axis and then be solution also. Therefore, a computational power larger it is necessary in order to find the critical value for the chemical potential which the gap equation has no solutions.
However, an analytic asymptotic limit at the expression I_T for $m^2=0$ is possible,

$$I_T[m^2=0,T,\mu]=-\frac{T^4}{\pi^2}\sum_{s=-1}^{+1}[\mathrm{Li}_4(e^{\frac{s\mu}{T}})+\mathrm{Li}_4(e^{-\frac{s\mu}{T}})], \tag{6.64}$$

where $\mathrm{Li}_n(z)=\sum_{i=1}^{+\infty}\frac{z^i}{i^n}$ is the polylogarithm function.

On the other hand, the convergence of the second term in (6.63) can be examined by the asymptotic expansion of the Bessel function

$$K_n(z)\approx\sqrt{\frac{\pi}{2z}}e^{-z}\left(\sum_{k=0}^{+\infty}\frac{a_k(n)}{z^k}\right),\quad |\mathrm{Arg}(z)|\leq\frac{3}{2}\pi. \tag{6.65}$$

Thus, if we called $J[m^2,T,\mu]$ only to the finite-temperature correction in (6.63), and using the expansion (6.65), we obtain

$$\begin{aligned} J[m^2,T,\mu] &= -\sum_{n=1}^{\infty}\frac{m^2T^2}{\pi^2n^2}\cosh\left(\frac{n\mu}{T}\right)K_2\left(\frac{nm}{T}\right) \\ &\approx -\sum_{n=1}^{+\infty}\frac{\sqrt{m^2}T^{5/2}}{2\pi^{3/2}n^2}\left[e^{\frac{n}{T}(\mu-\sqrt{m^2})}+e^{\frac{-n}{T}(\mu+\sqrt{m^2})}\right]\mathcal{O}\left(\frac{1}{n}\right). \end{aligned} \tag{6.66}$$

This expression tell us that if $(\mu_{cr}-\sqrt{m^2})>0$ then the expression to J do not diverge. The existence of this critical value for the chemical potential μ_{cr}, it is closely related to the existence of the critical chemical potential of the gap equation.

Moreover, probably this critical value μ_{cr} could be a signal of confinement zones for gluons, but it is worth noting that nowadays experiments at LHC or RHIC have not yet confirmed the presence or signal of free gluons, and therefore associate densities as chemical potentials it is not understood. However, recent works in quasiparticles model [79, 80, 81] have released a possible effective mass (due to the interactions between them) for gluons, and the formation of a gluonic Bose-Einstein condensate in heavy ion collisions.

6.3.4 Gap equation in presence of chemical potential at finite temperature

With the expression for the general trace I_T, it only remains to replace in (6.44) and find the gap equation correspondent. So, for values $N = 2$ and $D = 4$ in terms of I_T, we have

$$\varepsilon_v = \sum_{s=-1}^{+1} \left(-\frac{3}{g^2}(\lambda^2)^2 - \frac{3(\lambda^2)^2}{2(4\pi)^2}\left[\ln\left(\frac{\lambda^2}{\Lambda^2}\right) - \frac{3}{2}\right] + \frac{3}{2}\left(J[i\lambda^2,\mu,T] + J[-i\lambda^2,\mu,T]\right)\right) \tag{6.67}$$

where

$$J[m^2,\mu,T] = -\sum_{s=-1}^{+1}\sum_{n=1}^{\infty} \frac{m^2T^2}{\pi^2 n^2}\cosh\left(\frac{ns\mu}{T}\right) K_2\left(\frac{nm}{T}\right). \tag{6.68}$$

Taking the derivative of this effective potential with respect to λ^2, we obtain

$$1 + \frac{g^2}{2(4\pi)^2}\ln\left(\frac{\lambda^2}{\Lambda^2}\right) - \frac{g^2}{2(4\pi)^2} - \frac{1}{6}\left(\frac{\partial J}{\partial\lambda^2}[i\lambda^2,T,\mu] + \frac{\partial J}{\partial\lambda^2}[-i\lambda^2,\mu,T]\right) = 0. \tag{6.69}$$

If we now [35] define, λ_0 to be a solution to the gap equation at $T = 0$:

$$1 = \frac{g^2}{2(4\pi)^2}\left[1 - \ln\left(\frac{\lambda_0^2}{\Lambda^2}\right)\right], \tag{6.70}$$

then we can subtract this equation from the general gap equation (6.69), by a Pauli-Villars procedure, we get

$$\ln\left(\frac{\lambda^2}{\lambda_0^2}\right) = \frac{8\pi^2}{3}\sum_s\left(\frac{\partial J}{\partial\lambda^2}[i\lambda^2,T,\mu,s] + \frac{\partial J}{\partial\lambda^2}[-i\lambda^2,\mu,T,s]\right). \tag{6.71}$$

Therefore, now all integrations in (6.71) are convergent.

The solution for the gap equation corresponds to the intersection of the curves of each side in (6.71). We will see that the existence of solutions, it depends of the temperature, and the critical value $2\mu_c^2 > \lambda$. This analysis allows us see that the chemical potential affects non-trivially the gap equation, and in particular the Gribov mass which appears in the gluon propagator associated to the GZ action. Therefore, possible assumptions about confined zones for gluons can be established.

Chapter 7

Conclusions and Perspectives

In this thesis, it has been shown that the Gribov copies equation is affected by the presence of chemical potential. In particular, some examples of copies have been constructed on the T^3-topology in which the norm of the Gribov copies satisfying the adequate boundary conditions increases when the value of the chemical potential grows, and thereby less important it turns out be the Gribov problem from the path integral point of view.
The semi-classical Gribov gap equation in the chosen gauge fixing condition (called also Landau-De Witt gauge fixing condition) and the dependence of the Gribov mass parameter on the chemical potential, confirm consistently that the chemical potential affects both the symmetry as well as the propagator of the theory. In other words, we conclude that the larger is the value of the chemical potential, the smaller is the corresponding Gribov mass. Therefore, we could interpret this as a sign that the theory becomes "less confined" as the Gribov parameter reduces and the chemical potential increases.
In addition, this analysis is consistent with the Gribov-Zwanziger approach both in their "kinematic part" (corresponding to the Gribov copies equation) as the "dynamic part" (corresponding to the Gribov gap equation). From the technical point of view, a relevant result of the present thesis is the extension of the Gribov-Zwanziger approach in the presence of a background field, whose interpretation can be more general than a chemical potential.
On the other hand, the physical interpretation of a bosonic chemical potential in the case of confined particles (such a gluons) has not been discussed in details in the literature. However, these results show that, at least, the problem is mathematically consistent. In some sense, this thesis gives rise to a new edge on the problem of confinement of gluons.

Appendix A

Group theory and Lie algebra

It is an apparent fact that the nature exhibits many symmetries, both exact and approximate. A symmetry is an invariance property of a system under a set of transformations. Symmetry transformations of physical systems have properties analogous to those of a mathematical group. From this motivation, in this appendix we briefly discuss properties about Lie groups and their algebras, and in particular the group which we are working, the $SU(2)$ group. We evoke on group representations, especially irreducible unitary representations. More details can be found in many places, for example, in a book on unitary symmetry [82].

A.1 A little group theory

A group is a set G equipped with a binary operation $G \times G : (a, b) \mapsto ab$ such that the following properties hold:

- $a(bc) = (ab)c$. (Associative law).
- There is an $e \in G$ such that $ea = ae = a$ for all $a \in G$. (Existence of identity).
- For every $a \in G$ there is a $b \in G$ such that $ab = ba = e$. (Existence of inverses).

The number of elements of a group can be finite, in which case the group is called a finite group. If all the elements of a group commute with one another, the group is said to be *abelian*. Otherwise the group is *non-abelian*. A *subgroup* of a group is a subset of elements which is itself a group under the same multiplication law. Every group has at least two subgroups: itself and the group consisting only of the identity. These are called *improper subgroups*; any others are called *proper subgroups.*

An element a belonging to G is said to be *conjugate* to an element b in G if there exists an element u in G such that $a = ubu^{-1}$. Let H be a subgroup of G, and let h be in H and g be in G. Then, we realize the product elements $h' = ghg^{-1}$ for all h. Then h' form a group H' which is isomorphic to H. If, for all g in G, the elements of H and H' are identical, then H is called an *invariant* or self-conjugate subgroup of G.

A.2 Group representations

The representation of a group, as in quantum mechanics, fulfils the role of representing the physical states by column vectors. A *representation* is a homomorphism between the group and a group of

linear operators which operate on a vector space. For example, for a finite-dimensional matricial representation, the representation would be a homomorphism between the group and the group of matrices. When the representation is isomorphic, the group is then told *faithful*.
If we consider a G group with elements g, and the elements of the representation by $D(g)$, then a similarity transformation is a transformation that acts on a representation D, and on a vector V as

$$D' = UDU^{-1}, \quad V = UV, \tag{A.1}$$

where U is the transformation's matrix. It is important remark that it transformation maintains unaltered the algebra of the transformed system.

If a representation of dimension $n+m$ is *reducible*, then exist some basis such that $D(g)$ acquires the following form in blocks:

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}, \tag{A.2}$$

where $A(g)$ and $B(g)$ are square matrices of $n \times n$ and $m \times m$ dimensions, respectively. Otherwise, the representation is *irreducible*. We restrict ourselves to irreducible representations.
If exist a matrix that commutes with all the other matrices of the irreducible representation, then it would be proportional to identity. This statement is called Schur's lemma.
Now, let's see how it works the irreducible unitary representations on a concrete case. Let us consider a transition matrix (ϕ, ψ) with ϕ and ψ state vectors describing physical states. A unitary transformation U_a which acts over the vectors as $\phi' = U_a\phi$ and $\psi' = U_a\psi$, satisfies that

$$(\phi', \psi') = (U_a\phi, U_a\psi) = (U_a^{-1}U_a\phi, \psi) = (\phi, \psi). \tag{A.3}$$

Thus, unitary transformations in quantum mechanics leave invariant the transition matrixes.
An example more interesting is the case of unitary representations of groups within the Hamiltonian formalism. In fact, let us consider the expression

$$H\psi_n = E_n\psi_n. \tag{A.4}$$

If we operate on this equation by the right with U_a, we get

$$U_aH\psi_n = U_aHU_a^{-1}U_a\psi_n = E_nU_a\psi_n, \tag{A.5}$$

$$U_aHU_a^{-1}U_a\psi_n = E_nU_a\psi_n. \tag{A.6}$$

Now, if H transforms as

$$H' = U_aHU_a^{-1} \tag{A.7}$$

then Eq. (A.5) becomes

$$H'\psi_n' = E_n\psi_n'. \tag{A.8}$$

But because U_a is unitary, then H is invariant under it transformation. Thus, one obtains

$$HU_a = U_aH \Rightarrow [H, U_a] = 0, \tag{A.9}$$

where $[H, U_a] = HU_a - U_aH$ is called the *commutator* of H and U_a. Then (A.8) becomes simply

$$H\psi_n' = E_n\psi_n'. \tag{A.10}$$

It tells us that the wave functions ψ' also are eigenfunctions of H with the same energy levels E_n. In conclusion, if there is any operator that commutes with U_a, then all the members of the multiplet will have the same eigenvalues of that operator.

A.3 Lie groups

Unlike an ordinary group, a Lie group has a feature more interesting. A Lie group has an infinite number of parameters which continuously vary. More precisely, a Lie group G has two structures: G is a group and also is a smooth real and/or complex manifold. The Lie group does not require that G be connected, therefore any finite group is a 0-dimensional Lie group.
An example of a Lie group are the usual rotations in two dimensions. Here the angle that produces the rotations continuously vary over the interval $[0, 2\pi]$. We note that this group is compact because the interval is bounded. This does not occur to translations where the parameter vary without a bound.

Now we discuss the concepts of simple and semi-simple Lie groups. An oversimplified definition, which is adequate for our purposes, is that a Lie group is *simple* if it is non-abelian and has no proper invariant Lie subgroups. On other hand, it is *semi-simple* if is non-abelian and has no an abelian invariant Lie subgroup. The direct product of simple and/or semi-simple Lie groups is semi-simple.
For example the group of the standard model $SU(3) \times SU(2) \times U(1)$ is not semi-simple because it has an abelian invariant subgroup $U(1)$. However, the group $SU(3) \times SU(2)$ is semi-simple, where each group by separated, $SU(3)$ and $SU(2)$, are simple.

A.4 Lie algebras

A *Lie algebra* of a group constitutes the group elements which differ only infinitesimally from the identity. From these elements we can construct operators called *generators* which allow us to obtain an unitary representation of the group. In other words, we can obtain all the elements of the group which can be generated by continuous transformations from the identity. There is one generator for each parameter of the group.
When the generators are Hermitian, i.e., $H = H^T$, then an unitary representation U_a of an arbitrary group element is given by

$$U_a = e^{-i\sum_j a_j T_j}, \quad j = 1, 2, ..., r \tag{A.11}$$

where T_j is the set of generators of the group, and a_j are r real parameters.

It can be shown that the T_j form a Lie algebra satisfying the algebraic equations

$$[T_i, T_j] = i \sum_{k=1}^{r} c_{ij}^k T_k. \tag{A.12}$$

Here c_{ij}^k are constants called the *structure constants* of the group. The structure constants of a Lie algebra can differ with different choices of generators. To the case of an abelian group, all its structure constants are zero.

From the Eq. (A.12) one can see that the commutator of any two members of the algebra, i.e., generators of the Lie group, is a linear combination of the members of the Lie algebra. Thus, a representation of a Lie algebra would be a set of matrices which obey the commutation relations (A.12).
Moreover, one can define the *rank* of the group as the maximum number of commuting generators of a Lie group. It is important remark that k commuting generators of a Lie group can be simultaneously diagonalized in a matrix representation.

If a Lie group of rank k is semi-simple and compact, then one can construct from the members of its Lie algebra, k non-linear invariant operators which commute with every member of the algebra. These operators are called *Casimir operators*.
An example, in particular, is the rotation group in three dimensions $R(3)$. This group is characterized by 3 parameters (for example the Euler-angles). From the definition of rotations it is possible show that the number of parameters of $R(n)$ is $\frac{n(n-1)}{2}$. For our case, the 3-dimensional parameters are denoted by J_x, J_y , and J_z . They satisfy the Lie algebra

$$[J_x, J_y] = iJ_z, \quad \hbar = 1. \tag{A.13}$$

This group is rank one because none of the J_i commutes with any other. Moreover, it group is semi-simple (actually simple), so that it has one Casimir operator J^2 defined by

$$J^2 = J_x^2 + J_y^2 + J_z^2. \tag{A.14}$$

By Schur's lemma, if J^2 commutes with the members of a irreducible representation, then it would be a multiple of the unit matrix.
In three dimensions, the rotation's group satisfies

$$[J_i, J_j] = i\epsilon_{ij}^k J_k, \quad i, j, k = 1, 2, 3. \tag{A.15}$$

where $\epsilon_{ij}^k = -\epsilon_{ji}^k$ is completely antisymmetric in its indices, and $\epsilon_{123} = 1$. Therefore the structure constants of $R(3)$ are given by ϵ_{ijk}.

A.4.1 Cartan subalgebra

We want to construct a canonical form of commutation relations modeled on the case that we are interested, $SU(2)$ group. For this purpose we define the notion of Cartan subalgebra. This is a *maximal abelian subalgebra* of g, where g is a complex semisimple Lie algebra, such that all its elements are diagonalisable in the adjoint representation. A Cartan subalgebra H satisfy

- If $h_1, h_2 \in H$ then $[h_1, h_2] = 0$,
- for all $v \in g$, if $[v, h] = 0$ for all $h \in H$ then $v \in H$,
- for all $h \in H$, the operator $ad(h)$ is diagonalisable,

where $ad(\cdot)$ symbolize the adjoint representation of the linear transformations of the G group's Lie algebra. The two first conditions imply that H is a maximal commuting subalgebra of g. It is straightforward to construct a subalgebra satisfying those conditions, by induction, but it is non-trivial to satisfy the last condition. Moreover, all complex semi-simple Lie algebra admits a Cartan subalgebra.
Cartan subalgebras are not unique, however it can be shown that if H_1 and H_2 are two Cartan subalgebras of a matrix Lie algebra $\mathcal{L}(G)$ then there exists some $g \in G$ such that $H_1 = gH_2g^{-1}$. Hence the dimension of all Cartan subalgebras are equal.
Let H be a Cartan subalgebra of g, call l its dimension, it is independent of the choice of H and it is called the rank of g. As we will see more later, for the case $SU(2)$, this has rank 1, (the choice of J_z for example).

A.5 Unitary groups and algebras

An unitary group in n-dimensional $U(n)$ is the group of $n \times n$ matrices U_a satisfying

$$U_a^\dagger = U_a^{-1}, \quad \text{with} \quad (U_{ij})^\dagger = U_{ij}^*, \tag{A.16}$$

where a stands to the parameters of the group. When the matrix is unitary, there are n^2 relations among these numbers, so that $U(n)$ is characterized by n^2 parameters. For example, the electromagnetism's group $U(1)$, it is one-dimensional and is characterized by an unique parameter: the phase $e^{i\theta}$. On the other hand, the special unitary groups $SU(n)$ have matrices with determinants equal to the unity. This last fact provides another relation so that $SU(n)$ is characterized by $n^2 - 1$ parameters. This group is semi-simple and rank equal to $n - 1$, so that $SU(n)$ has $n - 1$ Casimir operators.

As we will see in the next section, the group $SU(n)$ has $n - 1$ so-called fundamental representations. From these, there are two n-dimensional if $n > 2$, and there is only one fundamental (two-dimensional) representation if $n = 2$.

A.5.1 $SU(2)$ group

The group $SU(2)$ is the set of all two dimensional, complex unitary matrices with unit determinant. This constraint of unitary determinant removes one more parameter. To understand this idea, it is better to construct the generators of this group.
In fact, let us consider the usual Pauli spin matrices σ_1, σ_2 and σ_3, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.17}$$

The Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k. \tag{A.18}$$

However, it is common write this matrices as $\tau_i = i\sigma_i$, such that

$$[\tau_i, \tau_j] = -2\epsilon_{ij}^k \tau_k. \tag{A.19}$$

These new $\tau-$matrices obey the following three fundamental 2×2 matrices

$$\tau_i \tau_j = -\delta_{ij}\mathbf{1} - i\epsilon_{ijk}\tau_k. \tag{A.20}$$

where $\mathbf{1}$ is the identity 2×2 matrix and ϵ_{abc} are the totally antisymmetric Levi-Civita symbols with $\epsilon^{123} = \epsilon_{123} = 1$.
From the definition in (A.17), these three matrices are hermitian $\tau_i = \tau_i^\dagger$ and traceless $\mathrm{Tr}(\tau_i) = 0$.
Still another representation of the Lie algebra of $SU(2)$ or 3-dimensional rotations is given by the two-dimensional matrices σ_+, σ_-, and σ_3, where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.21}$$

The matrix σ_+ is called raising operator and σ_- is called lowering operator because of their action on the eigenvectors of the operator σ_3. The matrices σ_+ and σ_- can be written in terms of σ_1 and σ_2 as follows:

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2}, \quad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2}. \tag{A.22}$$

The $SU(2)$ Cartan subalgebra has one commuting operator associate to τ_3. This significance that none rotations around any other axis commute with τ_3, so the Cartan subalgebra is one-dimensional.

As we said previously, the rotations are represented by the matrices of the form

$$e^{-i\theta\vec{\sigma}\cdot\hat{\theta}/2} = \cos\left(\frac{\theta}{2}\right)\mathbf{1} - i(\vec{\sigma}\cdot\hat{\theta})\sin\left(\frac{\theta}{2}\right). \tag{A.23}$$

These matrices are 2×2 complex unitary matrices with unitary determinant. The determinant is unitary because

$$\det\left(e^{-i\theta\vec{\sigma}\cdot\hat{\theta}/2}\right) = e^{\mathrm{Tr}(-i\theta\vec{\sigma}\cdot\hat{\theta}/2)} = e^0 = 1. \tag{A.24}$$

These elements are described by a set of three real parameters $(\theta_x, \theta_y, \theta_z)$, which

$$\vec{\sigma}\cdot\hat{\theta} = \begin{pmatrix} \hat{\theta}_z & \hat{\theta}_x - i\hat{\theta}_y \\ \hat{\theta}_x + i\hat{\theta}_y & -\hat{\theta}_z \end{pmatrix}. \tag{A.25}$$

Thus, this set of matrices are elements of a three-dimensional real vector space which can be identified as the space of physical vectors and, the three generators of rotations, σ_1, σ_2, σ_3, corresponding to the three components of a physical vector

$$\hat{S}_x = \frac{\hbar}{2}\sigma_1, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_2, \quad \hat{S}_z = \frac{\hbar}{2}\sigma_3. \tag{A.26}$$

Moreover, any normalized element of a complex two dimensional vector space $(\alpha \quad \beta)^T$ is also described by three real parameters, the real and imaginary parts of α and β with the constraint $|\alpha|^2 + |\beta|^2 = 1$. One can check easily, that under a $-\pi/2$ rotation about the $x-$axis, the expectation value for $\hat{S}_y$ for the rotated system should equal the expectation value $\hat{S}_z$ for the non-rotated system: $\langle\psi'|\hat{S}_y|\psi'\rangle = \langle\psi|\hat{S}_z|\psi\rangle$.

A.5.2 Representation of $SU(2)$ group

As we mentioned, the $SU(n)$ group also has a representation of n^2-1 dimensions, i.e., the same number of generators of the group. This representation is called the *adjoint representation*. Now we will construct the n-dimensional representations of the algebra $SU(n)$.

A.5.3 Multiplets of special unitary groups

In order to apply the n-dimensional $SU(n)$ generators, we define the column vectors $v_a, a = 1, 2, ..., n$, which are represented at an appropriate form such that the j-th row of v_a is equal to δ_{aj}. For example, for $SU(2)$ the v_a are

$$v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.27}$$

These vectors belong to the *first fundamental representation*. Then, for $SU(2)$, there is an unique fundamental representation.

We construct the multiplets of $SU(n)$ using the vectors v_a. In particle physics, one can design the vectors v_1 and v_2 to state with spin-up and spin-down, respectively, of a particle of $1/2$ spin (for example the quarks). For a system of several particles, we must consider the wave function $v(1)v(2)\cdots v(n)$, where $1, 2, ..., n$ denote the particle 1, particle 2, etc. On the other hand, we define the matrices M_{ab} of raising and lowering that operate on a certain state of a N-particle:

$$M_{ab} = \sum_{j=1}^{N} M_{ab}(j). \tag{A.28}$$

For simplicity, we start with only two particles. Let the normalized state $\psi_1 = u_1 u_1$, and operate on ψ_1 with the operator M_{ab}

$$\begin{aligned} M_{21}\psi_1 &= (M_{21}(1) + M_{21}(2))v_1 v_1, \\ &= (\sigma_-(1) + \sigma_-(2))v_1 v_1, \\ &= \sqrt{2}\psi_2, \end{aligned} \tag{A.29}$$

where ψ_2 is other normalized state. Analogously, if we again operate with M_{ab} but on the new state ψ_2, we obtain

$$M_{21}\psi_2 = \sqrt{2}u_2 u_2 = \sqrt{2}\psi_3. \tag{A.30}$$

If we operate on ψ_3 we get 0. Thus, if one starts with two doublets of $SU(2)$, we have obtained a triplet state ψ_i corresponding to spin or isospin 1. Now, one can construct a state ϕ_1 orthogonal to ψ_2, defined by

$$\phi_1 = \frac{(u_1 u_2 - u_2 u_1)}{2}. \tag{A.31}$$

So, if we operate on ϕ_1 we get 0. This state is called singlet, and it to corresponds to spin or isospin 0. We can obtain the eigenvalues of the diagonal operators J_3 and J_2 by directly operating on ψ_i and ϕ_1.

The coefficients multiplying the product of the wave functions $u_i u_j$ in the expressions for ψ_i and ϕ_i are known as *Clebsch-Gordan coefficients*. In this case we have considered, these Clebsch-Gordan coefficients unique, but in the case of the product of three or more wave functions, the Clebsch-Gordan coefficients can depend on somewhat arbitrary definitions of wave functions. We can see this as follows: If we start with the product $u_1 u_1 u_1$, we can use the lowering operator M_{21} to construct all the symmetric wave functions belonging to the same multiplet as $u_1 u_1 u_1$.

Moreover, the Clebsch-Gordan coefficients depend on what linear combination of these wave functions we choose. The choice in some instances is given by convention, but real questions of physics can influence what choice is convenient.

A.6 Isospin

In the preceding sections we saw how wave functions of some rotationally invariant system can be classified according to representations of the rotation group or $SU(2)$ group. Therefore, there exist an angular momentum operator acting on wave functions in quantum mechanics. However, in addition to orbital angular momentum, particles can have *intrinsic* angular momentum, a contribution to the angular momentum from the particle internal structure besides the motion of its center of gravity.
In principle, this can be understood by assuming that the particle has finite dimensions, so that its rotation around its center of gravity could be responsible for this extra contribution to the angular momentum, also called *spin*. This feature is known from classical mechanics (an example is a rotating top). This way of looking at intrinsic angular momentum is not free from troubles, but this does not concern us for the moment. Our starting point is that intrinsic angular momentum fits naturally within the formalism introduced in the previous sections.

All the elementary particles with the same spin and similar mass values, can be grouped in so-called *multiplets* (a multiplet is a set of n linearly-independent vectors, which can be choose orthonormal). For example,

$$\begin{aligned} &\text{nucleon-doublet} \quad (p, n), \\ &\text{pion-triplet :} \quad (\pi^+, \pi^0, \pi^-), \\ &\text{Delta-quadruplet :} \quad (\Delta^{++}, \Delta^+, \Delta^0, \Delta^-), \\ &\text{etc.} \end{aligned} \tag{A.32}$$

This structure can be explained by assuming that the subatomic world is quasi-invariant under what will be referred to as *isospin rotations*. The isospin rotations form a group, like the usual rotations, whose elements can be regarded as rotations in some "internal" three-dimensional space, where its states (isospin multiplets) transform according to irreducible representations of this group. Thus, the notion of isospin is analogous to the usual notion of spin, which is connected to rotations in ordinary 3-space.

A physical example occur in the atomic nucleus, where the force that binds the nucleons in an atomic nucleus is invariant under isospin rotations. As we know it is the *strong force*. However, the electromagnetic forces are not. For this reason the electric charges within one multiplet, it aren't all the same. The conservation of electric charge is exact in all the spacial directions. This is associated with conservation of isospin-"angular momentum" in the 3 direction, since electric charge is described by the *Gell-Mann-Nishijima relation*

$$Q = I_3 + \frac{1}{2}Y \tag{A.33}$$

where Y is the "hyper charge ", which has the same value for all states within one multiplet, and Y is equal to 1 for the nucleon and Δ multiplets, and 0 for the pion multiplet. Due to that isospin is related to strong interaction, then particles as quarks or gluons can be considered as different states of the same particle (proton, neutron, etc), but with isospin values related to the number of charge states.

Although it does not have the units of angular momentum and is not a type of spin, the formalism that describes it is mathematically similar to that of angular momentum in quantum mechanics, which means that it can be coupled in the same manner. For example, a proton-neutron pair can be coupled in a state of total isospin 1 or 0. It is a dimensionless quantity and the name derives from the fact that the mathematical structures used to describe it are very similar to those used to describe the intrinsic angular momentum (spin).

This term was derived from isotopic spin, a confusing term to which nuclear physicists prefer isobaric spin, which is more precise in meaning. Isospin symmetry is a subset of the flavour symmetry seen more broadly in the interactions of baryons and mesons. Isospin symmetry remains an important concept in particle physics, and a close examination of this symmetry historically led directly to the discovery and understanding of quarks and of the development of Yang-Mills theory.

Observation of the light baryons (those made of up, down and strange quarks) lead us to believe that some of these particles are so similar in terms of their strong interactions that they can be treated as different states of the same particle. In the modern understanding of quantum chromodynamics, this is because up and down quarks are very similar in mass, and have the same strong interactions. Particles made of the same numbers of up and down quarks have similar masses and are grouped together.

Appendix B

Path Integral in Field Theory

When Feynman was trying to formulate the path integral of quantum mechanics, he was inspired by Dirac's remark [83] which roughly states that $e^{iS/\hbar}$ corresponds to the transition amplitude, where S is the action. Dirac is indeed the forefather of the path integral approach to quantum mechanics. Due to the Lagrangian seems so the natural and effective in classical physics, it should have a counterpart in quantum mechanics, too. He indeed derived the possibility to compute the transition amplitude as a sum over histories with the Feynmanian exponential inserted in.

In this appendix we will try to explicate the path integral formulated by Feynman in a field theory [84]. We will start with a simple example of quantum mechanics, next will define the concept of "sum of histories".

B.1 Motivation

We consider the quantum description of one degree of freedom in one dimension. In terms of the operators $\hat{q}$ as the coordinates, and $\hat{p}$ as the momentum, then they obey the fundamental commutation relations

$$[\hat{q},\hat{q}] = [\hat{p},\hat{p}] = 0 \quad , \quad [\hat{q},\hat{p}] = i\hbar, \tag{B.1}$$

with $\hbar$ the Plancks' constant. The states of the system at a given time can be taken to be the position states $|q\rangle$ which satisfy

$$\hat{q}|q\rangle = q|q\rangle, \quad \hat{p} \mid q\rangle = -i\hbar\frac{\partial}{\partial q} \mid q\rangle, \quad \langle q|q'\rangle = \delta(q-q'), \quad \int dq|q\rangle\langle q| = 1. \tag{B.2}$$

From this, one can define a canonical transformation in quantum mechanics between the operators $(\hat{q},\hat{p})$ and $(\hat{Q},\hat{P})$, which does not change the form fundamental of the relations (B.2).
We can observe that $\langle q|\hat{q}|Q\rangle = q\langle q|Q\rangle$ and $\langle q|\hat{Q}|Q\rangle = Q\langle q|Q\rangle$. However, the operators $\hat{Q}$ and $\hat{q}$ need not commute so that the value of an arbitrary function $g(\hat{q},\hat{Q})$ may note be well defined. For example,

$$\langle q|f_1(\hat{q})f_2(\hat{Q})|Q\rangle = f_1(q)f_2(Q)\langle q|Q\rangle. \tag{B.3}$$

Hence we shall have functions that are separable as a function of $\hat{q}$ times a function of $\hat{Q}$, namely

$$\langle q|f(\hat{q},\hat{Q})|Q\rangle = f(q,Q)\langle q|Q\rangle. \tag{B.4}$$

From this, if one chooses

$$\langle q|Q\rangle = e^{-i/\hbar f(q,Q)} \tag{B.5}$$

then the mixed transformations are $\langle q|\hat{p}|Q\rangle = \partial f/\partial q\langle q|Q\rangle$ and $\langle q|\hat{P}|Q\rangle = -\partial f/\partial Q\langle q|Q\rangle$. Then, one now can find the equations

$$\hat{p} = \frac{\partial \hat{f}}{\partial q}, \quad \hat{P} = -\frac{\partial \hat{f}}{\partial Q}. \tag{B.6}$$

Our conclusion from this is that f is the quantum equivalent of the generating function of classic physical. Dirac calls it "corresponds to".

B.2 The Feynman's path integral

As saw in the previous section, Dirac tried that time and space variables were treated in an analogous form. Dirac proceeds, as in quantum mechanics, to apply these ideas to find the transition amplitude from the state $q = q'$ at t and $Q = q$ at T:

$$\langle q'(t)|q(T)\rangle \sim e^{\frac{i}{\hbar}\int_T^t dtL}. \tag{B.7}$$

Here $\sim$ denotes an assumption with no way to justify them. In order to better understanding, split up $(T-t)$ into N infinitesimal time intervals $t_a = t + a\epsilon$; $N\epsilon = (T-t)$. If we use the completeness relation for the states (B.2), we obtain

$$\langle q'_t|q_T\rangle = \int dq_1 dq_2 \cdots dq_{N-1}\langle q'_t|q_1\rangle\langle q_1|q_2\rangle \cdots \langle q_{N-1}|q_T\rangle. \tag{B.8}$$

Now our goal is to mean Eq.(B.7) as an equality. In fact, we assume this last (up to a constant) only for an infinitesimal time interval, i.e.,

$$\langle q'_t|q_{t+\delta t}\rangle = Ae^{-\frac{i}{\hbar}\delta t L(q'_t, q_{t+\delta t})} \tag{B.9}$$

with L as a function of q'_t and $q_{t+\delta t}$, because this is the equivalent formula of quantum mechanics. If now one use Eqs. (B.8) and (B.9) for the transition amplitude, yields

$$\langle q'_t|q_T\rangle = \lim_{N\to\infty} A^N \int \prod_{j=1}^{N-1} dq_j e^{-\frac{i}{\hbar}\int_T^t dtL(q,\dot{q})} \equiv \int \mathcal{D}q e^{-\frac{i}{\hbar}S(t,T,[q])}, \tag{B.10}$$

where A is a normalizing factor. This was exactly what Feynman proposed, and tells us that if we want to compute the probability amplitude for the particle to be at q' at time t, given that it was at q at time T, we must express it as the sum over all possible paths that start at q at T and end at q' at t weighted by the exponential of $-\frac{i}{\hbar}$ times the action evaluated for the particular path. We shall call to it a *path integral*, and the identifying notation in this expression is the script $\mathcal{D}$. In the quantum sense, the particle does not takes only one path to go from q to q' because all paths contribute to his trajectory.

B.3 The path integral in field theory

Our main goal is to generalize the previous section to a field theory. In order to obtain an analogy with quantum mechanics, we can describe the states of the system at a given time t by $|\phi(\vec{x})\rangle$. From

this, one needs to calculate the transition amplitude between two states at different times. The usual way that one identifies states of the system is add small perturbations to a zero order approximation of the theory, which the states can be easily recognized. Then, the effects of this small perturbation are computed. This procedure can perform only when one already has identified the full theory in terms of a small perturbation on a simple system. For example, in QCD it is believed that quarks are not physical particles but that bound states of quarks such as protons, π-mesons, etc., are physical. Therefore, we have to decide the size of the quark couplings among themselves. For example, we consider the two cases: if they are small, then quarks could serve as physical states, but if they are big, it is not good form to talk of quarks because quarks would tend to bind among themselves, and not appear as asymptotic states.
From this assumptions, we see that the physical states depends very much on the solution of the field theory, but whatever the states everyone agrees there must be a state of least energy, call it the *vacuum state*.

To start with our procedure, we can suppose the transition amplitude of the system from the vacuum state at $t = -\infty$ to the vacuum state at $+\infty$ in the presence of an arbitrary driving force $J(x)$. If we consider a self-interacting scalar field described by the action

$$S = \int d^4x \left(\frac{1}{2}\partial_\rho \phi \partial^\rho \phi - \frac{1}{2}m^2\phi^2 - V(\phi) \right) = \int d^4x \mathcal{L}(\phi, \partial_\rho \phi) \tag{B.11}$$

then the vacuum to vacuum amplitude is defined to be

$$\langle \Omega | \Omega \rangle_J \equiv W[J] = N \int \mathcal{D}\phi \mathcal{D}\pi e^{i\langle \pi\dot{\phi} - \mathcal{H} + J\phi \rangle} \tag{B.12}$$

where $\mathcal{H}$ is Hamiltonian density defined positive if $m^2 > 0$, π are the conjugate momenta associated to fields, and $V > 0$. This expression tells us that, the probability amplitude for the ground state $|0\rangle$ remaining unchanged under the action of the perturbation $J(x)$ in the interval $t - T$, namely for the process in which the perturbation does not induce excitations.
As we see, our strategy is to use these amplitudes to calculate the physical consequences of the theory. N is a constant and $\langle \cdots \rangle$ now it means integration over spacetime. If we now integrate over the momenta, it yields

$$W[J] = N' \int \mathcal{D}\phi e^{i\langle \frac{1}{2}\partial_\rho \phi \partial^\rho \phi - \frac{1}{2}m^2\phi^2 - V(\phi) + J\phi \rangle}. \tag{B.13}$$

To make disappear the oscillations [1] in the last integrand, define W at Euclidean space by setting $t = -i\tau$, which it reads

$$W_E[J] = N_E \int \mathcal{D}\phi e^{-\langle \frac{1}{2}\bar{\partial}_\rho \phi \bar{\partial}^\rho \phi - \frac{1}{2}m^2\phi^2 - V(\phi) + J\phi \rangle}. \tag{B.14}$$

Now, the integrand is negative and therefore Gaussian integrations can be to perform.

The generating functional generally is used to manufacture the Green's functions which are defined by the expansion

$$W[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle J_1 J_2 \cdots J_n G^n(1, 2, ..., n) \rangle_{1,2,\ldots,n} \tag{B.15}$$

[1] In other cases put a convergence factor as $e^{-\frac{1}{2}\epsilon\phi}$ for $\epsilon > 0$ is also a convenient way for depreciate the oscillations.

such that

$$G^n(1,2,...,n) = \frac{1}{i^n}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_n}W[J]\Big|_{J=0}. \tag{B.16}$$

with $J_k = J(x_k)$. The task is to compute the Green's functions to solve the equations of motion of the theory, and obtain the propagators of the particles as we discussed below.

B.4 Saddle-point method in the path integral

Following [69], let us consider a smooth function $f(x)$ which it is possible to expand around the stationary point x_0 as:

$$f(x) \approx f(x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots. \tag{B.17}$$

Now let us consider the integral

$$I := \int dx e^{-f(x)}. \tag{B.18}$$

Inserting Eq. (B.17) in (B.18), we obtain

$$\begin{aligned} I &= \int dx e^{-f(x_0)-\frac{1}{2}f''(x_0)(x-x_0)^2+\cdots}, \\ &= e^{-f(x_0)}\int dx e^{-\frac{1}{2}f''(x_0)(x-x_0)^2+\cdots}. \end{aligned} \tag{B.19}$$

Thus the last integral is a Gaussian which it is possible calculate iff $f''(x_0) > 0$. We want to perform the same previous computations on the generating functional defined as

$$W_E[J] = N_E\int \mathcal{D}\phi e^{-S_E[\phi,J]}, \tag{B.20}$$

where S_E is the Euclidean action of a simple scalar field

$$S_E[\phi,j] = \int d^4\bar{x}\left(\frac{1}{2}\bar{\partial}_\rho\phi\bar{\partial}^\rho\phi + \frac{1}{2}m^2\phi^2 + V(\phi) - J\phi\right). \tag{B.21}$$

If we expand S_E around a classic field configuration ϕ_0, we obtain

$$S_E[\phi,J] = S_E[\phi_0,J] + \langle\frac{\delta S_E}{\delta\phi}\Big|_{\phi=\phi_0}(\phi-\phi_0)\rangle + \frac{1}{2!}\langle\frac{\delta^2 S_E}{\delta\phi_1\delta\phi_2}\Big|_{\phi=\phi_0}(\phi-\phi_0)_1(\phi-\phi_0)_2\rangle_{12} + \cdots. \tag{B.22}$$

If we suppose that S_E is stationary at ϕ_0, then this configuration satisfies the equation of motion

$$\frac{\delta S_E}{\delta\phi}\Big|_{\phi=\phi_0} = -\Box\phi_0 + m^2\phi_0 - V'(\phi_0) - J = 0. \tag{B.23}$$

On the other hand, the second variation of S_E is given by

$$\frac{\delta^2 S_E}{\delta\phi_1\delta\phi_2}\Big|_{\phi=\phi_0} = \left(-\Box + m^2 + V''(\phi_0)\right)\delta(\bar{x}_1 - \bar{x}_2). \tag{B.24}$$

Therefore, the generating functional acquires the following form:

$$\begin{aligned} W_E[J] &\approx N_E e^{-S_E[\phi_0,J]} \int \mathcal{D}\phi e^{-\frac{1}{2}\langle \phi_1 \frac{\delta S_E}{\delta \phi}\Big|_{\phi=\phi_0} \phi_2\rangle_{12}}, \\ &= N_E e^{-S_E[\phi_0,J]} \int \mathcal{D}\phi e^{-\frac{1}{2}\langle \phi_1 A \phi_2\rangle_{12}}, \\ &\approx e^{-S_E[\phi_0,J]} \left(\det(A)\right)^{-1/2} \end{aligned} \tag{B.25}$$

with $A = [-\Box + m^2 V''(\phi_0)]\delta_{12}$. Utilizing the relation $\det(A) = e^{\mathrm{Tr}\ln A}$, the expression in Eq. (B.25) reads

$$W_E[J] \approx N'_E e^{-S_E[\phi_0,J] - \frac{1}{2}\mathrm{Tr}\ln(A)}. \tag{B.26}$$

The physical meaning of this last expression indicates that the first term $S_E[\phi_0, J]$ gives the Green's functions of the theory, and the second term $\sim \mathrm{Tr}\ln A$ is the first quantum correction of the Green's functions. Generally this expansion is known as *one-loop expansion* or semi-classical expansion. In conclusion, if one to insert carefully the Planck's constant in W_E, then the expansion of $Z_E[J]$ is a asymptotic series in $\hbar$, where the power $\hbar^1$ corresponds to the first quantum correction. Obviously, the expansions to greater orders in $\hbar$ are more difficult to calculate by means some regularization process.

B.5 The propagator

Our goal is to compute $W[J]$, given by

$$W[J] \equiv N \int \mathcal{D}\phi e^{i\langle \frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2 - \frac{i\epsilon}{2}\phi^2 + J\phi\rangle}, \tag{B.27}$$

where we had consider $V = 0$ and an infinitesimal parameter $\epsilon > 0$. For simplicity, we will work at the momenta space. If we define the four-dimensional Fourier transform as

$$\tilde{f}(p) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d^4x e^{-ip\cdot x} f(x), \tag{B.28}$$

then, we find

$$\begin{aligned} W[J] &= N e^{-\frac{i}{2}\int d^4p \frac{|\tilde{J}(p)|^2}{p^2 - m^2 + i\epsilon}} \int \mathcal{D}\phi' e^{i\langle \frac{1}{2}\partial\phi'\partial^\rho\phi' - \frac{1}{2}(m^2 - i\epsilon)\phi'^2\rangle} \\ &= W[0] e^{-\frac{i}{2}\int d^4p \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2 - m^2 + i\epsilon}}. \end{aligned} \tag{B.29}$$

The important fact is that we have succeeded in find the explicit form of $W[J]$. For that, we go back to coordinate space

$$W[J] = W[0] e^{-\frac{i}{2}\langle J_1 \Delta_{12} J_2\rangle_{12}}, \tag{B.30}$$

where Δ_{12} stands for $\Delta(x_1 - x_2)$:

$$\Delta_F(x - y) \equiv \Delta_{12} = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}. \tag{B.31}$$

The subscript F is because this way to insert the damping term $i\epsilon$ was performed by Feynmann, and therefore usually one call to Δ_F as the *Feynman propagator.*

We can easily calculate the Green's functions:

$$G^{(2)}(x_1, x_2) = \Delta_F(x_1 - x_2) \tag{B.32}$$

$$G^{(4)}(x_1, x_2, x_3, x_4) = -\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) - \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) - \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3), \quad \text{etc...} \tag{B.33}$$

Since $W[J]$ depends only on J^2 and the $G's$ with odd number of variables, vanishes. Also, one can find the $G^{(6)}, G^{(8)}$, etc, and all depending on $G^{(2)}$. Due to this, it would be apparently more convenient to set

$$W[J] = e^{iZ[J]} \tag{B.34}$$

and we obtain

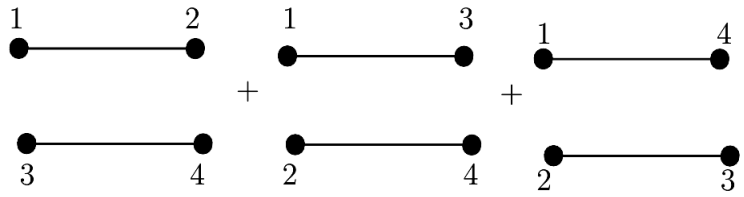
$$iZ[J] = \sum_n \frac{i^n}{n!} \langle G_c^{(n)}(1, ..., n) J_1, ..., J_n \rangle_{1,...,n} \tag{B.35}$$

The meaning of Δ_F is that solves the field equations of the field theory, and in particular to our case, is the Green's function of the operator $(\Box + m^2)$. In the view point physical, Δ_F identify a signals that propagates single particle and antiparticle states. In other words, the positive energy solutions of the Klein-Gordon equation are propagated forward in time, while negative energy solutions are propagated backwards in time.

Moreover, in our case the states are going to be particles with mass m^2, and we can interpret $G^{(2)}(x - y)$ as the amplitude for this particle to go from x to y. Feynman gave a diagrammatic representation in $x-$space, associating with $\Delta_F(x - y)$ a line connecting the two space-time points x and y:

$G^{(2)}(x - y) =$ 

For the others Green's function result

$G^{(4)}(x_1, x_2, x_3, x_4) =$ 

where the numbers $1, 2, 3, 4$ denotes the coordinates x_1, x_2, x_3, x_4, respectively. Analogously, one can interpret better these diagrams at momenta space by Fourier transformed and the conservation of momentum $p_1 + \cdots p_n = 0$.
Such diagrams are called tree diagrams, and it are useful to represents the Green's functions. Moreover, the main meaning of the tree diagrams is that there cannot any closed loop, since if there are loops we would be facing to quantum orders, i.e., order in $\hbar$.

Appendix C

Gribov copies with chemical potential

In this appendix we consider the derivations and properties of the equation of gauge-equivalent fields satisfying the same Landau-DeWitt gauge condition in the presence of chemical potential. The cases for the vacuum and a general field are shown by separate.

C.1 Chemical Gribov copies in the vacuum

Our aim is to calculate the condition for existence of Gribov's copies in the vacuum:

$$\bar{D}_\rho A^U_\rho = 0 \tag{C.1}$$

where is the "background" covariant derivative $\bar{D}^{ab}_\rho = \partial_\rho \delta^{ab} - i\mu\epsilon^{ab3}\delta_{\rho 0}$ and A^U_ρ is the transformed quantum field of the theory defined (in the vacuum case) by

$$A^U_\rho = U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho. \tag{C.2}$$

In terms of the $SU(2)$-functions (6.16), the pure-gauge field in (C.2) take the form

$$\begin{aligned} U^{-1}\partial_\rho U &= (Y^0(x^\rho)\mathbf{1} - Y^a(x^\rho)\tau_a)\partial_\rho(Y^0(x^\rho)\mathbf{1} + Y^b(x^\rho)\tau_b) \\ &= Y^0\partial_\rho Y^0\mathbf{1} + Y^0\partial_\rho Y^b\tau_b - Y^a\partial_\rho Y^0\tau_a - Y^a\partial_\rho Y^b\tau_a\tau_b. \end{aligned}$$

Using (A.20) and the identity in (6.16), we get

$$\begin{aligned} U^{-1}\partial_\rho U &= \left(\frac{1}{2}\partial_\rho\left((Y^0)^2 + Y^aY_a\right)\mathbf{1} + Y^0\partial_\rho Y^b\tau_b - Y^a\partial_\rho Y^0\tau_a + Y_a\partial_\rho Y_b\epsilon^{abc}\tau_c\right) \\ &= \left(Y^0\partial_\rho Y^c - Y^c\partial_\rho Y^0 + Y_a\partial_\rho Y_b\epsilon^{abc}\right)\tau_c. \end{aligned}$$

Therefore,

$$\Rightarrow U^{-1}\partial_\rho U = \left(Y^0\partial_\rho Y^c - Y^c\partial_\rho Y^0 + Y_a\partial_\rho Y_b\epsilon^{abc}\right)\tau_c. \tag{C.3}$$

For the term with the background field, we get

$$\begin{aligned} U^{-1}B_\rho U &= (Y^0(x^\rho)\mathbf{1} - Y^a(x^\rho)\tau_a)\frac{-i\mu}{g}\delta_{\rho 0}\tau_3(Y^0(x^\rho)\mathbf{1} + Y^b(x^\rho)\tau_b) \\ &= -\frac{i\mu}{g}\delta_{\rho 0}(Y^0(x^\rho)\mathbf{1} - Y^a(x^\rho)\tau_a)(Y^0(x^\rho)\mathbf{1}\tau_3 + Y^b(x^\rho)\tau_3\tau_b) \\ &= -\frac{i\mu}{g}\delta_{\rho 0}\left((Y^0)^2\tau_3 + Y^0Y^b\tau_3\tau_b - Y^aY^0\tau_a\tau_3 - Y^aY^b\tau_a\tau_3\tau_b\right). \end{aligned}$$

Utilizing the relation $\tau_a\tau_3\tau_b = -\tau_a\delta_{3b} - \epsilon_{3ab} - \delta_{a3}\tau_b + \tau_3\delta_{ab}$, then

$$\begin{aligned} U^{-1}B_\rho U &= -\frac{i\mu}{g}\delta_{\rho 0}\left[(Y^0)^2\tau_3 + Y^0Y^b(-\delta_{3b}\mathbf{1} - \epsilon_{3bc}\tau^c) - Y^aY^0(-\delta_{a3}\mathbf{1} - \epsilon_{a3c}\tau^c)\right. \\ &- \left. Y^aY^b(-\tau_a\delta_{3b} - \epsilon_{3ab} - \delta_{a3}\tau_b + \tau_3\delta_{ab})\right] \\ &= -\frac{i\mu}{g}\delta_{\rho 0}\left[(Y^0)^2\tau_3 - Y^0Y^b\epsilon_{3bc}\tau^c + Y^aY^0\epsilon_{a3c}\tau^c + Y^aY^3\tau_a + Y^aY^b\epsilon_{abc}\right. \\ &+ \left. Y^3Y^b\tau_b - Y^aY_a\tau_3\right]. \end{aligned} \tag{C.4}$$

The transformed field A^U then acquires the following form:

$$\begin{aligned} U^{-1}\partial_\rho U + U^{-1}B_\rho U - B_\rho &= \epsilon^{abc}Y_a\partial_\rho Y_b\tau_c + Y^0\partial_\rho Y^c\tau_c - Y^c\partial_\rho Y^0\tau_c \\ &-\frac{2i\mu}{g}\delta_{\rho 0}\left(Y^aY^0\epsilon_{a3c} + Y^3Y^c\right)\tau_c - \frac{2i\mu}{g}\delta_{\rho 0}(Y^0)^2\tau_3. \end{aligned} \tag{C.5}$$

In order to obtain a reduced expression for the Eq. (C.5), we must to introduce the generalized hedgehog *ansatz* defined in (6.22). Then, we get

$$U^{-1}\partial^\rho U = \left(\sin^2\alpha\epsilon^{abc}\hat{n}_a\partial^\rho\hat{n}_b + \hat{n}^c\partial^\rho\alpha + \frac{1}{2}\sin(2\alpha)\partial^\rho\hat{n}^c\right)\tau_c, \tag{C.6}$$

$$U^{-1}B^\rho U = -\frac{i\mu}{g}\delta^{\rho 0}(1 - 2\sin^2\alpha)\tau_3 - \frac{i\mu}{g}\delta^{\rho 0}\hat{n}^a\sin 2\alpha\epsilon_{a3c}\tau^c - 2\frac{i\mu}{g}\delta^{\rho 0}\sin^2(\alpha)\hat{n}^3\hat{n}^c\tau_c. \tag{C.7}$$

Applying the covariant background derivative $\bar{D}^{ab}_\rho$ on Eqs. (C.6) and (C.7), we obtain

$$\begin{aligned} \bar{D}_\rho(U^{-1}\partial^\rho U) &= \left(\epsilon^{abc}[\sin(2\alpha)\hat{n}_a\partial_\rho(\alpha)\partial^\rho(\hat{n}_b) + \sin^2(\alpha)\hat{n}_a\Box\hat{n}_b] + 2\cos^2(\alpha)\partial^\rho\hat{n}^c\partial_\rho\alpha + \hat{n}^c\Box\alpha\right. \\ &+ \left.\frac{1}{2}\sin(2\alpha)\Box\hat{n}^c + 2i\mu\delta_{\rho 0}\left(\epsilon^{3bc}\left[\hat{n}_b\partial_\rho(\alpha) + \frac{1}{2}\sin(2\alpha)\partial_\rho(\hat{n}_b)\right] - \sin^2(\alpha)[\hat{n}_3\partial^\rho\hat{n}^c - \hat{n}^c\partial^\rho\hat{n}_3]\right)\right)\tau_c, \end{aligned} \tag{C.8}$$

$$\begin{aligned} \bar{D}_\rho(U^{-1}B^\rho U) &= 2\frac{i\mu}{g}\delta^{\rho 0}\sin(2\alpha)\partial_\rho(\alpha)\tau_3 - \frac{2i\mu}{g}\delta^{\rho 0}\left(\epsilon_{a3c}\left[\cos(2\alpha)\partial_\rho(\alpha)\hat{n}^a + \frac{1}{2}\sin(2\alpha)\partial_\rho(\hat{n}^a)\right]\right. \\ &+ \left.2\sin(2\alpha)\partial_\rho(\alpha)\hat{n}_3\hat{n}_c + \sin^2(\alpha)[\partial_\rho(\hat{n}_3)\hat{n}_c + \partial_\rho(\hat{n}_c)\hat{n}_3] + 2\frac{\mu^2}{g}\left(\sin(2\alpha)\hat{n}_c + 2\sin^2(\alpha)\hat{n}^3\hat{n}^a\epsilon_{3ac}\right)\right)\tau^c \end{aligned} \tag{C.9}$$

Finally, the expression for the vacuum Gribov copies (C.1), it is obtained by means the sum of Eqs. (6.12), (C.9) and (C.10):

$$\Big(\epsilon^{abc}[\sin(2\alpha)\hat{n}_a\partial^\rho\alpha\partial_\rho\hat{n}_b+\sin^2\alpha\hat{n}_a\Box\hat{n}_b]+2\cos^2(\alpha)\partial^\rho(\hat{n}^c)\partial_\rho(\alpha)+\hat{n}^c\Box\alpha$$
$$+\frac{1}{2}\sin(2\alpha)\Box\hat{n}^c-\frac{2i\mu}{g}\delta_{\rho 0}\Big\{\cos(2\alpha)\partial^\rho(\alpha)\hat{n}_a\epsilon^{a3c}+\frac{1}{2}\sin(2\alpha)\partial^\rho(\hat{n}_a)\epsilon^{a3c}$$
$$+2\sin(2\alpha)\partial^\rho(\alpha)\hat{n}^3\hat{n}^c+\sin^2(\alpha)(\partial^\rho(\hat{n}^3)\hat{n}^c+\partial^\rho(\hat{n}^c)\hat{n}^3)\Big\}$$
$$+2i\mu\delta_{\rho 0}\left(\epsilon_{3bc}\left[\hat{n}^b\partial_\rho(\alpha)+\frac{1}{2}\sin(2\alpha)\partial_\rho(\hat{n}^b)\right]-\sin^2(\alpha)[\hat{n}_3\partial_\rho\hat{n}_c-\hat{n}_c\partial_\rho\hat{n}_3]\right)$$
$$+2\frac{\mu^2}{g}\left(\widehat{n}^c\sin(2\alpha)+2\widehat{n}_3\widehat{n}_a\epsilon^{3ac}\sin^2\alpha\right)\Big)\tau_c+\frac{2i\mu}{g}\delta_{\rho 0}\partial^\rho(\alpha)\sin(2\alpha)\tau_3=0 \tag{C.10}$$

where $\Box=\partial_\rho\partial^\rho$. We can see that this equation depends the metric the define the unitary vectors $\hat{n}^a$. Obviously, this equation will be more difficult to manipulate, depending on the metric that we choose.

C.2 Chemical Gribov copies in non-perturbative region

In this case, we only must calculate the term of the form $U^{-1}AU$, and then add it to the equation (C.1). In fact, let us consider the expression for the term $U^{-1}AU$ in terms of the generalized hedgehog ansatz:

$$U^{-1}A_\rho U=\frac{1}{g}\left(\cos(2\alpha)A_{\rho c}+\sin(2\alpha)A^b_\rho\hat{n}^a\epsilon_{abc}+2\sin^2(\alpha)A^b_\rho\hat{n}_b\hat{n}_c\right)\tau^c. \tag{C.11}$$

Thus, the application of the background covariant derivative $\bar{D}^a_\rho$, reads

$$\bar{D}_\rho(U^{-1}A^\rho U)=\frac{1}{g}\Big(2\partial_\rho(\alpha)[\cos(2\alpha)A^{\rho b}\hat{n}^a\epsilon_{abc}+\sin(2\alpha)A^{\rho b}\hat{n}_c\hat{n}_b-\sin(2\alpha)A^\rho_c]$$
$$+\partial_\rho(A^\rho_c)\cos(2\alpha)+[\sin(2\alpha)\hat{n}^a\epsilon_{abc}+2\sin^2(\alpha)\hat{n}_c\hat{n}_b]\partial_\rho(A^{\rho b})$$
$$+\sin(2\alpha)A^{\rho b}\partial_\rho(\hat{n}^a)\epsilon_{abc}+2\sin^2(\alpha)A^{\rho b}\partial_\rho(\hat{n}_c\hat{n}_b)\Big)\tau^c$$
$$+\frac{2i\mu}{g^2}\delta_{\rho 0}\left(\cos(2\alpha)A^c_\rho\epsilon_{3ca}+\sin(2\alpha)A_{\rho 3}\hat{n}_a-\sin(2\alpha)A_{\rho a}\hat{n}_3\right.$$
$$\left.+2\sin^2(\alpha)A_{\rho b}\hat{n}^b\hat{n}^c\epsilon_{3ca}\right)\tau^a \tag{C.12}$$

This last expression should be added to (C.10) to obtain the full expression of the equation of Gribov copies inside a non-perturbative functional region.

Appendix D

Brief Introduction to QCD

It is well known that gravity is governed by Einstein's general relativity, while the other three forces can be described, to an excellent degree, by a quantum field theory of quarks and leptons based on a framework consistent with Einstein's special theory of relativity and quantum mechanics: the so-called Standard Model (SM). In this appendix we only discuss the physics of *strong interaction*, their associated structure, and the interactions of hadronic matter.

D.1 Hadronic matter

What we usually observe in experimental apparatus are hadrons and nuclei which are bound states of basic building blocks. We understand well about what is "matter".
A meson is made of the quantum number of a quark and anti-quark pair. For example, the pion has the quantum number of $\bar{u}u - \bar{d}d$, $\bar{u}d$, and $\bar{d}u$, naturally color singlets. On the other hand, a baryon is made from the quantum numbers of **3** quarks, which can form a color singlet because the $SU(3)$ group multiplication rule says

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} \tag{D.1}$$

where **1** represents the color singlet. For example, a proton is made of two "up" quarks and one "down" quark. Baryons and mesons together are called *hadrons*, the bounds states of strong interactions. The lowest mass baryons are neutrons and protons, which are together called *nucleons*. There are attractive interactions between protons and neutrons, which are the residual color interactions, just like the Van der Waals forces between neutral atoms and molecules. These nuclear forces are responsible for binding the nucleons together to form the atomic nuclei, and the origin of the atomic energy.

D.2 Quarks and gluons

D.2.1 Quarks

The quarks are the fundamental particles that form the hadronic matter. Like the electrons, they are simple structureless (as far as we know) spin-1/2 particles. They can be described by Dirac spinors $\psi_\alpha(x)$ with four components $\alpha = 1, ..., 4$ as functions of the space-time. In the case that they does not interact witch each other, the quarks obey the free Dirac equation:

$$(i\partial\!\!\!/ - m)\psi(x) = 0, \quad \text{with} \quad \partial\!\!\!/ = \gamma^\rho \partial_\rho, \tag{D.2}$$

where γ^ρ are the four-dimensional Pauli matrices, and m is the free mass. Some states of this equation are the standard plane waves

$$\psi_{k,\lambda}(x) = f(k,\lambda)e^{-i(Et-\vec{x}\cdot\vec{k})}, \tag{D.3}$$

where $k^\rho = (E, \vec{k})$ and λ denote the four-momentum and polarization, respectively. The function $f(k\lambda)$ is the spin-dependent momentum-space wave function.
Although, free quarks have never been observed in laboratory, the physicists have discovered, theoretically, six flavors of quarks through various high-energy experiments: up(u), down(d), charm(c), strange(s), top(t) and bottom(b). These flavours organize themselves into families by means the weak interactions. "Up" and "down" form the first family (or generation), "charm" and "strange" the second, and "top" and "bottom" the third. These three families are basically repetitions of the same pattern (same quantum numbers) with unknown physical significance, but these six quarks are distinguished by their masses and associated "flavor" quantum numbers. For example, the electric charges of the up, charm and top quarks are all 2/3 of that of the proton, and the charges of the down, strange, and bottom are all $-1/3$.

D.3 Quark masses

The meaning of mass of the quarks requires some explanation more detailed, because we cannot measured directly it. Thus, we only can say that *the mass of a quark* is a parameter in the Lagrangian of the theory which describes the self-interaction of it, and that is not directly observable. As such, the mass parameter is much like a coupling constant in quantum field theory and is technically dependent on the momentum scale and the renormalization scheme. According to the Standard Model, the masses of the quarks are generated through a symmetry breaking in phase transitions of the electroweak interactions (a transition similar to that of a normal conductor to superconductor in condensed matter physics, in which an effective mass for the photon is produced). Although the discovery of the existence of Higgs bosons, the symmetry breaking is still under investigation in experiments at high-energy colliders.

D.4 Color charge

In the strong interactions, the quarks carry color charges. The *color charges* are the analogous of electric charge in Quantum Electrodynamics, but with important differences. Unlike electric charge, the color charge is a quantum vector charge, similar to angular momentum in quantum mechanics. The total color charge of a system must be obtained by combining the individual charges of the constituents according to group theoretic rules.

The quarks have three basic color-charge states, which can be labeled as $i = 1, 2, 3$, or red, green, and blue, mimicking three fundamental colors. These three color states form a basis in a 3-dimensional complex vector space. The color state can be rotated by 3×3 unitary matricial transformations which form a Lie group, the $SU(3)$ group. The 3-dimensional color space forms a fundamental representation of $SU(3)$. It is customary to label the color charges by the spaces of the $SU(3)$ representations, **3** in the case of a quark. The rules of adding together color charges follow those of adding representation spaces of the $SU(3)$ group. Moreover, the quarks, like the electron, have anti-particles, called *antiquarks*, often denoted by $\bar{q}$. The antiquarks have the same spin and mass as the quarks, but with opposite electric charges. The color charge of an antiquark is denoted by $\bar{\mathbf{3}}$, which is a representation space of $SU(3)$ where the vectors are transformed according to the complex

conjugate of an $SU(3)$ matrix.
From this, the confinement phenomenon can be considered as a color confinement, since the strong interactions don't allow states other than color-singlet or color-neutral. We can summary the features of these six quarks in the following table:

Quark Flavor:	up	down	charm	strange	top	bottom
Mass:	1.5-4 MeV	4-8 MeV	1.25 GeV	∼100 MeV	175 GeV	4.25 GeV
Charge:	2/3	-1/3	2/3	-1/3	2/3	-1/3

Table D.1: Quark masses in the $\bar{\text{M}}$S renormalization scheme at a scale of $\mu = 2$ GeV.

D.4.1 Gluons

The mediators in the strong interactions are called *gluons*. The gluons corresponds to the kinetic energy term in the QCD Lagrangian defined by

$$\mathcal{L}_g = -\frac{1}{4} F^{\rho\sigma a} F_{\rho\sigma a} \tag{D.4}$$

where $F_{\rho\sigma a}$ is the usual Yang-Mills field.

They have a similar function as photons in QED, which mediate electromagnetic interactions between charged currents. The gluons are massless particles of spin-1 with two polarization states (left-handed and right-handed). They are represented by a four-component gauge vector potential $A^\rho(x)$ with a Lorentz index $\rho = 0, 1, 2, 3$, just as in electromagnetism, then gluons are usually called gauge particles.
We know that there is only one type of photon mediating electromagnetic interactions, however in strong interactions there are 8 types of gluons mediating the quarks. This proliferation of gauge particles has to do with the $SU(3)$ color symmetry. A generic $SU(3)$ matrix requires 8 real parameters, usually written in the form

$$U = \exp\left(i \sum_{a=1}^{8} \theta^a \lambda_a / 2 \right) \tag{D.5}$$

where $\lambda_a/2$ are 3×3 hermitian matrices, the so-called generators of $SU(3)$ rotations. Commonly one refers to these matrices as the *Gell-Mann matrices*.
The Lagrangian associated to the quarks is invariant under the transformation

$$A \to U \left(A + i\partial \right) U^\dagger \tag{D.6}$$

which generalizes the gauge transformations in classical electromagnetism and reflects that the number of physical degrees of freedom associated with each gauge potential is just 2, those of massless spin-1 particles. This gauge symmetry generates a well-defined dynamics of the color charges. The 8 gluons are simply related to the 8 parameters of a general $SU(3)$ transformation. Moreover, because the gluons carry color charges, their self-interactions are a source of the key differences between QCD and QED.

A new feature in the interactions of quarks is that, these change its color from i to j by emitting or absorbing a gluon of color a coupled through $SU(3)$ generator $t^a_{ij} = \lambda^a_{ij}/2$.

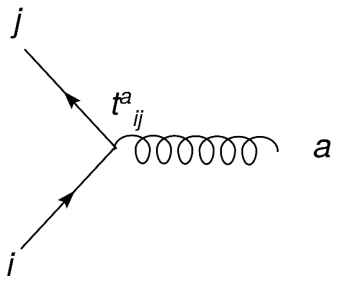


Figure D.1: The color of a quark can change from i to j by a gluon of color a

D.5 Asymptotic freedom

For a moment, we forget the fact that the free quarks do not exist and the interaction between quarks may not be calculable through simple one-gluon exchange.
Let us consider a quark of color i, exchanging a gluon with another quark of color j, scatters into a quark of color i', along with a quark of color j'. The scattering for this process is

$$S \sim (-igt^a_{i'i}\gamma^\rho)D_{\rho\sigma}(q)(-igt^a_{j'j}\gamma^\sigma), \tag{D.7}$$

where

$$D_{\rho\sigma}(q) = -i\frac{\delta_{\rho\sigma}}{q^2} \tag{D.8}$$

is the gluon propagator in the Feynman gauge. Here the momentum q of the gluon is space-like: $q^2 < 0$.
In the language of Feynman diagrams, this process can be viewed as

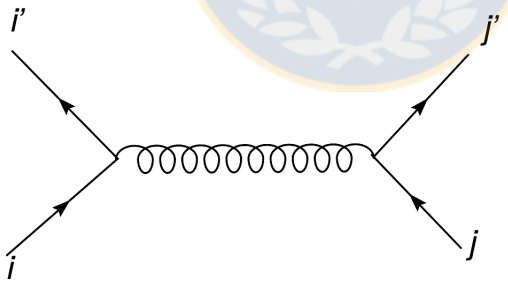


Figure D.2: Strong interaction of two quarks through one-gluon exchange.

Let us calculate the color factors for the two quarks in the conjugate 6-dimensional subspace $\bar{\mathbf{6}}$, and the conjugate 3-dimensional fundamental representation $\bar{\mathbf{3}}$ of $SU(3)$. If two quarks are in $\bar{\mathbf{6}}$, for example, we can take a specific case $i = j = i' = j' = 1$. By summing over a, the average color factor is $1/3$. Thus the interaction between the quarks is repulsive (analogous to the electric force). On the other hand, if the two quarks are in $\bar{\mathbf{3}}$, the color factor is $-2/3$, and the force between them is attractive. Thus, we conclude that the force between two quarks depend on the color states.

Although, the interactions between two quarks are similar to electric interactions, a striking property it happens in QCD: *the asymptotic freedom*. These phenomenon states that in strong interactions

physics, the interaction strength $\alpha_s = g^2/4\pi$ between quarks becomes smaller as the distance between them gets shorter [85].
From relativistic quantum mechanics point of view, the vacuum is not empty because it is just the lowest energy state of a field system and is filled with electrons of negative energies. Then, when a photon passes through the vacuum, it can induce transitions of an electron from negative to positive energy states, virtually creating a pair of electron and positron. This process is called *vacuum fluctuation*. Because of this, the interaction between two electrons in the vacuum becomes

$$F = \frac{\alpha_{em}(r^2)}{r^2} \tag{D.9}$$

where α_{em} is an effective fine structure constant, depending of r, or momentum transfer $q \sim 1/r$. This constant measures the interaction strength of the low-energy photon. For example in perturbation theory, this constant is given by

$$\alpha_{em}(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{\alpha_{em}(\mu_0)}{3\pi} \ln \frac{\mu^2}{\mu_0^2}} \tag{D.10}$$

where μ is a momentum scale, roughly corresponding to $1/r$. As we see, the interaction strength of the two electrons gets stronger as the distance between them becomes smaller. Therefore, QED becomes a strongly-coupled theory at very short distance scale.
On the other hand, in QCD the coupling constant have the following scale-dependence

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln(\mu/\Lambda_{QCD})} \tag{D.11}$$

with $\beta_0 = 11 - \frac{2}{3}n_f$ with n_f is the number of active quark flavor. This new coupling constant in QCD goes to 0 as the momentum scale $\mu \to \infty$ or the distance approaching 0. This strange behavior of the strong coupling has been verified in high-energy experiments to very high precision. The integration constant Λ_{QCD} is an intrinsic QCD scale, and it states the scale at which the coupling constant becomes large and the physics becomes non-perturbative, i.e., the scale for strong interactions physics. In the scheme $\bar{\text{MS}}$-scheme with 3 quark flavors

$$\Lambda_{QCD} \sim 250 MeV. \tag{D.12}$$

One must emphasize that the force between quarks does not get weaker at the shorter distance, despite the fact that the coupling does. In fact, the force still grows at short distance in an asymptotic free theory.

D.6 Color confinement

The confinement problem states that:

Any strongly interaction system at zero temperature and density must be a color singlet at distance scale larger than $1/\Lambda_{QCD}$.

This is a phenomenon at low-energy, and has as consequence that isolated quarks cannot exist in nature. It is important remark that the color confinement of QCD is a theoretical conjecture consistent with experimental facts. The problem can be understood with a simple analogy based on a spring. When a spring is stretched beyond the elastic limit, it breaks to produce two springs. In

the case of the quark pair, a new quark-antiquark pair will be created when pulled beyond certain distance. Part of the stretching energy goes into the creation of the new pair, and as a consequence, one cannot have quarks as free particles.
The example of above only is a sort of self-speculation of our own intuitions. One would think that for large distance scales the understanding of these interactions will be more easy. However, is very difficult at present time. The only way that we know how to solve QCD in the strong coupling regime is simulate the theory on a finite space-time lattice, or usually called *lattice QCD.* The lattice QCD can furnish important additional insight, but for multi-scale and/or time-evolution problems, the applicability of lattice methods is still severely restricted; the lattice formulation of QCD requires a Wick rotation to the usual space-time. The time-coordinate can then be treated on an equal footing with the other dimensions, but intrinsically Minkowskian problems, such as the time evolution of a system, are inaccessible. The limited size of current lattices also severely constrain the scale hierarchies that it is possible to "fit" between the lattice spacing and the lattice size.

D.7 Quark-gluon plasma

Although, one cannot observe particle as quarks or gluons in a isolated way, works in QCD and cosmology have released that at certain temperatures and thermodynamical densities, it could be possible have quarks (also gluons) interacting like free particles. The more popular model about this theme is the called Quark-Gluon Plasma.

It is believed that above 10^9K and/or pressures above 10^{32}Pa, the strong interaction to be the dominate interaction between the elementary constituents of the matter. Also to low temperatures and pressures above $10^{32}Pa$ the matter is describe by a degeneracy neutron gas, which it should exists in neutron stars. On the other hand, the neutron matter becomes in a nucleon gas if it is heated to temperatures of several MeV ($\sim$ 10MeV). It is expected that to high temperatures and pressures the nucleons gas could go through a phase transition to deconfinement state of the matter. It is due to the existence of the asymptotic freedom phenomenon in QCD. It tells us that strong interactions decreases to high temperatures. Such deconfinement state of the matter is known as quark-gluon plasma or QGP. The QGP has a transition at $\sim$ 200MeV ($\sim 2 \times 10^{12}$K). Here quarks and gluons are no confined in colorless particles.
Moreover, there are other properties of QCD predicting that to phase transitions should occur at high temperatures. However, like in QFTs, the spontaneously breaking of the chiral symmetry takes place to low temperatures, and his restoration to high temperatures becomes a sufficient condition to the existence of a phase transition. Although, it remains the open question whether the chiral symmetry transition and the deconfinement transition are or no the same one. Nowadays, only arguments of the lattice QCD have been able to provide signals about it.
The prediction of a deconfinement region was predicted to high temperatures and/or pressures [86, 87] after the discovery of asymptotic freedom [85]. In this region, the quarks and gluons interacts weakly and the system behave as an ideal ultra-relativistic gas. Therefore, the degrees of freedom correspond to the flavour numbers, spin states and, color and charge states. The name QGP was used to understood the matter like dissociated compounds. The question if the QGP phase transition takes place by means certain critical features of T and μ, it is solved by the intrinsic symmetries of the QCD Lagrangian.
To temperatures $\Lambda_{QCD} \leq T \leq$ charm mass, where the strong interaction is very weak, the QGP is an ideal gas, and therefore will be constituted by all the elementary particles: leptons, bosons and light quarks u, d, and s.
Nowadays, the most adequate theoretical method that tested QCD is the *lattice* which the usual

space-time is discretized. One of their results is that massless quarks shows a transition at baryionic chemical potential $\mu_B = 0$ in accordance to the spontaneous breaking of the chiral symmetry in QCD. Also lattice QCD computations have studied the parameters of the chiral and deconfinement transition showing that both transitions occur at the same temperature.
A full review about the QGP features, and their recent results can be found in Ref. [88]

Bibliography

[1] V.N. Gribov, *Quantization of non-Abelian gauge theories*, *Nuclear Phys.* **B139** (1978).

[2] D. Zwanziger, Nucl. Phys. B **323** (1989) 513; ibid. **399** (1993) 477.

[3] P. Arnold, *Int. J. Mod. Phys. E* **16** (2007) 2555 [arXiv:0708.0812 [hep-ph]]. U. W. Heinz, *Nucl. Phys. A* **721** (2003) 30 [arXiv:nucl-th/0212004]. L. McLerran, *Nucl. Phys. A* **787** (2007) 1.

[4] K. Yagi, T. Hatsuda, Y. Miake, *Quark-Gluon Plasma: From Big Bang to Little Bang* , Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (No. 23), October, 2008.

[5] B. C. Barrois, *Nucl. Phys. B* **129** (1977) 390.

[6] F. Weber, Prog. Part. *Nucl. Phys.* **54** (2005) 193; D. H. Rischke, *Prog. Part. Nucl. Phys.* **52** (2004)197 (2004); M. Alford, *Prog. Theor. Phys. Suppl.* **153** (2004) 1; M. Buballa, *Phys.Rept.* **407** (2005) 205; I. Shovkovy, *Found. Phys.* **35** (2005) 1309.

[7] A. D. Linde, *Rept. Prog. Phys.* **42** (1979) 389.

[8] V.A. Matveev, V.A. Rubakov, A.N. Tavkhelidze, and M.E. Shaposhnikov, *Sov. Phys. Usp.* **31** (1988), Usp. Fiz. Nauk, **156** (1988) 253; V.A. Rubakov and M.E. Shaposhnikov, *Phys. Usp.* **39** (1996) 461, Usp. Fiz. Nauk 166 (1996) 493.

[9] J. Kapusta, *Phys. Rev. D* **24** (1981) 426.

[10] L. D. Faddeev and V. N. Popov, *Phys. Lett.* **25B**, 30 (1967).

[11] C. N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev.* **96** (1954), 191–195.

[12] E. Witten, "Some Exact Multipseudoparticle Solutions Of Classical Yang-Mills Theory," *Phys. Rev. Lett.* **38**, 121 (1977).

[13] F. Canfora, M. Kurkov, L. Rosa, P. Vitale, *The Gribov problem in Noncommutative QED*, arXiv:1505.06342v1 [hep-th].

[14] I.V. Tyutin, LEBEDEV-75-39 (1975), arXiv:0812.0580 [hep-th].

[15] C. Becchi, A. Rouet, R. Stora, *Ann. Phys.* **98**, 287–321 (1976).

[16] C. Becchi, A. Rouet, R. Stora, *Commun. Math. Phys.* **42**, 127–162 (1975).

[17] T. Kugo, I. Ojima, Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[18] F.J. Dyson, *Phys. Rev.* **75**, 1736–1755 (1949).

[19] J.S. Schwinger, *Proc. Natl Acad. Sci.* **37**, 452–455 (1951).

[20] J.S. Schwinger, *Proc. Natl Acad. Sci.* **37**, 455–459 (1951).

[21] M. Q. Huber, *On Gauge Fixing Aspects of the Infrared Behavior of Yang-Mills Green Functions*, Springer Theses, Physics, Particle and Nuclear Physics, 2012.

[22] O. Piguet, S.P. Sorella, *Lect. Notes Phys.* M **28**, 1–134 (1995).

[23] T. Kugo, S. Uehara, *Nucl. Phys.* B **197**, 378 (1982).

[24] A. Das, *Lectures on Quantum Field Theory*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2008.

[25] I. M. Singer, *Commun. Math. Phys.* **60** (1978) 7.

[26] Wilson, K.G., *Confinement of quarks. Phys. Rev.* D **10**, 2445–2459 (1974).

[27] F. Lenz, J. W. Negele, M. Thies, *Phys. Rev.* D **69**, 074009 (2004) [arXiv:hep-th/0306105].

[28] T. C. Kraan, P. van Baal, *Nucl. Phys.* B **533**, 627 (1998) [arXiv:hep-th/9805168].

[29] K. M. Lee, C. h. Lu, *Phys. Rev.* D **58**, 025011 (1998) [arXiv:hep-th/9802108].

[30] F. S. Henyey, *Phys. Rev. D* **20** (1979) 1460.

[31] P. van Baal, Nucl. Phys. B **369**, 259–275 (1992).

[32] D. Zwanziger, Phys. Rev. D **69**, 016002 (2004), arXiv:hep-ph/0303028.

[33] M.A.L. Capri, A.J. Gomez, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella, Phys. Rev. D **79**, 025019 (2009), arXiv:0811.2760 [hep-th].

[34] G. Dell'Antonio, D. Zwanziger, *Every gauge orbit passes inside the Gribov horizon*, *Comm. Math. Phys.* **138** (1991) 291–299.

[35] R.F. Sobreiro, S.P. Sorella, *Introduction to the Gribov ambiguities in Euclidean Yang–Mills theories.* arXiv:hep-th/0504095.

[36] D. Dudal, S. P. Sorella, N. Vandersickel, H. Verschelde, *Phys. Rev. D* **79**, 121701 (2009).

[37] D. Zwanziger, N. Vandersickel, *The Gribov Problem and QCD Dynamics*, *Phys. Rep.*, **20**, 175-251, 2012.

[38] R. Alkofer, L. von Smekal, *Phys. Rept.* **353**, 281 (2001) [hep-ph/0007355].

[39] L. Baulieu, D. Dudal, M. S. Guimaraes, M. Q. Huber, S. P. Sorella, N. Vandersickel and D. Zwanziger, *Phys. Rev. D* **82** (2010) 025021; arXiv:1009.5846 [hep-th]].

[40] S. P. Sorella, D. Dudal, M. S. Guimaraes, N. Vandersickel, (2010) arXiv:1102.0574 [hep-th].

[41] A. Cucchieri, T. Mendes, PoS **LAT2007**, 297 (2007). [arXiv:0710.0412 [hep-lat]].

[42] A. Cucchieri, T. Mendes, *Phys. Rev. Lett.* **100**, 241601 (2008). [arXiv:0712.3517 [hep-lat]].

[43] A. Cucchieri, T. Mendes, *Phys. Rev.* **D81**, 016005 (2010). [arXiv:0904.4033 [hep-lat]].

[44] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel and H. Verschelde, *Phys. Rev. D* **78**, 065047 (2008).

[45] D. Dudal, O. Oliveira, N. Vandersickel, *Phys. Rev. D* **81**, 074505 (2010). [arXiv:1002.2374 [hep-lat]].

[46] D. Zwanziger, arXiv:0904.2380 [hep-th], 2009.

[47] K. Fujikawa, *Nucl. Phys.* B **223** (1983), 218.

[48] K. Fujikawa, *Gribov Problem and BRST Symmetry*, 1995, arXiv:hep-th/9511124.

[49] M. A. L. Capri, A. J. Gomez, M. S. Guimares *et al.*, Phys. Rev. **82**, 105019 (2010). [arXiv:1009.4135 [hep-th]].

[50] S. P. Sorella, *Phys. Rev.* **D80**, 025013 (2009). [arXiv:0905.1010 [hep-th]].

[51] K. -I. Kondo, [arXiv:0905.1899 [hep-th]].

[52] D. Dudal, N. Vandersickel, [arXiv:1010.3927 [hep-th]].

[53] L. Dolan and R. Jackiw, *this issue, Phys. Rev. D* **9**, 3320 (1974).

[54] S. Weinberg, *this issue, Phys. Rev. D* **9**, 3357 (1974).

[55] J. I. Kapusta, C. Gale, "*Finite-Temperature Field Theory: Principles and Applications*", Cambridge University Press, Cambridge, 2006.

[56] M. Le Bellac, *Thermal Field Theory*, (Cambridge University Press, Cambridge, 2000).

[57] J. I. Kapusta and Gale, *Finite-Temperature Field Theory: Principles and Applications*, (Cambridge University Press., Cambridge, 2006).

[58] R.P. Feynman and A.R. Hibbs, *'Quantum Mechanics and Path Integrals (McGrawHill)'* , New York, 1965).

[59] C. W. Bernard, *Phys. Rev. D* **9**, 3312 (1974).

[60] W. Dittrich, M. Reuter, *Selected Topics in Gauge Theories*, Lectures Notes in Physics. Springer-Verlag, Tubingen, (1985).

[61] N. Jacobson, *Lie Algebras*, New York, Chapter III, pp.58 (1979).

[62] B. S. DeWitt, *Phys. Rev.* **162** (1967) 1195; *Dynamical Theory of Groups and Fields (Gordon and Breach, New York, 1965)*; in Quantum Gravity 2, ed. C. J. Isham, R. Penrose and D. W. Sciama (Oxford University Press, New York, 1981), pp. 449.

[63] G. 't Hooft, *in Acta Universitatus Wratislavensis No. 368, XIIth Winter School of Theoretical Physics in Karpacz, February-March, 1975; also in Functional and Probabilistic Methods in Quantum Field Theory, Vol. I.*

[64] L. F. Abbott, *Nucl. Phys.* B **185**, 189 (1981); *Acta Physica Polonica* B **13**, 33 (1982).

[65] S. J. Gates, M. T. Grisaru, M. Rocek, W. Siegel, *' Superspace Or One Thousand And One Lessons In Supersymmetry', Reading, Usa: Benjamin/Cummings (1983) 548 P. (Frontiers In Physics, 58).*

[66] L. F. Abbott, *Nucl. Phys.* **B185** (1981).

[67] S. Ichinose, M. Omote, *Nucl. Phys.* **B203** (1982); D. M. Capper, A. MacLean, *Nucl. Phys.* **203** (1982) 413.

[68] P. A. Grassi, *Nucl. Phys.* **B560** 499 (1999).

[69] P. Ramond, *Field Theory: A Moder Primer*, Frontiers in Physics Series, Vol 74, 2nd Edition, 1981.

[70] S. Weinberg: *The Quantum Theory of fields v.2*, (Cambridge University Press., 1996), pp. 95-109.

[71] M. Luscher, *Phys. Lett. l18* B (1982) **391**; Nucl. Phys. B **219** (1983) 233.

[72] F. Canfora, P. Salgado-Rebolledo, *Phys. Rev. D* **87**, 045023 (2013).

[73] M. Luscher, *Phys. Lett. l18* **B** (1982) 391; *Nucl. Phys. B* **219** (1983) 233.

[74] M. Guimaraes, S. Sorella, *J. Math. Phys.* **52**, 092302 (2011).

[75] K. Yagi, T. Hatsuda, Y. Miake, Quark-Gluon Plasma: From Big Bang to Little Bang , Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (No. 23), pp. 60-64, October, 2008.

[76] F. Canfora, P. Pais and P. Salgado-Rebolledo, *Eur. Phys. J.* C **74** (2014) 2855.

[77] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, Inc. (1979).

[78] F. Canfora, D. Dudal, I. Justo, P. Pais, L. Rosa, V. Vercauteren, *Eur. Phys. J. C.* **75**, 326 (2015).

[79] A. Peshier, D. Giovannoni, *IOP Science*, Journal of Physics: Conference Series **668** (2016) 012076.

[80] Blaizot J. P., Liao J. and McLerran L. 2013 *Nucl. Phys.* **A 920** 58.

[81] Peshier A., Kampfer B., Pavlenko O. P. and Soff G. 1996 *Phys. Rev* **D 54** 2399.

[82] D. B. Lichtenberg, *Unitary Symmetry and Elementary Particles*, 2nd Edition, 1978.

[83] P. Dirac, *The Lagrangian in Quantum Mechanics*, November 19, 1932.

[84] R. P. Feynman, *Space-Time Approach to Non-relativistic Quantum Mechanics*, *Rev. Mod. Phys.*, vol. 20, 1948.

[85] D. J. Gross and F. Wilczek, Ultraviolet behavior of non-abelian gauge theories, Phys. Rev. Lett. 30 (1973), 1343–1346.

[86] J.C. Collins, M.J. Perry. *Superdense Matter: Neutrons or Asymptotically Free Quarks?* Phys. Rev. Let., **34**, p.p. 1353, 1975.

[87] N. Cabibbo, G. Parisi *Exponential hadronic spectrum and quark liberation.* Phys. Let. B, **59**, p.p. 67, 1975.

[88] G. Martinez, *Advances in Quark Gluon Plasma*, arXiv:1304.1452 [nucl-ex].