Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
Doctorado en Ciencias Físicas
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Dipartimento di Fisica
Dottorato di ricerca in Fisica
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By

Diego Sebastián Molina Peñafiel

Thesis Directors : Dr. Patricio Salgado
: Dr. Laura Andrianopoli

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Thesis Directors : **Dr. Patricio Salgado**
Departamento de Física
Universidad de Concepción, Chile

: **Dr. Laura Andrianopoli**
Dipartimento di Fisica
Politecnico di Torino, Italia

Comission : **Dr. Mario Trigiante**
Dipartimento di Fisica
Politecnico di Torino, Italia

: **Dr. Riccardo D 'Auria**
Dipartimento di Fisica
Politecnico di Torino, Italia

: **Dr. Fernando Izaurieta**
Departamento de Física
Universidad de Concepción, Chile

: **Dr. Ricardo Carocaa**
Departamento de Física
Universidad Católica de la Santísima Concepción, Chile

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Abstract

This thesis is about a new methd to perform the S-expansion procedure and studies in extended supergravites.

In the present work, the concept of zero-reduction has been extended reproducing a generalized Inönü-Wigner contraction. This involves an infinite abelian semigroup $S_E^{(\infty)}$ and the removal of an ideal subalgebra. We refer about the use of theorem VII.2 of the reference [20] which serves to construct the topological invariant of a respective S-expanded algebra from another of which the bilinear form is known, therefore is an extension and generalization of the mentioned theorem. This procedure allows to develop the dynamics and construct the Lagrangians of several theories. This work reproduces the results already presented in the literature, concerning the generalized Inönü-Wigner contraction, and also gives some new features. Moreover, it gives a connection between the contraction processes and the expansion methods introduced in [17], which was an open question already mentioned in [12].

Also is shows one of the interesting applications, which is to obtain a particular hidden Maxwell superalgebra underlying supergravity in four dimensions. Thus we have written the hidden Maxwell superalgebra in the Maurer-Cartan formalism, and then, we have considered the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, in order to show the way in which the trivial boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator, appearing in the hidden Maxwell superalgebra. These extensions contain terms which involve the cosmological constant. Interestingly, the presence of these terms depends strictly on the form of the extra fermionic generators appearing in this hidden extension of $D = 4$ supergravity.

Besides, we have reviewed some concepts of the S-expansion. We show especially, how S-expansion procedure affects the geometry of a Lie group: was found how the magnitude of a vector change and the angle between two vectors. About the kind of algebra, after apply an S-expansion, it is a non-simple Lie algebra. Then, considering resonance and reduction, we built an analytic method able to give us the multiplication table(s) of the set(s) involved in an S-expansion process for reaching a target Lie (super)algebra from a starting one, after having chosen properly the partitions over subspaces of the considered (super)algebras.

Furthermore, we study in the context of ungauged supergravities, the symmetry under the kinetic part in the action and the realization to a global symmetry group G. Moreover, we show the use of the solution generating technique. As a first step in the unfinished research we work with the AdS metric with the

global group $G_{2(2)}$,which it is the global symmetry group of the 3D description of $N = 2$ supergravity coupled to a vector multiplet [13], where we found the charges. Considering that a solution with cosmological constant is always gauged in extended supergravities [36], it is planned to investigate the applicability of the solution generating technique on gauged theories asymptotically flat, i.e AdS metric.

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Introduction

Figuras grababan en las rocas,
Hacían fuego, ¡también hacían fuego!
Yo soy el Individuo.
Me preguntaron que de dónde venía.
Contesté que sí, que no tenía planes determinados,
Contesté que no, que de allí en adelante.

a part from Soliloquio del individuo[a]*, Poemas y Antipoemas de Nicanor Parra (1965)*

[a] I saw they made certain things, They asked me where I came from. I answered yes, that I had no definite plans, I answered no, that from here on out.

Gauge theories and gravity

The Yang Mills theory (YM) is a *gauge theory* that has provided excellent results in the unification of three fundamental interactions. With this we refer to the *standard model* of particle physics (SM), which is a relativistic field theory where the three forces are mediated by spin-1 particles which respect invariance under local transformations (i.e. space-time dependent) of some suitable internal symmetry group. Gauge theories provide a re-normalizable description of these fields and of their coupling to matter. The *Brout-Englert-Higgs mechanism* (BEH) of spontaneous symmetry breaking then allows the interaction fields to have an effective mass without spoiling the renormalization property of the theory.

On the other hand, the theory of *general relativity*, is the geometric theory of gravitation published by Albert Einstein which elegantly describes the gravitational interaction of massive bodies. Is the fourth force which, unlike the others, does not constitute a genuine gauge theory, such as the YM does.

In a Yang-Mills theory, where ${F_a}^{\mu\nu}$ is the field strength and the metric $g_{\mu\nu}$ of the space-time its a non-dynamical background [1],

$$S_{YM} = -\frac{1}{4}\int d^4x g_{\mu\rho}g_{\nu\sigma}F^{a\rho\sigma}{F_a}^{\mu\nu}, \tag{0.0.1}$$

but in the Einstein equations the actions depends of $g_{\mu\nu}$ and its derivatives,

$$S_g = \int \sqrt{-g}Rd^4x, \tag{0.0.2}$$

where lies the dynamics. This is an important difference between the two theories. Einstein's gravity can be viewed as the gauge theory of the Poincaré group and the graviton as the gauge boson associated with the local translation generators P_a. That means that while a gauge group in an internal symmetry acts on internal degrees of freedom, the gauge group in gravity describes external, i.e. spacetime, symmetries. As long as there is no dynamics along the internal directions, there is dynamics along the external ones: The dependence of the fields on the space-time coordinates is not the result of some (unphysical) gauge transformation, but is dictated by the field equations where torsion vanish (*on-shell* invariance).

In order to unify gravity with the other interactions in a unique theory, it is necessary to put together, the internal symmetries with the space-time symmetries. A good candidate for this purpose is *supersymmetry.* The basic idea is simple. As the color group $SU(3)$ of the standard model reshapes the color states of a given quark flavor among them and the $SU(2)$ group combines the fields of the weak doublets, supersymmetry has the ability to interchange bosons and fermions.

Is it possible to have fermions and bosons in a symmetric theory? The answer turned out to be negative, for the Coleman-Mandula theorem [2], whose conclusions were that any group of bosonic symmetries of the S-matrix must be the direct product of the Poincaré group with an internal symmetry group, so that, if Poincaré and internal symmetries were to combine, the S-matrices for all processes would be zero. Or in other words; "there are no generators of bosonic symmetries that generate symmetry transformations connecting fields of different spins." That is, "There are no bosonic generators that generate supersymmetry". However, the results of this theorem were related to its strong assumptions, and one of them was that the considered algebra was a *Lie algebra.* In fact, it turns out that generalizing the notion of Lie algebra to a graded Lie algebra, the theorem could be evaded [3]. A *graded Lie Algebra* is an algebra that contains, besides standard bosonic generators $\mathbf{b}$, also fermionic generators $\mathbf{f}$, which correspond to *supersymmetric generators* obeying anticommutation relations. Symbolically

$$[\mathbf{b},\mathbf{b}] \propto \mathbf{b}, \quad [\mathbf{b},\mathbf{f}] \propto \mathbf{f}, \quad \{\mathbf{f},\mathbf{f}\} \propto \mathbf{f}.$$

Its generators comprise Lorentz transformations J_{ab}, space-time translations P_a, generators B_r of internal (compact) symmetry group, and also, fermionic generators Q_α. The Q's generate the so-called *superalgebra* which involve the *supersymmetry.* Supersymmetric theories differ in the amount of supersymmetry, namely in the number $\mathcal{N}$ of the supersymmetry generators Q, and in the field content which should correspond to multiplets of *super group* (SG). $\mathcal{N}$ supersymmetry generators define an $\mathcal{N}$-extended supersymmetry, because it has Q^A-generators, where $A = 1, \ldots, \mathcal{N}$ where $\mathcal{N}$ is the number of supersymmetries in the theory.

The general transformation laws read:

$$\delta_\epsilon \mathbf{b} = \bar{\epsilon}\ \mathbf{f}, \qquad \delta_\epsilon \mathbf{f} = \epsilon\ \partial \mathbf{b},$$

where ϵ is the infinitesimal supersymmetry parameter, carrying spinorial indices, and ∂ is for a space-time derivative. Since the commutator of two supersymmetry transformations amounts to an operator which is proportional to the space-time derivative

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \propto (\bar{\epsilon}_1 \gamma^{\mu} \epsilon_2)\, \partial_{\mu} \mathbf{b}.$$

Using this result, we can immediately deduce the important fact that a supersymmetric theory will necessarily be invariant under translations. This link between supersymmetry and the Lorentz group is recorded in the algebra in form of the anticommutator of two supersymmetry charges, modulo action of internal symmetry operators Z, which is expressed schematically as*

$$\{Q, Q\} \propto P + Z,$$

which are central charges of the superalgebra.

The corresponding superalgebra of infinitesimal generators is therefore called *super-Poincaré algebra*. The *fermionic* generators Q allow for a non-trivial interplay between space-time and internal symmetries. Moreover, having a spin quantum number, the action of Q on a states varies by $1/2$ its spin, so that:

$$Q|\mathbf{f}\rangle = |\mathbf{b}\rangle, \quad Q|\mathbf{b}\rangle = |\mathbf{f}\rangle.$$

The supersymmetry generators belong to the spin-$1/2$ representation of the Lorentz group as well as to a representation of the internal compact group. A supersymmetry transformation generated by the Q-generators, sends a bosonic state into a fermionic one and vice-versa.

This implies that, modulo internal transformations Z and the combination of two subsequent supersymmetry transformations amounts to a space-time translation. Moreover Q commutes with P, and therefore with the mass operator $m^2 = P^2/c^2$. As a consequence of this, irreducible representations of SG (supermultiplets), comprise one-particle states with the same mass but different spins. This is certainly a desirable feature if we ultimately aim at unifying all fundamental forces of nature together with matter. In fact, the gravitational force is mediated by the spin-2 graviton while the other interactions by spin-1 vector bosons and matter is made of spin-$1/2$ particles.

However the Poincaré superalgebra is not the only scenario to obtain a theory of supergravity, as well as AdS or Poincaré is not the only Lie Algebra to generate a theory of gravity. In the first part of the present thesis we will cover a spectrum of Lie algebras and superalgebras where gravity and supergravity unfolds. In the second part we focus on the (super) ungauged gravity and a technique on it visualizing a possible link with a gauged theory.

Thesis plan

As previously mentioned, throughout this thesis two areas will be approached, the two with respect to gauge theories involving (super)gravity. In the first part, in section 1.3, will we study some properties of the S-expansion, and we apply the knowledge into the technique to control what kind of (super)algebra we want to obtain, specifically in section 1.4, when it is show a formal way to find the connection between different (super)algebras through the determination of a

*We have been omitted the bar on the second generator Q; $\overline{Q}$.

(semi)group. The reproduced results in section 1.5 are known in the literature, but there are also some new ones.

In chapter 2 we extend the usual S-expansion procedure. Explicitly, we realize the generalization of an Inönü-Wigner contraction as an infinite S-expansion whit an Ideal-subtraction. Several examples are shown in section 2.6.

In chapter 3 we work in four dimension in the framework of free differential algebra finding new hidden super-algebras. Thus, we have written the hidden Maxwell superalgebra in the Maurer-Cartan formalism, and we have then considered the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, in order to show the way in which the boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator appearing in the hidden Maxwell superalgebra. These extensions contain terms which involve the cosmological constant.

In the second part of the thesis, chapters 4 and 5, the study carried out on *extended supergravities*, studying the scalar manifold where the coset space of the symmetries lies and application of the *solution generating technique*, which is a powerfull method to obtain a lot of information of a *ungauged supergravity*. Using dimensional reduction and dualization over the fields in a particular solution, we can get the charges and when we apply over these a particular transformation we find another new ones. Examples of application are considered and the line of research to follow in that area is shown.

Part I

Applications of S-Expanded Lie (super)algebras in (super)gravity

Chapter 1

S-Expansion Process and its Geometry

" I am certain, absolutely certain, that, at some point in the future, these theories will be recognized as fundamental. If I wish to create such an understanding sooner, it's because, among other things, I could then do ten times more."

Sophus Lie, in a letter to Mayer in 1884.

1.1 Motivation

Understanding the importance of searching in various fields of mathematics have important physical implications, that is why in the first part of this thesis we focus on the study, analysis and techniques to find Lie algebras from existing ones. In particular we analyse the S-expansions and contraction.

Its not hard to justified the study of Lie algebras, in the last years there are more and more applications that everybody can find in the modern physics. That is why we can analyze different regimes of physics when we work with two different algebras. Even more, the relation of two given Lie (super)algebras among themselves, and in particular, the derivation of new (super)algebras from other ones, is a problem of great interest, in both Mathematics and Physics, since it involves the problem of mixing (super)algebras, which is a non-trivial way of enlarging spacetime symmetries. Then, it is justified, the study and the motivation to devote time and energy in paying attention to the transformations that allow to find diverse Lie algebras.

A very known method to link two different algebras is the *contraction*, an idea presented by Segal in 1951 [4] and Inönü and Wigner in 1953 [6] (see also [7]). Consist in obtain from a given Lie group a different (non-isomorphic) Lie group through a group contraction with respect to a continuous subgroup of it. That amounts to a limiting operation on a parameter of the Lie algebra, altering the structure constants of this Lie algebra in a nontrivial singular manner, under suitable circumstances (*standard Inönü-Wigner contraction*). Then, Weimar-Woods generalized this procedure using parameters of arbitrary degree, the so-called *generalized Inönü-Wigner contraction* [8,9]

These contractions are important because explain formally why some theories arise as a limit regime of more “complete” theories (see [10] and references therein), and maybe the best known are the passage from the Poincaré algebra to the Galilei algebra, in the limit where the speed of light approaches infinity [6]. Similarly, the (Anti-)de Sitter algebra can be contracted to the Poincaré algebra in the limit where the radius of the universe is large. Other examples include the $\mathfrak{so}(3)$ algebra of rotations on a sphere, which contracts rotations into translations for small angular displacements, and the dynamical algebra $\mathfrak{sp}(2n, R)$ of the harmonic oscillators in n dimensions, which contracts to the $\mathfrak{u}(n) \oplus \mathfrak{hn}(2n)$ algebra, where $\mathfrak{hn}(2n)$ is the m'th Heisenberg-Weyl algebra, which describe collective excitations at low energy. From these examples, it can be seen that the existence of an “effective” theory is often the results of a contraction.

Another method to connect different (super)algebras is the expansion procedure, introduced for the first time in [16], and subsequently studied under different scenarios in [17–19]. In 2006, a natural outgrowth of the power series expansion method was proposed (see Ref.s [20]), which is based on combining the structure constants of the initial (super)algebra with the inner multiplication law of a discrete set S with the structure of a semigroup, in order to define the Lie bracket of a new S-expanded (super)algebra.

An additional advantage of in the work done in [20], and its relation with the standard Inönü-Wigner contraction process is discussed in chapter one and two of the present thesis. We will study the S-expansion method to be able to determine when it is possible to apply it relating two different algebras and we will analyze also their limitation. This last will serve to extend the process of S-expansion in such a way that it reproduces the same results of a generalized known Inönü-Wigner contraction.

On the other side, supergravity theories have, in various spacetime dimensions $4 \leq D \leq 11$, a field content that generically includes the metric, the gravitino, a set of 1-form gauge potentials, and $(p+1)$-form gauge potentials of various $p \leq 9$, and they are discussed in the context of *Free Differential Algebras* (FDAs). In particular, in the framework of FDAs, the structure of $D = 11$ supergravity, first constructed in [21]; then it was reconsidered in [22], adopting the superspace geometric approach. In the same paper, the supersymmetric FDA was also investigated in order to see whether the FDA formulation could be interpreted in terms of an ordinary Lie superalgebra in its dual Maurer-Cartan formulation. This was proven to be true, and the existence of a hidden superalgebra underlying the theory was presented for the first time. In fact, in [22], the authors proved that the FDA underlying $D = 11$ supergravity can be traded with a Lie superalgebra which contains, besides the Poincaré superalgebra, also new bosonic 1-forms and a nilpotent fermionic generator Q's, necessary for the closure of the superalgebra.

Focused in this context, we obtain a *hidden Maxwell superalgebra* in four dimensions by performing an infinite S-expansion with subsequent ideal subtraction of the *hidden AdS-Lorentz superalgebra* underlying $D = 4$ supergravity. We also display a map in order to show the way in which the hidden AdS-Lorentz superalgebra and the Maxwell superalgebra can be obtained. This map also offers the links, consisting in infinite S-expansion and ideal subtraction, between other superalgebra in four dimensions.

1.2 Quick review of S-expansion process

In the following, we give a review of the concepts of S-expansion, reduction and resonance [20, 23].

The S-expansion procedure consists in combining the structure constants of a Lie algebra $\mathfrak{g}$ with the inner multiplication law of a semigroup S, to define the Lie bracket of a new S-expanded algebra $\mathfrak{g}_S = S \times \mathfrak{g}$.

Let $S = \{\lambda_\alpha\}$, with $\alpha = 1, ..., N$, be a *finite*, abelian semigroup with *two-selector* $K_{\alpha\beta}{}^{\gamma}$ defined by

$$K_{\alpha\beta}{}^{\gamma} = \begin{cases} 1, & \text{when } \lambda_\alpha \lambda_\beta = \lambda_\gamma, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.1}$$

Let $\mathfrak{g}$ be a Lie algebra with basis $\{T_A\}$ and structure constants $C_{AB}{}^{C}$, defined by the commutation relations

$$[T_A, T_B] = C_{AB}{}^{C}\, T_C. \tag{1.2.2}$$

Denote a basis element of the direct product $S \times \mathfrak{g}$ by $T_{(A,\alpha)} = \lambda_\alpha T_A$, and consider the induced commutator

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] \equiv \lambda_\alpha \lambda_\beta \left[T_A, T_B\right]. \tag{1.2.3}$$

It can be showed (see Ref. [20]) that the product

$$\mathfrak{g}_S = S \times \mathfrak{g} \tag{1.2.4}$$

corresponds to the Lie algebra given by

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}\, T_{(C,\gamma)}, \tag{1.2.5}$$

whose structure constants can be written as

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}. \tag{1.2.6}$$

The product $[\,\cdot\,,\,\cdot\,]$ defined in (1.2.5) is also a Lie product, because it is linear, antisymmetric, and satisfies the Jacobi identity. This product defines a new Lie algebra (characterized by $(\mathfrak{g}_S, [\cdot,\cdot])$), which is called *$S$-expanded Lie algebra*. This implies that, for every abelian semigroup S and Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{g}_S$ obtained through the product (1.2.4) is also a Lie algebra, with a Lie bracket given by (1.2.5).

Reduced algebras

The concept of *reduction* of Lie algebras, and in particular *0_S-reduction*, was introduced in [20]. It involves the extraction of a smaller algebra from a S-expanded Lie algebra $\mathfrak{g}_S$, when certain conditions are met.

Let us consider a Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g} = V_0 \oplus V_1$, where V_0 and V_1 are two subspaces respectively given by $V_0 = \{T_{a_0}\}$ and $V_1 = \{T_{a_1}\}$. When $[V_0, V_1] \subset V_1$, that is when the commutation relations between generators present the following form

$$[T_{a_0}, T_{b_0}] = C_{a_0 b_0}{}^{c_0} T_{c_0} + C_{a_0 b_0}{}^{c_1} T_{c_1}, \tag{1.2.7}$$

$$[T_{a_0}, T_{b_1}] = C_{a_0 b_1}{}^{c_1} T_{c_1}, \tag{1.2.8}$$

$$[T_{a_1}, T_{b_1}] = C_{a_1 b_1}{}^{c_0} T_{c_0} + C_{a_1 b_1}{}^{c_1} T_{c_1}. \tag{1.2.9}$$

It can be seen hat the structure constants ${C_{a_0b_0}}^{c_0}$ satisfy the Jacobi identity themselves, and therefore

$$[T_{a_0}, T_{b_0}] = {C_{a_0b_0}}^{c_0} T_{c_0} \tag{1.2.10}$$

itself corresponds to a Lie algebra, which is called *reduced algebra* of $\mathfrak{g}$ *. The authors also thought to call it: *Forced algebra.* As we will see later, this name may be more appropriate.

Let see how a reduced algebra emerge from a particular S-expansion. Let us consider an abelian semigroup S and the S-expanded (super)algebra $\mathfrak{g}_S = S \times \mathfrak{g}$. When the semigroup S has a *zero element* $0_S \in S$ (also denoted with the symbol $\lambda_{0_S} \equiv 0_S$ in the literature), this element plays a peculiar role in the S-expanded (super)algebra, as shown in [20]. In fact, we can split the semigroup S into non-zero elements λ_i, $i = 0, ..., N$, and a zero element $\lambda_{N+1} = 0_S = \lambda_{0_S}$. The zero element λ_{0_S} is defined as one for which

$$\lambda_{0_S}\lambda_\alpha = \lambda_\alpha\lambda_{0_S} = \lambda_{0_S}, \tag{1.2.11}$$

for each $\lambda_\alpha \in S$. Under this assumption, we can write $S = \{\lambda_i\} \cup \{\lambda_{N+1} = \lambda_{0_S}\}$, with $i = 1, ..., N$ (the Latin index run only on the non-zero elements of the semigroup). Then, the two-selector satisfies the relations

$$\begin{aligned}
{K_{i,N+1}}^{j} &= {K_{N+1,i}}^{j} = 0, \\
{K_{i,N+1}}^{N+1} &= {K_{N+1,i}}^{N+1} = 1, \\
{K_{N+1,N+1}}^{j} &= 0, \\
{K_{N+1,N+1}}^{N+1} &= 1,
\end{aligned}$$

from the viewpoint of multiplication rules,

$$\lambda_{N+1}\lambda_i = \lambda_{N+1}, \tag{1.2.12}$$

$$\lambda_{N+1}\lambda_{N+1} = \lambda_{N+1}. \tag{1.2.13}$$

Therefore, for $\mathfrak{g}_S = S \times \mathfrak{g}$ we can write the commutation relations

$$\left[T_{(A,i)}, T_{(B,j)}\right] = {K_{ij}}^{k} {C_{AB}}^{C} T_{(C,k)} + {K_{ij}}^{N+1} {C_{AB}}^{C} T_{(C,N+1)}, \tag{1.2.14}$$

$$\left[T_{(A,N+1)}, T_{(B,j)}\right] = {C_{AB}}^{C} T_{(C,N+1)}, \tag{1.2.15}$$

$$\left[T_{(A,N+1)}, T_{(B,N+1)}\right] = {C_{AB}}^{C} T_{(C,N+1)}. \tag{1.2.16}$$

If we now compare these commutation relations with (1.2.7), (1.2.8), and (1.2.9), we see clearly that

$$\left[T_{(A,i)}, T_{(B,j)}\right] = {K_{ij}}^{k} {C_{AB}}^{C} T_{(C,k)} \tag{1.2.17}$$

are the commutation relations of a *reduced* Lie algebra generated by $\{T_{(A,i)}\}$, whose structure constants are ${K_{ij}}^{k} {C_{AB}}^{C}$.

The reduction procedure, in this particular case, is equivalent to the imposition of the condition

$$T_{A,N+1} = \lambda_{0_S} T_A \equiv 0_S T_A = 0. \tag{1.2.18}$$

*Let us observe that, in general, a reduced algebra does *not* correspond to a subalgebra (see Ref. [20] for further details).

We can notice that, in this case, the reduction abelianize large sectors of the (super)algebra, and that for each j satisfying ${K_{ij}}^{N+1} = 1$ (that is $\lambda_{0_S}\lambda_j = \lambda_{N+1}$), we have

$$\left[T_{(A,i)}, T_{(B,j)}\right] = 0. \tag{1.2.19}$$

The above considerations led the authors of [20] consider that, whenever we have an abelian semigroup with a zero element $0_S \in S$, and let $\mathfrak{g}_S = S \times \mathfrak{g}$ be an S-expanded algebra, the algebra obtained by imposing the condition

$$\lambda_{0_S} T_A \equiv 0_S T_A = 0 \tag{1.2.20}$$

on $\mathfrak{g}_S$ (or on a subalgebra of it) generate a 0_S-reduced algebra of $\mathfrak{g}_S$ (or of the subalgebra). Futhermore, as we will see next, when a 0_S-reduced algebra presents a structure which is *resonant* with respect to the structure of the semigroup involved in the S-expansion process, the procedure takes the name of 0_S*-resonant-reduction*.

Resonant subalgebras for a semigroup

Another way to get smaller algebras from $S \times \mathfrak{g}$, which strongly depends on the structure of the semigroup involved in the process is through resonance, it will be described below.

Let $\mathfrak{g} = \bigoplus_{p\in I} V_p$ be a decomposition of $\mathfrak{g}$ into subspaces V_p, where I is a set of indices. For each $p, q \in I$ it is always possible to define the subsets $i_{(p,q)} \subset I$, such that

$$[V_p, V_q] \subset \bigoplus_{r\in i_{(p,q)}} V_r, \tag{1.2.21}$$

where the subsets $i_{(p,q)}$ store the information on the subspace structure of $\mathfrak{g}$.

Now, let $S = \bigcup_{p\in I} S_p$ be a subset decomposition of the abelian semigroup S, such that

$$S_p \cdot S_q \subset \bigcup_{r\in i_{(p,q)}} S_r, \tag{1.2.22}$$

where the product $S_p \cdot S_q$ is defined as

$$S_p \cdot S_q = \{\lambda_\gamma \mid \lambda_\gamma = \lambda_{\alpha_p}\lambda_{\alpha_q},\ \text{with } \lambda_{\alpha_p} \in S_p, \lambda_{\alpha_q} \in S_q\} \subset S. \tag{1.2.23}$$

When such subset decomposition $S = \bigcup_{p\in I} S_p$ exists, then, we say that this decomposition is in *resonance* with the subspace decomposition of $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_{p\in I} V_p$.

The resonant subset decomposition is essential in order to extract, systematically, subalgebras from the S-expanded algebra $\mathfrak{g}_S = S \times \mathcal{G}$, as it was enunciated and proven with the following theorem in Ref. [20] (This theorem corresponds to Theorem IV.2 of Ref. [20]. Its proof can be found in the same reference [20]):

Theorem 1. *Let $\mathfrak{g} = \bigcup_{p\in I} V_p$ be a subspace decomposition of $\mathfrak{g}$, with a structure described by equation (1.2.21), and let $S = \bigcup_{p\in I} S_p$ be a resonant subset decomposition of the abelian semigroup S, with the structure given in equation (1.2.22). Define the subspaces of $\mathfrak{g}_S = S \times \mathfrak{g}$ as*

$$W_p = S_p \otimes V_p, \quad p \in I. \tag{1.2.24}$$

Then,

$$\mathfrak{g}_R = \bigoplus_{p\in I} W_p \tag{1.2.25}$$

is a subalgebra of $\mathfrak{g}_S = S \times \mathfrak{g}$, called resonant subalgebra of $\mathfrak{g}_S$.

Invariant tensor

Given a Lie Algebra and a discrete abelian semigroup, the method allows us to construct, directly, new Lie Algebras with their corresponding non-trivial invariant tensors [20, 23, 24].

Theorem VII.2 of Ref. [20]: Let S be an abelian semigroup with nonzero elements λ_i, $i = 0, \ldots, N$ and $\lambda_{N+1} = 0_S$. Let $\mathfrak{g}$ be a Lie (super)algebra of basis $\{T_A\}$, and let $\langle T_{A_0} \cdots T_{A_n} \rangle$ be an invariant tensor for $\mathfrak{g}$. The expression

$$\langle T_{(A_1,i_1)}, \ldots T_{(A_n,i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n}{}^{j} \langle T_{A_1}, \ldots T_{A_n} \rangle \tag{1.2.26}$$

where α_j are arbitrary constants, corresponds to an invariant tensor for the 0_S-reduced algebra obtained from $\mathfrak{g}_S = S \times \mathfrak{g}$. Proof: the proof may be found in section 4.5 of Ref. [20].

1.3 Geometrical aspects of S-expansion Procedure

In order to know certain limits and applicability of the simple S-expansion process through semigroups (simple means without resonance or reduction) we will look at their effects at geometric levels, such as the Killing Cartan (KC) metric and the dimension of a respective algebra.

1.3.1 Visualizing the S-expansion through Killing-Cartan inner product

The commutator of an arbitrary element $Z = z^a T_a$ in some algebra is

$$[Z, T_a] = R(Z)_a{}^b T_b. \tag{1.3.1}$$

where $R(X)_a{}^b$ play the role of constant structure and correspond to the adjoint or regular representation of Z.

Let X be a vector of the vector space $\mathfrak{g}$, with generators $T_a \in \mathfrak{g}$, $a = 1, \ldots, N$. The Killing-Cartan form of a Lie algebra is a symmetric bilinear form given by [11]:

$$\begin{aligned}
(X, X) =& \mathrm{Tr}\left(R(X)R(X)\right) \\
=& \mathrm{Tr}\left(v^a R(T_a) v^b R(T_b)\right) \\
=& \mathrm{Tr} v^a v^b \left(R(T_a)^c_d R(T_b)^d_c\right) \\
=& \mathrm{Tr} v^a v^b \left(C_a\right)^c_d \left(C_b\right)^d_c .
\end{aligned} \tag{1.3.2}$$

The inner product of Killing-Cartan provides information about the geometry of the manifold of the group in a neighborhood of identity. The information is obtained in terms of compactness, not compact or nilpotency of the group. This information can be extrapolated to the rest of the manifold using the fact that a Lie group is a "geodesically complete" manifold, i.e., we can reconstruct it completely through the process of exponentiation of algebra. The vector space of the Lie algebra can be divided into three subspaces under the Cartan-Killing inner product. The inner product is positive-definite, negative-definite, and identically zero. These three subspaces are denoted by

$$\mathfrak{g} = V_+ + V_- + V_0. \tag{1.3.3}$$

The subspace V_0 is a subalgebra of $\mathfrak{g}$. It is the largest nilpotent invariant subalgebra of $\mathfrak{g}$. Under exponentiation, this subspace maps onto the maximal nilpotent invariant subgroup in the original Lie group.

The subspace V_- is also a subalgebra of $\mathfrak{g}$. It consists of compact operators. That is, the exponential of this subspace is a subset of the original Lie group that is parameterized by a compact manifold. It also forms a subalgebra in $\mathfrak{g}$ (not invariant).

Finally, the subspace V_+ is not a subalgebra of $\mathfrak{g}$. It consists of noncompact operators. The exponential of this subspace is parameterized by a noncompact submanifold in the original Lie group.

The decomposition of an algebra $\mathfrak{g}$ in these subspaces implies that in the quadratic form (X, X) of the group, there are summands with different signs representing the spaces denoted as V_-, V_+, and V_0.

The character of an algebra, denoted with the symbol χ, measures the degree of compactness of the manifold of the group within a limited range of integer values. The character of an algebra is defined as follows [11]:

$$\chi = \begin{pmatrix} \text{Number of} \\ \text{non-compact} \\ \text{generators} \end{pmatrix} - \begin{pmatrix} \text{Number of} \\ \text{compact} \\ \text{generators} \end{pmatrix}, \tag{1.3.4}$$

which is the trace of the normalized Cartan-Killing form. In the following we will analyze the product of KC under the procedure of S-expansion and we will obtain interesting information.

If X_S is a vector of the vector space $S \times \mathfrak{g}$ with S-expanded generators $T_{(\alpha,a)}$, then we introduce the inner product as the Killing-Cartan product

$$(X_S, X_S) \equiv \operatorname{tr}\left(R(X_S)R(X_S)\right) \tag{1.3.5}$$

where of course $R(X_S)$ is the adjoint representation of X_S, just like in (1.3.2). Therefore, we have

$$\begin{aligned}
(X_S, X_S) =& \operatorname{tr}\left(R(X)R(X)\right) \\
=& v^{(\alpha,a)} v^{(\beta,b)} R(T_{(\alpha,a)})^{(\delta,d)}_{(\gamma,c)} R(T_{(\alpha,a)})^{(\gamma,c)}_{(\delta,d)} \\
=& v^{(\alpha,a)} v^{(\beta,b)} K^{\delta}_{\alpha\gamma} K^{\gamma}_{\beta\delta} (C_a)^d_c (C_b)^c_d \\
=& v^{(\alpha,a)} v^{(\beta,b)} K^{\delta}_{\alpha\gamma} K^{\gamma}_{\beta\delta} \left(T_a, T_b\right),
\end{aligned}$$

where $T_a \in \mathfrak{g}$. Therefore, we see that the product of Killing-Cartan of $\mathfrak{g}$ undergoes a change due to the presence of the K-selectors.

Because of the Killing-Cartan product, of the original algebra appears immersed in the Killing-Cartan product of the expanded algebra, the calculations are simplified. Since a metric is a symmetric bilinear form, we can use the spectral theorem, to obtain the corresponding diagonal metric. This theorem states that every real symmetric matrix is diagonalizable in $\mathbb{R}$.

This means that exist a transformation such that $X \longrightarrow \tilde{X}$ allows to write $\left(\tilde{X}, \tilde{X}\right) = 0,\ \forall a \neq b$. So that

$$\left(\tilde{X}, \tilde{X}\right) = \tilde{v}^{(\alpha,a)}\tilde{v}^{(\beta,b)}\left(\tilde{X}_a, \tilde{X}_b\right), \tag{1.3.6}$$

therefore the Killing-Cartan product of the expanded algebra, takes the form

$$\left(\tilde{X}, \tilde{X}\right)_S = \tilde{v}^{(\alpha,a)}\tilde{v}^{(\beta,b)}K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}\left(\tilde{X}_a, \tilde{X}_b\right) \neq 0, \quad \text{when } a = b. \tag{1.3.7}$$

However not all matrices are diagonalizable. The decomposition of a matrix in Jordan canonical form is a decomposition that generalizes the notion of diagonalization. The interesting thing about this decomposition is that every matrix can be brought into its canonical form, i.e., any matrix A can be written in the form $A = SJS^{-1}$, where J is known as Jordan matrix.

This means that for each value of the index "a", the values that can take the pair of indices $\{\alpha, \beta\}$ will play the role of labeling new coordinates unlike the index set $\{\gamma, \delta\}$ whose role is of geometrical nature. For this reason we use the indices $\{i, j\}$ to denote $\{\alpha, \beta\}$. Thus, we have

$$\left(\tilde{X}, \tilde{X}\right)_S = \sum_{a}^{\dim(\mathfrak{g})=N} \sum_{i,j,\gamma,\delta}^{P} \tilde{v}^{(i,a)}\tilde{v}^{(j,a)}K_{i\gamma}^{\delta}K_{j\delta}^{\gamma}\left(\tilde{X}_a, \tilde{X}_b\right) \tag{1.3.8}$$

From now on, we use the Einstein convention of repeated indices to avoid excessive notation.

Note that although the original metric is diagonal, the expanded metric doesnt need to be diagonal. This is because the coordinates of the new $S \times G$ vector space, denoted as $\tilde{v}^{(i,a)}$, has indices of the semigroup S. To see this, we define

$$M_K = K_{i\gamma}^{\delta}K_{j\delta}^{\gamma}. \tag{1.3.9}$$

This is a matrix of $P \times P$ dimensions which contains, through its entries (i, j), the multiplication rules of the semigroup (where greek indices run from 0 to P, representing the semigroup elements). Thus, we have for the KC-form of the S-expanded algebra:

$$(X_S, X_S) = \left(\hat{v}^{(i,a)}\right)_{1\times(P\cdot N)} (g_S)_{ab} \left(\hat{v}^{(i,a)}\right)_{(P\cdot N)\times 1} \tag{1.3.10}$$

where

$$(g_S)_{ab} = \begin{pmatrix} \varrho_1(M_K) & & 0 \\ & \ddots & \\ 0 & & \varrho_N(M_K) \end{pmatrix}_{(P\cdot N)\times(P\cdot N)} \tag{1.3.11}$$

is the new metric tensor.

To simplify the notation we can summarize by writing the two Killing-Cartan form; the first one (we write the diagonal form without lose generality), has the form

$$(X, X) = \hat{v}^a g_{aa} \hat{v}^a, \tag{1.3.12}$$

where g_{ab} is the metric tensor and the KC-form for the S-expanded algebra has the form

$$(X_S, X_S) = \hat{v}^{(i,a)}(g_S)_{ab}\hat{v}^{(j,a)}. \tag{1.3.13}$$

where $(g_S)_{ab}$ is a kind of direct product of the M_K matrix and g_{ab}, because basically every entry of the original metric tensor is multiplied by the matrix M_K;

$$g_{ab} \longrightarrow (g_S)_{ab} = M_K \otimes g_{ab}. \tag{1.3.14}$$

From here we can take some interesting properties of the subspaces involved in S-expansion [25]. For example, we can study the change in the signature of the metric tensor with respect to the signature of the original metric. If we denote as l the number of the entries which has positive signature and m which has negative signature, $(+_1, +_2, \ldots, +_l, -_1, -_2, \ldots, -_m)$ and $l + m = N$, where is good remember that N is the dimension of the semigroup (quantity of elements), we see how the signature can change of the metric tensor with the S-expansion. There will be a change of sign for each of the N matrices M_K. This means that the dimension of the original algebra, as well as the dimension of its subspaces plays a crucial role in the study of the final signature of the metric. If the dimension of the metric is $N = 2n$ for some $n \in N$ and $p \in \mathbb{N}$, then, there will be, necessarily, an even number of sign changes in the diagonal: Even numbers form a semigroup under multiplication and similarly for $N = 2n + 1$.

Before analyzing how subspaces change, we will remember and define certain quantities that we will need next

$$\begin{aligned}
\text{Dimension of } \mathfrak{g} =& N, \\
\text{Number of elements of the semigroup } S =& P, \\
\text{Number of negative eigenvalues of } M_K =& Q \\
\text{Numer of null eigenvalues on } M_K =& H.
\end{aligned}$$

In the number of diagonal elements of the S-expanded metric tensor $(g_S)_{ab}$ the zeros that come from the original algebra should be considered, which may have nilpotent subalgebras or subspaces.

Following the analysis performed in the reference [25], the dimensions of the new subspaces of $S \times \mathfrak{g}$ are related to (1.3.3) as

$$\text{Ran}(V_-)_S = \text{Ran}(V_-)\,(P - H - Q) + \text{Ran}(V_+)Q, \tag{1.3.15}$$

$$\text{Ran}(V_+)_S = \text{Ran}(V_+)\,(P - H - Q) + \text{Ran}(V_-)Q, \tag{1.3.16}$$

$$\text{Ran}(V_0)_S = \text{Ran}(V_0) + [\text{Ran}(V_+) + \text{Ran}(V_-)]\,H. \tag{1.3.17}$$

where $(V_-)_S$, $(V_+)_S$ and $(V_0)_S$ are the subspaces of $\mathfrak{g}_S$ and V_+, V_- and V_0 the subspaces of $\mathfrak{g}$. These are three very simple equations that tell us about some features that has to have the semigroup to produce the S-expansion between two algebras of which we know its size and those of its subspaces. These have implications in the character of the algebra χ (see eq. (1.3.4)):

$$\chi_S = \chi\,(P - H - 2Q)\,. \tag{1.3.18}$$

Then, the matrix M_K has to have these requirement, and that allows us to think that maybe it is possible search a good (semi)group to stablish the link between two algebras, but this can work only in low dimensions matrices M_K and as we see later, the standard S-expansion is very limited in comparison if include also resonance and reduction.

Effects on a vector

An interesting consequence of the above property is that $K_{\alpha\gamma}^{\delta}K_{\alpha\delta}^{\gamma}$ affects the measurement of the length of the original basis vectors $X_a \in \mathfrak{g}$. Indeed, let be X_Φ a basis vector of $\mathfrak{g}_S = S \times \mathfrak{g}$, where $\Phi = (\alpha, a)$ are the indices of the S-expanded algebra

$$
\begin{aligned}
\parallel X_\Phi \parallel =& \sqrt{(X_\Phi, X_\Phi)} = \sqrt{\operatorname{Tr}(R(X_\Phi)R(X_\Phi))} \\
=& \sqrt{R(X_\Phi)_\Omega^{\ \Theta} R(X_\Phi)_\Theta^{\ \Omega}} = \sqrt{(C_\Phi)_\Omega^{\ \Theta}(C_\Phi)_\Theta^{\ \Omega}} \\
=& \sqrt{(C_{(\alpha,a)})_{(\gamma,c)}^{\ \ (\beta,b)}(C_{(\alpha,a)})_{(\beta,b)}^{\ \ (\gamma,c)}} \\
=& \sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}\sqrt{C_{ac}^{\ \ b}C_{ab}^{\ \ c}} = \sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}\sqrt{g_{aa}} \\
=& \sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}} \parallel X_a \parallel
\end{aligned}
$$

The general case:

$$
\parallel X_\Phi \parallel = v^{(\alpha,a)}\sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}} \parallel X_a \parallel . \tag{1.3.19}
$$

This means that the metric tensor components experience a rescaling.

The S-expansion procedure affects the angle between two vectors in the $V_+ + V_- = (S \times \mathfrak{g})/V_0 \subseteq S \times \mathfrak{g}$ space. Therefore,

$$
\begin{aligned}
\cos\theta_S =& \frac{(X_\Phi, X_\Phi)}{\parallel (X_\Phi, X_\Phi) \parallel^2} \\
=& \frac{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}{\sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}\sqrt{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}} \cos\theta .
\end{aligned}
$$

In general, we find that

$$
(v^{(\alpha,a)}X_{(\alpha,a)}, v^{(\beta,b)}X_{(\beta,b)}) = v^{(\alpha,a)}v^{(\beta,b)} \frac{K_{\alpha\gamma}^{\delta}K_{\beta\delta}^{\gamma}}{\sqrt{K_{\alpha\gamma}^{\delta}K_{\alpha\delta}^{\gamma}}\sqrt{K_{\beta\gamma}^{\delta}K_{\beta\delta}^{\gamma}}} \cos\theta . \tag{1.3.20}
$$

from we can see that the following conditions:

$$
\sqrt{K_{\alpha\gamma}^{\delta}K_{\alpha\delta}^{\gamma}} \cdot \sqrt{K_{\beta\gamma}^{\delta}K_{\beta\delta}^{\gamma}} \neq 0 \tag{1.3.21}
$$

must be fulfilled. These results lead to define a number which depends on the composition law of the semigroup.

1.3.2 No-simplicity of S-expanded algebra

There are another restrictions that must be satisfied in the process of pure S-expansion. A simple Lie algebra is one that has no nontrivial ideals and cannot be expressed as a direct sum of other Lie algebras. This fact leads to the important result that a Lie algebra obtained by S-expansion of another Lie algebra cannot be simple.

To see this, we consider a square matrix $K_{\alpha\beta}^{\gamma}$ of order and range P, where P will have nonzero eigenvalues and can be expressed in the diagonal form (eigenvalues in the diagonal). This matrix is a faithful 3-dimensional matrix representation of any abelian, discrete, and finite semigroup. In the other hand, Adj($\mathfrak{g}$) is the adjoint representation of some algebra. Then, we have that, for the adjoint representation for $\mathfrak{g}_S = S \times \mathfrak{g}$ we must write

$$K_{\alpha\beta}^{\gamma} \otimes \mathrm{Adj}(\mathfrak{g}) = \begin{pmatrix} \varrho_0\left(K_{\alpha\beta}^{\gamma}\right) & & 0 \\ & \ddots & \\ & & \varrho_{P-1}\left(K_{\alpha\beta}^{\gamma}\right) \end{pmatrix} \tag{1.3.22}$$

$$= \varrho_0\left(K_{\alpha\beta}^{\gamma}\right) \oplus \cdots \oplus \varrho_{P-1}\left(K_{\alpha\beta}^{\gamma}\right) \equiv \mathrm{Adj}\left(\mathfrak{g}_S\right). \tag{1.3.23}$$

This means that if Adj($\mathfrak{g}$) is the adjoint representation of an arbitrary Lie algebra $\mathfrak{g}$ and if $K_{\alpha\beta}^{\gamma}$ is a faithful matrix representation of an abelian, discrete, and finite semigroup S, then Adj($\mathfrak{g}_S$) is the adjoint representation of a non-simple Lie algebra, given by the direct sum of P Lie algebras $\mathfrak{g}$ (which can be simple or not). This will occur when the rank of the matrix $K_{\alpha\beta}^{\gamma}$ is equal to the number of elements of the semigroup S.

This result leads to state the following.

Theorem 2. *If S is a finite, discrete, and abelian semigroup and if $\mathfrak{g}$ is an arbitrary Lie algebra, then the product space $S \times \mathfrak{g}$ is a non-simple Lie algebra consisting of the direct sum of P original Lie algebras $\mathfrak{g}$, where P is the number of elements of the semigroup S.*

Proof. The case in which this is not valid is only if the matrix has a lower rank than P, then will have two equal or proportional rows or alternatively a third row that is a linear combination of other linearly independent rows.

But, we see in the next that it is impossible for rows or columns to repeat themselves. we can write $K_{\alpha\beta}^{\gamma}$ in the form

$$(K_\alpha)\,\beta^\gamma = \begin{pmatrix} (K_r)_0^0 & \cdots & (K_u)_0^x & \cdots & (K_x)_0^{P-1} \\ \vdots & \ddots & & & \vdots \\ (K_s)_i^0 & \cdots & (K_v)_i^x & & (K_y)_i^{P-1} \\ \vdots & & & \ddots & \vdots \\ (K_s)_{P-1}^0 & \cdots & (K_v)_{P-1}^x & & (K_y)_{P-1}^{P-1} \end{pmatrix} \tag{1.3.24}$$

We will use the indices i, j, k, r with $j \neq r$ (e.g., $j < r$). The operation between the ith and jth element of the semigroup, results in the kth element of the semigroup. The corresponding matrix element is $(K_i)_k^j$, which is located in the jth row and kth column of the matrix. If there is an ith row equal or proportional to it, then there is also an element of the form $C(K_j)_i^r$, which is not in the kth column because if the rows were equal, then the element in the row belonging to the kth column will have the form $(K_i)_k^i$, for which $k \neq r$. This has the consequence that $\lambda_i\lambda_j = \lambda_k$ and $\lambda_j\lambda_i = \lambda_r$ with λ_k, λ_r implying that $\lambda_i\lambda_j \neq \lambda_j\lambda_i$.

This contradicts the condition of abelian semigroup S. Therefore, the matrix has no equal (or proportional) rows. Similarly, it is proved that the matrix does not have equal (or proportional) columns. □

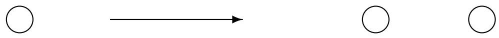

Figure 1.1: Dynkin diagram in the left correspond to $\mathfrak{so}(3)$ and in the right the $\mathfrak{so}(4)$. The row means the S-expansion procedure. Source: self made.

Example: $\mathfrak{so}(3)$ to $\mathfrak{so}(4)$

In the theorem 2, from the point of view of Dynkin diagram [11], the S-expansion procedure just copy the Dynkin diagram P times. A good interesting example is $\mathfrak{so}(3)$ and $\mathfrak{so}(4)$. $\mathfrak{so}(4)$ is precisely the repetition of $\mathfrak{so}(3)$, see Figure 1.1.

In fact, the repetition of Dynkin diagrams always happens in a S-expansion procedure. Then, we can reproduce the same result with a general group of P elements with the simple rule $\lambda_i \lambda_i = \lambda_i$, which correspond to have $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ P-times, and the big difference if we use another semigroup in a S-expansion procedure will be only the change of basis.

Then, we rediscover the (semi)groups which work for the S-expansion between $\mathfrak{so}(3)$ and $\mathfrak{so}(4)$. We consider first that

$$\begin{aligned} \chi_{\mathfrak{so}(3)} &= -3 & \chi_{\mathfrak{so}(4)} &= -6 \\ \mathrm{Ran}(V_-)_{\mathfrak{so}(3)} &= -3 & \mathrm{Ran}(V_-)_{\mathfrak{so}(4)} &= -6 \\ \mathrm{Ran}(V_+)_{\mathfrak{so}(3)} &= 0 & \mathrm{Ran}(V_+)_{\mathfrak{so}(4)} &= 0 \\ \mathrm{Ran}(V_0)_{\mathfrak{so}(3)} &= 0 & \mathrm{Ran}(V_0)_{\mathfrak{so}(4)} &= 0 \end{aligned}$$

Then, using equations (1.3.15), (1.3.16), (1.3.17) and (1.3.18) we have

$$P = 2, \quad H = 0, \quad Q = 0.$$

Then, we must consider

$$M_K = \begin{pmatrix} K^{\alpha}_{1\beta} K^{\beta}_{1\alpha} & K^{\alpha}_{1\beta} K^{\beta}_{2\alpha} \\ K^{\alpha}_{2\beta} K^{\beta}_{1\alpha} & K^{\alpha}_{2\beta} K^{\beta}_{2\alpha} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d = 0, 1, 2$. From the condition (1.3.21)

$$(a + c)(a + c) \neq 0$$

then $a, c \neq 0$. Also the eigenvalues are positives $(Q, H = 0)$, then

$$0 < \varrho_{1,2} = \frac{1}{2} a + \frac{1}{2} c \pm \sqrt{a^2 - 2ac + 4b^2 + c^2},$$

such that

$$ac > b^2.$$

Even with these conditionsthere are many matrices. But basically we have two kinds:

$$(M_K)_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad (M_K)_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.3.25}$$

These matrices are the only matrices that generates two different tables of semigroup; that is, respect associativity.

We can apply an analitiyc procedure as shown in [25] or, a good alternative, is to use a computer program to find the matrices that have the specific requirement and the respective table of the semigroup. Following [25] we found two different semigroups which connect $\mathfrak{so}(3)$ with $\mathfrak{so}(4)$. These groups are

$$\mathbb{Z}_2 \text{ and } S_E^0. \tag{1.3.26}$$

The respective semigroups tables are:

$$\begin{array}{c|cc} \cdot & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 \end{array}\,, \qquad \begin{array}{c|cc} \cdot & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_1 \\ \lambda_2 & \lambda_2 & \lambda_1 \end{array}\,. \tag{1.3.27}$$

The above outcomes allow to state: *The semigroup which leads from a Lie algebra to another by S-expansion method is not necessarily unique.*

The $\mathbb{Z}_2$ generate the S-expanded algebra with different commutators rules, but the change of basis between these two sistem is

$$J'_a = J_a - 2P_a, \qquad P'_a = P_a. \tag{1.3.28}$$

The $S_E^{(0)}$ semigroup is the disjoint semigroup, because follows the same rules of disjunction.

1.4 Finding the suitable (semi)group for a resonance reduced procedure

In the last section we see how the geometry changes of a S-expanded algebra, involving a strong restriction about the final algebra is a non-simple algebra (see Theorem 2). Neverless, the process to find a new algebra using S-expansion is more rich if we consider resonance and reduction (see chapter 1).

In this section we describe an analytic method able to give the multiplication table(s) of the set(s) involved in an S-expansion process (with either resonance or 0_S-resonant-reduction) for reaching a target Lie (super)algebra from a starting one, after having properly chosen the partitions over subspaces of the considered (super)algebras. This analytic method gives us a simple set of expressions to find the subset decomposition of the set(s) involved in the process. Then, we use the information obtained from both, the initial (super)algebra and the target one for reaching the multiplication table(s) of the mentioned set(s). Finally, we check associativity with an auxiliary computational algorithm, in order to understand whether the obtained set(s) can describe semigroup(s) or just abelian set(s) connecting two (super)algebras. We also give some interesting examples of application, which check and corroborate our analytic procedure and also generalize some result already presented in the literature.

The first step to make this construction is consider a finite Lie (super)algebra $\mathfrak{g}$, which can be decomposed into N subspaces V_A, with $A = 0, 1, \ldots, N-1$, and can be written as their direct sum, namely $\mathfrak{g} = \bigoplus_A \tilde{V}_A$. Then, let us consider the algebra to which we want to go, the target Lie (super)algebra $\mathfrak{g}_{RR}$ (not confuse with *target manifold*) (where the label S_{RR} stands for "S-expanded, ($_S$-)resonant-reduced"), which can analogously be decomposed into N subspaces

V_A, with $A = 0, 1, \ldots, N-1$, and can be written as their direct sum, namely $\mathfrak{g}_{RR} = \oplus_A V_A$.

We will denote each of this subsets with S_{Δ_A}, where the composed index Δ_A expresses both, the cardinality (number of elements) of each subsets (capital Greek index, Δ), and the subspace associated (capital Latin index, $A, B, C, \ldots$). The association between the subsets and the (super)algebra subspaces is unique (under the *resonance* condition), and we will see that, for each value of A, we will have a unique value for the corresponding index Δ. This is the reason why we are using this composite index.

Thus, let us consider the decomposition of the set $\tilde{S}$ in terms of its subsets:

$$\tilde{S} = \sqcup_{\Delta_A} S_{\Delta_A}, \tag{1.4.1}$$

where with the symbol $\sqcup$ we mean the disjoint union of sets.

We can now use this general decomposition and perform a 0_S-*resonant-reduced* process *, linking the original Lie (super)algebra $\mathcal{G}$ and the target one $\mathcal{G}_{S_{RR}}$. In this way, we get

$$\begin{aligned} \mathcal{G}_{S_{RR}} = \tilde{V}_0 \oplus \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_{N-1} = \\ = (S_{\Delta_0} \otimes V_0) \oplus (\{\lambda_{0_S}\} \otimes V_0) \oplus \\ \oplus (S_{\Delta_1} \otimes V_1) \oplus (\{\lambda_{0_S}\} \otimes V_1) \oplus \\ \oplus \cdots \oplus \left(S_{\Delta_{N-1}} \otimes V_{N-1}\right) \oplus (\{\lambda_{0_S}\} \otimes V_{N-1}), \end{aligned} \tag{1.4.2}$$

Since we can factorize the zero element, the above relation can be simply rewritten as

$$\begin{aligned} \mathcal{G}_{S_{RR}} &= \tilde{V}_0 \oplus \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_{N-1} = \\ &= \left[(S_{\Delta_0} \otimes V_0) \oplus (S_{\Delta_1} \otimes V_1) \oplus \cdots \oplus \left(S_{\Delta_{N-1}} \otimes V_{N-1}\right)\right] \oplus (\{\lambda_{0_S}\} \otimes \mathcal{G}). \end{aligned} \tag{1.4.3}$$

As we have said above, equation (1.4.3) comes from the study of a 0_S-resonant-reduced process, which we can be written in a more formal way as

$$\begin{aligned} \mathcal{G}_{S_{RR}} &= \tilde{V}_0 \oplus \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_{N-1} = \qquad (1.4.4) \\ &= \left[\tilde{S} \ominus \left(\sqcup_{\Delta_{A\neq 0}} S_{\Delta_A} \oplus \lambda_{0_S}\right)\right] \otimes V_0 \oplus \\ &\quad \oplus \left[\tilde{S} \ominus \left(\sqcup_{\Delta_{A\neq 1}} S_{\Delta_A} \oplus \lambda_{0_S}\right)\right] \otimes V_1 \oplus \\ &\quad \oplus \cdots \oplus \left[\tilde{S} \ominus \left(\sqcup_{\Delta_{A\neq N-1}} S_{\Delta_A} \oplus \lambda_{0_S}\right)\right] \otimes V_{N-1} = \\ &= \bigoplus_{T=0}^{N-1} \left[\tilde{S} \ominus \left(\sqcup_{\Delta_{A\neq T}} S_{\Delta_A} \oplus \lambda_{0_S}\right)\right] \otimes V_T, \end{aligned}$$

*A process which involves only resonance would be a simpler one, and it will be briefly treated in the following.

where we have denoted with $\oplus$ and $\ominus$ the direct sum and subtraction over subsets, respectively.

From expression (1.4.4), taking into account the dimensions of the subspaces involved in the partitions of the considered (super)algebras, the following system of $N+1$ equations result:

$$\dim\left(\tilde{V}_i\right) = \dim\left(V_i\right)\left(\tilde{P} - 1 - \sum_{A\neq i}^{N-1} \Delta_A\right), \tag{1.4.5}$$

$$\tilde{P} = \sum_{A}^{N-1} \Delta_A + 1, \tag{1.4.6}$$

where $i = 0, \ldots, N-1$ and in the last expression we have $\tilde{P} \geq P$ (let us remember that P is the total number of elements of the set $\tilde{S}$), and the $+1$ contribution is given by the presence of the zero element λ_{0_S}.

We can rewrite the system above in the following simpler form (which comes directly from relation (1.4.3)):

$$\begin{cases} \dim\left(\tilde{V}_i\right) = \dim\left(V_i\right)\left(\Delta_i\right), \\ \tilde{P} = \sum_{A}^{N-1} \Delta_A + 1. \end{cases} \tag{1.4.7}$$

If this system admits a solution (which, if exists, is *unique*), then we will immediately know, for construction, that it is possible to reach a S-expanded, 0_S-resonant-reduced (super)algebra $\mathcal{G}_{S_{RR}}$ starting from the initial Lie (super)algebra $\mathcal{G}$ with the considered partition over subspaces, and we will also know the way in which the elements of $\tilde{S}$ are distributed into different subsets, *i.e.* the cardinality of the subsets associated with the subspaces of the initial Lie (super)algebra.

In fact, knowing the dimensions of the partitions of both the initial and the target (super)algebra, the system (1.4.7) can be solved with respect to the variables

$$\tilde{P},\ \Delta_A,\ A = 0, ..., N-1, \tag{1.4.8}$$

and the solution (1.4.8) admits only values in $\mathbb{N}$.

We can observe that the system (1.4.7) admits solution if and only if the dimensions of the subspaces of the target (super)algebra are proportional (multiples) to the dimensions of the respective subspaces of the initial one, and this is the reason why, if the system (1.4.7) admits a solution, this solution is trivially unique. Furthermore, this system admits solutions only if the number of subspaces in the partition of the target (super)algebra is equal to the number of subspaces in the partition of the starting one. These considerations offer a criterion to properly choose a partition over subspaces for both the initial and the target Lie (super)algebras, namely:

1. The number of subspaces in the partition of the target (super)algebra must be equal to that of the starting (super)algebra;

2. The dimensions of the subspaces of the target (super)algebra must be multiples of the dimensions of the respective subspaces of the initial one.

Once these two conditions over the partitions are met, one is able to develop our analytic method and find all the semigroup(s), **with respect to the chosen partitions**, linking the considered (super)algebras.

We observe that the system (1.4.7) can also be solved when considering an S-expansion including just a *resonant* processes, since it also contains the subsystem in which we can clearly see that we are now considering the variable $\tilde{P} = \sum_A^{N-1} \Delta_A$ *without* the $+1$ contribution, whose presence was due to the inclusion of the zero element λ_{0_S}. In this case, the solution to the system is *unique* again, and the considerations done for the 0_S-resonant-reduced case still hold.

At this point, we know the cardinality of each of the subsets of the set $\tilde{S}$ involved in the process. Now we can understand something more about the multiplication rules of the set $\tilde{S}$, by studying the adjoint representation of the initial Lie (super)algebra with respect to the partition over subspaces.

The subspaces of the initial (super)algebra, we can now write, according to the usual S-expansion procedure (as it was done in [20]), the relations

$$\begin{aligned}&[(S_{\Delta_A} \otimes V_A) \oplus (\{\lambda_{0_S}\} \otimes V_A), (S_{\Delta_B} \otimes V_B) \oplus (\{\lambda_{0_S}\} \otimes V_B)] = \\ &= \left(K_{(\Delta_A)(\Delta_B)}^{(\Delta_C)} (C)_{AB}^{C} \right) [(S_{\Delta_C} \otimes V_C) \oplus (\{\lambda_{0_S}\} \otimes V_C)] \,,\end{aligned} \tag{1.4.9}$$

where we have also taken into account the presence of the zero element in the set $\tilde{S}$, since we have considered a 0_S-resonant-reduction process. A process involving only resonance would be a simpler one, and it would require a similar (but simpler) analysis, since, in that case, one would relax the reduction condition. Here, the composite index Δ_A, Δ_B, and Δ_C label, as said before, the cardinality of the different subsets (labeled with the capital Greek index Δ), uniquely associated with the different subspace partitions (labeled with capital Latin index).

If the subspaces involving the linear combination of generators were coupled with different elements of $\tilde{S}$, we would have

$$\begin{aligned}&[(\{\lambda_{\alpha,\Delta_A}\} \otimes V_A) \oplus (\{\lambda_{0_S}\} \otimes V_A), (\{\lambda_{\beta,\Delta_B}\} \otimes V_B) \oplus (\{\lambda_{0_S}\} \otimes V_B)] = \\ &= \left(K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_1,\Delta_C)} (C)_{AB}^{C} \right) [(\{\lambda_{\gamma_1,\Delta_C}\} \otimes V_C) \oplus (\{\lambda_{0_S}\} \otimes V_C)] + \\ &+ \left(K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_2,\Delta_C)} (C)_{AB}^{C} \right) [(\{\lambda_{\gamma_2,\Delta_C}\} \otimes V_C) \oplus (\{\lambda_{0_S}\} \otimes V_C)] + \\ &\qquad\vdots \\ &+ \left(K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_n,\Delta_C)} (C)_{AB}^{C} \right) [(\{\lambda_{\gamma_n,\Delta_C}\} \otimes V_C) \oplus (\{\lambda_{0_S}\} \otimes V_C)] \,,\end{aligned} \tag{1.4.10}$$

where with $\lambda_{\alpha,\Delta_A}$ we denote an arbitrary element λ_α contained in the subset S_A, associated with the subspace V_A, with cardinality Δ, and where

$$K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_1,\Delta_C)} \neq K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_2,\Delta_C)} \neq \cdots \neq K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma_n,\Delta_C)}. \tag{1.4.11}$$

Equations (1.4.10) and (1.4.11) would mean that different two-selectors were associated with the same resulting element, which would break the uniqueness of the internal composition law of the set $\tilde{S}$. Thus, since the composition law associates each couple of elements in $\tilde{S}$ with a *unique* element of the set $\tilde{S}$, we can finally say that

$$\begin{aligned}&[(\{\lambda_{\alpha,\Delta_A}\}\otimes V_A)\oplus(\{\lambda_{0_S}\}\otimes V_A)\,,(\{\lambda_{\beta,\Delta_B}\}\otimes V_B)\oplus(\{\lambda_{0_S}\}\otimes V_B)]=\\&=\left(K_{(\alpha,\Delta_A)(\beta,\Delta_B)}^{(\gamma,\Delta_C)}(C)_{AB}^{C}\right)[(\{\lambda_{\gamma,\Delta_C}\}\otimes V_C)\oplus(\{\lambda_{0_S}\}\otimes V_C)]\,.\end{aligned} \tag{1.4.12}$$

It can also be said that, when the commutator of two generators of the original Lie (super)algebra falls into a linear combination involving more than one generator, the intersection between the subsets of the set $\tilde{S}$ could be a non-empty set, which means that the same element(s) will appear in more than one subset of the set $\tilde{S}$.

We may observe that equation (1.4.9) can be rewritten in a simple form (due to the fact that the left hand side produces commutation relations that trivially conduce to the zero element of the set $\tilde{S}$), which reads

$$[S_{\Delta_A}\otimes V_A,S_{\Delta_B}\otimes V_B]=\left(K_{(\Delta_A)(\Delta_B)}^{(\Delta_C)}(C)_{AB}^{C}\right)[(S_{\Delta_C}\otimes V_C)\oplus(\{\lambda_{0_S}\}\otimes V_C)]\,, \tag{1.4.13}$$

so as to highlight the information we need to know about the multiplication rules between the elements in the set $\tilde{S}$.

Now, we can proceed with the development of our analytic method. The relation (1.4.9) gives us a first view on the multiplication rules between the elements of the set $\tilde{S}$, since it tells us the way in which the different subsets of $\tilde{S}$ combine among each other, this mean that

$$(S_{\Delta_A}\cup\{\lambda_{0_S}\})\cdot(S_{\Delta_B}\cup\{\lambda_{0_S}\})\subset S_{\Delta_C}\cup\{\lambda_{0_S}\}, \tag{1.4.14}$$

where the product "$\cdot$" is the internal product of the set $\tilde{S}$, and thus between its subsets. According to the relation in (1.4.13), equation (1.4.14) can also be rewritten as

$$S_{\Delta_A}\cdot S_{\Delta_B}\subset S_{\Delta_C}\cup\{\lambda_{0_S}\}. \tag{1.4.15}$$

Thus, we have considered all the information coming from the starting (super)algebra $\mathfrak{g}$, and we have obtained a first view on the multiplication rules of the elements of the subsets of $\tilde{S}$. Now we can exploit the information coming from the target (super)algebra, in order to fix some detail on the multiplication rules and to construct the whole multiplication table describing the set $\tilde{S}$. This step is based on the following *identification criterion*.

1.4.1 Identification criterion

It is now necessary to understand the structure of the whole multiplication table of the set $\tilde{S}$. For this purpose, the other pieces of information we need to know, come from the target Lie (super)algebra. In fact, at this point, we already know the composition laws between the subsets of S, that is why we can write now the following identification between the S-expanded generators of the initial Lie (super)algebra and the generators of the target one:

$$\tilde{T}_A=T_{A,\alpha}\equiv\lambda_\alpha T_A, \tag{1.4.16}$$

where T_A are the generators included in the subspace V_A of the starting (super)algebra and $\tilde{T}_A$ are the generators in the subspace $\tilde{V}_A$ of the target (super)algebra, and where $\lambda_\alpha\in\tilde{S}$ is a general element of the set $\tilde{S}$. But to get the rules of multiplication table we use (1.4.16) with a different notation; For each element of the set $\tilde{S}$, associating each element of each subset with the generators in the subspace related to the considered subset, that is, in our notation,

$$\tilde{T}_A = \lambda_{(\alpha,\Delta_A)} T_A, \tag{1.4.17}$$

where $\lambda_{(\alpha,\Delta_A)} \equiv \lambda_\alpha \in S_{\Delta_A}$. With the identification (1.4.16), we can link the commutation relations between the generators of the target (super)algebra with the commutation relations of the S-expanded ones, and, factorizing the elements of the set $\tilde{S}$, we have the chance of fixing the multiplication relations between these elements. To this aim, we first observe that for the target (super)algebra we can write the commutation relations

$$\left[\tilde{T}_A, \tilde{T}_B\right] = \tilde{C}_{AB}{}^C \tilde{T}_C, \tag{1.4.18}$$

where $\tilde{T}_A$, $\tilde{T}_B$, and $\tilde{T}_C$ are the generators in the subspaces $\tilde{V}_A$, $\tilde{V}_B$, and $\tilde{V}_C$ of the partition over the target Lie (super)algebra, respectively ($A, B, C \in \{0, ..., N-1\}$). Here, with $\tilde{C}_{AB}{}^C$ we denote the structure constants of the target Lie (super)algebra, that is $\tilde{C}_{AB}{}^C \equiv C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)}$, in the usual notation. Then, by following the usual S-expansion procedure (see Ref. [20]), since for the initial (super)algebra we can write

$$[T_A, T_B] = C_{AB}{}^C\, T_C, \tag{1.4.19}$$

where we have adopted the same notation used in the case of the target (super)algebra, and where $C_{AB}{}^C$ are the structure constants of the initial Lie (super)algebra, we are able to write the relations (1.2.5). We also report them here for completeness:

$$\left[T_{(A,\alpha)}, T_{(B,\beta)}\right] = K_{\alpha\beta}{}^\gamma C_{AB}{}^C\, T_{(C,\gamma)}, \tag{1.4.20}$$

namely

$$[\lambda_\alpha T_A, \lambda_\beta T_B] = K_{\alpha\beta}{}^\gamma C_{AB}{}^C\, \lambda_\gamma T_C, \tag{1.4.21}$$

where the two-selector is defined by (1.2.1).

We now write the structure constants of the target (super)algebra in terms of the two-selector and of the structure constants of the starting one, namely, reporting equation (1.2.6) here for completeness,

$$\tilde{C}_{AB}{}^C \equiv C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^\gamma C_{AB}{}^C, \tag{1.4.22}$$

and we exploit the identification (1.4.16) in order to write the commutation relations of the target (super)algebra (1.4.18) in terms of the commutation relations between the S-expanded generators of the starting one, factorizing the elements of the set $\tilde{S}$ out of the commutators. In this way, we get the following relations:

$$[\lambda_\alpha T_A, \lambda_\beta T_B] = K_{\alpha\beta}{}^\gamma C_{AB}{}^C \lambda_\gamma T_C, \quad \longrightarrow \quad \lambda_\alpha \lambda_\beta\, [T_A, T_B] = K_{\alpha\beta}{}^\gamma C_{AB}{}^C \lambda_\gamma T_C. \tag{1.4.23}$$

If we now compare the commutation relations (1.4.23) with the ones of the starting (super)algebra in (1.4.19), we are able to deduce something more about the multiplication rules between the elements of $\tilde{S}$, that is:

$$\lambda_\alpha \lambda_\beta = \lambda_\gamma. \tag{1.4.24}$$

We have to repeat this procedure for all the commutation rules of the target (super)algebra, in order to sculpt the multiplication rules between the elements of the set $\tilde{S}$.

We observe that, during this process, the possible existence of the zero element in the set $\tilde{S}$, namely λ_{0_S}, can play a crucial role, since, in the case in which the commutation relations of the target (super)algebra read

$$\left[\tilde{T}_A, \tilde{T}_B\right] = 0, \tag{1.4.25}$$

and at the same time from the initial (super)algebra we have

$$[T_A, T_B] \neq 0, \tag{1.4.26}$$

putting all together the relations

$$[\lambda_\alpha T_A, \lambda_\beta T_B] = \lambda_\alpha \lambda_\beta [T_A, T_B] = C_{AB}{}^{C} \lambda_\gamma T_C = 0 \tag{1.4.27}$$

and (1.4.26), we can conclude that

$$\lambda_\alpha \lambda_\beta = \lambda_{0_S}. \tag{1.4.28}$$

Thus, at the end of the whole procedure, we are left with the complete multiplication table(s) describing the set(s) $\tilde{S}$ involved in the S-expansion (with either resonance or 0_S-resonant-reduction) process for moving from an initial Lie (super)algebra to a target one.

The final step consist in checking that $\tilde{S}$ is indeed an abelian semigroup. This is done by checking the *associativity* of the multiplication table(s) (one of the properties required by a set to be defined as a semigroup is, in fact, the associative property). But as we will see in section 1.5.1 this is questionable.

1.5 Examples of application considering geometrical aspects

In this section, we give some example of application of the analytic method previously developed. We start with a simple example involving the Bianchi Type I and the Bianchi Type II algebras, and then we move to more complicated cases. In particular, the last example presented in this section involves the supersymmetric Lie algebra $osp(32/1)$ and the hidden superalgebra underlying $D = 11$ supergravity, largely discussed in [22, 26].

1.5.1 From the BTI algebra to the BTII

We apply the method previously explained in order to find the possible semigroup(s) leading from the non-trivial Bianchi Type I algebra (BTI) to the Bianchi Type II (BTII) one. To this aim, we first of all analyze the structures of the initial algebra and of the target one. The only commutator different from zero for the BTI algebra is

$$[X_1, X_2] = X_1, \tag{1.5.1}$$

where X_1 and X_2 are the generators of the BTI algebra. For the BTII algebra, instead, we have

$$[Y_1, Y_2] = 0, \tag{1.5.2}$$
$$[Y_1, Y_3] = 0, \tag{1.5.3}$$
$$[Y_2, Y_3] = Y_1, \tag{1.5.4}$$

where Y_1, Y_2, and Y_3 are the generators of the BTII algebra.

Let us consider the following subspaces partition for the BTI algebra:

$$[V_0, V_0] \subset V_0, \tag{1.5.5}$$
$$[V_0, V_1] \subset V_0 \oplus V_1, \tag{1.5.6}$$
$$[V_1, V_1] \subset V_0, \tag{1.5.7}$$

where we have set

$$V_0 = \{0\} \cup \{X_2\}, \quad V_1 = \{X_1\}. \tag{1.5.8}$$

Similarly, we can write the subspaces partition for the target BTII algebra:

$$\left[\tilde{V}_0, \tilde{V}_0\right] \subset \tilde{V}_0, \tag{1.5.9}$$
$$\left[\tilde{V}_0, \tilde{V}_1\right] \subset \tilde{V}_0 \oplus \tilde{V}_1, \tag{1.5.10}$$
$$\left[\tilde{V}_1, \tilde{V}_1\right] \subset \tilde{V}_0, \tag{1.5.11}$$

where we have denoted with $\tilde{V}_A$, $A = 0, 1$, the subspaces related to the target algebra and where we have defined

$$\tilde{V}_0 = \{0\} \cup \{Y_3\}, \quad \tilde{V}_1 = \{Y_1,\ Y_2\}. \tag{1.5.12}$$

Let us observe that, in this way, we have the same partition structure both for the initial algebra and for the target one.

We now follow the steps described in the analytic procedure of Section 1.4, in order to obtain the possible abelian set(s) (with respect to the chosen partitions) leading from the BTI algebra to the BTII one (see appendix A.1). Then, following the identification criterion described in subsection 1.4.1, we have obtained the "strong" multiplication rules

$$\lambda_c \lambda_a = \lambda_b, \tag{1.5.13}$$
$$\lambda_b \lambda_a = \lambda_{0_S}. \tag{1.5.14}$$

Let us notice that these multiplication rules are consistent with those previously obtained along the procedure, when we have exploited the information coming from the initial BTI algebra.

We are now able to write the following multiplication tables for the sets $\tilde{S}$'s involved in the procedure:

	λ_a	λ_b	λ_c	λ_{0_S}
λ_a	$\lambda_{a,0_S}$	λ_{0_S}	λ_b	λ_{0_S}
λ_b	λ_{0_S}	$\lambda_{a,0_S}$	$\lambda_{a,0_S}$	λ_{0_S}
λ_c	λ_b	$\lambda_{a,0_S}$	$\lambda_{a,0_S}$	λ_{0_S}
λ_{0_S}	λ_{0_S}	λ_{0_S}	λ_{0_S}	λ_{0_S}

(1.5.15)

We can now perform the following identification:

$$\lambda_a = \lambda_2, \quad \lambda_b = \lambda_3, \quad \lambda_c = \lambda_1, \quad \lambda_{0_S} = \lambda_4. \tag{1.5.16}$$

Thus, we can rewrite tables (1.5.15) as follows (where the elements are written in the usual order):

	λ_1	λ_2	λ_3	λ_4
λ_1	$\lambda_{2,4}$	λ_3	$\lambda_{2,4}$	λ_4
λ_2	λ_3	$\lambda_{2,4}$	$\lambda_{2,4}$	λ_4
λ_3	$\lambda_{2,4}$	$\lambda_{2,4}$	$\lambda_{2,4}$	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

(1.5.17)

These are the multiplication tables of the possible sets $\tilde{S}$'s involved in the S-expansion,0_S-resonant-reduced procedure from the BTI algebra to the BTII one. Here we see clearly that the tables described in (1.5.17) also include abelian sets that cannot be defined as semigroup, since they do not possess the associative property. Fortunately, for both the BTI and BTII algebras, each term of the Jacobi identity is equal to zero (thus, the Jacobi identity is trivially satisfied), and thus each possible combination of elements in (1.5.17) is valid for describing an expansion procedure involving both resonance and reduction, without the necessity of requiring associativity. Thus, the multiplication tables (1.5.17) generalize the result previously obtained in [27].

But, as the S-expansion procedure has the requarment the use of a associative set of discrete element, we need to find the table(s) in (1.5.15) which describe semigroup(s) exploiting the required property of associativity and also fix the degeneracy on it. The calculation is rather tedious to be performed by hand, and we have done it with a computational algorithm. For completeness, here we report only the significant relations for checking associativity by hand and understanding which are the semigroups in (1.5.15):

$$(\lambda_c\lambda_c)\lambda_b = \lambda_c(\lambda_c\lambda_b) \quad \Rightarrow \quad \lambda_c\lambda_b = \lambda_{0_S}, \tag{1.5.18}$$

$$(\lambda_c\lambda_a)\lambda_a = \lambda_c(\lambda_a\lambda_a) \quad \Rightarrow \quad \lambda_a\lambda_a = \lambda_{0_S}, \tag{1.5.19}$$

$$(\lambda_c\lambda_b)\lambda_b = \lambda_c(\lambda_b\lambda_b) \quad \Rightarrow \quad \lambda_b\lambda_b = \lambda_{0_S}. \tag{1.5.20}$$

After having checked associativity, we are thus left with the only degeneracy

$$\lambda_c\lambda_c = \lambda_{a,0_S}. \tag{1.5.21}$$

We can now substitute the index $a, b, c, 0_S$ with numbers. We perform again the identification (1.5.16), and we write the multiplication tables thus obtained in terms of λ_i, with $i = 1, 2, 3, 4$, in the usual order:

	λ_1	λ_2	λ_3	λ_4
λ_1	$\lambda_{2,4}$	λ_3	λ_4	λ_4
λ_2	λ_3	λ_4	λ_4	λ_4
λ_3	λ_4	λ_4	λ_4	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

(1.5.22)

We observe that the abelian, commutative and associative tables (1.5.22) include the multiplication table of the semigroup S_{N2} described in [27], namely

$$\begin{array}{c|cccc} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \end{array} \tag{1.5.23}$$

which represents a possible semigroup for moving form a BTI algebra to a BTII algebra, through a 0_S-resonant-reduction procedure. The degeneracy appearing in (1.5.22) (namely $\lambda_1\lambda_1 = \lambda_{2,4}$) shows us that there are two possible semigroups able to give the same result (one of them is the same described it [27], S_{N2}, while the other one is a new result that we have obtained with our analytic procedure).

We have thus given an example in which the method described in Section 1.4 allows us to find the semigroups for moving from the BTI algebra to a S-expanded, 0_S-resonant-reduced one (BTII), once the partitions over subspaces have been properly chosen.

We can now try to achieve the same result, by considering an S-expansion with only a *resonant* structure (relaxing the reduction condition). To this aim, we study the system (1.4.7), following appendix A.2 we can thus conclude that, in this case, our analytic method shows us the necessity of including a 0_S-reduction to the resonant process too. Thus, we can reach the following multiplication rules:

$$\begin{aligned} \lambda_a\lambda_a &= \lambda_a, \\ \lambda_c\lambda_a &= \lambda_b, \\ \lambda_{b,c}\lambda_{b,c} &= \lambda_a, \\ \lambda_b\lambda_a &= \lambda_{0_S}, \end{aligned} \tag{1.5.24}$$

and the multiplication table, after having performed the identification

$$\lambda_a = \lambda_2, \quad \lambda_b = \lambda_3, \quad \lambda_c = \lambda_1, \quad \text{with the extra zero element } \lambda_{0_S} = \lambda_4, \tag{1.5.25}$$

reads

$$\begin{array}{c|ccc} & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_2 & \lambda_4 \\ \lambda_3 & \lambda_2 & \lambda_4 & \lambda_2 \end{array} \longrightarrow \begin{array}{c|cccc} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_2 & \lambda_4 \\ \lambda_2 & \lambda_3 & \lambda_2 & \lambda_4 & \lambda_4 \\ \lambda_3 & \lambda_2 & \lambda_4 & \lambda_2 & \lambda_4 \\ \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \end{array} \tag{1.5.26}$$

Let us finally observe that table (1.5.26), which is abelian (but *not* associative), is included in the multiplication tables (1.5.17), previously obtained in the context of 0_S-resonant-reduction.

And if we choose another distribution of the subspaces?

The first step to find the corresponding semigroup is defining the subspaces in the two algebras. Therefore, we will see now what happen if we modify the equation (1.5.12) in the form

$$\tilde{V}_0 = \{0\} \cup \{Y_3\}, \qquad \tilde{V}_1 = \{Y_1,\ Y_3\}. \tag{1.5.27}$$

We consider now

$$\lambda_a X_2 = Y_2 \qquad\qquad \lambda_b X_1 = Y_1 \qquad (1.5.28)$$
$$\lambda_c X_1 = Y_3.$$

The procedure is exactly the same as before with the only modification in the multiplication table when we use the identification criterion. Once again our attention lies in the commutators and in this case the most important situation is present in

$$[Y_2, Y_3] = [\lambda_a X_2, \lambda_c X_1] = \lambda_a \lambda_c [X_2, X_1], \qquad (1.5.29)$$

since the commutator is equal to $-X_1$, and for the same condition A.1.8 we have the only difference with the table 1.5.15

$$\lambda_a \lambda_c = -\lambda_b.$$

This is clearly outside of the standard definition of (semi) group, but as we shall see below, we follow the idea suggested in [20] and we use a ring in a expansion instead of a semigroup, since the ring contemplates two mathematical operations $(+, \cdot)$. Thus, we could now be in the presence that the method proposed in [28] shows the spectrum of possibilities that is needed to expand an algebra through the direct product of the same with some other mathematical structure.

Let us briefly comment on what happens if we extend S-expansion by considering a ring instead of a semigroup. Let the next multiplication table with two binary operations $(\cdot, +)$ where we introduce *Grassmann variables*:

$\cdot$	λ_a	λ_b	λ_c
λ_a	0_S	λ_c	λ_b
λ_b	$-\lambda_c$	0_S	λ_a
λ_c	$-\lambda_b$	$-\lambda_a$	0_S

(1.5.30)

Then, if we have the Heisenberg Algebra $\mathfrak{h}_3$ whose commutators are

$$[T_1, T_2] = T_3, \qquad (1.5.31)$$

and we use resonance in the S-expansion, where $T_1 \to \lambda_a T_1$, $T_2 \to \lambda_b T_2$ and $T_3 \to \lambda_c T_3$, we obtain

$$\{T_1, T_2\} = T_3. \qquad (1.5.32)$$

That is, one could get a superalgebra from a Lie algebra. I would only analyze the non-triviality of Bianchi's identities.

1.5.2 Hidden superalgebra and $osp(32/1)$ $D = 11$ supergravity

With this example, we move to *superalgebras*, and in particular we concentrate on the supersymmetric Lie algebra $osp(32/1)$ and on the hidden superalgebra underlying supergravity in eleven dimensions.

Simple supergravity in $D = 11$ was first constructed in [21]. The bosonic field content of $D = 11$ supergravity is given by the metric $g_{\mu\nu}$ and by a 3-index antisymmetric tensor $A_{\mu\nu\rho}$ $(\mu, \nu, \rho, ... = 0, 1, ..., D-1)$; The theory also presents a single Majorana gravitino Ψ_μ in the fermionic sector. By dimensional reduction (as it was shown in [32]), the theory yields $\mathcal{N}=8$ supergravity in four dimensions, which is considered a possibly viable unification theory of all interactions.

An important task to accomplish was the identification of the supergroup underlying the theory, and allowing the unification of all elementary particles in a single supermultiplet, since a supergravity theory whose supergroup is unknown is an incomplete one.

The need for a supergroup was already felt by the inventors of the theory, and in [21] the authors proposed $osp(32/1)$ as the most likely candidate. However, the field $A_{\mu\nu\rho}$ of the Cremmer-Julia-Scherk theory is a 3-form rather than a 1-form, and therefore it cannot be interpreted as the potential of a generator in a supergroup.

The structure of this same theory was then reconsidered in [22, 26], in the Free Differential Algebra (FDA) framework, using the superspace geometric approach. In [22], the supersymmetric FDA was also analyzed in order to see whether the FDA formulation could be interpreted in terms of an ordinary Lie superalgebra (in its dual Maurer-Cartan formulation), introducing the notion of Cartan integrable systems. This was proven to be true, and the existence of a hidden superalgebra underlying the theory was presented for the first time (the authors got a dichotomic solution, consisting in two different supergroups, whose 1-form potentials can be alternatively used to parametrize the 3-form).

This hidden superalgebra includes, as a subalgebra, the super-Poincaré algebra of the eleven-dimensional theory, but it also involves two extra bosonic generators $Z^{ab}, Z^{a_1\cdots a_5}$ $(a, b, \cdots = 0, 1, \cdots 10)$, commuting with the 4-momentum P_a and having appropriate commutators with the $D=11$ Lorentz generators J_{ab}. The generators that commute with all the superalgebra but the Lorentz generators can be named "almost central".

Furthermore, to close the algebra, an extra nilpotent fermionic generator Q' must be included. In the following, we will replace the notation in [22, 26] as follows

$$Z^{ab} \rightarrow \tilde{Z}^{ab}, \tag{1.5.33}$$

$$Z^{a_1\ldots a_5} \rightarrow \tilde{Z}^{a_1,\ldots,a_5}, \tag{1.5.34}$$

$$Q' \rightarrow \tilde{Q}', \tag{1.5.35}$$

in order to be able to recognize the generators of the target superalgebra from the generators of starting one, as we have previously done along the paper.

The bosonic generators Z^{ab} and $Z^{a_1\cdots a_5}$ were understood as p-brane charges, sources of dual potentials [35, 38]. The role played by the extra fermionic generator Q' was much less investigated, and the most relevant contributions were given first in [39], and then in particular in [40], where the results in [22] were further analyzed and generalized.

In [26], the authors have shown that, as the generators of the hidden super Lie algebra span the tangent space of a supergroup manifold, then, in the geometrical approach, the fields are naturally defined in an enlarged manifold, corresponding to the supergroup manifold, where all the invariances of the FDA are diffeomorphisms, generated by Lie derivatives.

The extra spinor 1-form involved in the construction of the hidden superalgebra allows, in a dynamical way, the diffeomorphisms in the directions spanned by the almost central charges to be particular gauge transformations, so that one obtains the ordinary superspace as the quotient of the supergroup over the fiber subgroup of gauge transformations.

Now we will show that, with the analytic method developed in Section 1.4, we are able to find the semigroup which is involved in the S-expansion (0_S-resonant-reduction) procedure for moving from the original $osp(32/1)$ Lie algebra to the hidden superalgebra underlying supergravity in eleven dimensions.

This achievement tell us that the method described in [22,26], which is based on the development of the FDA in terms of 1-forms (the Maurer-Cartan formulation of the FDA has a dual description in terms of commutation relations of the considered Lie algebra, as it is shown in [41]), lead to the same result (that is, to the same hidden superalgebra) that can be found performing a S-expansion (0_S-resonant-reduction) procedure from $osp(32/1)$, with an appropriate semigroup. We will display the multiplication table of the mentioned semigroup in the following, and we will see that it is the semigroup $S_E^{(3)}$, which satisfies the multiplication rules

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leqslant 4, \\ \lambda_4, & \text{when } \alpha+\beta > 4. \end{cases} \tag{1.5.36}$$

The same result was previously achieved in [20], where the authors showed how to perform a S-expansion from $osp(32/1)$ to a D'Auria-Fré-like superalgebra (with the same structure of the D'Auria-Fré superalgebra, but with different details), using $S_E^{(3)}$ as semigroup. This analogy confirms and corroborates the analytic method developed in the present work.

In this example, we also analyze the link between $osp(32/1)$ and another superalgebra included in the dichotomic solution found in [22, 26], in which the translations and the fermionic generators, respectively denoted by $\tilde{P}_a$ and $\tilde{Q}$, commute. We will see that the supersymmetric Lie algebra $osp(32/1)$ and this particular hidden superalgebra are linked by a S-expansion (0_S-resonant-reduction) procedure, in which the semigroup involved in the process is the semigroup $S_E^{(2)}$, which satisfies the multiplication rules

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leqslant 3, \\ \lambda_3, & \text{when } \alpha+\beta > 3. \end{cases} \tag{1.5.37}$$

We now want to find the correct semigroup leading from $osp(32/1)$ to the hidden superalgebra underlying $D = 11$ supergravity through our analytic method. Let us start from collecting the useful information coming from the starting algebra $osp(32/1)$. The generators of $osp(32/1)$ are, with respect to the Lorentz subgroup $SO(1,10) \subset osp(32/1)$, the following set of tensors (or spinors)

$$\{P_a, J_{ab}, Z_{a_1\ldots a_5}, Q_\alpha\}, \tag{1.5.38}$$

where J_{ab}, P_a, Q_α can be respectively interpreted as the Lorentz, translations and supersymmetry generators, and where $Z_{a_1\ldots a_5}$ is a 5-index skew-symmetric generator associated with the physical $A_{\mu\nu\rho}$ field appearing in $D = 11$ supergravity.

Now we have to take into account the information coming from the target superalgebra, that is the hidden superalgebra underlying the eleven-dimensional supergravity [22, 26]. The generators of the mentioned superalgebra are given by the set

$$\left\{ \tilde{P}_a, \tilde{J}_{ab}, \tilde{Z}_{ab}, \tilde{Z}_{a_1 \ldots a_5}, \tilde{Q}_\alpha, \tilde{Q}'_\alpha \right\}, \tag{1.5.39}$$

where $\tilde{Z}_{ab}, \tilde{Z}_{a_1 \cdots a_5}$ are two extra bosonic generators, and where $\tilde{Q}'$ is an extra fermionic generator that controls the gauge symmetry of the theory and allows the closure of the algebra.

In the following, we display the detailed calculations for moving from the supersymmetric Lie algebra $osp(32/1)$ to the hidden superalgebra underlying $D = 11$ supergravity, through a 0_S-resonant-reduction procedure, and we show how to find the set(s) involved in the process, once the partitions over subspaces for both the considered superalgebras have been properly chosen.

The generators of $osp(32/1)$ are

$$\{P_a, J_{ab}, Z_{a_1 \ldots a_5}, Q_\alpha\}. \tag{1.5.40}$$

The commutations relation between these generators can be written as

$$\begin{aligned}
[P_a, P_b] &= J_{ab}, \\
\left[J^{ab}, P_c\right] &= \delta^{ab}_{ec} P^e, \\
\left[J^{ab}, J_{cd}\right] &= \delta^{abf}_{ecd}\ J^e_{\ f}, \\
[P_a, Z_{b_1 \cdots b_5}] &= -\frac{1}{5!} \epsilon_{ab_1 \cdots b_5 c_1 \cdots c_5} Z^{c_1 \cdots c_5}, \\
\left[J^{ab}, Z_{c_1 \cdots c_5}\right] &= \frac{1}{4!} \delta^{abe_1 \cdots e_4}_{dc_1 \cdots c_5} Z^d_{\ e_1 \cdots e_4}, \\
[Z^{a_1 \cdots a_5}, Z_{b_1 \cdots b_5}] &= \eta^{[a_1 \cdots a_5][c_1 \cdots c_5]} \epsilon_{c_1 \cdots c_5 b_1 \cdots b_5 e} P^e + \delta^{a_1 \cdots a_5 e}_{db_1 \cdots b_5} J^d_{\ e} + \\
&\quad - \frac{1}{3!3!5!} \epsilon_{c_1 \cdots c_{11}} \delta^{a_1 \cdots a_5 c_4 c_5 c_6}_{d_1 d_2 d_3 b_1 \cdots b_5} \eta^{[c_1 c_2 c_3][d_1 d_2 d_3]} Z^{c_7 \cdots c_{11}}, \\
[P_a, Q] &= -\frac{1}{2} \Gamma_a Q, \\
[J_{ab}, Q] &= -\frac{1}{2} \Gamma_{ab} Q, \\
[Z_{abcde}, Q] &= -\frac{1}{2} \Gamma_{abcde} Q, \\
\{Q^\rho, Q^\sigma\} &= -\frac{1}{2^3} \left[\left(\Gamma^a C^{-1}\right)^{\rho\sigma} P_a - \frac{1}{2} \left(\Gamma^{ab} C^{-1}\right)^{\rho\sigma} J_{ab} \right] + \\
&\quad - \frac{1}{2^3} \left[\frac{1}{5!} \left(\Gamma^{abcde} C^{-1}\right)^{\rho\sigma} Z_{abcde} \right],
\end{aligned} \tag{1.5.41}$$

where $C_{\rho\sigma}$ is the charge conjugation matrix and Γ_a, Γ_{ab}, Γ_{abcde} are the Dirac matrices in eleven dimensions.

Let us perform the following subspaces partition for the $osp(32/1)$ algebra:

$$[V_0, V_0] \subset V_0, \tag{1.5.42}$$
$$[V_0, V_1] \subset V_1, \tag{1.5.43}$$
$$[V_0, V_2] \subset V_2, \tag{1.5.44}$$
$$[V_1, V_1] \subset V_0 \oplus V_2, \tag{1.5.45}$$
$$[V_1, V_2] \subset V_1, \tag{1.5.46}$$
$$[V_2, V_2] \subset V_0 \oplus V_2, \tag{1.5.47}$$

where we have set $V_0 = \{J_{ab}\}$, $V_1 = \{Q_\alpha\}$, and $V_2 = \{P_a, Z_{a_1\dots a_5}\}$. Thus, the dimensions of the internal decomposition of $osp(32/1)$ read

$$\dim(V_0) = \underbrace{55}_{J_{ab}}, \tag{1.5.48}$$

$$\dim(V_1) = \underbrace{32}_{Q_\alpha}, \tag{1.5.49}$$

$$\dim(V_2) = \underbrace{11}_{P_a} + \underbrace{462}_{Z_{a_1\cdots a_5}} = 473. \tag{1.5.50}$$

The generators of the superalgebra underlying $D = 11$ supergravity are given by the set

$$\{\tilde{P}_a, \tilde{J}_{ab}, \tilde{Z}_{ab}, \tilde{Z}_{a_1\dots a_5}, \tilde{Q}_\alpha, \tilde{Q}'_\alpha\}. \tag{1.5.51}$$

These generators satisfy the following commutation relations:

$$\begin{aligned}
\{\tilde{Q}, \bar{\tilde{Q}}\} &= -\left(\mathrm{i}\Gamma^a \tilde{P}_a + \frac{1}{2}\Gamma^{ab}\tilde{Z}_{ab} + \frac{\mathrm{i}}{5!}\Gamma^{a_1\dots a_5}\tilde{Z}_{a_1\dots a_5}\right), \qquad (1.5.52)\\
\{\tilde{Q}', \bar{\tilde{Q}}'\} &= 0\,,\\
\{\tilde{Q}, \bar{\tilde{Q}}'\} &= 0\,,\\
\left[\tilde{Q}, \tilde{P}_a\right] &= -2\mathrm{i}\begin{pmatrix}5\\0\end{pmatrix}\Gamma_a\tilde{Q}'\,,\\
\left[\tilde{Q}, \tilde{Z}^{ab}\right] &= -4\Gamma^{ab}\tilde{Q}'\,,\\
\left[\tilde{Q}, \tilde{Z}^{a_1\dots a_5}\right] &= -2\,(5!)\mathrm{i}\begin{pmatrix}\frac{1}{48}\\ \frac{1}{72}\end{pmatrix}\Gamma^{a_1\dots a_5}\tilde{Q}'\,,\\
\left[\tilde{J}_{ab}, \tilde{Z}^{cd}\right] &= -8\delta^{[c}_{[a}\tilde{Z}_{b]}^{\;d]}\,,\\
\left[\tilde{J}_{ab}, \tilde{Z}^{c_1\dots c_5}\right] &= -20\delta^{[c_1}_{[a}\tilde{Z}_{b]}^{\;c_2\dots c_5]}\,,\\
\left[\tilde{J}_{ab}, \tilde{Q}\right] &= -\Gamma_{ab}\tilde{Q}\,,\\
\left[\tilde{J}_{ab}, \tilde{Q}'\right] &= -\Gamma_{ab}\tilde{Q}'\,,\\
\left[\tilde{P}_a, \tilde{Q}'\right] &= \left[\tilde{Z}_{ab}, \tilde{Q}'\right] = \left[\tilde{Z}_{a_1\dots a_5}, \tilde{Q}'\right] = \left[\tilde{P}_a, \tilde{P}_b\right] = 0,\\
\left[\tilde{J}^{ab}, \tilde{P}_c\right] &= \delta^{ab}_{ec}\tilde{P}^e,\\
\left[\tilde{J}^{ab}, \tilde{J}_{cd}\right] &= \delta^{abf}_{ecd}\;\tilde{J}^e_{\;f},\\
\left[\tilde{Z}_{ab}, \tilde{Z}_{bc}\right] &= \left[\tilde{Z}_{ab}, \tilde{Z}_{a_1\dots a_5}\right] = \left[\tilde{Z}_{ab}, \tilde{P}_c\right] = \left[\tilde{P}_a, \tilde{Z}_{a_1\dots a_5}\right] = \left[\tilde{Z}_{a_1\dots a_5}, \tilde{Z}_{b_1\dots b_5}\right] = 0,
\end{aligned}$$

where the free parameter E_2 appearing in Ref. [26] has been consistently fixed to the value 1 (this is due to the possibility of fixing the normalization of the differential form associated with the extra fermionic generator $\tilde{Q}'$).

We observe that the above algebra actually describes two superalgebras, due to the degeneracy appearing in the commutation relation (1.5.53), from which we see clearly that the generators $\tilde{Q}$ and $\tilde{P}_a$ can also commute. In the following, we will discuss the S-expansion, 0_S-resonant-reduced procedure for both these superalgebras.

Let us also observe that, in the description of the hidden superalgebra, the coefficients are written following the notation and conventions presented in Ref.s [22, 26], while, when considering the supersymmetric $osp(32/1)$ Lie algebra, we have adopted the notation presented in Ref. [20]. However, the coefficients appearing in the mentioned algebras are not relevant to our discussion, since we just need to know the *structure* of the algebras for applying our analytic method (see appendix A.3).

Then, we have

$$\begin{array}{c|ccccc} & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_3 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \end{array} \tag{1.5.53}$$

which is the multiplication table describing the semigroup $S_E^{(3)}$, that, as it was also shown in [20], is exactly the semigroup leading, through a S-expansion procedure (0_S-resonant-reduction), from the $osp(32/1)$ algebra to the hidden superalgebra described in [22, 26]. Thus, we have shown that our analytic method immediately allows us to discover that these two superalgebras can be linked through a S-expansion procedure (0_S-resonant-reduction), involving the semigroup $S_E^{(3)}$.

We now make some consideration on the case in which

$$\left[\tilde{Q}, \tilde{P}_a\right] = 0, \tag{1.5.54}$$

that is one of the commutation relations the other superalgebra presented in [22]. In this case, from the relation

$$\left[\tilde{P}_a, \tilde{Q}\right] = \left[\lambda_e P_a, \lambda_c Q\right] = \lambda_e \lambda_c \left[P_a, Q\right] = 0 \quad \rightarrow \quad \lambda_e \lambda_c = \lambda_{0_S}, \tag{1.5.55}$$

we observe that we have to fix

$$\lambda_b = \lambda_e, \tag{1.5.56}$$

and also

$$\lambda_d = \lambda_{0_S}, \tag{1.5.57}$$

in order to have consistent multiplication rules. Thus, following the usual procedure, we can build the multiplication table of the set $\tilde{S}$, which in this case reads

$$\begin{array}{c|cccc} & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_3 \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \end{array} \tag{1.5.58}$$

This is exactly the multiplication table describing the semigroup $S_E^{(2)}$, which satisfies the multiplication rules

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leqslant 3, \\ \lambda_3, & \text{when } \alpha + \beta > 3. \end{cases} \tag{1.5.59}$$

In Ref. [20], the authors showed that $S_E^{(2)}$ is the semigroup allowing the S-expansion from $osp(32/1)$ to the M-algebra (the algebra of the M-theory). We have now shown that the same result is reproduced when we are dealing with a S-expansion (0_S-resonant-reduction) from $osp(32/1)$ to a particular subalgebra of the hidden superalgebra obtained in [22, 26] (that is, in the case in which $\tilde{Q}$ and $\tilde{P}_a$ commute). However, this particular subalgebra can be obtained with S-expansion from $osp(32/1)$ only if it coincides with the M-algebra. In fact, in this case the extra-fermionic generator of the target hidden superalgebra goes to zero.

Thus, a strong relation between the D'Auria-Fré superalgebra and the M-algebra is evident. Both of them, as it was shown in [20], can be reached with a S-expansion from $osp(32/1)$, respectively with the semigroup $S_E^{(2)}$ and $S_E^{(3)}$ (and this fact furnished us another corroboration of the analytic method developed in Section 1.4).

Chapter 2

Generalized Inönü-Wigner Contraction as S-Expansion and Ideal Subtraction

"Walking in a straight line one can not get very far."

The Little Prince - Antoine de Saint-Exupéry.

In the present chapter it shows a new prescription for S-expansion, involving an infinite abelian semigroup $S_E^{(\infty)}$, with subsequent subtraction of a suitable infinite ideal, in "replacement" of the zero element present in the usual S-expansion. According to the literature, the S-expansion procedure involving a finite semigroup can reproduce a standard Inönü-Wigner contraction. This, is a generalization of the finite S-expansion procedure, and it allows to reproduce a generalized Inönü-Wigner contraction via infinite S-expansion between two different algebras, extending the results presented in the literature. With these tools we can then write the invariant tensors of the target algebras in terms of those of the starting ones.

2.1 Inönü-Wigner Contraction

Historically, the first extensively studied kind of contractions of Lie algebras, after Segal introduced the general notion of contractions [4], was the class of Saletan (linear) contractions [5]. But contractions of Lie algebras became known as a tool of theoretical physics after Inönü and Wigner [6, 7] on an important specific subclass of linear contractions, motivated by the fact that classical mechanics is a limit of relativistic mechanics, Inönü and Wigner searched the connection between the respective groups. They consider the whole class of linear contractions but they erroneously claimed in [6] that any linear contraction is diagonalizable. Even though Inönü and Wigner corrected their considerations in the next paper [7], they proceeded to exclusively study diagonalizable linear contractions, which due to their contribution are now called Inönü-Wigner contractions.

The aim of the paper [6] was the theorem I, p. 513. But better is the equivalent definition written in [9]:

Any Inönü-Wigner contraction of a Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{g}'$ is associated with a subalgebra of $\mathfrak{g}$, say $\mathfrak{s}$, and starting with an arbitrary subalgebra of the algebra $\mathfrak{g}$ one can construct an Inönü-Wigner contraction of this algebra. In the contracted algebra $\mathfrak{g}'$ there exists an Abelian ideal $\mathcal{I}$ such that the quotient algebra constructed with $\mathfrak{g}$ and $\mathcal{I}$ is isomorphic to $\mathfrak{s}$.

The IW-contraction has a lot of applications in Physics, among which the relevant cases of the Galilei algebra as an Inönü-Wigner contraction of the Poincaré algebra, and the Poincaré algebra as a contraction of the de Sitter algebra [10]. Both of them, in particular, involve two universal constants: the velocity of light c and the cosmological constant Λ, respectively.

Standard Inönü-Wigner contraction

We now consider the so-called *standard Inönü-Wigner contraction.*

Definition 1. ***(see Ref. [10])*** *Let X_i, $i = 1, 2, \ldots, n$, be a set of basis vectors for a Lie algebra $\mathfrak{g}$. Let a new set of basis vectors Y_i, $i = 1, 2, \ldots, n$, be related to the X_i's by*

$$Y_j = U(\varepsilon)_j{}^i X_i, \quad U(\varepsilon = 1)_j{}^i = \delta_j{}^i, \quad det\,[U(\varepsilon = 0)_j{}^i] = 0. \tag{2.1.1}$$

The structure constants of the Lie algebra $\mathfrak{g}$ with respect to the new basis are given by

$$[Y_i, Y_j] = C_{ij}{}^k(\varepsilon) Y_k. \tag{2.1.2}$$

When the limit

$$\lim_{\varepsilon \longrightarrow 0} C_{ij}{}^k(0) = C'_{ij}{}^k \tag{2.1.3}$$

exists and is well defined, the new structure constants $C'_{ij}{}^k$ characterize a Lie algebra that is not isomorphic to the original one. This procedure is called standard Inönü-Wigner contraction.

In other words, if we consider the symmetric cosets of simple Lie algebras, namely $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h}$ closes a subalgebra and $\mathfrak{p}$ is a complementary subspace, with commutation relations of the form

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \tag{2.1.4}$$
$$[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \tag{2.1.5}$$
$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} \oplus \mathfrak{p}, \tag{2.1.6}$$

the Inönü-Wigner contraction of $\mathfrak{g} \longrightarrow \mathfrak{g}'$ involves the matrix $U(\varepsilon)$:

$$U(\varepsilon) = \begin{pmatrix} I_{dim(\mathfrak{h})} & 0 \\ 0 & \varepsilon I_{dim(\mathfrak{p})} \end{pmatrix}, \tag{2.1.7}$$

where I is the identity matrix, and $dim(\mathfrak{h})$ and $dim(\mathfrak{p})$ stand for the dimension of $\mathfrak{h}$ and $\mathfrak{p}$, respectively. We can thus write

$$\begin{pmatrix} \mathfrak{h}' \\ \mathfrak{p}' \end{pmatrix} = \begin{pmatrix} I_{dim(\mathfrak{h})} & 0 \\ 0 & \varepsilon I_{dim(\mathfrak{p})} \end{pmatrix} \begin{pmatrix} \mathfrak{h} \\ \mathfrak{p} \end{pmatrix}. \tag{2.1.8}$$

Therefore, after having performed the limit $\varepsilon \longrightarrow 0$, the contracted algebra $\mathfrak{g}'$ becomes a semidirect product $\mathfrak{g}' = \mathfrak{h}' \ltimes \mathfrak{p}' = \mathfrak{h} \ltimes \mathfrak{p}'$, and we can write the following commutation relations:

$$[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}', \tag{2.1.9}$$

$$[\mathfrak{h}', \mathfrak{p}'] \subset \mathfrak{p}', \tag{2.1.10}$$

$$[\mathfrak{p}', \mathfrak{p}'] = 0, \tag{2.1.11}$$

from which we can see that $\mathfrak{p}'$ is now an abelian sector.

The dimension of $\mathfrak{g}$ and $\mathfrak{g}'$ is the same, and that happens in all kind of contractions. For other side, all contraction processes come down, ultimatly, to a study of the properties of the Lie bracket under singular transformations. Also, the original algebra may be simple, semisimple, or nonsemisimple [10]. The contracted algera is necessarily nonsemisimple.

Generalized Inönü-Wigner contraction

The procedure involved in the standard Inönü-Wigner contraction can be extended to the concept of *generalized Inönü-Wigner contraction*, in the sense intended in Ref.s [8, 9, 42].

Definition 2. ***(see Ref.s [8, 9, 42])*** *Let $\mathfrak{g}$ be a non-simple algebra which can be written as a sum of $n+1$ subspaces (sets of generators) V_i, $i = 0, 1, \dots, n$,*

$$\mathfrak{g} = \bigoplus_{i=0}^{n+1} V_i = V_0 \oplus V_1 \oplus \cdots \oplus V_n, \tag{2.1.12}$$

such that the following Weimar-Woods conditions [8, 9] are satisfied:

$$[V_p, V_q] \subset \bigoplus_{s \leq p+q} V_s, \quad p, q = 0, 1, \dots, n. \tag{2.1.13}$$

The conditions in (2.1.13) imply that V_0 is a subalgebra of $\mathfrak{g}$. The contraction of $\mathfrak{g}$ can be obtained after having performed a proper rescaling on the generators of each subspace [8, 9, 42], and once a singular limit for the contraction parameter have been considered, namely by considering

$$V'_i = \varepsilon^{a_i} V_i, \quad i = 0, 1, \dots, n, \tag{2.1.14}$$

with the choice of the powers a's providing a finite limit of the contracted algebra if $\varepsilon \longrightarrow 0$. This procedure is called generalized Inönü-Wigner contraction.

We observe that the contracted algebra has the same dimension of the starting one, and that the case $n = 1$ reproduces the case of the standard Inönü-Wigner contraction. In Ref. [42], the author analyzed the $n \geq 2$ cases.

The authors, in the first paper [8] they put the restriction of $n_j \geq 0$. But in [9] they allow the case where $U(0)$ does not exist, then the exponents n_j can become negative. These contractions made a brief appearance in the literature under the name of p-contractions. They were introduced in [14, 15] specifically for the (unsuccessful) search for Lie algebras, other than the de Sitter algebras, which contract into the Poincar é algebra. However the negative exponents did not play any role, and indeed were incorrectly ignored [9].

2.2 (Gen.) IW-contraction as S-Expansion case

In the following, we show that a standard Inönü-Wigner contraction can be reproduced with a S-expansion procedure (with resonance and 0 S -reduction), and we then explain how to reproduce a generalized Inönü-Wigner contraction with a S-expansion involving an infinite abelian semigroup $S_E^{(\infty)}$ (infinite S-expansion) and the subsequent subtraction of an ideal.

2.2.1 Standard Inönü-Wigner contraction as a S-expansion with reduction

Just as we see in section 1.2, the reduction generates the standard IW-contraction.

Let us consider a symmetric coset $\mathfrak{g}$ with a subalgebra $\mathfrak{h}$ and a complementary subspace $\mathfrak{p}$, namely $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, with commutation relations of the form [10]

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p}, \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{h} \oplus \mathfrak{p}. \end{aligned} \tag{2.2.1}$$

The standard Inönü-Wigner contraction of $\mathfrak{g} \longrightarrow \mathfrak{g}'$ involves the following transformation:

$$\begin{pmatrix} \mathfrak{h}' \\ \mathfrak{p}' \end{pmatrix} = \begin{pmatrix} I_{dim(\mathfrak{h})} & 0 \\ 0 & \lambda I_{dim(\mathfrak{p})} \end{pmatrix} \begin{pmatrix} \mathfrak{h} \\ \mathfrak{p} \end{pmatrix} \tag{2.2.2}$$

where I is the identity matrix, and $dim(\mathfrak{h})$ and $dim(\mathfrak{p})$ stand for the dimension of $\mathfrak{h}$ and $\mathfrak{p}$, respectively. Here, λ is the contraction parameter (previously denoted as ε). Then, the commutation relations of $\mathfrak{g}' = \mathfrak{h}' \ltimes \mathfrak{p}' = \mathfrak{h} \ltimes \mathfrak{p}'$ are well defined for all values of λ, including the singular limit $\lambda \longrightarrow 0$:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, & & & [\mathfrak{h}', \mathfrak{h}'] &\subset \mathfrak{h}', \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p}, & &\xrightarrow{\mathfrak{h}'=\mathfrak{h},\ \mathfrak{p}'=\lambda\mathfrak{p},\ \lambda\longrightarrow 0} & [\mathfrak{h}', \mathfrak{p}'] &\subset \mathfrak{p}', \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{h} \oplus \mathfrak{p}, & & & [\mathfrak{p}', \mathfrak{p}'] &= 0. \end{aligned} \tag{2.2.3}$$

This particular case of Inönü-Wigner contraction, namely the standard one, can be reproduced with a S-expansion procedure involving 0_S-resonant-reduction, developed in [20]. We now show that the standard Inönü-Wigner contraction can be reproduced with a finite S-expansion involving the semigroup $S_E^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$ (where the zero element of the semigroup is $\lambda_{0_S} = \lambda_2$), described by the multiplication table

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_2
λ_2	λ_2	λ_2	λ_2

(2.2.4)

In [20], this is done by considering a proper partition over the subspaces of the starting Lie algebra and by multiplying its generators by the elements of the semigroup $S_E^{(1)}$. Then, the resonant-reduced, S-expanded Lie algebra $\mathfrak{g}_{S_{RR}} = \mathfrak{h}' \oplus \mathfrak{p}'$, where $\mathfrak{h}' = \lambda_0 \mathfrak{h}$ and $\mathfrak{p}' = \lambda_1 \mathfrak{p}$, satisfies the following commutation relations:

$$\begin{aligned}[\lambda_0\mathfrak{h}, \lambda_0\mathfrak{h}] \subset & \lambda_0\mathfrak{h}, & (2.2.5)\\ [\lambda_0\mathfrak{h}, \lambda_1\mathfrak{p}] \subset & \lambda_1\mathfrak{p}, & (2.2.6)\\ [\lambda_1\mathfrak{p}, \lambda_1\mathfrak{p}] = & 0. & (2.2.7)\end{aligned}$$

Here, we see clearly that the role of the zero element $\lambda_2 = \lambda_{0_S}$ is to turn to zero each multiplicand (see section 1.2). We can easily see that by rewriting $\lambda_0\mathfrak{h}$ and $\lambda_1\mathfrak{p}$ in terms of $\mathfrak{h}'$ and $\mathfrak{p}'$, we arrive to equation (2.2.3). Thus, we can conclude that the standard Inönü-Wigner contraction can be seen as a S-expansion (involving 0_S-resonant-reduction) performed with the semigroup $S_E^{(1)}$.

2.2.2 Extension to the generalized Inönü-Wigner contraction case

In Ref. [9], the author claimed that any diagonal contraction is equivalent to a generalized Inönü-Wigner contraction, with integer parameter powers. These contractions are produced by diagonal matrices of the form $U(\varepsilon)_{jj} = \varepsilon^{n_j}$, $n_j \in \mathbb{Z}$. Following Ref. [9], a contraction $\mathfrak{g} \xrightarrow{U(\varepsilon)_{ij}} \mathfrak{g}'$ is called *generalized Inönü-Wigner contraction* if the matrix $U(\varepsilon)$ has the form

$$U(\varepsilon)_{ij} = \delta_{ij}\varepsilon^{n_j}, \quad n_j \in \mathbb{R}, \quad \varepsilon > 0, \quad i, j = 1, 2, \ldots, N, \tag{2.2.8}$$

with respect to a basis of generators $\{T_{a_1}, T_{a_2}, \ldots, T_{a_N}\}$.

By following Ref. [42], the generalized Inönü-Wigner contraction can be performed when we consider a non-simple algebra $\mathfrak{g}$ decomposed into $n+1$ sets $V^{(i)}$ of generators $T_a^{(i)}$ $(i = 0, 1, \ldots n)$, namely

$$\mathfrak{g} = V^{(0)} \oplus V^{(1)} \oplus \cdots \oplus V^{(n)}, \tag{2.2.9}$$

where the following conditions are satisfied

$$\left[V^{(i)}, V^{(j)}\right] \subset \bigoplus_{k \leq i+j} V^{(k)}. \tag{2.2.10}$$

The generalized Inönü-Wigner contraction of (2.2.9) is obtained by properly rescaling each generator $T_a^{(i)}$ by a power of the contraction parameter ε (see Ref. [42]), namely

$$T_a^{(i)} \in V^{(i)} \longrightarrow \varepsilon^{a_i} T_a^{(i)} \in V'^{(i)}, \tag{2.2.11}$$

where the choice of a_i provides finite limits of the contracted algebra when $\varepsilon \longrightarrow 0$. The choice of the exponent a_i is crucial, and in this context a resonant S-expansion would offer suitable choices, by multiplying each generator by appropriate powers of the parameter ε, in order to reach a suitable structure for the resulting algebra. Moreover, an S-expansion procedure involving 0_S-resonant-reduction has the property of properly cutting the algebra through the aforementioned zero element of the semigroup involved in the procedure, reproducing a contraction. However, despite its great potential, a *finite* S-expansion procedure *cannot* reproduce a generalized Inönü-Wigner contraction.

With this in mind, in our method, we consider an *infinite* S-expansion procedure (with resonance and reduction), namely a S-expansion which involves an *infinite semigroup* $S_E^{(\infty)}$ (*i.e.* with a semigroup with an infinite number of elements) *.

Then, instead of multiplying the zero element of a finite semigroup by a single generator of the starting algebra, we confer the role of "generating zeros" to a particular set of generators in the initial algebra, associating them with different elements of the infinite semigroup $S_E^{(\infty)}$. Thus, this set of generators becomes an *ideal* (ideal subalgebra, see Ref. [44]) of the S-expanded algebra reached by performing the S-expansion with the infinite semigroup $S_E^{(\infty)}$, and we can then subtract this ideal from the S-expanded algebra, in order to reproduce the same result which would be obtained by having performed a generalized Inönü-Wigner contraction.

We observe that the algebra we end up after the ideal subtraction, in general, is *not* a subalgebra of the starting algebra.

In this context, we are taking into account the following operation on a given algebra $\mathcal{A}$:

$$\mathcal{A} \ominus \mathcal{I} = \mathcal{A}_0, \tag{2.2.12}$$

where $\mathcal{A}_0$ generates a *coset space*, and where $\mathcal{I}$ is an *ideal* (ideal subalgebra) of $\mathcal{A}$, namely a subalgebra of $\mathcal{A}$ that satisfies the property $[\mathcal{A}, \mathcal{I}] \subset \mathcal{I}$ (see Ref. [44] for further details); It is an algebra related to a normal subgroup $H \subset G$, as we will shortly explain below.

2.3 Ideal subalgebra

In the following, we extend the concept of *normal subgroup* of a group to the concept of *ideal subalgebra* of an algebra (see Ref. [44] for the notation adopted and for further details on the definitions and properties of normal subgroups, homomorphisms, and ideals).

Definition 3. *Let us now consider a Lie group G and a normal subgroup H of G. Let the Lie algebra $\mathcal{A}$ be the algebra associated with the Lie group G (via exponentiation), and let the subalgebra $\mathcal{I}$ of $\mathcal{A}$ be the subalgebra associated with the normal subgroup H. Then, $\mathcal{I}$ is an ideal (ideal subalgebra) of $\mathcal{A}$, and we can write (see Ref. [44])*

$$[\mathcal{A}, \mathcal{I}] \subset \mathcal{I}. \tag{2.3.1}$$

Let us now consider the homomorphism $\varphi : G \longrightarrow G/H$, where G is a Lie group and H a normal subgroup of G. Let $\mathcal{A}$ be the Lie algebra associated with the Lie group G.
By definition (see Ref. [44]), $H = ker\varphi$, and G/H is a Lie group. Let $\hat{\mathcal{A}}$ be the Lie algebra associated with the Lie group G/H. Then, φ induces a homomorphism $\hat{\varphi} : \mathcal{A} \longrightarrow \hat{\mathcal{A}}$ between algebras, such that, if $a \in \mathcal{A}$, then $\varphi(e^a) = e^{\hat{\varphi}(a)}$. Since, from Definition 3, $\forall a \in \mathcal{I}$ we have $e^a \in H$, this now implies $\varphi(e^a) = e' \in G/H \iff \hat{\varphi}(a) = 0$. Consequently, $\hat{\varphi}(\mathcal{I}) = \{0\}$.

*The semigroup $S_E^{(\infty)}$ is an extension and generalization of the semigroups of the type $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, endowed with the following multiplication rules: $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N + 1$, and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N + 1$.

The algebra $\hat{\mathcal{A}}$ is isomorphic to the coset space $\mathcal{A} \ominus \mathcal{I}$, whose elements are the equivalence classes $[a] = \{a' \in \mathcal{A} : a' - a \in \mathcal{I}\}$.

Let us now consider a representation ρ: $G \longrightarrow Aut(V)$ * of G, where V is a vector space. This defines a representation of G/H provided on the vector space V. It has a trivial action on each vector $\Psi \in V$:

$$\rho(H)\Psi = \Psi. \tag{2.3.2}$$

Then, by defining, $\forall\, [g] \in G/H$,

$$\rho([g]) = \rho(g), \tag{2.3.3}$$

since

$$\begin{aligned} \rho(g \cdot h)\Psi =& \rho(g)\rho(h)\Psi \quad (2.3.4)\\ =& \rho(g)\Psi, \quad \forall \Psi \in V, \quad (2.3.5) \end{aligned}$$

we can say that $\rho(G/H) = \rho(G)$ is the property we need in order to require

$$\rho(\mathcal{I}) = 0. \tag{2.3.6}$$

In this way, ρ provides a representation of $\mathcal{A} \ominus \mathcal{I}$ such that:

$$\rho(\mathcal{A}) = \rho(\mathcal{A} \ominus \mathcal{I}). \tag{2.3.7}$$

Thus, if we now write

$$\mathcal{A} \ominus \mathcal{I} = \mathcal{A}_0, \tag{2.3.8}$$

where $\mathcal{A}_0$ is a coset space, we can say that $\rho(\mathcal{A}_0)$ is homomorphic to $\rho(\mathcal{A})$, and we can finally write

$$\rho(\mathcal{A})\Psi = \rho(\mathcal{A}_0)\Psi, \quad \forall \Psi \in V. \tag{2.3.9}$$

2.4 Infinite S-expansion with ideal subtraction

In the following, we explain our method involving an infinite S-expansion with subsequent ideal subtraction, which offers a new prescription for S-expansion, allowing it to reproduce a generalized Inönü-Wigner contraction.

We first of all generate an algebra with the structure written in (2.2.9) as a *loop-like Lie algebra* (see Ref. [45] for further details). We consider the set $(\mathbb{N}, +)$ *, that presents the same multiplication rules of the general semigroup $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, namely $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq N+1$, and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha + \beta > N + 1$ †.

Now, let $\{\lambda^i\}_{i=0}^{\infty} \equiv \{\lambda^i\}_0^{\infty} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots\}$ be the infinite discrete set of elements of the semigroup $S_E^{(\infty)}$. With this assumption, the infinite $S_E^{(\infty)}$-expanded algebra can be rewritten as

$$\begin{aligned} \mathfrak{g}_S^{\infty} &= \{\lambda_n\}_{n=0}^{\infty} \times \mathfrak{g} \\ &= \{\lambda_n\}_0^{\infty} V_0 \oplus \{\lambda_n\}_0^{\infty} V_1 \oplus \cdots \oplus \{\lambda_n\}_0^{\infty} V_n, \end{aligned} \tag{2.4.1}$$

*Here, $Aut(V)$ is the group of the automorphisms on the vector space V.

*The loop algebra introduced in Ref. [45] was constructed by considering $(\mathbb{Z}, +)$. In our work we restrict to $(\mathbb{N}, +)$, and we leave the extension to $(\mathbb{Z}, +)$ to the future.

†Different semigroups of the type $S_E^{(N)}$ have been used and discussed in several works on S-expanded algebras, among which [20, 47–53].

where we have taken into account the decomposition (2.2.9)of the algebra $\mathfrak{g}$.

At this point, it is possible to separate each subspace appearing in (2.4.1) into two parts, so that a first part belongs to the algebra we want to reach by reproducing a generalized Inönü-Wigner contraction, while the part which is left belongs to an infinite ideal subalgebra of the algebra $\mathfrak{g}_S^\infty$, and can be thus taken apart.

For reaching this result, we part the infinite semigroup $S_E^{(\infty)}$ into the following sets of elements: The sets $\{\lambda_{a_0}, \lambda_{a_1}, \ldots, \lambda_{a_m}\}_i$ are finite ones, they are associated with the subspace V_i through the index i, and contain certain selected elements in increasing order $a_0 < a_1 < \ldots < a_m$; The other sets are infinite ones and can be written as $\left(\{\lambda_{b_l}\}_{l=0}^\infty\right)_i$ (this sets, a priori, can also include the elements appearing in the sets $\{\lambda_{a_0}, \lambda_{a_1}, \ldots, \lambda_{a_m}\}_i$; Their contents must be chosen case by case). Then, the resonant multiplication between $S_E^{(\infty)}$ and an arbitrary subspace V_i of $\mathfrak{g}$, where $i = 0, 1, 2, \ldots, n$ (see equation (2.4.1)), can be written as

$$\left[\{\lambda_{a_0}, \lambda_{a_1}, \ldots, \lambda_{a_m}\}_i \cup \left(\{\lambda_{b_l}\}_{l=0}^\infty\right)_i\right] \times V_i. \tag{2.4.2}$$

We now define the spaces $W^{(i)}$ and $W^{\mathcal{I}(i)}$ as follows ($\forall$ i, each $W^{(i)}$ and $W^{\mathcal{I}(i)}$ must be in resonance with the subspace V_i associated, under resonance condition):

$$\begin{aligned} W^{(i)} &= \{\lambda_{a_0}, \lambda_{a_1}, \ldots, \lambda_{a_m}\}_i \times V_i, \\ W^{\mathcal{I}(i)} &= \left(\{\lambda_{b_l}\}_{l=0}^\infty\right)_i \times V_i, \end{aligned} \tag{2.4.3}$$

such that

$$\left[\{\lambda_{a_0}, \lambda_{a_1}, \ldots, \lambda_{a_m}\}_i \cup \left(\{\lambda_{b_l}\}_{l=0}^\infty\right)_i\right] \times V_i = W^{(i)} \oplus W^{\mathcal{I}(i)}. \tag{2.4.4}$$

Then, we have

$$\mathfrak{g}_S^\infty = \bigoplus_i W^{(i)} \oplus \bigoplus_i W^{\mathcal{I}(i)}, \tag{2.4.5}$$

and thus

$$\mathfrak{g}_S^\infty = \mathfrak{g}_S \oplus \mathcal{I}, \tag{2.4.6}$$

where

$$\mathfrak{g}_S = \bigoplus_i W^{(i)}, \text{ and } \mathcal{I} = \bigoplus_i W^{\mathcal{I}(i)}. \tag{2.4.7}$$

In this way, we have reproduced the structure displayed in (2.2.12). Thus, adopting the notation previously presented in Subsection 2.3, we can write

$$\mathfrak{g}_S = \mathfrak{g}_S^\infty \ominus \mathcal{I}, \tag{2.4.8}$$

where $\mathfrak{g}_S$ is the Lie algebra obtained through an infinite S-expansion with subsequent ideal subtraction, which reproduces the structures required by a generalized Inönü-Wigner contraction process (considering natural powers of the contraction parameter).

We observe that each particular choice of the elements included into the ideal can lead to different contracted algebras.

The case of the standard Inönü-Wigner contraction from an algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ to an algebra $\mathfrak{g}_S = \mathfrak{h}' \ltimes \mathfrak{p}' = \mathfrak{h} \ltimes \mathfrak{p}'$ (see the structure developed in (2.1.7)) can be easily reproduced by performing our infinite S-expansion procedure with ideal subtraction, with the following choice:

$$\begin{aligned} \mathfrak{g}_S &= \{\lambda_0 \mathfrak{h}, \lambda_1 \mathfrak{p}\}\,, \\ \mathcal{I} &= \{\lambda_2 \mathfrak{p}, \lambda_2 \mathfrak{h}, \lambda_3 \mathfrak{p}, \lambda_3 \mathfrak{h}, \ldots\}\,. \end{aligned} \tag{2.4.9}$$

Hereafter, the dots "..." appearing into the ideal denote that the same structure is repeated to infinite, considering all the other elements of the infinite semigroup.

2.5 Invariant tensor of the contracted algebra

Till now, we have reproduced the process involved in a generalized Inönü-Wigner contraction through a new prescription for S-expansion, which consists in performing an infinite S-expansion procedure with subsequent ideal subtraction.

We can now consider Theorem VII.1 developed in Ref. [20], which tell us that the components of the invariant tensor of a target algebra obtained through a (finite) S-expansion can be written in terms of those of the initial algebra, and extend it to the case of an infinite S-expansion with ideal subtraction.

In Ref. [45, 46], the authors showed how to construct the invariant tensor of the form $\langle T_{A_1} \ldots T_{A_N} \rangle$ of a loop algebra, and in Ref. [20] the authors developed a theorem on invariant tensors in the S-expansion context (Theorem VII.1 of Ref. [20]), which can be extended to the present work as follows:

Theorem 1. *Let $\langle T_{A_1} \ldots T_{A_N} \rangle$ be an invariant tensor of an algebra $\mathfrak{g}$, being T_{A_i} the N generators ($i = 1, 2, \ldots, N$) of $\mathfrak{g}$, and let the algebra $\mathfrak{g}_S^{\infty)} = S_E^{(\infty)} \times \mathfrak{g}$ be the one constructed by infinite S-expansion involving the abelian semigroup $S_E^{(\infty)}$. Denote the generators of $\mathfrak{g}_S^{\infty}$ as $\lambda_{a_i} T_{A_i} \equiv T_{A_i}^{a_i}$, $a = 0, 1, 2, \ldots \infty$, and define a particular ideal $\mathcal{I}$ of $\mathfrak{g}_S^{\infty}$. The invariant tensor of the algebra $\mathfrak{g}_S = \mathfrak{g}_S^{\infty} \ominus \mathcal{I}$ can be then written in the form*

$$\langle T_{A_1}^{a_1} \ldots T_{A_N}^{a_N} \rangle = \sum_{m=0}^{n_{max}} \alpha^m \delta_m^{a_1 + a_2 + \ldots + a_N} \langle T_{A_1} \ldots T_{A_N} \rangle, \tag{2.5.1}$$

where the α^m's are arbitrary constants and where n_{max} denotes the greatest index between those of the semigroup elements (namely, $\lambda_{n_{max}}$) whose product with the generators of $\mathfrak{g}$ is contained into the algebra $\mathfrak{g}_S$ (and is thus excluded from the ideal).

The proof of Theorem 1 can be easily developed by considering Theorem VII.2 of Ref. [20] and by applying it to the invariant tensor of a loop algebra (see Ref. [45, 46]). The map between the case of a loop algebra and the case of an infinite S-expansion with subsequent ideal subtraction is simply given by the replacement

$$\sum_{m=-\infty}^{+\infty} \longrightarrow \sum_{m=0}^{n_{max}}, \tag{2.5.2}$$

since we have to take into account the ideal subtraction.

2.6 Examples of application

In the following, we develop some examples of application, in which we replicate the (generalized) Inönü-Wigner contraction of some (super)algebras presented in the literature through our infinite S-expansion approach with ideal subtraction. We also find the invariant tensors of some of the mentioned (super)algebras.

2.6.1 $\mathfrak{so}(3) \longrightarrow \mathfrak{iso}(2,1)$

We consider $\mathfrak{g} = \mathfrak{so}(3)$ algebra wich the commutators

$$\begin{aligned}[L_1, L_2] &= -L_3, \\ [L_2, L_3] &= -L_1, \\ [L_3, L_1] &= -L_2.\end{aligned}$$

And the semigroup $S = \{\lambda_0 \lambda_1 \lambda_2\}$ with the table

$$\begin{array}{c|ccc} \cdot & \lambda_0 & \lambda_1 & \lambda_2 \\ \hline \lambda_0 & \lambda_1 & \lambda_0 & \lambda_2 \\ \lambda_1 & \lambda_0 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \end{array}, \tag{2.6.1}$$

and Following [20] the resonant S-expanded algebra

$$\mathfrak{g} = \oplus_{i=1}^{2} S_i \otimes \{V_i\}$$

where

$$\begin{aligned} S_1 &= \{\lambda_0, \lambda_2\}, & V_1 &= \{L_1, L_2\} \\ S_2 &= \{\lambda_1, \lambda_2\}, & V_2 &= \{L_3\}. \end{aligned}$$

We see that

$$[\lambda_2 L_A . \lambda_i L_B] = \lambda_2 L_C$$

where λ_i is any element of S, so the zero element causes the Ideal $\mathcal{I} = \{\lambda_2 L_1, \lambda_2 L_2, \lambda_2 L_3\}$. With the subtraction of this and renamed the generators as $\lambda_0 L_1 = P_2,\ \lambda_1 L_2 = P_1$ and $L_3 = L_3$ we get

$$\begin{aligned}[L_3, P_1] &= -P_2, \\ [L_3, P_2] &= P_1, \\ [L_1, L_2] &= 0.\end{aligned}$$

In this case its not necesary a infinite semigroup but its not immposible neather, e.i.

$$\begin{array}{c|ccccc} \cdot & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \dots \\ \hline \lambda_0 & \lambda_1 & \lambda_0 & \lambda_2 & \lambda_3 & \\ \lambda_1 & \lambda_0 & \lambda_2 & \lambda_3 & \lambda_4 & \dots \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \\ \lambda_3 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \dots \\ \vdots & & \vdots & & \vdots & \end{array} \tag{2.6.2}$$

so the usual contraction is a possible case of a subtraction of a complete infinite structure

$$\mathcal{I} = \{\lambda_2 L_1, \lambda_2 L_2, \lambda_2 L_3, \lambda_4 L_1, \lambda_4 L_2, \lambda_4 L_3, \lambda_5 L_1, \ldots\} \tag{2.6.3}$$

generated by a the infinite semigroup.

2.6.2 (Non)-symmetric cosets and group manifolds

In the literature, the standard Inönü-Wigner contraction has been usually applied to symmetric cosets of simple Lie algebras *, *i.e.* to cosets of Lie algebras which can be written as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where the following commutation relations hold:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p}, \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{h}. \end{aligned} \tag{2.6.4}$$

After having performed a standard Inönü-Wigner contraction ($\mathfrak{h}' = \mathfrak{h}'$, $\mathfrak{p}' = \varepsilon \mathfrak{p}$, and $\varepsilon \longrightarrow 0$) on $\mathfrak{g}$, we get the contracted commutation relations

$$\begin{aligned} [\mathfrak{h}', \mathfrak{h}'] &\subset \mathfrak{h}', \\ [\mathfrak{h}', \mathfrak{p}'] &\subset \mathfrak{p}', \\ [\mathfrak{p}', \mathfrak{p}'] &= 0. \end{aligned} \tag{2.6.5}$$

The standard Inönü-Wigner contraction abelianizes the last commutator ($[\mathfrak{p}', \mathfrak{p}'] = 0$), and thus produces an algebra that is non-isomorphic to the starting one. In the case in which

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{h} \oplus \mathfrak{p}, \tag{2.6.6}$$

that is, when we are taking into account a non-symmetric coset, the standard Inönü-Wigner contraction still abelianizes this commutator.

With the method presented in this paper, we can also obtain a particular *group manifold* starting from a symmetric coset, after having performed the infinite S-expansion procedure with ideal subtraction. In fact, if we S-expand $\mathfrak{g}$ by using $S_E^{(\infty)}$, and we then choose the ideal as follows:

$$\mathcal{I} = \{\lambda_2 \mathfrak{p}, \lambda_3 \mathfrak{p}, \lambda_3 \mathfrak{h}, \ldots\}, \tag{2.6.7}$$

while

$$\mathfrak{g}_S = \{\lambda_0 \mathfrak{h}, \lambda_1 \mathfrak{p}, \lambda_2 \mathfrak{h}\}, \tag{2.6.8}$$

after having properly renamed the generators, we see clearly that we end up with an algebra $\mathfrak{g}_S = \mathfrak{g}_S^\infty \ominus \mathcal{I}$ which is associated with a group manifold, namely

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p}, \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{p}. \end{aligned} \tag{2.6.9}$$

*A symmetric coset in Physics is defined, in general, as a space whose curvature is covariantly constant. In particular, the most notable symmetric spaces considered in General Relativity are the Minkowski space, the de Sitter space, and the anti-de Sitter space. An introduction to symmetric spaces can be found in Ref.s [54, 55].

2.6.3 Particular example involving the AdS algebra

We now consider the anti-de Sitter (AdS) algebra in three dimensions, whose commutation relations read:

$$\begin{aligned}\left[\tilde{J}_a, \tilde{J}_b\right] =& \epsilon_{abc}\tilde{J}_c,\\ \left[\tilde{J}_a, \tilde{P}_b\right] =& \epsilon_{abc}\tilde{P}_c,\\ \left[\tilde{P}_a, \tilde{P}_b\right] =& \epsilon_{abc}\tilde{J}_c.\end{aligned} \tag{2.6.10}$$

Then, we apply the infinite S-expansion procedure with ideal subtraction described in the present work, considering

$$\begin{aligned}\mathcal{I} =& \left\{\lambda_2\tilde{P}_a, \lambda_3\tilde{P}_a, \lambda_3\tilde{J}_{ab}, \dots\right\},\\ \mathfrak{g}_S =& \left\{\lambda_0\tilde{J}_{ab}, \lambda_1\tilde{P}_a, \lambda_2\tilde{J}_{ab}\right\}.\end{aligned}$$

Subsequently, we rename the generators as follows:

$$\lambda_0\tilde{J}_a = J_a, \tag{2.6.11}$$

$$\lambda_1\tilde{P}_a = P_a, \tag{2.6.12}$$

$$\lambda_2\tilde{J}_a = Z_a. \tag{2.6.13}$$

We then consider $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p} = \{J_a\}\oplus\{P_a, Z_a\}$, where $\mathfrak{h} = \{J_a\}$ and $\mathfrak{p} = \{P_a, Z_a\}$, and we end up with a the following commutation relations:

$$\begin{aligned}[\mathfrak{h}, \mathfrak{h}] \subset& \mathfrak{h},\\ [\mathfrak{h}, \mathfrak{p}] \subset& \mathfrak{p},\\ [\mathfrak{p}, \mathfrak{p}] \subset& \mathfrak{p}.\end{aligned} \tag{2.6.14}$$

This algebra respect Bianchi's identity. Therefore, we have reached a particular group manifold (in which $\mathfrak{p}$ is nilpotent and involves a redefinition of the connection) starting from a symmetric coset, by applying our method of infinite S-expansion with ideal subtraction.

2.6.4 From AdS algebra to all Maxwell algebras type

In the following, we perform an infinite S-expansion with ideal subtraction, in order to reach different algebras (depending on the choice of the ideal) starting from an AdS algebra, showing that the new prescription developed in the present work is an extension of the finite S-expansion procedure, and that it can reproduce the Inönü-Wigner contraction.

The AdS algebra $\mathfrak{so}(D-1,2)$ has the following set of generators: $\left\{\tilde{J}_{ab}, \tilde{P}_a\right\}$. Let us now consider $S_E^\infty = \{\lambda_\alpha\}_{\alpha=0}^\infty$ and perform the multiplication:

$$\mathfrak{g}_S^\infty = \{\lambda_\alpha\}_{\alpha=0}^\infty \times \left\{\tilde{J}_{ab}, \tilde{P}_a\right\}. \tag{2.6.15}$$

By following Ref.s [45, 46, 48] (we also adopt the same notation and subspaces partition of Ref. [48]), we perform an infinite resonant S-expansion process with the infinite semigroup $S_E^{\infty} = \{\lambda_\alpha\}_{\alpha=0}^{\infty}$. The new generators are defined and identified as follows:

$$\begin{aligned} J_{(ab,2k)} =& \lambda_{2k}\tilde{J}_{ab}, \\ P_{(a,2k+1)} =& \lambda_{2k+1}\tilde{P}_a, \end{aligned} \tag{2.6.16}$$

where, in our case, $k = 0, 1, \ldots, \infty$, and satisfy the commutation relations

$$\begin{aligned} \left[J_{(ab,2k)}, J_{(cd,2k)}\right] =& \eta_{bc}J_{(ad,4k)} - \eta_{ac}J_{(bd,4k)} - \eta_{bd}J_{(ac,4k)} + \eta_{ad}J_{(bc,4k)}, \\ \left[J_{(ab,2k)}, P_{(c,2k+1)}\right] =& \eta_{bc}P_{(a,4k+1)} - \eta_{ac}P_{(b,4k+1)}, \\ \left[P_{(a,2k+1)}, P_{(b,2k+1)}\right] =& J_{(ab,2k+2)}. \end{aligned} \tag{2.6.17}$$

We can then subtract an ideal $\mathcal{I}$ in order to reach an algebra $\mathfrak{g}_S$

$$\mathfrak{g}_S = \mathfrak{g}_S^{\infty} \ominus \mathcal{I} \tag{2.6.18}$$

In the following, we analyze how the choice of different particular ideals leads to different algebras, starting from the same infinite S-expanded algebra.

Poincaré algebra $\mathfrak{iso}(D-1,1)$

The Poincaré algebra $\mathfrak{iso}(D-1,1)$ can be obtained as a contraction of the AdS algebra $\mathfrak{so}(D-1,2)$. If we now consider the infinite AdS S-expanded algebra (2.6.17) and perform a particular choice of the ideal, we can reach the Poincaré algebra, thus reproducing a contraction via an infinite S-expansion with subsequent ideal subtraction.

Let us thus perform the following choice (we exclude the case $k = 0$ from the ideal, see (2.6.16)):

$$\mathfrak{g}_S^{\infty} = \mathfrak{g}_S \oplus \mathcal{I} = \left\{J_{(ab,0)}, P_{(a,1)}\right\} \oplus \left\{J_{(ab,2)}, P_{(a,3)}, J_{(ab,4)}, P_{(a,5)}, \ldots\right\}, \tag{2.6.19}$$

where the ideal reads $\mathcal{I} = \left\{J_{(ab,2)}, P_{(a,3)}, J_{(ab,4)}, P_{(a,5)}, \ldots\right\}$. Then, after having subtracted the ideal $\mathcal{I}$ and after having renamed the generators in the following way: $J_{(ab,0)} = J_{ab}$, $P_{(a,1)} = P_a$, we obtain the Poincaré algebra $\mathfrak{iso}(D-1,1)$, which can be written in terms of the following commutation relations:

$$\begin{aligned} [J_{ab}, J_{cd}] =& \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \\ [J_{ab}, P_c] =& \eta_{bc}P_a - \eta_{ac}P_b, \\ [P_a, P_b] =& 0. \end{aligned} \tag{2.6.20}$$

We have thus reached a Poincaré algebra from an AdS algebra by performing an infinite S-expansion with ideal subtraction, reproducing a contraction.

Maxwell-like algebra $\mathcal{M}_5$

In Ref.s [47–49], the authors shown that standard odd-dimensional General Relativity theories can be obtained from Chern-Simons gravity theories invariant under Maxwell algebras, while standard even-dimensional General Relativity theories emerge as limits of Born-Infeld theories invariant under certain subalgebras of the Maxwell algebras.

The Maxwell algebras [52], denoted by $\mathcal{M}_m$, can be obtained as a finite S-expansion (with resonance and reduction) from the anti-de Sitter algebra $\mathfrak{so}(D-1,2)$, using semigroups of the type $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, which are endowed with the multiplication rules $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha+\beta \leq N+1$, and $\lambda_\alpha \lambda_\beta = \lambda_{N+1}$ if $\alpha+\beta > N+1$.

The reduction procedure involved in the construction of the Maxwell algebras takes into account the presence of a zero element in the semigroup. This zero element is defined as $\lambda_{0_S} = \lambda_{N+1}$, and depends of the number N in the semigroup.

We can now reach the Maxwell-like algebra $\mathcal{M}_5$ by performing an ideal subtraction on the infinite S-expanded AdS algebra (2.6.17). In fact, if we now perform the following choice with respect to (2.6.16) (adopting the same notation and subspaces partition of Ref. [52]):

$$\mathfrak{g}_S^\infty = \mathfrak{g}_S \oplus \mathcal{I} = \left\{ J_{(ab,0)}, P_{(a,1)}, J_{(ab,2)}, P_{(a,3)} \right\} \oplus \left\{ J_{(ab,4)}, P_{(a,5)}, J_{(ab,6)}, P_{(a,7)}, \dots \right\}, \tag{2.6.21}$$

where the second set correspond to the ideal $\mathcal{I}$, then, after having subtracted the ideal $\mathcal{I}$ and after having renamed the generators in the following way: $J_{(ab,0)} = J_{ab}$, $P_{(a,1)} = P_a$, $J_{(ab,2)} = Z_{ab}$, and $P_{(a,3)} = Z_a$, we finally end up with the Maxwell-like algebra $\mathcal{M}_5$, endowed with the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ab} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{2.6.22}$$
$$[Z_{ab}, J_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{2.6.23}$$
$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{2.6.24}$$
$$[J_{ab}, Z_b] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{2.6.25}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{2.6.26}$$
$$[P_a, P_b] = Z_{ab}, \tag{2.6.27}$$
$$[Z_{ab}, Z_{cd}] = [Z_a, Z_b] = 0. \tag{2.6.28}$$

We have thus reached the Maxwell-like algebra $\mathcal{M}_5$ by performing an infinite S-expansion with ideal subtraction on an AdS algebra.

Invariant tensor of the Maxwell-like algebra in $D = 3$

In the three-dimensional case, considering equation (2.5.1), the components of the invariant tensor of the Maxwell-like algebra different from zero are the following ones:

$$\langle J_{ab} J_{cd} \rangle = \alpha^0 \left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd} \right), \qquad \langle J_{ab} P_c \rangle = \alpha^1 \epsilon_{abc}, \tag{2.6.29}$$
$$\langle Z_{ab} J_{cd} \rangle = \alpha^2 \left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd} \right), \qquad \langle J_{ab} Z_c \rangle = \alpha^3 \epsilon_{abc}, \tag{2.6.30}$$
$$\langle Z_{ab} P_c \rangle = \alpha^3 \epsilon_{abc}, \qquad \langle P_a P_b \rangle = \alpha^2 \eta_{ab}, \tag{2.6.31}$$

where the α's are arbitrary constants.

One could now construct a Lagrangian for a gravitational theory, as it was done in Ref.s [49, 56].

2.6.5 D=3 algebras: Poincaré, Bargmann and Newton

In the following, we apply our method involving an infinite S-expansion with subsequent ideal subtraction, in order to relate different algebras in three dimensions, which have been objects of great interest in the literature [57–65], namely the Poincaré and Galilean algebras, the Bargmann algebra, and Newton-Hooke algebra in $D = 3$.

The Bargmann algebra in three dimensions, $\mathfrak{b}(2,1)$, is the Galilean algebra [58, 64] in $D = 3$ augmented with a central generator M, and can be obtained by performing a contraction on $\mathfrak{iso}(2,1) \oplus \mathfrak{g}_M$, where $\mathfrak{iso}(2,1)$ is the Poincaré algebra in three dimensions, and where $\mathfrak{g}_M$ is a commutative subalgebra spanned by a central generator M (see Ref. [59]).

Some algebras involving the presence of the cosmological constant were deeply studied in the past years. The most relevant symmetry groups with the presence of a cosmological constant Λ are the de Sitter and the anti-de Sitter groups, that are related to relativistic symmetries which also involved the velocity of the light in the vacuum, c. In the non-relativistic limit, that is, when $c \longrightarrow \infty$ and $\Lambda \longrightarrow 0$, with $c^2\Lambda$ finite (see Ref. [65]), the de Sitter (dS) and the anti-de Sitter (AdS) groups become the so-called *Newton-Hooke groups* (the analogous of the Galilean groups in the presence of a universal cosmological repulsion or attraction).

The Newton-Hooke algebra in three dimensions, $\mathfrak{nh}(2,1)$, can be obtained as a contraction of the de Sitter (dS) or anti-de Sitter (AdS) algebra (see Ref [65] for further details).

Bargmann algebra

Let us now consider a central extension (with central generator M) of the Poincaré algebra in $D = 3$ (this extension of the Poincaré algebra is a commutative algebra $\mathfrak{g}_M$ spanned by the central generator M), in order to apply our procedure and obtain the Bargmann in three dimensions, $\mathfrak{b}(2,1)$, by performing an infinite S-expansion with ideal subtraction.

We thus start by defining $J_{12} = -J$, $J_{i0} = G_i$, and $P_0 = H \longrightarrow H + M$. We then write the following partition over subspaces:

$$V_0 = \{J, H\}, \tag{2.6.32}$$
$$V_1 = \{P_i, G_i\}, \tag{2.6.33}$$
$$V_2 = M. \tag{2.6.34}$$

After that, we perform an infinite S-expansion with $S_E^{(\infty)}$, and, following the procedure described in Subsection 2.4, we write

$$W^{(0)} = \lambda_0 \{J, H\}, \qquad W^{\mathcal{I}(0)} = \{\lambda_{2n}\}_{n=1}^{\infty} \times \{J, H\}, \tag{2.6.35}$$
$$W^{(1)} = \lambda_1 \{P_i, G_i\}, \qquad W^{\mathcal{I}(1)} = \{\lambda_{2n+1}\}_{n=1}^{\infty} \times \{P_i, G_i\}, \tag{2.6.36}$$
$$W^{(2)} = \lambda_2 M, \qquad W^{\mathcal{I}(2)} = \{\lambda_{2n}\}_{n=2}^{\infty} \times \{M\}. \tag{2.6.37}$$

Then, the target algebra $\mathfrak{g}_S$, which is, in this case, the Bargmann algebra in three dimensions, $\mathfrak{b}(2,1)$, is given by $\mathfrak{g}_S = \mathfrak{g}_S^{\infty} \ominus \mathcal{I}$, being

$$\mathcal{I} = \bigoplus_{i=0}^{2} W^{\mathcal{I}(i)} \tag{2.6.38}$$

the ideal that must be subtracted.

Thus, after having properly renamed the generator as follows: $\lambda_0 J = \hat{J}$, $\lambda_0 H = \hat{H}$, $\lambda_1 P_i = \hat{P}_i$, $\lambda_1 G_i = \hat{G}_i$, and $\lambda_2 M = \hat{M}$, we finally get

$$\begin{aligned}\left[\hat{J}, \hat{J}\right] &\propto \hat{J}, \\ \left[\hat{G}_i, \hat{P}_j\right] &= -\delta_{ij} \hat{M}, \\ \left[\hat{G}_i, \hat{J}\right] &= \epsilon_{ij} \hat{G}_j, \\ \left[\hat{G}_i, \hat{H}\right] &= -\hat{P}_i, \\ \left[\hat{P}_i, \hat{J}\right] &= \epsilon_{ij} \hat{P}_j,\end{aligned} \tag{2.6.39}$$

that are the commutation relations of the Bargmann algebra $\mathfrak{b}(2,1)$.

Newton-Hooke algebra

We now perform an infinite S-expansion with ideal subtraction on the AdS algebra in $D = 3$, in order to reach the Newton-Hooke algebra [65] $\mathfrak{nh}(2,1)$.

In $D = 3$, being

$$J_a = \frac{1}{2} \epsilon_{abc} J^{bc}, \qquad P^a = J^{a3}, \tag{2.6.40}$$

we can write the three-dimensional AdS algebra as follows:

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = \epsilon_{abc} J^c. \tag{2.6.41}$$

We now consider a new basis:

$$\tilde{J} = -J_0, \qquad K_i = -\epsilon_{ij} J_j, \qquad \tilde{H} = -P_0. \tag{2.6.42}$$

In this new basis, we can write:

$$\begin{aligned}&[K_i, K_j] = \epsilon_{ij} \tilde{J}, && \left[K_i, \tilde{H}\right] = P_i, && \left[\tilde{H}, P_i\right] = K_i, \\ &[K_i, P_j] = \delta_{ij} \tilde{H}, && \left[P_i, \tilde{J}\right] = \epsilon_{ij} P_j. && [P_i, P_j] = -\epsilon_{ij} \tilde{J}, \\ &\left[K_i, \tilde{J}\right] = \epsilon_{ij} K_j, && && \end{aligned} \tag{2.6.43}$$

Following Ref. [65], we now consider two central extensions, namely S and M. We then write the change of basis

$$H = \tilde{H} - M, \qquad J = \tilde{J} - S, \tag{2.6.44}$$

and we perform the following subspaces partition:

$$V_0 = \{J, H\}, \qquad V_1 = \{P_i, K_i\}, \qquad V_2 = \{M, S\}. \tag{2.6.45}$$

We can now apply the procedure developed in Subsection 2.4, which leads us to:

$$W^{(0)} = \lambda_0 \{J, H\}, \qquad W^{\mathcal{I}(0)} = \{\lambda_{2n}\}_{n=1}^{\infty} \times \{J, H\}, \tag{2.6.46}$$

$$W^{(1)} = \lambda_1 \{P_i, K_i\}, \qquad W^{\mathcal{I}(1)} = \{\lambda_{2n+1}\}_{n=1}^{\infty} \times \{P_i, K_i\}, \tag{2.6.47}$$

$$W^{(2)} = \lambda_2 \{M, S\}, \qquad W^{\mathcal{I}(2)} = \{\lambda_{2n}\}_{n=2}^{\infty} \times \{M, S\}. \tag{2.6.48}$$

Then, the commutators of $\mathfrak{g}_S = \mathfrak{g}_S^\infty \ominus \mathcal{I}$, being

$$\mathcal{I} = \bigoplus_{i=0}^{2} W^{\mathcal{I}(i)} \tag{2.6.49}$$

the ideal, can be finally written as follows:

$$\begin{aligned}
\left[\hat{K}_i, \hat{K}_j\right] &= \epsilon_{ij}\hat{S}, & \left[\hat{K}_i, \hat{P}_j\right] &= \delta_{ij}\hat{M}, & \left[\hat{K}_i, \hat{J}\right] &= \epsilon_{ij}\hat{K}_j \\
\left[\hat{P}_i, \hat{P}_j\right] &= -\epsilon_{ij}\hat{S}, & \left[\hat{H}, \hat{P}_i\right] &= \hat{K}_i, & \left[\hat{P}_i, \hat{J}\right] &= \epsilon_{ij}\hat{P}_j, \\
\left[\hat{K}_i, \hat{H}\right] &= \hat{P}_i,
\end{aligned} \tag{2.6.50}$$

where we have defined $\lambda_0 J = \hat{J}$, $\lambda_0 H = \hat{H}$, $\lambda_1 P_i = \hat{P}_i$, $\lambda_1 K_i = \hat{K}_i$, $\lambda_2 M = \hat{M}$, and $\lambda_2 S = \hat{S}$. These last commutation relations are those of the Newton-Hooke algebra $\mathfrak{nh}(2,1)$.

We have thus reached the Newton-Hooke algebra in three dimensions by performing an infinite S-expansion with subsequent ideal subtraction starting from the AdS algebra.

2.6.6 Non-standard super-Maxwell in $D = 3$ from the AdS algebra $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$

The Maxwell superalgebra can be obtained as an Inönü-Wigner contraction of the AdS-Lorentz superalgebra, which is an S-expansion of the AdS Lie algebra [56]. The non-standard Maxwell algebra introduced in Ref.s [66, 67] can be recovered by performing a suitable Inönü-Wigner contraction of the supersymmetric extension of the AdS-Lorentz algebra (see Ref. [42]).

In the following, we reproduce the non-standard Maxwell superalgebra in three dimensions, by performing an infinite S-expansion with $S_E^{(\infty)}$ on the AdS superalgebra, and subsequently removing an ideal. We also write the components of the invariant tensor of the target algebra in terms of those of the AdS superalgebra.

Let us thus consider the AdS superalgebra in three dimensions, $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$, with $\mathcal{N} = 1$, generated by $\tilde{J}_{ab}$, $\tilde{P}_a$, and $\tilde{Q}_\alpha$, which satisfy the following commutation relations:

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{2.6.51}$$

$$\left[\tilde{J}_{ab}\tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \tag{2.6.52}$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \tilde{J}_{ab}, \tag{2.6.53}$$

$$\left[\tilde{P}_a, \tilde{Q}_\alpha\right] = \frac{1}{2}\left(\Gamma_a \tilde{Q}\right)_\alpha, \tag{2.6.54}$$

$$\left[\tilde{J}_{ab}, \tilde{Q}_\alpha\right] = \frac{1}{2}\left(\Gamma_a \tilde{Q}\right)_\alpha, \tag{2.6.55}$$

$$\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} = -\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab} - 2\left(\Gamma^a C\right)_{\alpha\beta}\tilde{P}_a\right]. \tag{2.6.56}$$

After having performed an infinite S-expansion with subsequent ideal subtraction (see Appendix B.1 for further details on this calculation), we reach a new superalgebra, $[\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)] \ominus \mathcal{I}$, which corresponds to the non-standard Maxwell superalgebra described in [42], namely

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{2.6.57}$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{2.6.58}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{2.6.59}$$
$$[P_a, P_b] = Z_{ab}, \tag{2.6.60}$$
$$[J_{ab}, Q_\alpha] = \frac{1}{2} (\Gamma_{ab} Q)_\alpha, \tag{2.6.61}$$
$$[P_a, Q_\alpha] = 0, \tag{2.6.62}$$
$$[Z_{ab}, Z_{cd}] = 0, \tag{2.6.63}$$
$$[Z_{ab}, P_c] = 0, \tag{2.6.64}$$
$$[Z_{ab}, Q_\alpha] = 0, \tag{2.6.65}$$
$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left(\Gamma^{ab} C\right)_{\alpha\beta} Z_{ab}. \tag{2.6.66}$$

We have thus reached the non-standard Maxwell superalgebra described in Ref. [42], by performing an infinite S-expansion with ideal subtraction on the AdS superalgebra.

Invariant tensor of the non-standard Maxwell superalgebra in $D = 3$

We write the components of the invariant tensor of the AdS superalgebra as follows (according with what was done in Ref. [51]):

$$\langle \tilde{J}_{ab} \tilde{J}_{cd} \rangle = \tilde{\mu}_0 (\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}), \tag{2.6.67}$$
$$\langle \tilde{J}_{ab} \tilde{P}_c \rangle = \tilde{\mu}_1 \epsilon_{abc}, \tag{2.6.68}$$
$$\langle \tilde{P}_a \tilde{P}_b \rangle = \tilde{\mu}_0 \eta_{ab}, \tag{2.6.69}$$
$$\langle \tilde{Q}_\alpha \tilde{Q}_\beta \rangle = (\tilde{\mu}_0 - \tilde{\mu}_1) C_{\alpha\beta}, \tag{2.6.70}$$

where $\tilde{\mu}_0$ and $\tilde{\mu}_1$ are arbitrary constants.

If we now use (2.5.1), we see that the non-zero components of the invariant tensor for the non-standard Maxwell superalgebra can be written as

$$\langle J_{ab} J_{cd} \rangle = \alpha^0 \langle \tilde{J}_{ab} \tilde{J}_{cd} \rangle = \alpha^0 \tilde{\mu}_0 (\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}) \equiv \tilde{\alpha}^0 (\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}), \tag{2.6.71}$$
$$\langle J_{ab} Z_{cd} \rangle = \alpha^2 \langle \tilde{J}_{ab} \tilde{J}_{cd} \rangle = \alpha^2 \tilde{\mu}_0 (\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}) \equiv \tilde{\alpha}^2 (\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}), \tag{2.6.72}$$
$$\langle J_{ab} P_c \rangle = \alpha^1 \langle \tilde{J}_{ab} \tilde{P}_c \rangle = \alpha^1 \tilde{\mu}_1 \epsilon_{abc} \equiv \tilde{\alpha}^1 \epsilon_{abc}, \tag{2.6.73}$$
$$\langle P_a P_b \rangle = \alpha^2 \langle \tilde{P}_a \tilde{P}_b \rangle = \alpha^2 \tilde{\mu}_0 \eta_{ab} \equiv \tilde{\alpha}^2 \eta_{ab}, \tag{2.6.74}$$
$$\langle Q_\alpha Q_\beta \rangle = \alpha^2 \tilde{\mu}_0 C_{\alpha\beta} \equiv \tilde{\alpha}^2 C_{\alpha\beta}, \tag{2.6.75}$$

where we have defined

$$\tilde{\alpha}^0 \equiv \alpha^0 \tilde{\mu}_0, \tag{2.6.76}$$
$$\tilde{\alpha}^1 \equiv \alpha^1 \tilde{\mu}_1, \tag{2.6.77}$$
$$\tilde{\alpha}^2 \equiv \alpha^2 \tilde{\mu}_0, \tag{2.6.78}$$

and where we have taken into account the ideal subtraction. The α's are arbitrary constants.

2.6.7 PP-wave (super)algebra in $D = 11$ from the $AdS_4 \times S^7$ (super)algebra

In Ref. [16], the authors showed that the (super-)PP-wave algebra in eleven dimensions can be obtained through a generalized Inönü-Wigner contraction of the $AdS_4 \times S^7$ (super)algebra in $D = 11$ (Penrose limit, see Ref. [68]).

In the following, we will reach the same result by performing an infinite S-expansion with ideal subtraction. Indeed, our prescription for an infinite S-expansion with ideal subtraction is able to reproduce a generalized Inönü-Wigner contraction.

PP-wave algebra from the $AdS_4 \times S^7$ algebra

The $AdS_4 \times S^7$ algebra in $D = 11$ can be written in the following traditional form (see Ref. [16]):

$$[P_a, P_b] = 4J_{ab}, \qquad [P_{a'}, P_{b'}] = 4J_{a'b'}, \tag{2.6.79}$$
$$[J_{ab}, P_c] = 2\eta_{bc} P_a, \qquad [J_{a'b'}, P_{c'}] = 2\eta_{b'c'} P_{a'}, \tag{2.6.80}$$
$$[J_{ab}, J_{cd}] = 4\eta_{ad} J_{bc}, \qquad [J_{ab}, J_{cd}] = 4\eta_{ad} J_{bc}, \tag{2.6.81}$$

where the vector index of AdS_4 is $i = 0, 1, 2, 3$, and that of S^7 is $a' = 4, 5, 6, 7, 8, 9, \sharp$, according with the notation adopted in [16].

Following Ref. [16], we define the light cone components of the momenta P's and boost generators P^*'s as

$$P_\pm \equiv \frac{1}{\sqrt{2}} \left(P_\natural \pm P_0 \right), \tag{2.6.82}$$
$$P_m = \left(P_i, P_{i'} \right), \tag{2.6.83}$$
$$P^*_m = \left(P^*_i = J_{i0}, P^*_{i'} = J_{i'\natural} \right). \tag{2.6.84}$$

Thus, we can write the commutation relations of $AdS_4 \times S^7$ as follows:

$$\begin{aligned}
[P_i, P_+] &= 2\sqrt{2} P^*_i, & [P_{i'}, P_+] &= -\frac{1}{\sqrt{2}} P^*_i, \\
[P^*_i, P_+] &= -\frac{1}{\sqrt{2}} P_i, & [P^*_{i'}, P_+] &= \frac{1}{\sqrt{2}} P_{i'}, \\
[P_i, P_-] &= -2\sqrt{2} P^*_i, & [P_{i'}, P_-] &= -\frac{1}{\sqrt{2}} P^*_{i'}, \\
[P^*_i, P_-] &= \frac{1}{\sqrt{2}} P_i, & [P^*_{i'}, P_-] &= \frac{1}{\sqrt{2}} P_{i'}, \\
[P^*_i, P_j] &= -\frac{1}{\sqrt{2}} \eta_{ij} \left(P_- - P_+ \right), & [P^*_{i'}, P_{j'}] &= -\frac{1}{\sqrt{2}} \eta_{i'j'} \left(P_- + P_+ \right),
\end{aligned} \tag{2.6.85}$$

$$\begin{aligned}
[P_i, P_j] &= 4J_{ij}, & [P_{i'}, P_{j'}] &= -J_{i'j'}, \\
\left[P^*_i, P^*_j\right] &= J_{ij}, & \left[P^*_{i'}, P^*_{j'}\right] &= -J_{i'j'}, \\
[J_{ij}, P_k] &= 2\eta_{jk} P_i, & [J_{i'j'}, P_{k'}] &= 2\eta_{j'k'} P_{i'}, \\
[J_{ij}, P^*_k] &= 2\eta_{jk} P^*_i, & [J_{i'j'}, P^*_{k'}] &= 2\eta_{j'k'} P^*_{i'}, \\
[J_{ij}, J_{kl}] &= 4\eta_{il} J_{jk}, & [J_{i'j'}, J_{k'l'}] &= 4\eta_{i'l'} J_{j'k'}.
\end{aligned}$$

Before applying the infinite S-expansion method with subsequent ideal subtraction, we consider the following subspaces partition of the $AdS_4 \times S^7$ algebra:

$$\begin{aligned} V_0 =& \{P_-, J_{ij}, J_{i'j'}\} \,, \\ V_1 =& \{P_i, P_{i'}, P^*_i, P^*_{i'}\} \,, \\ V_2 =& P_+ . \end{aligned} \tag{2.6.86}$$

Following the procedure developed in Subsection 2.4, we now perform an infinite S-expansion with the semigroup $S_E^{(\infty)}$ on the algebra $\mathfrak{g} = AdS_4 \times S^7$, considering a resonant structure between the subsets partition and the following subsets decomposition of the semigroup $S_E^{(\infty)}$:

$$\begin{aligned} S_0 =& \{\lambda_0\} \cup \{\lambda_{2n}\}_{n=1}^{\infty} \,, \\ S_1 =& \{\lambda_1\} \cup \{\lambda_{2n+1}\}_{n=1}^{\infty} \,, \\ S_2 =& \{\lambda_2\} \cup \{\lambda_{2n}\}_{n=2}^{\infty} \,. \end{aligned} \tag{2.6.87}$$

The resonant structure is given by

$$S_0 \times V_0 = \lambda_0 V_0 \oplus \{\lambda_{2n}\}_{n=1}^{\infty} \times V_0, \tag{2.6.88}$$

$$S_1 \times V_1 = \lambda_1 V_1 \oplus \{\lambda_{2n+1}\}_{n=1}^{\infty} \times V_1, \tag{2.6.89}$$

$$S_2 \times V_2 = \lambda_2 V_2 \oplus \{\lambda_{2n}\}_{n=2}^{\infty} \times V_2. \tag{2.6.90}$$

Then, the infinite S-expanded (resonant) algebra can be written as

$$\mathfrak{g}_S^{\infty} = (S_0 \times V_0) \oplus (S_1 \times V_1) \oplus (S_2 \times V_2) \,. \tag{2.6.91}$$

We then choose the following infinite ideal:

$$\mathcal{I} = (\{\lambda_{2n}\}_{n=1}^{\infty} \times V_0) \oplus (\{\lambda_{2n+1}\}_{n=1}^{\infty} \times V_1) \oplus (\{\lambda_{2n}\}_{n=2}^{\infty} \times V_2) \,, \tag{2.6.92}$$

and we perform the ideal subtraction on the infinite S-expanded algebra.

The most relevant step consists in writing the commutation relations

$$[\lambda_1 P^*_i, \lambda_1 P_j] = -\frac{1}{\sqrt{2}} \eta_{ij} \left(\lambda_2 P_- - \lambda_2 P_+\right) , \tag{2.6.93}$$

$$[\lambda_1 P^*_{i'}, \lambda_1 P_{j'}] = -\frac{1}{\sqrt{2}} \eta_{i'j'} \left(\lambda_2 P_- - \lambda_2 P_+\right) , \tag{2.6.94}$$

where $\lambda_2 P_-$ belongs to the ideal, while $\lambda_2 P_+$ does not. This is the reason why a finite S-expansion in this case would *not* work.

Thus, after the ideal subtraction, we end up with the algebra $\mathfrak{g}_{PP}$ (the PP-wave algebra):

$$\mathfrak{g}_{PP} = \mathfrak{g}_S^{\infty} \ominus \mathcal{I}. \tag{2.6.95}$$

In fact, if we now rename the generators as follows: $\lambda_0 J_{ij} = \hat{J}_{ij}$, $\lambda_0 J_{i'j'} = \hat{J}_{i'j'}$, $\lambda_0 P_- = \hat{P}_-$, $\lambda_1 P_i = \hat{P}_i$, $\lambda_1 P_{i'} = \hat{P}_{i'}$, $\lambda_1 P^*_i = \hat{P}^*_i$, $\lambda_1 P^*_{i'} = \hat{P}^*_{i'}$, $\lambda_2 P_+ = \hat{P}_+$, we can finally write the PP-wave algebra (see Ref. [69]) in terms of the commutation relations

$$\begin{aligned}
\left[\hat{P}_i,\hat{P}_-\right] &= -2\sqrt{2}\hat{P}^*_i, & \left[\hat{P}_{i'},\hat{P}_-\right] &= -\frac{1}{\sqrt{2}}\hat{P}^*_i,\\
\left[\hat{P}^*_m,\hat{P}_-\right] &= -\frac{1}{\sqrt{2}}\hat{P}_m, & \left[\hat{P}^*_m,\hat{P}_n\right] &= -\frac{1}{\sqrt{2}}\eta_{mn}\hat{P}_+,\\
\left[\hat{J}_{mn},\hat{J}_{pq}\right] &= 4\eta_{mq}\hat{J}_{np}, & \left[\hat{J}_{mn},\hat{P}^*_p\right] &= 2\eta_{np}\hat{P}^*_m,\\
\left[\hat{J}_{mn},\hat{J}_{pq}\right] &= 4\eta_{mq}\hat{J}_{np},
\end{aligned} \tag{2.6.96}$$

where we have adopted the notation of Ref. [69].

We have thus reached the PP-wave algebra in $D = 11$ with an infinite S-expansion with ideal subtraction, starting from the $AdS_4 \times S^7$ algebra, reproducing, in this way, a generalized Inönü-Wigner contraction.

Super-PP-wave algebra from Super-$AdS_4 \times S^7$

We now extend the previous analysis to the supersymmetric case. We thus consider the addition of fermionic generators to the $AdS_4 \times S^7$ algebra, and then perform an infinite S-expansion with consequent ideal subtraction.

Following Ref. [69], the commutators involving the supercharges Q's can be decomposed in the following way:

$$Q = Q_+ + Q_-, \quad Q_\pm = Q_\pm \mathcal{P}_\pm, \tag{2.6.97}$$

using the light cone projection operators

$$\mathcal{P}_\pm = \frac{1}{2}\Gamma_\pm\Gamma_\mp, \tag{2.6.98}$$

$$\Gamma_\pm \equiv \frac{1}{\sqrt{2}}\left(\Gamma_\natural \pm \Gamma_0\right), \tag{2.6.99}$$

where the Γ's are the gamma matrices in eleven dimensions.

Now we add $Q_\pm$ into the subspaces partition, before applying the ideal subtraction, and we thus write

$$\begin{aligned}
V_0 &= \{P_-, J_{mn}, Q_-\},\\
V_1 &= \{P_m, P^*_m, Q_+\},\\
V_2 &= P_+.
\end{aligned} \tag{2.6.100}$$

In this way, after the ideal subtraction (in analogy to what has been done in the bosonic case, see Appendix B.2 for further details on this calculation), we finally get

$$
\begin{aligned}
\left[\hat{P}_-,\hat{Q}_+\right] &= -\frac{3}{2\sqrt{2}}\hat{Q}_+ I,\\
\left[\hat{P}_-,\hat{Q}_-\right] &= -\frac{1}{2\sqrt{2}}\hat{Q}_- I\\
\left[\hat{P}_i,\hat{Q}_-\right] &= \frac{1}{\sqrt{2}}\hat{Q}_+\Gamma^- I\Gamma_i,\\
\left[\hat{P}_{i'},\hat{Q}_-\right] &= \frac{1}{2\sqrt{2}}\hat{Q}_+\Gamma^- I\Gamma_{i'},\\
\left[\hat{P}^*_m,\hat{Q}_-\right] &= \frac{1}{2\sqrt{2}}\hat{Q}_+\Gamma_m\Gamma^-,\\
\left[\hat{J}_{mn},\hat{Q}_\pm\right] &= \frac{1}{2}\hat{Q}_\pm\Gamma_{mn},\\
\left\{\hat{Q}_+,\hat{Q}_+\right\} &= -2\mathcal{C}\Gamma^+\hat{P}_+,\\
\left\{\hat{Q}_-,\hat{Q}_-\right\} &= -2\mathcal{C}\Gamma^-\hat{P}_- - \sqrt{2}\mathcal{C}\Gamma^- I\Gamma^{ij}\hat{J}_{ij} + \frac{1}{\sqrt{2}}\mathcal{C}\Gamma^- I\Gamma^{i'j'}\hat{J}_{i'j'},\\
\left\{\hat{Q}_+,\hat{Q}_-\right\} &= \left(-2\mathcal{C}\Gamma^m\hat{P}_m - 4\mathcal{C}I\Gamma^i\hat{P}^*_i - 2\mathcal{C}I\Gamma^{i'}\hat{P}^*_{i'}\right)\hat{\mathcal{P}}_-,
\end{aligned}
\tag{2.6.101}
$$

where we have again adopted the notation defined in [69]. These commutation relations correspond to those of the super-PP-wave algebra (see Ref. [69] for further details).

We have thus reached the super-PP-wave algebra in $D = 11$ with an infinite S-expansion with ideal subtraction, starting from the super-$AdS_4 \times S^7$, reproducing, in this way, a generalized Inönü-Wigner contraction on a supersymmetric algebra.

2.7 Comments about the requirements on S-expansion

The associativity present in the (semi)group of an S-expansion is a requirement to maintain the Jacobi identity in the algebra generated. With the associativity it is possible to factorize the elements from the new structure constants to maintain the value of zero through Bianchi identity of the initial algebra. However there are occasions where Jacobi identity is zero on its own, therefore it is not necessary to impose this condition of associativity.

As explained in ref. [28] to find the sets of elements that perform a certain S-expansion we must discard the non-associative sets, work that it is possible do with a computer program. But doing this we may be excluding some correct sets, when the Jacobi identity is still zero whitout associativity in the set.

A good example is 1.5.1 explained above. In this example we develop with the application of our analytic method that, in order to reach the target Bianchi Type II algebra from the Bianchi Type I algebra, the structure of semigroup is not necessary. This is due to the fact that, in that case, the Jacobi identities of both the mentioned algebras are trivially satisfied (each term of the Jacobi identities is equal to zero).

Curiously, all the examples in what we work with the technique of analytical analysis has resulted in sets of elements that always generates an algebra that respect the identity of Jacobi. This motivates to think that perhaps the exposed method always leads to obtain a set of elements that meet the requirements to build an algebra.

Since the found set has a resonance structure with the subspaces of the initial algebra, it follows the same closure rules between the subspaces of final algebra. On the other hand the signs of multiplication are exposed when the expanded generators are written in function of the old ones. This could justify the phenomenon, but it requires a much more thorough analysis that will not be analyzed in this thesis. As a conclusion this establishes a generality to the S-expansion process from the requirement that it be an abelian magma, and from there, using the exposed method to find the necessary magma, semigroup or group, all always abelian.

On the other hand, the author in [20] already mentioned that the requirement of abelianidad, when relaxing it, it was possible to obtain an super algebra from an algebra, due to the possibility to transform the commutators into anticommutadores using variables of Grassman [23].

Following the work done in [70], the generalized IW-contraction is reproduced as a infinite resonant S-expansion using at the same time an ideal-reduction, which generalizes the 0_S-reduction and answers some of the open questions described in [71] . At the same time, in ref. [72] The authors reproduce a generalized IW-contraction using Grassman parameters, which leads one to think of relaxing the abelianity at the level of algebras, not only of super-algebras, is obviously possible, as we see in the end of example 1.5.1.

Furthermore, even if we consider different kind of structure that multiples an arbitrary algebra, remains the natural question: What is the meaning of this?. The most close answer is given by the most known contractions, because in many example the parameter has a physical meaning. What means that maybe the knowing parameters has besides a matematical structure. Of course, this text is only motivational for future research

Chapter 3

Superalgebras and FDA on Supergravity

Wisdom is not obvious. You must see the subtle and notice the hidden to be victorious.

Sun Tzu, The Art of War

3.1 Introduction

Supergravity theories in various spacetime dimensions $4 \leq D \leq 11$ have a field content that generically includes the metric, the gravitino, a set of 1-form gauge potentials, and $(p+1)$-form gauge potentials of various $p \leq 9$, and they are discussed in the context of Free Differential Algebras (FDAs).

In particular, in the framework of FDAs [29], the structure of $D = 11$ supergravity, first constructed in [21], was then reconsidered in [22], adopting the superspace geometric approach. In the same paper, the supersymmetric FDA was also investigated in order to see whether the FDA formulation could be interpreted in terms of an ordinary Lie superalgebra in its dual Maurer-Cartan formulation *. This was proven to be true, and the existence of a hidden superalgebra underlying the theory was presented for the first time. In fact, in [22], the authors proved that the FDA underlying $D = 11$ supergravity can be traded with a Lie superalgebra which contains, besides the Poincaré superalgebra, also new bosonic 1-forms and a nilpotent fermionic generator Q', necessary for the closure of the superalgebra.

Later, in [73], the authors wonder whether eleven dimensional supergravity can be decontracted into a non-abelian (gauged) model. This problem was reduced to that of finding an algebra whose contraction yields the $D = 11$ algebra of [22]. In the same paper, they also considered the $D = 4$ case, in order to explain their approach through a toy-model. However, the four dimensional gauged case results interesting, since its algebraic form (presented in [73]) corresponds to an AdS-Lorentz-like superalgebra, presented and largely discussed

*The supergroup structure allows a deeper understanding of the symmetry and topological properties of the theory.

in [74], where the authors explored the supersymmetry invariance of an extension of minimal $D = 4$ supergravity in the presence of a non-trivial boundary, presenting the explicit construction of the $\mathcal{N} = 1$, $D = 4$ AdS-Lorentz supergravity bulk Lagrangian in the rheonomic framework. In particular, they developed a peculiar way to introduce a generalized supersymmetric cosmological term to supergravity. Then, by studying the supersymmetry invariance of the Lagrangian in the presence of a non-trivial boundary, they interestingly found that the supersymmetric extension of a Gauss-Bonnet like term is required in order to restore the supersymmetry invariance of the full Lagrangian.

In [26], the authors clarified the role of the nilpotent fermionic generator Q' introduced in [22] by looking at the gauge properties of the theory. They found that its presence is necessary, in order that the extra 1-forms of the hidden superalgebra give rise to the correct gauge transformations of the p-forms of the FDA. In particular, in its absence, the extra bosonic 1-forms do *not* enjoy gauge freedom, but generate, together with the supervielbein, new directions of an enlarged superspace, so that the FDA on ordinary superspace is no more reproduced.

On the group theoretical side, in [20], the authors developed the so-called S-expansion procedure, which is based on combining the inner multiplication law of a discrete set S with the structure of a semigroup, with the structure constants of a Lie algebra $\mathfrak{g}$. The new, larger Lie algebra thus obtained is called S-expanded algebra, and it is written as $\mathfrak{g}_S = S \times \mathfrak{g}$.

There are two facets applicable in the S-expansion method, which offer great manipulation on (super)algebras, *i.e. resonance* and *reduction*. The role of *resonance* is that of transferring the structure of the semigroup to the target (super)algebra; Meanwhile, *reduction* plays a peculiar role in cutting the (super)algebra properly, thanks to the existence of a zero element in the set involved in the procedure.

From the physical point of view, several (super)gravity theories have been largely studied using the S-expansion approach, enabling the achievement of several results over recent years (see Ref.s [27,47–50,52,53,75–93]). Furthermore, in [28], an analytic method for S-expansion was developed. This method is able to give the multiplication table(s) of the (abelian) set(s) involved in an S-expansion process for reaching a target Lie (super)algebra from a starting one, after having properly chosen the partitions over subspaces of the considered (super)algebras. A complete review of S-expansion can be found in [20] and [28].

In [70], the authors proposed a new prescription for S-expansion, involving an infinite abelian semigroup $S_E^{(\infty)}$, with subsequent subtraction of a suitable infinite ideal. Their approach is a generalization of the finite S-expansion procedure, and it allows to reproduce a generalized Inönü-Wigner contraction via infinite S-expansion between two different algebras.

In this work, we obtain a particular hidden Maxwell superalgebra in four dimensions by performing an infinite S-expansion with subsequent ideal subtraction of the hidden AdS-Lorentz superalgebra underlying $D = 4$ supergravity. We also display a map in order to show the way in which the hidden AdS-Lorentz superalgebra and the Maxwell superalgebra can be obtained. This map also offers the links, consisting in infinite S-expansion and ideal subtraction, between other superalgebra in four dimensions. We then adopt the Maurer-Cartan (and FDA) formalism and we consider the paremetrization of the 3-form $A^{(3)}$, whose field strength is a 4-form $F^{(4)} = dA^{(3)} + \ldots$, modulo fermionic bilinears, in terms of 1-forms, and we show how the (trivial) boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator appearing in the hidden Maxwell superalgebra underlying supergravity in four dimensions. This extension involves the cosmological constant. Interestingly, the presence of these terms strictly depends on the form of the extra fermionic generator appearing in the hidden superMaxwell-like extension of $D = 4$ supergravity.

This chapter is organized as follows: In Section 3.2, we perform the S-expansion of different superalgebras describing and underlying $D = 4$ supergravity, and we also display a map which links different superalgebras in four dimensions. In Section 3.3, we write some of the superalgebras presented in Section 3.2 in the Maurer-Cartan formalism and, in particular, we consider a gauged version of $D = 4$ supergravity, which hidden structure corresponds to the Maxwell superalgebra. We then write the parametrization of the 3-form $A^{(3)}$ in this context and we show that the (trivial) boundary contribution $dA^{(3)}$ can be naturally extended with the addition of terms involving the cosmological constant.

3.2 Infinite S-expansion and ideal subtraction of superalgebras in four dimensions

It is well known that we can construct several theories in four dimensions by choosing different amount of physical (and unphysical) fields, invariant under different superalgebras. One of the simplest case is the Poincaré superalgebra $\overline{osp(1|4)}$, which is abelian in the momenta. On the other side, the (Anti-)de Sitter ($(A)dS$) algebra is characterized by the fact that the translations commute with Lorentz transformations. In Ref. [94], the authors presented a geometric formulation involving the AdS structure group (the AdS one), known as *MacDowell-Mansouri action*. The generalization of their work consists in considering the supergroup $Osp(\mathcal{N}|4)$.

In Ref. [22], was presented a particular superalgebra, now known as *hidden superalgebra*, which includes, as a subalgebra, the super-Poincaré algebra, and also involves two extra bosonic generators Z_{ab}, commuting with the generators P_a. Furthermore, an extra nilpotent fermionic generator Q'_α must be included (in order to satisfy the closure of the superalgebra). It is then possible to consider a *Hidden AdS-Lorentz superalgebra*, when the commutators between the momenta is equal to a Lorentz-like generator, which will be referred as to Z_{ab}. Finally, we should mentioned that the introduction of a second fermionic generator has been considered in the literature; This feature lead the authors of [53] to consider *Maxwell superalgebras* for constructing actions for supergravity theories.

We will now consider the same types of superalgebras in *four* dimensions. Each of these superalgebras gives rise to the construction of an action for a supergravity theory. The existence of connections between different physical theories motivates to look for connections between the superalgebras underlying these theories. In the following, we present a scheme that displays the various relations between the mentioned superalgebras, involving infinite S-expansion and ideal subtraction. We first consider a "toy model" superalgebra in four dimensions described in [73], namely the *AdS*-Lorentz-like superalgebra with an extra fermionic generator, which is the hidden superalgebra underlying the *AdS* supergravity theory in $D = 4$. This algebra will be named *hidden AdS-Lorentz superalgebra*, and can be written as

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
[Q_\alpha, Z_{ab}] &= -(\gamma_{ab} Q)_\alpha - (\gamma_{ab} Q')_\alpha, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
[Q_\alpha, P_a] &= -i(\gamma_a Q)_\alpha - i(\gamma_a Q')_\alpha \qquad [P_a, P_b] = -Z_{ab}, \\
[J_{ab}, Q_\alpha] &= -(\gamma_{ab} Q)_\alpha, \quad [J_{ab}, Q'_\alpha] = -(\gamma_{ab} Q')_\alpha, \quad [Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
\{Q_\alpha, Q_\beta\} &= -i(\gamma^a C)_{\alpha\beta} P_a - \frac{1}{2}(\gamma^{ab} C)_{\alpha\beta} Z_{ab}, \qquad \{Q'_\alpha, Q'_\beta\} = 0,
\end{aligned} \tag{3.2.1}$$

where C stands for the charge conjugation matrix and γ_a and γ_{ab} are Dirac matrices in four dimensions. Let us notice that the Lorentz type algebra generated by $\{J_{ab}, Z_{ab}\}$ is a subalgebra of the above superalgebra. In [74], the authors explored the supersymmetry invariance of an extension of minimal $D = 4$ supergravity in the presence of a non-trivial boundary, and they presented the explicit construction of the $\mathcal{N} = 1$, $D = 4$ *AdS*-Lorentz supergravity bulk Lagrangian in the rheonomic framework. In particular, they developed a peculiar way to introduce a generalized supersymmetric cosmological term in supergravity. The starting algebra they considered was exactly the *AdS*-Lorentz superalgebra (3.2.1), with a consistent truncation of the fermionic generator Q'_α *. In other words, the hidden *AdS*-Lorentz superalgebra can be consistently viewed as an extension of the *AdS*-Lorentz algebra described in [74], with the inclusion of an extra fermionic generator Q'_α.

The extra fermionic generator Q'_α contained into the hidden *AdS*-Lorentz superalgebra presents the structure of an abelian ideal: $[\,\cdot\,, Q'_\alpha] = 0 \longrightarrow \mathcal{I} = \{Q'_\alpha\}$ (see Ref. [70] for further details). A technique proposed by the authors of [70], which consists in a new prescription for S-expansion, involving an infinite abelian semigroup $S^{(\infty)}_E$, with subsequent subtraction of a suitable infinite ideal †, allows to obtain the *AdS*-Lorentz superalgebra, generated only by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$, studied in [74] from the hidden *AdS*-Lorentz superalgebra, generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha, Q'_\alpha\}$, described in [73], by taking into account the existence of the ideal structure mentioned above. This is done by subtracting the ideal, namely

$$\mathfrak{g}_{AdS\text{-Lorentz}} = \mathfrak{g}_{\text{Hidden } AdS\text{-Lorentz}} \ominus \mathcal{I}. \tag{3.2.2}$$

On the other hand, we can reach a *(hidden) Maxwell superalgebra* [42,43] in four dimensions by starting from the hidden *AdS*-Lorentz superalgebra (3.2.1).

*Let us observe that the authors of [74] adopted the Maurer-Cartan formalism in their work, where the superalgebra generators are properly associated to 1-forms.

†Their approach is a generalization of the finite S-expansion procedure, and it allows to reproduce a generalized Inönü-Wigner contraction with an infinite S-expansion with subsequent ideal subtraction.

Thus, following [70], we can perform a S-expansion with the infinite abelian semigroup $S_E^{(\infty)}$, involving a resonant structure:

$$S_E^{(\infty)} \times \mathfrak{g}_{\text{Hidden } AdS\text{-Lorentz}}. \tag{3.2.3}$$

The resonance condition can be written as

$$\begin{aligned}
\{\lambda_{2n}\}_{n=0}^{\infty} \times V_0 &= \{\lambda_{2n}\}_{n=0}^{\infty} \{J_{ab}\}\,,\\
\{\lambda_{2n+1}\}_{n=0}^{\infty} \times V_1 &= \{\lambda_{2n+1}\}_{n=0}^{\infty} \times \{Q_\alpha, Q'_\alpha\}\,,\\
\{\lambda_{2n}\}_{n=1}^{\infty} \times V_2 &= \{\lambda_{2n}\}_{n=1}^{\infty} \times \{Z_{ab}, P_a\}\,.
\end{aligned} \tag{3.2.4}$$

Then, we perform the subtraction of the following particular ideal:

$$\begin{aligned}
\mathcal{I} &= \bigoplus_{\beta} W_\beta^{\mathcal{I}}, \quad \beta = 0,1,2, \\
W_0^{\mathcal{I}} &= \{\lambda_{2n}\}_{n=2}^{\infty} \times \{J_{ab}\}\,,\\
W_1^{\mathcal{I}} &= \{\lambda_{2n+1}\}_{n=2}^{\infty} \times \{Q_\alpha, Q'_\alpha\}\,,\\
W_2^{\mathcal{I}} &= \{\lambda_{2n}\}_{n=3}^{\infty} \times \{Z_{ab}, P_a\}\,.
\end{aligned} \tag{3.2.5}$$

After having properly renamed (following the convention of Ref. [70]) the new generators, we end up with the (hidden) Maxwell superalgebra, which is thus given by

$$\mathfrak{g}_{\text{Maxwell Superalgebra}} = \left(S_E^{(\infty)} \times \mathfrak{g}_{\text{Hidden } AdS\text{-Lorentz}}\right) \ominus \mathcal{I}. \tag{3.2.6}$$

We also observe that, analogously to the cases of the hidden AdS-Lorentz and of the AdS-Lorentz superalgebras in four dimensions, the same procedure adopted for that cases allows to reproduce the Poincaré superalgebra from the hidden Poincaré superalgebra introduced and studied in [22, 73] ‡. In fact, by starting from the hidden Poincaré superalgebra, generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha, Q'_\alpha\}$, which can be written as

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc},\\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc},\\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, & [P_a, P_b] &= 0,\\
[Q_\alpha, P_a] &= -i(\gamma_a Q')_\alpha & [J_{ab}, Q_\alpha] &= -(\gamma_{ab}Q)_\alpha,\\
[Q_\alpha, Z_{ab}] &= -(\gamma_{ab}Q')_\alpha, & [J_{ab}, Q'_\alpha] &= -(\gamma_{ab}Q')_\alpha,\\
\{Q_\alpha, Q_\beta\} &= -i(\gamma^a C)_{\alpha\beta}P_a - \frac{1}{2}(\gamma^{ab}C)_{\alpha\beta}Z_{ab}, & \{Q'_\alpha, Q'_\beta\} &= 0,
\end{aligned} \tag{3.2.7}$$

‡It is well known that the Poincaré and the AdS superalgebras are related by Inönü-Wigner contraction, *i.e.* by rescaling and consequently considering a particular limit for the generators, namely $P_a \longrightarrow \sigma^2 P_a$, and $Q_\alpha \longrightarrow \sigma Q_\alpha$, where σ is the rescaling parameter. It is also possible to end up with the non-standard Maxwell superalgebra by starting from the AdS superalgebra or Maxwell superalgebra (see Ref. [70] for further details). In the same way, by performing the rescaling $Z_{ab} \longrightarrow \sigma Z_{ab}$, $P_a \longrightarrow \sigma P_a$, $Q_\alpha \longrightarrow \sigma Q_\alpha$, and $Q_\alpha \longrightarrow \sigma^2 Q_\alpha$, and a consequent consistent limit (Inönü-Wigner contraction) of the hidden AdS-Lorentz algebra (3.2.1), we obtain the hidden Poincaré superalgebra.

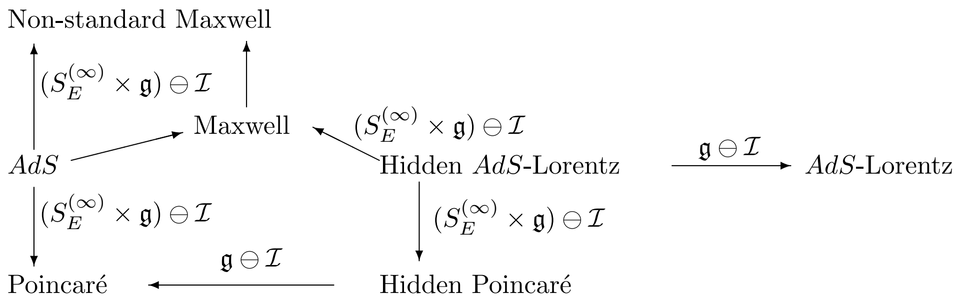

Figure 3.1: Map between different superalgebras linked through infinite S-expansion and/or ideal subtraction. Source: self made.

and considering the ideal $\mathcal{I} = \{Z_{ab}, Q'_\alpha\}$, we can reach the standard Poincaré superalgebra in four dimensions generated only by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$, namely

$$\mathfrak{g}_{\text{Poincaré superalgebra}} = \mathfrak{g}_{\text{hidden Poincaré superalgebra}} \ominus \mathcal{I}. \tag{3.2.8}$$

In Figure 3.1, we have collected and summarized the relationships between the mentioned superalgebras. The link between this superalgebras involve infinite S-expansion and ideal subtraction [§].

3.3 Hidden Maxwell superalgebra in the Maurer-Cartan formalism

There are two dual ways of describing a (super)algebra: The first one is provided by the (anti)commutation relations between the generators; The second one is instead provided by the so-called Maurer-Cartan equations. These two descriptions are equivalent and dual each other.

The generators T_A's, which form a basis of the tangent space $T(\mathcal{M})$ of a manifold $\mathcal{M}$, satisfy the (anti)commutation relations of the (super)algebra and the (super) Jacobi identity. The same information is enclosed in the Maurer-Cartan equations, which read

$$d\sigma^A = -\frac{1}{2} C^A{}_{BC} \sigma^B \sigma^C, \tag{3.3.1}$$

where σ^A stands for the forms involved into the Maurer-Cartan equations, and where $C^A{}_{BC}$ are the coupling constants.

As we can see, the Maurer-Cartan equations are written in terms of the dual forms σ^A's of the generators T_A's, which are related through the expression

$$\sigma^A(T_B) = \delta^A_B, \tag{3.3.2}$$

[§]Let us remind that both the standard and the generalized Inönü-Wigner contractions are reproducible through S-expansion: The standard Inönü-Wigner contraction can be reproduced with a finite S-expansion, while the generalized one can be reproduced through an infinite S-expansions with subsequent ideal subtraction.

up to normalization factors *.

We now consider the Maurer-Cartan equations associated with the superalgebras in $D = 4$ presented in [73]. In the case of $\overline{osp(4|1)}$ (Poincaré superalgebra), we have

$$\begin{aligned} R^{ab} =&0, \\ DV^a =&\frac{i}{2}\bar{\Psi}\gamma^a\Psi, \\ D\Psi =&0, \end{aligned} \tag{3.3.3}$$

where γ^a, as said before, are the four-dimensional gamma matrices, and where $D = d + \omega$ is the Lorentz covariant exterior derivative. Here we have fixed the normalization of DV^a to $\frac{i}{2}$, according to the usual convention. The closure ($d^2 = 0$) of this superalgebra is trivially satisfied.

In the AdS case, instead, we have that the anticommutator of the generators Q's falls into the Poincaré traslations and the Lorentz rotations, generating non-vanishing value of the curvature, namely

$$\begin{aligned} R^{ab} =&\alpha e^2 V^a V^b + \beta e\bar{\Psi}\gamma^{ab}\Psi, \\ DV^a =&\frac{i}{2}\bar{\Psi}\gamma^a\Psi, \\ D\Psi =&\frac{i}{2}e\gamma_a\Psi V^a, \end{aligned} \tag{3.3.4}$$

where $e = 1/2l$ corresponds to the inverse of the AdS radius †, and where we have fixed the normalization of $D\Psi$ to $\frac{i}{2}$. Here, α and β are constants. In order to fix them, we require the d^2-closure of the superalgebra ($d^2 = 0$); In this way we get $\beta = \frac{1}{2}\alpha$ and, after having fixed the normalization $\alpha = -1$, we can thus write $\beta = -\frac{1}{2}$.

We observe that in the limit $e \to 0$ we correctly get the Maurer-Cartan equations in the flat (*i.e.* Minkowski) space.

As shown in [22], with the introduction of a nilpotent fermionic generator Q', which also appears in the supergroup called hidden supergroup, we can write the hidden Poincaré (3.2.7) and the hidden AdS-Lorentz (3.2.1) (see Ref. [73]) superalgebras in terms of the corresponding respective Maurer-Cartan equations. For the hidden Poincaré case (3.2.7), we have

$$R^{ab} =0, \tag{3.3.5}$$

$$DV^a =\frac{i}{2}\bar{\Psi}\gamma^a\Psi, \tag{3.3.6}$$

$$D\Psi =0, \tag{3.3.7}$$

$$DB^{ab} =\frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, \tag{3.3.8}$$

$$D\eta =\frac{i}{2}\delta\gamma_a\Psi V^a + \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab}. \tag{3.3.9}$$

*See the maps between the two formalism presented [26] for further details.

†The scaling constant e is the charge associated with the cosmological constant switched on in the AdS curved space.

Here, δ and ϵ are two *arbitrary* parameters. In fact, requiring the closure of the superalgebra, and in particular of $D\eta$, we simply get the identity $0 = 0$, which leads the solution to be given in terms of two free parameters, namely δ and ϵ. In particular, for reaching this result we have made use of the following Fierz identities in four dimensions:

$$\Psi\gamma_a\bar{\Psi}\gamma^a\Psi = 0, \tag{3.3.10}$$

,

$$\Psi\gamma_{ab}\bar{\Psi}\gamma^{ab}\Psi = 0. \tag{3.3.11}$$

In this superalgebra the Lorentz curvature is zero: $R^{ab} = 0$. However, we have a non-trivial Lorentz-like contribution given by $DB^{ab} = \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi$.

We observe, as it was done in [73], that in $D = 4$ we also have a particular subalgebra of the above hidden Poincaré one, which can be obtained through an Inöü-Wigner contraction on the hidden AdS-Lorentz superalgebra (3.2.1). In fact, let us remind that, as observed in [73], we do not even need the 1-form B^{ab} to find an underlying group for the Cartan Integrable System (CIS) in the four dimensional Minkowski space. Indeed, the general solution contains as special case the subalgebra whose associated Maurer-Cartan equations can be written as

$$R^{ab} = 0, \tag{3.3.12}$$

$$DV^a = \frac{i}{2}\bar{\Psi}\gamma^a\Psi, \tag{3.3.13}$$

$$D\Psi = 0, \tag{3.3.14}$$

$$D\eta = \frac{i}{2}\gamma_a\Psi V^a, \tag{3.3.15}$$

which endowed the CIS with a 3-form $A^{(3)} = -i\bar{\Psi}\gamma_a\eta V^a$.

As previously shown in Section 3.2, we can write a (hidden) Maxwell superalgebra in four dimensions, by starting from the hidden AdS-Lorentz one (3.2.1). For completeness, in the following we report the Maurer-Cartan equations associated with the hidden AdS-Lorentz superalgebra (3.2.1) (which were displayed in [73]):

$$R^{ab} = 0, \tag{3.3.16}$$

$$DV^a = \frac{i}{2}\bar{\Psi}\gamma^a\Psi - eB^{ab}V_b, \tag{3.3.17}$$

$$D\Psi = \frac{i}{2}e\gamma_a\Psi V^a + \frac{e}{4}\gamma_{ab}\Psi B^{ab}, \tag{3.3.18}$$

$$DB^{ab} = \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi - eB^{ac}B_c^{\ b} + eV^aV^b, \tag{3.3.19}$$

$$D\eta = \frac{i}{2}\gamma_a\Psi V^a + \frac{1}{4}\gamma_{ab}\Psi B^{ab}, \tag{3.3.20}$$

where the parameters have been fixed by requiring the closure of the superalgebra and properly fixing the normalization of the 1-form η (see Ref. [73] for further details) ‡.

‡We observe, reminding [73], that in the hidden AdS-Lorentz superalgebra we can write $D\eta = \frac{1}{e}\Lambda$ and $D\Psi = \Lambda$, where Λ is the 2-form that reads $\Lambda = \frac{i}{2}e\gamma_\Psi V^a + \frac{1}{4}e\gamma_{ab}\Psi B^{ab}$.

We now write the Maurer-Cartan equations associated with this Maxwell superalgebra in $D=4$, namely

$$\begin{aligned} R^{ab} &= 0, && (3.3.21)\\ DV^a &= \frac{i}{2}\bar{\Psi}\gamma^a\Psi, && (3.3.22)\\ D\Psi &= 0, && (3.3.23)\\ DB^{ab} &= \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi, && (3.3.24)\\ D\tilde{B}^{ab} &= \alpha e\bar{\Psi}\gamma^{ab}\eta + \beta e B^{ac}B_c{}^b + \gamma e V^a V^b, && (3.3.25)\\ D\eta &= \frac{i}{2}\delta\gamma_a\Psi V^a + \frac{1}{2}\varepsilon\gamma_{ab}\Psi B^{ab}. && (3.3.26) \end{aligned}$$

Once again, we must require the closure $d^2=0$ of the superalgebra. In this way, from the first Maurer-Cartan equation we get $\delta\alpha=\gamma$, and $\beta=-2\alpha\epsilon$. We now choose the normalization $\alpha=1$ and $\delta=1$. We can thus write $\gamma=1$ and $\beta=-2\epsilon$, being ϵ a free parameter. We observe that the Lorentz curvature is again zero: $R^{ab}=0$. In this case, we have two non-trivial Lorentz-like contributions, namely $DB^{ab}=\frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi$ and $D\tilde{B}^{ab}=\alpha e\bar{\Psi}\gamma^{ab}\eta+\beta e B^{ac}B_c{}^b+\gamma e V^a V^b$. Then, we can finally write

$$\begin{aligned} R^{ab} &= 0,\\ DV^a &= \frac{i}{2}\bar{\Psi}\gamma^a\Psi,\\ D\Psi &= 0,\\ DB^{ab} &= \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi,\\ D\tilde{B}^{ab} &= e\bar{\Psi}\gamma^{ab}\eta + \beta e B^{ac}B_c{}^b + eV^aV^b,\\ D\eta &= \frac{i}{2}\gamma_a\Psi V^a + \frac{1}{2}\varepsilon\gamma_{ab}\Psi B^{ab}, \end{aligned} \qquad (3.3.27)$$

where $\beta=-2\epsilon$. This superalgebra is the (hidden) Maxwell superalgebra underlying supergravity in four dimensions.

We observe that, with the choice $\beta=\epsilon=0$, we get the following subalgebra:

$$\begin{aligned} R^{ab} &= 0,\\ DV^a &= \frac{i}{2}\bar{\Psi}\gamma^a\Psi,\\ D\Psi &= 0,\\ DB^{ab} &= \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi,\\ D\tilde{B}^{ab} &= e\bar{\Psi}\gamma^{ab}\eta + eV^aV^b,\\ D\eta &= \frac{i}{2}\gamma_a\Psi V^a. \end{aligned} \qquad (3.3.28)$$

Now, one may wonder how to write the parametrization of the 3-form $A^{(3)}$ in $D=4$ supergravity, once the above relations are given. In the following, we will write the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms (also the "hidden" ones, namely the 1-forms which do *not* represent physical fields), both for the hidden Maxwell superalgebra (3.3.27) and for its subalgebra (3.3.28). We will then study the particular structures of the (trivial) boundary contribution $dA^{(3)}$ in the four-dimensional "gauged" case.

3.3.1 Gauged extensions of $dA^{(3)}$

We start our analysis by considering the hidden Maxwell superalgebra in four dimensions [§] and its structure (3.3.27). Then, we write the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, both for the hidden Maxwell superalgebra and for its particular subalgebra displayed above, and we study the different structures of the (trivial) boundary contribution $dA^{(3)}$.

The four-dimensional hidden Maxwell superalgebra is generated by the following set of generators:

$$\left\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Q_\alpha, Q'_\alpha\right\}. \tag{3.3.29}$$

Let us now consider the (hidden) Maxwell superalgebra valued curvatures, which are defined by

$$R^{ab} = d\omega^{ab} - \omega^a{}_c\omega^{cb}\,, \tag{3.3.30}$$

$$R^a = DV^a - \frac{i}{2}\bar{\Psi}\gamma^a\Psi\,, \tag{3.3.31}$$

$$F^{ab} = DB^{ab} - \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi\,, \tag{3.3.32}$$

$$\tilde{F}^{ab} = D\tilde{B}^{ab} - e\bar{\Psi}\gamma^{ab}\eta - \beta e B^{ac}B_c{}^b - eV^aV^b\,, \tag{3.3.33}$$

$$\rho = D\Psi\,, \tag{3.3.34}$$

$$\sigma = D\eta - \frac{i}{2}\gamma_a\Psi V^a - \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab}, \tag{3.3.35}$$

where $D = d + \omega$ is the Lorentz covariant exterior derivative. In the four-dimensional Minkowski space, one can also write

$$F^{(4)} = dA^{(3)} - \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^aV^b\,, \tag{3.3.36}$$

where the 4-form $F^{(4)}$ is trivially given in terms of a boundary contribution.

Now, our aim is that of writing the deformation to the 4-form $F^{(4)}$ induced by the presence the cosmological constant in the hidden Maxwell superalgebra underlying $D = 4$ supergravity.

We can write the Maurer-Cartan equations in four dimensions for the Maxwell superalgebra, by setting the curvatures to zero, namely

[§] That is to say, the superalgebra we have previously obtained through infinite S-expansion with subsequent ideal subtraction by starting from the hidden AdS-Lorentz superalgebra

$$R^{ab} = d\omega^{ab} - \omega^{a}_{\ c}\omega^{cb} = 0\,, \tag{3.3.37}$$

$$R^{a} = DV^{a} - \frac{i}{2}\bar{\Psi}\gamma^{a}\Psi = 0\,, \tag{3.3.38}$$

$$F^{ab} = DB^{ab} - \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi = 0\,, \tag{3.3.39}$$

$$\tilde{F}^{ab} = D\tilde{B}^{ab} - e\bar{\Psi}\gamma^{ab}\eta - \beta e B^{ac}B_{c}^{\ b} - eV^{a}V^{b} = 0\,, \tag{3.3.40}$$

$$\rho = D\Psi = 0\,, \tag{3.3.41}$$

$$\sigma = D\eta - \frac{i}{2}\gamma_{a}\Psi V^{a} - \frac{1}{2}\epsilon\gamma_{ab}\Psi B^{ab} = 0\,, \tag{3.3.42}$$

which simply lead to the expression (3.3.27).

Now, as done in the $D = 11$ and $D = 7$ supergravity cases in [22] and [26], respectively, we can write the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms. We first of all observe that, since $dA^{(3)}$ is a boundary contribution in four dimensions, we expect a topological form for the parametrization of $A^{(3)}$. We thus start by writing

$$A^{(3)} = \frac{1}{e}B^{ab}DB_{ab} + \frac{1}{e}\tilde{B}^{ab}D\tilde{B}_{ab} - 2\bar{\eta}D\eta. \tag{3.3.43}$$

This particular parametrization will give rise to a "topological" structure for the boundary contribution $dA^{(3)}$. In fact, if we now compute $dA^{(3)}$, we simply get the following expression:

$$\begin{aligned} dA^{(3)} &= \frac{1}{e}d(B^{ab}DB_{ab}) + \frac{1}{e}d(\tilde{B}^{ab}D\tilde{B}_{ab}) - 2d(\bar{\eta}D\eta) = \\ &= \frac{1}{e}DB^{ab}DB_{ab} + \frac{1}{e}D\tilde{B}^{ab}D\tilde{B}_{ab} - 2D\bar{\eta}D\eta, \end{aligned} \tag{3.3.44}$$

which automatically satisfies the closure requirement $d^2 = 0$. If we now substitute the Maurer-Cartan equations (3.3.27) in the expression (3.3.44), we get

$$\begin{aligned} dA^{(3)} = &\frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^{a}V^{b} + e\bar{\Psi}\gamma_{ab}\eta\bar{\Psi}\gamma^{ab}\eta + 2\beta e\bar{\Psi}\gamma_{ab}\eta B^{ac}B_{c}^{\ b} + 2e\bar{\Psi}\gamma_{ab}\eta V^{a}V^{b} + \\ &+ 2\beta e B^{ac}B_{c}^{\ b}V_{a}V_{b} - 2i\epsilon\bar{\Psi}\gamma_{a}\Psi B^{ab}V_{b} + \epsilon^{2}\bar{\Psi}\gamma_{ac}\Psi B^{ab}B^{c}_{\ b}. \end{aligned} \tag{3.3.45}$$

In the limit $e \to 0$, the expression (3.3.45) reduces to

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^{a}V^{b} - 2i\epsilon\bar{\Psi}\gamma_{a}\Psi B^{ab}V_{b} + \epsilon^{2}\bar{\Psi}\gamma_{ac}\Psi B^{ab}B^{c}_{\ b}. \tag{3.3.46}$$

We observe that, interestingly, this solution does *not* reduce to the four-dimensional Minkowski flat space limit when $e \to 0$. However, if we now consider the particular solution $\beta = \epsilon = 0$, which conduces to the subalgebra (3.3.28) of the hidden Maxwell superalgebra in four dimensions (as mentioned in the previous section), we see clearly that, interestingly, this particular solution leads to

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^a V^b + e\bar{\Psi}\gamma_{ab}\eta\bar{\Psi}\gamma^{ab}\eta + 2e\bar{\Psi}\gamma_{ab}\eta V^a V^b, \tag{3.3.47}$$

which exactly reproduces the Minkowski FDA with

$$dA^{(3)} = \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^a V^b \tag{3.3.48}$$

in the limit $e \to 0$. Thus, the particular subalgebra (3.3.28) of the Maxwell superalgebra (3.3.27) underlying supergravity in four dimensions can be written as

$$\begin{aligned}
R^{ab} &= 0\,, \\
DV^a &= \frac{i}{2}\bar{\Psi}\gamma^a\Psi\,, \\
DB^{ab} &= \frac{1}{2}\bar{\Psi}\gamma^{ab}\Psi\,, \\
D\tilde{B}^{ab} &= e\bar{\Psi}\gamma^{ab}\eta + eV^a V^b\,, \\
D\Psi &= 0\,, \\
D\eta &= \frac{i}{2}\gamma_a\Psi V^a\,, \\
dA^{(3)} &= \frac{1}{2}\bar{\Psi}\gamma_{ab}\Psi V^a V^b + e\bar{\Psi}\gamma_{ab}\eta\bar{\Psi}\gamma^{ab}\eta + 2e\bar{\Psi}\gamma_{ab}\eta V^a V^b,
\end{aligned} \tag{3.3.49}$$

where, by setting $\beta = \epsilon = 0$, we have withdrawn the B^{ab}-contributions in $D\tilde{B}^{ab}$ and $D\eta$ ¶.

For the sake of completeness, in the following we also report the complete parametrization of the 3-form $A^{(3)}$ written in terms of 1-forms:

$$A^{(3)} = \frac{1}{2e}\bar{\Psi}\gamma_{ab}\Psi B^{ab} + \bar{\Psi}\gamma_{ab}\Psi\tilde{B}^{ab} + \beta\tilde{B}_{ab}B^{ac}B_c{}^b + \tilde{B}_{ab}V^a V^b - i\bar{\Psi}\gamma_a\eta V^a - \epsilon\bar{\Psi}\gamma_{ab}\eta B^{ab}, \tag{3.3.50}$$

where the topological structure, however, is *not* evident. We have made it manifest in (3.3.43) and, subsequently, in (3.3.44). By setting $\beta = \epsilon = 0$ in (3.3.50), we obtain

$$A^{(3)} = \frac{1}{2e}\bar{\Psi}\gamma_{ab}\Psi B^{ab} + \bar{\Psi}\gamma_{ab}\Psi\tilde{B}^{ab} + \tilde{B}_{ab}V^a V^b - i\bar{\Psi}\gamma_a\eta V^a. \tag{3.3.51}$$

We finally observe that the parametrization (3.3.50) can be rewritten in the following simple form:

$$A^{(3)} = \tilde{B}_{ab}V^a V^b + \beta\tilde{B}_{ab}B^{ac}B_c{}^b - i\bar{\Psi}\gamma_a\eta V^a + \bar{\Psi}\gamma_{ab}\left(\frac{1}{2e}\Psi B^{ab} + \eta\tilde{B}^{ab} - \epsilon\eta B^{ab}\right), \tag{3.3.52}$$

¶The existence of a subalgebra with $D\eta = \frac{i}{2}\gamma_a\Psi V^a$ was already discussed in the Poincaré case in Ref. [73] and also mentioned in the present work.

where we remind that $\beta = -2\epsilon$, or in the even simpler one:

$$A^{(3)} = \tilde{B}_{ab}V^aV^b + \beta\tilde{B}_{ab}B^{ac}B_c^{\ b} - i\bar{\Psi}\gamma_a\eta V^a + \bar{\Psi}\gamma_{ab}\left[\left(\frac{1}{2e}\Psi - \epsilon\eta\right)B^{ab} + \eta\tilde{B}^{ab}\right], \tag{3.3.53}$$

which shows us that the parametrization we have considered in the present work is given in terms of 1-forms structures that are pretty similar to the ones appearing in the ("standard") parametrization of $A^{(3)}$ adopted in the Minkowski $D = 11$ case in [22], and later in [73].

Part II

Solutions on Extended Supergravities

Chapter 4

Extended Supergravities Ungauged

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected.

P.A.M. Dirac in the famous paper of Quantised Singularities in the Electromagnetic Field, 1931 [95]

An ungauged theory has a lot of drawbacks but they are really interesting. First of all they are plagued, at the classical level, by the presence of massless scalar fields, some of which are related to the moduli of the internal manifold ($\mathscr{M} = \mathscr{M}_D \times \mathscr{M}_{\text{int}}$), which describe fluctuations in its shape and size. Massless scalars coupled to gravity would produce effects which are not observed in our universe. Moreover such fields enter interaction terms in the Lagrangian, thus spoiling the predictiveness of the theory, being their v.e.v. not fixed by any dynamics [34]. On the other side, the classical theory features no scalar potential which could fix the values of these scalars on the vacuum. Aside from this serious shortcoming, ungauged supergravities are interesting. First of all they provide the general framework from which to construct more realistic models through the *gauging procedure*, and secondly these theories feature, at the classical level, a rich structure of global symmetries which were conjectured [35] to encode the known string/M-theory dualities.

Here we present the most general bosonic Lagrangian wich describe stationary solutions in an extended, ungauged $D = 4$ supergravity and then we apply the so-called solution-generating-technique, over stationary solutions, which only work in ungauged theories. Neverless, an ungauged gravity theory has the property of only admit a supersimmetric Minkowski vaccum, i.e. asymptotically flat. Then, a simple case as the AdS metric looks good but it is not possible since has a vector potencial [36]. However, we perform the mentioned procedure over the AdS metric to start the research in this context.

4.1 Ungauged Supergravity

Ungauged supergravity correspond to a supergravity model in which the vector fields are not monimally coupled to any other field in the theory. Vector fields transform under an abelian group and there are no charged fields. Moreover, the only admitted vacuum of the theory is a supersymmetric Minkowski vacuum. Than means, we can consider any solutions which are (locally) asymptotically flat.

We will study the bosonic sector of the theory, the total structure being to a large extent determined by supersymmetry. The bosonic sector consists of the graviton field, n_v vector fields and n_s scalar fields. The possible couplings are constrained by the request of supersymmetry and diffeomorphism invariance. The scalar fields are described by a non-linear σ-model, that is they are coordinates of a non-compact target space $\mathscr{M}_{scal}$, a Riemannian differentiable manifold. The σ-model action turns out to be invariant under the action of global isometries of the scalar manifold, i.e. the isometry group of the manifold is a global symmetry.

In the $\mathcal{N} = 1$ case, which is the minimal supergravity, the scalar manifold $\mathscr{M}_{scal}$ scal describes the scalar fields in the chiral multiplets. Strictly speaking this is a complex manifold of *Hodge-Kähler* type, which is a particular kind of Kähler manifold [37] in which the Kähler transformations act on the fermion fields as $U(1)$-transformations, which are the $\mathcal{N} = 1$ $U(1)$ R-symmetry transformations. Consistency of such transformations on the fermion and gravitino fields (similar to that yielding the Dirac quantization of the electric charge) imposes a constraint on the geometry of the Kähler manifold. The structure of the bosonic Lagrangian is completely fixed by the following independent data: the Kähler potential $\mathcal{K}(z, \bar{z})$ associated with the manifold, a holomorphic superpotential $W(z)$, andof the matrices $\mathcal{I}_{\Lambda\Sigma}$, $\mathcal{R}_{\Lambda\Sigma}$, defining the vector kinetic part and which are constrained by supersymmetry to be holomorphic functions of the complex scalar fields: $\mathcal{I}_{\Lambda\Sigma}(z)$, $\mathcal{R}_{\Lambda\Sigma}(z)$. The σ-model action reads [34]:

$$\mathcal{L}_{scal} = eG_{\alpha\bar{\beta}}(z, \bar{z})\partial_\mu z^\alpha \partial^\nu z^{\bar{\beta}}. \tag{4.1.1}$$

If the theory is gauged, that is a subgroup of the isometry group G of the scalar manifold is promoted to local internal symmetry, additional terms, as mentioned above, appear in the supersymmetry transformation laws and in the Lagrangian, which also affect the scalar potential (through additional *D-terms*). For the sake of completeness we write the most general $\mathcal{N} = 1$ potential [34]:

$$V(z, \bar{z}) = e^{\mathscr{G}} \left(G^{\alpha\bar{\beta}} \frac{\partial}{\partial z^\alpha} \mathscr{G} \frac{\partial}{\partial \bar{z}^{\bar{\beta}}} \mathscr{G} - 3 \right) + \frac{1}{4} \mathcal{I}^{-1\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma, \tag{4.1.2}$$

where $\mathscr{G} = \mathcal{K}(z, \bar{z}) + \log(|W(z, \bar{z})|^2)$ and $\mathcal{P}_\Lambda(z, \bar{z})$ are real quantities depending on the choice of the gauged isometries. They are the moment maps associated with these isometries, in terms of which the holomorphic Killing vectors $k^a_\Lambda(z)$ are expressed as follows: $k^a_\Lambda(z) = iG^{\alpha\bar{\beta}} \frac{\partial}{\partial \bar{z}} \mathcal{P}_\Lambda$.

In extended supergravities, i.e. $\mathcal{N} > 1$, multiplets start becoming large enough as to accommodate both scalar and vector fields. This feature has important implications on the mathematical structure of the models, since in addition to put in the same level to different entities on the theory (as supersymmetry does with bosons and fermions) poses strong constraints on the (non-minimal) scalar-vector couplings in the Lagrangian. Given the scalar manifold

$\mathscr{M}_{scal}$, supersymmetry fixes the couplings, up to a choice of the frame related to the geometric structure of the scalar manifold. Moreover, with each point on the manifold a symmetric symplectic matrix $\mathcal{M}(\phi)$ is associated, and with each isometry transformation on the same manifold is associated a corresponding constant symplectic matrix $\mathbf{S}^M_N(g)$. Finally, global isometry transformations on the scalar fields induce, by supersymmetry, global transformations on the vector fields that act as electricmagnetic transformations on the vector field strengths and their magnetic duals, defining the on-shell global symmetries of the theory. This corresponds to the so-called dual symmetry.

4.2 Extended Supergravities

Extended supergravity theories $\mathcal{N} > 1$ describing the supergravity multiplet, consisting of the graviton and of $\mathcal{N}$ gravitino fields Ψ_i where $i = 1 \ldots, \mathcal{N}$, coupled to a number of vector and matter multiplets. The consistent definition of a number $\mathcal{N}$ of massless gravitino fields on a curved space-time requires, for each of them, the decoupling of the spin-1/2 longitudinal modes, which in turn follows from the invariance of the theory under a transformation of the form

$$\Psi^i_\mu \longrightarrow \Psi^i_\mu + \partial_\mu \epsilon^i \tag{4.2.1}$$

that is under $\mathcal{N}$-independent supersymmetries. Thus a consistent theory containing $\mathcal{N}$ massless gravitinos is an $\mathcal{N}$-extended supergravity.

Supersymmetry constrains the form of the Lagrangian, i.e. the structure of its kinetic terms, mass terms, couplings and scalar potential. The larger the amount $\mathcal{N}$ of supersymmetry, the more stringent these constraints. The theory is characterized by a bosonic sector and a fermionic one. Once the former is given, the latter is completely fixed by supersymmetry. Here are some general common features of the bosonic sector of a supergravity Lagrangian.

We consider the most general Lagrangian for bosonics with

- n_s scalar fields $\phi^s(x)$,
- n_v vector fields A^Λ_μ $(\Lambda = 1, \ldots, n_v)$
- and the graviton field $V_\mu{}^a$.

Let us consider the simpler case of an *ungauged* supergravity in four dimensions, namely of a supergravity model in which the vector fields are not minimally coupled to any other field. The general form of the supergravity action describing the only bosonic sector is:

$$\frac{1}{e}\mathcal{L} = \frac{R}{2} - \frac{1}{2}G_{rs}(\phi)\partial_\mu\phi^r\partial^\mu\phi^s + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\mu\nu\sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma} \tag{4.2.2}$$

where

$$e = e_4 = \sqrt{g_{\mu\nu}}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The scalar field $\phi^s(x^\mu)$ is described by a *σ-model*, with coordinates in the target space $\mathscr{M}_{scal}$ with the positive definite metric $G_{rs}(\phi)$. This maniforld is non-compact*, Riemannian, differentiable n_s dimensional. The σ-model action is clearly invariant under the action of global space-time independent isometries of the scalar manifold. Indeed, if G is the isometry group of $\mathscr{M}_{scal}$, a generic element of it will map the scalar fields $\phi = (\phi^s)$ in new ones $\phi'^s = g \star \phi^s = \phi'^s(\phi`t)$, wich in general are non-linear functions

$$\forall g \in G : \longrightarrow g \star \phi = \phi'(\phi) : \quad G_{s't'}(\phi'(\phi)) \frac{\partial \phi'^{s'z}}{\partial \phi^s} \frac{\partial \phi'^{t'}}{\partial \phi^t} = G_{st}(\phi). \tag{4.2.3}$$

The group G can be promoted to a *on-sheel global symmetry group*, that means a symmetry of the field equations and Bianchi identities, provided its (non-linear) action on the scalar fields (5.2.3) is combined with an electric-magnetic duality transformation on the vector field strengths and their magnetic duals, obtaining a complete action σ-model type on-shell invariant.

4.3 Scalar Manifolds of extended supergravity

For $\mathcal{N} = 1$ the $\mathscr{M}_{scal}$ describes the scalar fields in the chiral multiplet. Is a complex manoifolds of Hodge-kähler type. In the case of $\mathcal{N} = 2$ models allow for a class of homogeneous scalar manifolds, while in all $\mathcal{N} > 2$ models supersymmetry constrains the scalar manifold to be *homogeneous symmetric manifold*.

A manifold $\mathscr{M}$ is said *homogeneous* if acts *transitively*, that means if a point in space can be achieved by another by the action of a group G^*. Then, any point p can be reached from the origin O through a (not necessarily unique) element of the isometry group G. We define the transitive action to be a *left action*† denoted by a star:

$$\forall p \in \mathscr{M}, \ \exists g_p \in G \mid p = g_p \star O \tag{4.3.1}$$

However these parameters are "redundant", or equivalent, the action of G on $\mathscr{M}$ may not be *free*, in the sense that the element g_p is not unique, because for any $p \in \mathscr{M}$ there may be a subgroup H_p of G wich leaves p invariant:

$$H_p \star p = p. \tag{4.3.2}$$

This group is called *isotropy group* of p, is a subgroup $H \subset G$ wich when applied to a point on the variety, it is remains unchanged, for that is called also

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* Semisimple if it is a direct sum of simple Lie algebras, i.e., non-abelian Lie algebras $\mathfrak{g}$ whose only ideals are $\{0\}$ and $\mathfrak{g}$ itself. Equivalent definitions are also: the Killing form, $\kappa(x, y) = tr(ad(x)ad(y))$ is non-degenerate, $\mathfrak{g}$ has no non-zero abelian ideals, $\mathfrak{g}$ has no non-zero solvable ideals. The radical (maximal solvable ideal) of $\mathfrak{g}$ is zero.

Roughly speaking, a compact space is in some sense finite and a noncompact space is not finite. The sphere is compact because you can covered only with to subcover, but the plane $\mathbb{R}^2$ it doesnt because it is not possible covered with a finite gap.

* A good example of this is the sphere, where any point (x, y, z) belonging to the sphere can be taken to another point (x', y', z') by the transitive action of an element belonging to the group $SO(3)$.

†By left action we mean that for any $g_1 g_2 \in G$ and $p \in \mathscr{M}$, we have $g_1 \star (g_2 \star p) = (g_1 g_2) \star p$

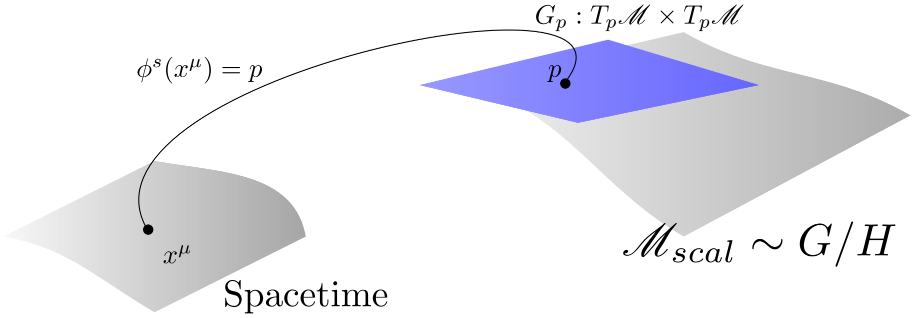

Figure 4.1: The diagram represents the nature of the variety manifold $\mathscr{M}_{scal}$ of scalar fields. The mapping G_p generates the tangent space in p where the vielbein are defined through $G(\phi) = G_{rs} d\phi^r \otimes d\phi^r$, $d\phi^r(V_i) = V_i{}^r(\phi)$. Source: self made.

stabilizer. Given a point p in $\mathscr{M}$ and an element g_p of G mapping O to p, any other element differing from g_p by the right multiplication by an element of H will still map O to p:

$$\forall g' \in G \ ; \ g' = g_p h \ (h \in H) \ : \ g' \star O = (g_p h) \star O = g_p \star (h \star O) = g_p \star O = p.$$

If we denote by $gH = \{gh \in G | h \in H\}$ the left coset of H in G, there is a one-to-one correspondence between the points of the homogeneous manifold $\mathscr{M}$ and left cosets gH:

$$p \in \mathscr{M} \leftrightarrow g_p H \subset G. \tag{4.3.3}$$

Denoting by G/H the set of all left cosets of H in G, there is therefore a bijection between $\mathscr{M}$ and G/H. This coset is basically the elements that are not present in H inside of G, that means the two can be identified:

$$\mathscr{M} \sim G/H. \tag{4.3.4}$$

G/H is called *coset manifold*. Then, homogeneous spaces can be treated as coset manifolds, that means, we can compute all geometric quantities of $\mathscr{M}$ like connection, curvature, geodesics etc... on G/H. Note also that the coset space G/H is not a group since in general H is not a *normal subgroup*[‡].

A generic element g of G is defined by a number of continuous parameters given by $\dim(G)$. Through right multiplication by an element of H, we may fix a number $\dim(G)$ of these parameters, so that each left-coset depends on a minimum number of parameters given by the diferential of two spaces. This number turns out to be the dimension of $\mathscr{M}$:

$$\dim(\mathscr{M}) = \dim(G) - \dim(H).$$

[‡]A normal subgroup is a subgroup which is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup H of a group G is normal in G if and only if $gH = Hg$ for all g in G; i.e., the sets of left and right cosets coincide. We analyze the definition of an ideal in section 2.3

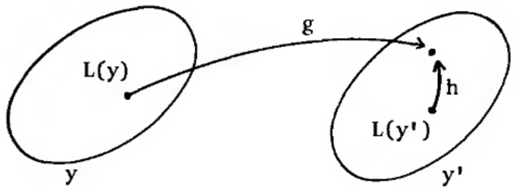

Figure 4.2: This diagram shows the relationship between two coordinate representations on the same point. Image from the book of *Castellani Leonardo, D'Auria Riccardo. (1991). Supergravity and Superstrings.*

Lets denote like ϕ^s the parameters of $\mathscr{M}$, wich is a D-plet $\phi = (\phi^1, \ldots, \phi^D)$, obtained upon fixing the right-action of H. We can choose a representative group element denoted by $\mathbb{L}(\phi^s) \in G$. Under left multiplication by $g \in G$. We therefore describe each point of $\mathscr{M}$ in terms of a *coset representative* $\mathbb{L}(\phi^s)$:

$$p \in \mathscr{M} \quad \leftrightarrow \quad \mathbb{L}(\phi^s) \in g_p H \subset G. \tag{4.3.5}$$

They provide a parametrization of $\mathscr{M}$ and depend on how this fixing is performed, namely which representative $\mathbb{L}(\phi^s)$ of each coset $g_p H$ is taken to represent the corresponding point p of $\mathscr{M}$.

Let $g \in G$ be an isometry of the manifold, p a point of coordinates $\phi = \phi^s$ and $p' = g \star p$ the transformed of p through g, of coordinates $\phi^s = g \star \phi = \phi'^s(\phi^t)$. Now, since both $g\mathbb{L}(\phi)$ and $\mathbb{L}(g \star \phi)$ represent the same point p', they must belong to the same left-coset (see figure 4.2) so that we can write:

$$g\mathbb{L}(\phi) = \mathbb{L}(g \star \phi) h(\phi, g). \tag{4.3.6}$$

The element $h(g, \phi)$ is called *compensator* and in general depends on g and on the point p of coordinates ϕ.

In general G need not be a semisimple Lie group. Homogeneous manifolds occurring in supergravity theories are non-compact, simply-connected, negative-curvature spaces. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of the groups G and H, respectively. We can split the former as follows:

$$\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{K}. \tag{4.3.7}$$

Where $\mathfrak{H}$ is a Subalgebra,

$$[\mathfrak{H}, \mathfrak{H}] \subseteqq \mathfrak{H}, \tag{4.3.8}$$

$$[\mathfrak{H}, \mathfrak{K}] \subseteqq \mathfrak{K}, \tag{4.3.9}$$

$$[\mathfrak{K}, \mathfrak{K}] \subseteqq \mathfrak{H} \oplus \mathfrak{K}, \quad \text{If it is only} = \mathfrak{H}, \text{ its called } \textit{Symmetric}. \tag{4.3.10}$$

The above adjoint action of $\mathfrak{H}$ on $\mathfrak{K}$ defines a *representation* of the group H. Indeed, if $\{H_r\}$ denotes a basis of $\mathfrak{H}$ and $\{K_s\}$ a basis of $\mathfrak{K}$, the Jacobi identities give:

$$\left[H_u, K_{\underline{s}}\right] = C_{u\underline{s}}{}^{\underline{t}} K_{\underline{t}} = -\left(H_u\right)_{\underline{s}}{}^{\underline{t}} K_{\underline{t}} \tag{4.3.11}$$

where the structure constant $C_{u\underline{s}}{}^{t} = C_{u\underline{s}}{}^{t} = (H_u)_{\underline{s}}^{\underline{t}}$ define an adjoint representation of H, denoted by $\mathscr{R}_K$ of the generators (H_u). Considering $h \in H$ we have

$$h^{-1} K_{\underline{s}} h = h_{\underline{s}}{}^{\underline{t}} K_{\underline{t}}, \tag{4.3.12}$$

where $h_{\underline{s}}{}^{t}$ represent the element h in the (adjoint) representation $\mathscr{R}_K$. $\mathfrak{K}$ Can be viewed as tha *tangent space* to G/H a the origin.

When the space is symmetric (4.3.10) is defined in general as a space invariant under parallel translations with curvature covariantly constant. Symmetric, simply-connected spaces are also homogeneous.

4.3.1 Cartan decomposition

For non-compact, simply-connected symmetric spaces with negative curvature there exists a transitive *semisimple*, *non-compact* isometry group G and H is its *maximal compact subgroup*. In any given matrix representation of G, one can choose a basis in which H is represented by anti-hermitian matrices and $\mathfrak{K}$ by hermitian ones. This basis is called the *Cartan basis*.

$$H \in \mathfrak{H} \longrightarrow H^\dagger = -\, H, \tag{4.3.13}$$

$$K \in \mathfrak{K} \longrightarrow K^\dagger = K. \tag{4.3.14}$$

Properties (4.3.9) and (4.3.10) clearly follow from commutation rules, since the commutator of an anti-hermitian with a hermitian generator is hermitian, while that of two hermitian generators is anti-hermitian. In the corresponding basis $T_A = \{H_r, K_s\}$ of generators of $\mathfrak{g}$, condition (4.3.10) reads

$$\left[K_{\underline{s}}, K_{\underline{t}}\right] = C_{\underline{st}}{}^{u} H_u. \tag{4.3.15}$$

In this basis a coset representative $\mathbb{L}(\phi)$ is well defined: if $\{K_s\}$ denote a basis of $\mathfrak{K}$ of hermitian matrices we can write:

$$\mathbb{L}(\phi) = e^{\phi^s K_s}. \tag{4.3.16}$$

This parametrization is called *Cartan parametrization* [55]. It is defined in terms of the coordinates ϕ^s, that transform under H (isotropy group of the origin $\phi_0^s = 0$) in a linear way, namely in the representation defined by the adjoint action of H on the space $\mathfrak{K}$. To see this we consider (4.3.6) and use h instead of g together with equations (4.3.12) and (4.3.16),

$$\begin{aligned} \mathbb{L}(h \star \phi) =& h\mathbb{L}(\phi)h^{-1} = h e^{\phi^s K_s} h^{-1} = e^{\phi^s h K_s h^{-1}}, \\ =& e^{\phi^s}(h_s^t K_t) = e^{\phi'^t K_t}, \\ \Longrightarrow& \ \phi'^s = (h \star \phi)^s = \phi^t (h^{-1})_t{}^s. \end{aligned} \tag{4.3.17}$$

4.3.2 Solvable decomposition

As discussed before, in $N = 2$ admits non-homogeneous, homogeneous and homogeneous-symmetric scalar manifolds, while the scalar manifolds of $N > 2$ supergravities are only of normal type. For this, they admit a transitive *solvable Lie group* of isometries who action on $\mathscr{M}$ is free.

Locally is described by a *solvable algebra* $\mathscr{S}$

$$G_{Solv} = \exp(\mathscr{S}). \tag{4.3.18}$$

The definition for a solvable algebra is an algebra $\mathscr{S}$ who $D^k \mathscr{S} = 0$ for some $k > 0$. Where

$$D\mathfrak{g} \equiv [\mathfrak{g}, \mathfrak{g}]\,, \tag{4.3.19}$$

$$D^n\mathfrak{g} \equiv \left[D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}\right]. \tag{4.3.20}$$

In a suitable basis of a given representation, all the elements of a solvable Lie group or a solvable Lie algebra are described by upper (or lower) triangular matrices. This is very important as we see in section 5.3.

Since there is a transitive solvable group $G_{Solv}$of isometries with a free action on the homogeneous scalar manifolds $\mathscr{M}$, it is possible to choose a representative $\mathbb{L}_s(\phi_p)$ in each left coset g_pH, by suitably fixing the right-action of H, so that

$$\{\mathbb{L}_s(\phi_p)\}_{p\in\mathscr{M}} = G_S. \tag{4.3.21}$$

In other words the manifold M is isometric to a solvable Lie group.

$$\mathscr{M} \sim G_S. \tag{4.3.22}$$

once we fix on the tangent space to G_{Solv} at the origin the metric of $\mathscr{M}$ on the tangent space at the corresponding point. This description defines a parametrization called the *solvable parametrization* of $\mathscr{M}$.

Both the solvable and the Cartan parametrizations for symmetric cosets are global parametrizations of the scalar manifold. For symmetric manifolds, the solvable Lie group G_{Solv} is defined by the *Iwasawa decomposition* [55] of the non-compact semisimple group G_{ss} with respect to H, according to which there is a unique decomposition of a generic element g of G_{ss} as the product of an element s of G_{Solv} and an element h of H:

$$\forall g \in G_{ss} \quad \Longrightarrow \quad g = sh\ |s \in G_{ss}, \quad h \in H. \tag{4.3.23}$$

and this defines a unique coset representative $\mathbb{L}_s$ for each point of the manifold $\mathscr{M}$!.

The solvable parametrization is useful when the four dimensional supergravity is described as resulting from the Kaluza-Klein reduction of a higher dimensional supergravity on some internal compact manifold. The solvable coordinates directly describe dimensionally reduced fields and moreover this parametrization makes the shift symmetries of the metric manifest. The drawback of such description is that $\mathscr{S}$ does not define the carrier of a representation of $\mathfrak{K}$ as $\mathfrak{H}$ does, namely Eq. (4.3.9) does not hold for $\mathscr{S}$:

$$[\mathfrak{H}, \mathscr{S}] \nsubseteq \mathscr{S}. \tag{4.3.24}$$

In what follows we shall restrict ourselves to symmetric cosets of which we can give a description either in terms of Cartan coordinates or of solvable coordinates.

4.4 Vielbein and connection on symmetric cosets

Let us see how vielbein and connection can be defined on a symmetric coset. Let $\mathbb{L}(\phi)$ be a coset representative corresponding to a generic parametrization. We can construct the left-invariant one form $\Omega = \mathbb{L}^{-1}d\mathbb{L}$ which is a 1-form on G/H with value in $\mathfrak{g}$. Expanding in the Cartan basis we have

$$\{T_A\} = \{P_{\underline{s}}, H_u\}.$$

We consider the invariant 1-form;

$$\Omega(\phi) = \sigma^A(\phi)T_A = \mathbb{L}^{-1}(\phi)d\mathbb{L}(\phi) = V^{\underline{s}}(\phi)P_{\underline{s}} + \omega^u(\phi)H_u = V(\phi) + \omega(\phi), \tag{4.4.1}$$

where

$$\Omega(\phi) = \Omega_s(\phi) d\phi^s, \qquad V^{\underline{t}} = V_s^{\underline{t}} d\phi^s,$$
$$V(\phi) = V^{\underline{s}}(\phi) P_{\underline{s}}, \qquad \omega(\phi) = \omega^u(\phi) H_u$$

and we use the underline indices $\underline{s}, \underline{t}, \ldots$ as *rigid indices* to label the basis $P_{\underline{s}}$ of the tangent space G/H to the group manifold defining a representation $\mathscr{P}$ of H, just like (4.3.11), and should not be confused with the *curved indices* $s, t, \ldots$ labeling the coordinates ϕ^t, i.e. the labeling fields.

To continue we need to define the τ automorphism. Let G be a non-compact real form of some compact Lie group. There is an involutive* automorphism such that

$$H = \{h \in G : \tau(h) = h\} \tag{4.4.2}$$

is the maximal compact subgroup of G and the coset space G/H is non-compact Riemannian symmetric space.

Only in the Cartan parametrization $\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{K}$ using (4.4.2) wich has

$$\tau(V) = -V, \tag{4.4.3}$$
$$\tau(\omega) = \omega, \tag{4.4.4}$$

the scalar fields carry rigid indices. We can know who these quantities transform under G if we look $\mathbb{L} = g\mathbb{L}(\phi)h^{-1}$ (see (4.3.6))

$$\begin{aligned}\Omega(g \star \phi) =& \sigma^A(\phi)\mathbb{L}^{-1}(\phi)g^{-1}d\left(g\mathbb{L}(\phi)h^{-1}\right)\\ =& h\mathbb{L}g^{-1}gd\left(\mathbb{L}(\phi)h^{-1}\right)\\ =& h\mathbb{L}^{-1}(\phi)\left(d\mathbb{L}(\phi)\right)h^{-1} + hdh^{-1}\end{aligned} \tag{4.4.5}$$

Then,

$$V^{\underline{s}}(g \star \phi) P_{\underline{s}} + \omega(g \star \phi) = hV(\phi)h^{-1} + h\omega(\phi)h^{-1} + hdh^{-1} \tag{4.4.6}$$

Since hdh^{-1} is the invariant one-form of $\mathfrak{h}$, it has value in this algebra. Then

$$V(g \star \phi) = hV(\phi)h^{-1}, \tag{4.4.7}$$
$$\omega(g \star \phi) = h\omega(\phi)h^{-1} + hdh^{-1}. \tag{4.4.8}$$

ω can be interpreted as connection for H whereas V transforms H-covariantly and both are G-invariant. So it is possible to see the clear analogy to the fielbein and spin conection.

4.4.1 The metric and current

Given any invariant scalar product $\langle \cdot\cdot \rangle$ on $\mathfrak{g}$ such that $G:\ T_p\mathscr{M} \times T_p\mathscr{M} \longrightarrow \mathbb{R}$ we can define an invariant metric on G/H by

$$\langle VV \rangle = G(\phi) = ds^2 = G_{st} d\phi^s d\phi^t. \tag{4.4.9}$$

Where G_{rs} is the metric of $\mathscr{M}$. With reference to matrix representation of G we define $\kappa_{\underline{ab}}$ as the restriction of the Cartan-Killing metric of $\mathfrak{g}$ to $\mathfrak{K}$:

$$\kappa_{\underline{st}} \equiv k\mathrm{Tr}\left(K_{\underline{s}} K_{\underline{t}}\right), \tag{4.4.10}$$

*That is, $\tau \neq 1$ and $\tau^2 = 1$.

being k is a representation-dependent normalization constant. Then,

$$\begin{aligned} ds^s =& G_{st} d\phi^s d\phi^t, \\ =& V_r^{\underline{k}} V_s^{\underline{s}} k_{\underline{st}} d\phi^s d\phi^t, \\ =& k\mathrm{Tr}\left(V_r V_s\right) d\phi^s d\phi^t, \\ =& k\mathrm{Tr}\left(V^2\right). \end{aligned} \tag{4.4.11}$$

The G-invariance of this metric immediately follows from (4.4.7) since $\forall g \in G$: $ds^s(g \star \phi) = k = k\mathrm{Tr}\left((V(g \star \phi))^2\right) = k\mathrm{Tr}\left(hV(\phi)^2 h^{-1}\right) = k\mathrm{Tr}\left(V^2\right) = ds^s$.

The σ-model Lagrangian density can be written in the following form

$$\mathcal{L}_{scal} = e\frac{k}{2}\mathrm{Tr}\left(V_r(\phi)V_s(\phi)\right)\partial_\mu \phi^r \partial^\mu \phi^t. \tag{4.4.12}$$

For other side the automorphism τ (see (4.4.2)) provides us with a canonical embedding of G/H in G,

$$\mathbb{L} \longrightarrow \mathcal{M} = \tau(\mathbb{L}^{-1})\mathbb{L}, \quad \tau(\mathcal{M}) = \mathcal{M}^{-1}, \tag{4.4.13}$$

where $\mathcal{M}$ is a H-invariant and transform coraviantly under G

$$\mathcal{M} \longrightarrow \tau(g)\mathcal{M}g^{-1}, \quad g \in G. \tag{4.4.14}$$

The current is constructed with this invariant:

$$J = \frac{1}{2}\mathcal{M}^{-1}d\mathcal{M}. \tag{4.4.15}$$

Then

$$V = \mathbb{L}^{-1}D\mathbb{L} = \frac{1}{2}\mathbb{L}\mathcal{M}^{-1}dM\mathbb{L}^{-1} \equiv \mathbb{L}J\mathbb{L}^{-1} \tag{4.4.16}$$

where $D\mathbb{L} = d\mathbb{L} - \omega\mathbb{L}$ is the covariant derivative of $\mathbb{L}$.

The line element (4.4.9) can be writed in terms of $\mathcal{M}$

$$d\phi^i d\phi^j G_{ij}(\phi) = \frac{1}{4}\langle \mathcal{M}^{-1}d\mathcal{M}, \mathcal{M}^{-1}dM\rangle, \tag{4.4.17}$$

and the σ-model field equations can be rewritten in the equivalent forms

$$D^\alpha V_\alpha = 0, \quad \text{or} \quad \nabla^\alpha J_\alpha = 0, \quad \text{or} \quad \nabla(\mathcal{M}^{-1}\partial_\alpha \mathcal{M}) = 0,$$

where $D_\alpha V_\beta = \nabla V - [\omega, V]$ As the reference [96] says $\mathcal{M}$ and P are very close analogues of the metric and moving frame, the tetrad, in general relativity. Because $\mathcal{M}$ it is enough to formulate the σ-model and this remains true if we include vector fields. Neverthless the "moving frame" P with a "triangular" gauge choice yields a very convenient and simple parametrization of G/H. The situation changes if we add fermions fieldsto some models like in $\mathcal{N} = 1$ supergravity, because these fermion fields transform with some representation of H and not of G. The moving frame P is, therefore, necessary in order to describe these fermions.

4.4.2 On-shell Duality

As we shall see below, the G group can be promoted to a global symmetry group of field equations and Bianchi identities (i.e. global symmetry group on-shell) provided it has (non-linear) The action in the scalar fields of the previous equation is combined with a transformation of duality and magnetic electricity in the field intensities and their dual magnetic.

Consider an extended ungauged supergravity theory with homogeneous symmetric scalar manifold described in terms of the bosonic Lagrangian. Let us define the *dual field strengths*

$$G_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} \, {}^*\left(\frac{\partial \mathcal{L}_4}{\partial F^{\Lambda}_{\rho\sigma}}\right). \tag{4.4.18}$$

$$\begin{aligned}
\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial F^{\Lambda}_{\rho\sigma}}\mathcal{L} =& - \epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}\delta^{\rho\sigma}_{\mu\nu} I_{\Lambda\Sigma}(\phi) F^{\Sigma\mu\nu} + \frac{1}{2}\mathcal{R}_{\Lambda\Sigma}(\phi)\, {}^*F^{\Lambda\mu\nu}\right)\\
=& - \epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}\right)\mathcal{I}_{\Lambda\Sigma}(\phi) F^{\Sigma\mu\nu} + \frac{1}{2}\mathcal{R}_{\Lambda\Sigma}(\phi)\, {}^*F^{\Lambda\mu\nu}\right)\\
=& - \epsilon_{\mu\nu\rho\sigma}\left(2\mathcal{I}_{\Lambda\Sigma}(\phi) F^{\mu\nu} + \frac{1}{2}R_{\Lambda\Sigma}(\phi)\, {}^*F^{\Lambda\mu\nu}\right)\\
=& - \mathcal{I}_{\Lambda\Sigma}(\phi)\, {}^*F^{\mu\nu} + \mathcal{R}_{\Lambda\Sigma}(\phi) F^{\Lambda\mu\nu}
\end{aligned} \tag{4.4.19}$$

For Maxwell theory we consider

$$\mathcal{I}_{\Lambda\Sigma} = -1, \; \mathcal{R}_{\Lambda\Sigma} = 0, \; G_{\mu\nu} = {}^*F_{\mu\nu}.$$

The equations of motion for the scalar and vector fields are

$$\mathscr{D}_{\mu}\left(\partial^{\mu}\phi^{s}\right) = \frac{1}{4}G^{st}\left[F^{\lambda}_{\mu\nu}\partial_t I_{\Lambda\Sigma}F^{\Sigma\mu\nu} + F^{\Lambda}_{\mu\nu}\partial_t R_{\Lambda\Sigma}\, {}^*F^{\Sigma\mu\nu}\right] \tag{4.4.20}$$

$$\nabla_{\mu}\left({}^*F^{\Lambda\mu\nu}\right) = 0; \quad \nabla_{\mu}\left({}^*G^{\Lambda\mu\nu}\right) = 0 \tag{4.4.21}$$

where

$$\partial_s \equiv \frac{\partial}{\partial\phi^s}$$

. We note that

$$\nabla_{\mu}(\;)^{\nu} = \partial_{\mu}(\;)^{\nu} + \Gamma^{\nu}_{\mu\rho}(\;)^{\rho}$$

is the covariant derivative containing the Levi-Civita conection on space-time, while $\mathscr{D}_{\mu}$ also contains to the Levi-Civita conection $\widetilde{\Gamma}^{s}_{t_1 t_2}$ on $\mathscr{M}_{scal}$ defined by:

$$\mathscr{D}_{\mu}(\partial_{\nu}\phi^{s}) \equiv \nabla_{\mu}(\partial_{\nu}\phi^{s}) + \widetilde{\Gamma}^{s}_{t_1 t_2}\partial_{\mu}\phi^{t_1}\partial_{\nu}\phi^{t_2}.$$

From (4.4.18) we have

$$\begin{aligned}
{}^*F^{\Lambda} =& \left(\mathcal{I}^{-1}\right)^{\Lambda\Sigma}\left(\mathcal{R}_{\Sigma\Pi}F^{\Pi} - G_{\Sigma}\right),\\
{}^*G_{\Lambda} =& (RI^{-1}\mathcal{R} + \mathcal{I})_{\Lambda\Sigma}F^{\Sigma} - (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma}G_{\Sigma}.
\end{aligned}$$

Is convenient define a $2n_s$-dimensional vector $\mathbb{F} = \mathbb{F}^{M}$ of two forms:

$$\mathbb{F} = \begin{pmatrix} F \\ G \end{pmatrix}.$$

where the dual form is easy to write

$$^*\mathbb{F} = \begin{pmatrix} ^*F \\ ^*G \end{pmatrix} = -\mathbb{C}\mathcal{M}(\phi^s)\mathbb{F}, \tag{4.4.22}$$

$$\mathbb{C} \equiv \mathbb{C}^{MN} = \mathbb{C}_{MN} = \equiv \begin{pmatrix} \mathbf{0}_{n_v} & \mathbf{1}_{n_v} \\ -\mathbf{1}_{n_v} & \mathbf{0}_{n_v} \end{pmatrix} \tag{4.4.23}$$

and where $\mathcal{M}(\phi)$ reads:

$$\mathcal{M}(\phi) = \begin{pmatrix} (\mathcal{R}\mathcal{I}^{-1}\mathcal{R} + \mathcal{I})_{\Lambda\Sigma} & -(\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Gamma} \\ -(\mathcal{I}^{-1}\mathcal{R})^{\Xi}{}_{\Sigma} & (\mathcal{I}^{-1})^{\Xi\Gamma} \end{pmatrix} \tag{4.4.24}$$

resulting in a symmetric, negative-definite matrix, function of the scalar fields. In matrix notation, the Maxwell equations can then be recast in the following equivalent forms:

$$\nabla_\mu \left(^*\mathbb{F}^{\mu\nu}\right) = 0 \;\Leftrightarrow\; \nabla_\mu \left(\mathbb{C}\mathcal{M}(\phi)\mathbb{F}\right) = 0 \;\leftrightarrow\; d\mathbb{F} = 0. \tag{4.4.25}$$

where we have used the matrix notation and suppressed the indices $M, N, \ldots$.

The field equations depending on the vector field strengths can be rewritten in terms of this matrix $\mathcal{M}$ and of its derivatives. The scalar field equations (4.4.20) can be rewritten

$$\mathscr{D}_\mu(\partial^\mu \phi^s) = \frac{1}{8} G^{st} \mathbb{F}^T_{\mu\nu} \partial_s \mathcal{M}(\phi) \mathbb{F}^{\mu\nu}.$$

The Einstein equations are:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = T^{(S)}_{\mu\nu} + T^{(V)}_{\mu\nu}. \tag{4.4.26}$$

where the energy-momentum tensors for the scalar and vector fields can be cast in the following general form

$$T^{(S)}_{\mu\nu} = G_{st}(\phi)\partial_\mu \phi^r \partial_\nu \phi^s - \frac{1}{2} g_{\mu\nu} G_{rs}(\phi) \partial_\rho \phi^r \partial^\rho \phi^s \tag{4.4.27}$$

$$T^{(V)}_{\mu\nu} = F^T_{\mu\rho} I F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} \left(F^T_{\rho\sigma} I F^{\rho\sigma}\right). \tag{4.4.28}$$

The vector fields energy-momentum tensors can be expressed en terms of $\mathcal{M}(\phi)$ and$\mathbb{F}$ as

$$T^{(V)}_{\mu\nu} = \frac{1}{2} \left(\mathbb{F}_{\mu\rho}\right)^T \mathcal{M}(\phi) \mathbb{F}_\nu{}^\rho. \tag{4.4.29}$$

Since in (4.4.26) we have

$$R = G_{st} \partial_\rho \phi^s \partial^\rho \phi^t \tag{4.4.30}$$

the Einstein equation can be finally recast in the form:

$$R_{\mu\nu} = G_{st}(\phi) \partial_\mu \phi^s \partial^\rho \phi^t. \tag{4.4.31}$$

Returning to action (4.2.2), the isometry group G is a global symmetry only of the scalar kinetic term, since, in general, it alters the action for the vector fields as a consequence of the scalar field-dependence of the coupling constants $\mathcal{I}(\phi)$

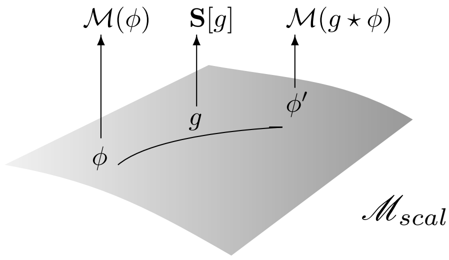

Figure 4.3: The scalar manifold of a symplectic geometric structure is associate with each point ϕ on the manifold the symmetric symplectic $2n_v \times 2n_v$ matrix $\mathcal{M}(\phi)_{MN}$, at the same time each isometry transformation $g \in G$ on the same manifold is associated with a $2nv \times 2nv$ matrix $\mathbf{S}[g]_N^{\ M}$. Source: self made.

and $\mathcal{R}(\phi)$. For other side, the while the Maxwell equations $\bigtriangledown_\mu({}^*\mathbb{F}^{\Lambda\mu\nu}) = 0$ are invariant under linear transformation in $\mathbb{F}$, $\mathcal{G}$ do not, and the equations (4.4.22).

One remarkable feature of extended supergravity theories is the fact that the global invariance of the scalar kinetic term (described by $\mathcal{G}$) can be extended to a global symmetry of the full set of equations of motion and Bianchi identities, though not in general of the whole action [97]. This is possible because, in extended supergravities, supersymmetry connects scalar with vector fields. One of the consequences, is that transformations on the scalars imply transformations on the vector field strengths F^Λ and their duals G^Λ: this follows from the definition on the scalar manifold of a symplectic geometric structure which associates with each point ϕ on the manifold the symmetric symplectic $2n_v \times 2n_v$ matrix $\mathcal{M}(\phi)_{MN}$. At the same time, each isometry transformation $g \in G$ on the same manifold is associated with a constant symplectic $2n_v \times 2n_v$ matrix $\mathbf{S}[g] \equiv \mathbf{S}[g]_N^{\ M}$ such that:

$$\mathcal{M}(g \star \phi) = \mathbf{S}[g]^{-T}\mathcal{M}(\phi)\mathbf{S}[g]^{-1}. \tag{4.4.32}$$

Recall that a symplectic matrix A is determined by $A \in Sp(2n_V, \mathbb{R})$, $A^T\mathbb{C}A = A\mathbb{C}A^T = \mathbb{C}$, where

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0}, \end{pmatrix}$$

where the symplectic invariant matrix is defined in (4.4.23). The symmetric matrix $M(\phi)$ plays the role of A and satisfies the properties

$$\mathcal{M}(\phi)_{MN}\mathbb{C}^{PL}\mathcal{M}(\phi)_{LN} = \mathbb{C} \implies \mathcal{C}(\phi)^{-1} = -\mathbb{C}\mathcal{M}(\phi)\mathbb{C}, \tag{4.4.33}$$

The correspondence between $g \in G$ and $\mathbf{S}[g]$ defines a symplectic representation of the group G, that is an embedding S of the group G inside $Sp(2n_v, \mathbb{R})$:

$$G \hookrightarrow \mathrm{Sp}(2n_v, \mathbb{R}) \implies g \in G \longrightarrow \mathbf{S}[g] \in \mathrm{Sp}(2n_v, \mathbb{R}) \tag{4.4.34}$$

with the general properties defining a representation and a symplectic matrix.

$$\mathcal{M}(\phi) \longrightarrow \mathcal{M}'(\phi) = E^T\mathcal{M}(\phi)E. \tag{4.4.35}$$

This suggests that the definition of the matrix $\mathcal{M}(\phi)_{Mn}$ is built-in in the mathematical structure of the scalar manifold. The matrices $\mathcal{I}(\phi)$ and $\mathcal{R}(\phi)$ entering the action are then defined in terms of $\mathcal{M}(\phi)$. The only freedom left consists in the choice of the basis of the symplectic representation (*symplectic frame*), which amounts to a change in the definition of $\mathcal{M}(\phi)$ by a constant symplectic transformation E:

$$\mathcal{M}(\phi) \longrightarrow \mathcal{M}'(\phi) = E^T \mathcal{M}(\phi) E. \tag{4.4.36}$$

The form of the action is affected by the above transformation, in particular we have a change in the coupling of the scalar fields to the vectors. At the ungauged level, this only amounts to a (non-perturbative) redefinition of the vector field strengths and their duals, having no physical implication. In the presence of a gauged theory, where vectors are minimally coupled to the other fields, the symplectic frame becomes physically relevant and may lead to different vacuum-structures defined by the scalar potential.

We emphasize here that the existence of this symplectic structure on the scalar manifold is a general feature of all extended supergravites.

For homogeneous manifolds, the isometry group G has a symplectic, $2n_v$-dimensional representation $\mathbf{S}$ and we can express $\mathcal{M}(\phi)$ in terms of the coset representative:

$$\mathcal{M}(\phi)_{MN} = \mathbb{C}_{MP}\mathbb{L}^P{}_L\mathbb{L}^R{}_L\mathbb{L}_{RN} \Longleftrightarrow \mathcal{M}(\phi) = \mathbb{C}\mathbf{S}[\mathbb{L}(\phi)]\mathbf{S}[\mathbb{L}(\phi)]^T\mathbb{C}, \tag{4.4.37}$$

$\mathbb{L}^P_L$ are the entries of the symplectic matrix $\mathbf{S}[L(\phi)]$ associated with $L(\phi)$ as an element of G. Being a representation, $\mathbf{S}$ is a homomorphism, and eq. (4.3.6) can be written in terms of symplectic matrices as:

$$\mathbf{S}[g]\mathbf{S}[\mathbb{L}(\phi)] = \mathbf{S}[g \star \phi]\mathbf{S}[h(g,\phi)]$$

Using these properties is possible to see that

$$\mathcal{M}(g \star \phi) = \mathbf{S}[g]^{-T}\mathcal{M}(\phi)\mathbf{S}[g]^{-1}.$$

This tells us that $\mathcal{M}$ is H-invariant, that means, do not depends of the representative coset, but only in the point ϕ of the manifold, as it should be. In fact:

$$\begin{aligned}\mathcal{M}(\phi) \xrightarrow{\mathbb{L}\longrightarrow\mathbb{L}\cdot h} (\mathcal{M}(\phi))' =&\mathbb{C}\mathbf{S}[\mathbb{L}(\phi)\cdot h]\mathbf{S}[h\cdot\mathbb{L}(\phi)]^T\mathbb{C},\\ =&\mathbb{C}\mathbf{S}[\mathbb{L}(\phi)]\mathbf{S}[h]\mathbf{S}[h]^T\mathbf{S}[\mathbb{L}(\phi)]^T\\ =&\mathbb{C}\mathbf{S}[\mathbb{L}(\phi)]\mathbf{S}[h]\mathbf{S}[h]^{-1}\mathbf{S}[\mathbb{L}(\phi)]^T\\ =&\mathcal{M}(\phi).\end{aligned} \tag{4.4.38}$$

namely the matrix $\mathcal{M}(\phi)$ is H-invariant and depends only on the choice of ϕ, i.e. it is consistently defined on G/H.

We can now study the simultaneous action of G on the scalar fields and on the field strength vector $\mathbb{F}_{\mu\nu}$:

$$g \in G : \begin{cases} \phi^r & \longrightarrow & g\star\phi^r \\ \mathbb{F}_{\mu\nu} & \longrightarrow & \mathbf{S}[g]^M{}_N\mathbb{F}^N_{\mu\nu} \end{cases} \tag{4.4.39}$$

is a symmetry of the field equations. But it must also leave invariant the transformed fields, to see this we consider

$$\begin{aligned}\mathbf{S}[g]^{-1}\ {}^*\mathbb{F}' =& -\mathbb{C}\mathbf{S}[g]^T\mathcal{M}(g\star\phi)\mathbf{S}[g]\mathbf{S}[g]^{-1}\mathbb{F}' = -\mathbf{S}[g]^{-1}\mathbb{C}\mathcal{M}(g\star\phi)\mathbb{F}',\\ \Longrightarrow\ {}^*\mathbb{F} =& -\mathbb{C}\mathcal{M}(g\star\phi)\mathbb{F}'.\end{aligned} \tag{4.4.40}$$

Then, is invariant.

On the other side, the invariance of the scalar and Einstein equations can be visualized if we can see the invariance in the term that they have in common:

$$\left(\mathbb{F}'_{\mu\nu}\right)^T \mathcal{M}'(\phi)\mathbb{F}'_{\rho\rho} = \mathbb{F}^{-T}_{\mu\nu}\mathbf{S}[g]^T\mathbf{S}[g]^{-T}\mathcal{M}(g\star\phi)\mathbf{S}[g]^{-1}\mathbf{S}[g]\mathbb{F}_{\rho\sigma}, \tag{4.4.41}$$

$$= \left(\mathbb{F}_{\mu\nu}\right)^T \mathcal{M}(\phi)\mathbb{F}_{\rho\rho}. \tag{4.4.42}$$

and directly implies the invariance of $T^{(V)}_{\mu\nu}$ and the covariance of the scalar field equation. Moreover, the duality invariance of the space-time metric and of the scalar action under (4.4.39) imply the same property for the Einstein tensor and for the scalar energy-momentum tensor $T^{(}_{\mu\nu}S)$.

In summary, the bosonic equations derived in the above discussion can be written in the manifestly G-invariant form:

$$\textbf{Scalar equations}: \quad \mathscr{D}_\mu\left(\partial^\mu\phi^s\right) = \frac{1}{8}G^{st}\mathbb{F}^T_{\mu\nu}\partial_s\mathcal{M}(\phi)\mathbb{F}^{\mu\nu} \tag{4.4.43}$$

$$\textbf{Einstein equations}: \quad \mathcal{R}_{\mu\nu} = G_{st}(\phi)\partial_\mu\phi^s\partial_\nu\phi^t + \frac{1}{2}\mathbb{F}^T_{\mu\rho}\mathcal{M}(\phi)\mathbb{F}_\nu{}^\rho \tag{4.4.44}$$

$$\textbf{Maxwell equations}: \quad d\mathbb{F} = 0 \iff {}^*\mathbb{F} = -\mathbb{C}\mathcal{M}(\phi^s)\mathbb{F} \tag{4.4.45}$$

where we have obviously omitted the fermionic fields.

The action of G on the field strengths and their magnetic duals is defined by the symplectic embedding $\mathbf{S}[g]$, and can be seen as a *generalized electric-magnetic duality* transformation (see [98]) which promotes the isometry group of the scalar manifold to a global symmetry group of the field equations and Bianchi identities. It is a generalization of the $U(1)$-duality invariance of the standard Maxwell theory.

$$\begin{pmatrix} F_{\mu\rho} \\ {}^*F_{\mu\rho} \end{pmatrix} \xrightarrow{U(1)} = \begin{pmatrix} F'_{\mu\rho} \\ {}^*F'_{\mu\rho} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} F_{\mu\rho} \\ {}^*F_{\mu\rho} \end{pmatrix} \tag{4.4.46}$$

For this reason G is called the *duality group* of the classical theory. In our case

$$\begin{pmatrix} F^\lambda \\ \mathcal{G}\lambda \end{pmatrix} \xrightarrow{\mathbf{S}[g]} = \begin{pmatrix} F^\lambda \\ \mathcal{G}_\lambda \end{pmatrix} = \begin{pmatrix} A^\lambda{}_\Sigma & A^{\Lambda\Sigma} \\ C_{\Lambda\Sigma} & D_\lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} F^\lambda \\ \mathcal{G}_\lambda \end{pmatrix} \tag{4.4.47}$$

We emphasize that these are not a symmetry of the action but on-shell symmetry, because correspond to a symmetry on field equations and Bianchi identities.

On a charged dyonic solution, we define the electric and magnetic charges as the integrals:

$$e_\Lambda \equiv \frac{1}{4\pi}\int_{S^2}\mathcal{G}_\Lambda = \frac{1}{8\pi}\int_{S^2}G_{\Lambda\mu\nu}dx^\mu\wedge dx^\nu,$$

$$m^\Lambda \equiv \frac{1}{4\pi}\int_{S^2}F^\lambda = \frac{1}{8\pi}\int_{S^2}F^\Lambda_{\mu\nu}dx^\mu\wedge dx^\nu.$$

we define the symplectic vector

$$\Gamma = \Gamma^M = \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} = \frac{1}{4\pi}\int_{S^2}\mathbb{F}. \tag{4.4.48}$$

These are the quantized charges, namely they satisfy the Dirac-Schwinger-Zwanziger quantization condition for dyonic particles (see Apendix (C)) [95, 99, 100]:

$$\Gamma_2^T\mathbb{C}\Gamma_1 = m_2e_{1\Lambda} - m_1e_{2\Lambda} = 2\pi\hbar cn, \quad n\in\mathbb{Z}. \tag{4.4.49}$$

Fermions

Let us discuss the general features of the fermionic sector, its symmetries and the couplings of the fermions fields to the bosons. We have seen that the vector fields and the scalar fields transform under the action of the group G, isometry group of the scalar manifold. More precisely this group has a global symplectic (duality) action on the vector of electric field strengths and their magnetic duals, while it acts on the scalar fields as an isometry group. Just as the fermion fields (including the graviton), transform covariantly with respect to the isotropy group of space-time (local Lorentz transformations), they have a well defined transformation property only with respect to the isotropy group H of the scalar manifold. In all extended supergravities this group has the following form [101]:

$$H = G_R \times H_{matter}, \tag{4.4.50}$$

where G_R is the automorphism of the supersymmetry algebra (the R-symmetry group), while H matter is a compact Lie group acting on the matter multiplets.

The coupling of the bosons to the fermionic fields is also fixed by the geometry of $\mathscr{M}_{scal}$. In particular, in the models with an homogeneous scalar manifold, this coupling is fixed by the coset representative $\mathbb{L}(\phi)$.

Let us recall that (4.3.6) states that the matrix $\mathbb{L}(\phi)$ is acted to the left by G and to the right by the compensator element in H

$$G \rightsquigarrow \mathbb{L}(\phi) \leftsquigarrow H. \tag{4.4.51}$$

The matrix $\mathbb{L}(\phi)$ therefore "mediates" between objects, like the bosonic fields, transforming directly under G and other objects, like the fermionic fields, transforming only under H.

This means that we can construct G-invariant terms by contracting $\mathbb{L}$ to the left by bosons (scalars, vectors and their derivatives), and to the right by fermion bilinears

This means that we can construct G-invariant quantities coupling (in suitable ways) the bosonic fields (including their derivatives) to the fermions through $\mathbb{L}(\phi)$, that is, symbolically, considering the contraction

$$(\partial \mathbf{b}) \cdot \mathbb{L}(\phi) \cdot \mathbf{f} = \mathbf{d}(\phi, \partial \mathbf{b}) \cdot \mathbf{f}. \tag{4.4.52}$$

This scalar-dependent matrix determines the coupling of bosons and fermions in the Lagrangian and in the equations of motion. The fermions, in other words, couple to composite objects - that we denoted $\mathbf{d}(\phi, \partial \mathbf{b})$ - obtained by "dressing" the derivatives of bosonic fields by scalar fields through the matrix $\mathbb{L}(\phi)$ [34]. Then, these objects transform only through the corresponding compensating transformations $h(\phi, g) \in H$, as the scalars and vectors transform under G (see (4.3.6)). This tell us that the trasformations of all fermion fields is obtained by means of $h(\phi, g)$, namely we can define the action of G over all the fields of the theory as:

$$g \in G : \left\{ \begin{array}{ccc} \phi^r & \longrightarrow & g \star \phi^r \\ \mathbb{F}_{\mu\nu} & \longrightarrow & \mathbf{S}[g]^M{}_N \mathbb{F}^N_{\mu\nu} \\ \mathbf{f} & \longrightarrow & \mathbf{f} = h(\phi, g) \star \mathbf{f} \end{array} \right. \tag{4.4.53}$$

Now one can construct a manifestly H-invariant Lagrangian using the fermion fields and the composite fields $\mathbf{d}(\phi, \partial b)$. Moreover, H-covariance of the supersymmetry transformations mentioned in the introduction implies that the supersymmetry variations for the fermion fields can be written as:

$$\delta_\epsilon \mathbf{f} = (\phi, \partial \mathbf{b})\epsilon. \tag{4.4.54}$$

The fields transforming in representations of G_R are therefore either the fermions or the composite fields $\mathbf{d}(\phi, \partial\mathbf{b})$, but not the scalar fields ϕ^s and the vector fields A_μ^Λ directly, since the latter are always real fields. The composite objects $\mathbf{d}(\phi, \partial\mathbf{b})$ can be imagined as the actual bosonic fields that can be measured, at spatial infinity, on a solution.

Chapter 5

Solution-generating technique

In every atom of the universe domains there are infinite solar systems.

The adornment of the Great Flower, ancient Buddhist writing

5.1 Introduction

In this section we present the so-called *solution-generating techniques* [102,103]. Basically what is done is that solutions in $D = 4$ supergravity are also solutions to an *Euclidean theory in three dimensions*, formally obtained by a Kaluza-Klein type compaction of the $D = 4$ parent model along the time-direction [96] and dualizing all the vector fields into scalars fields. The resulting $D = 3$ theory is a sigma-model coupled to gravity and has the desirable feature of having a global $G_{(3)}$ symmetry group (duality group) larger than the original $G_{(4)}$ group of the four-dimensional model, i.e. $G_{(4)} \subset G_{(3)}$. The extra symmetries can be used to generate new (hidden) four-dimensional solutions from known ones. These symmetries, for instance, include the Harrison transformations which can generate electric and magnetic charges when acting on a neutral solution like the Schwarzschild or the Kerr black hole. The relevant physical properties of stationary black holes in four dimensions are thus conveniently described by the orbits of such solutions with respect to the action of the $D = 3$ global symmetry group $G_{(3)}$.

5.2 $D = 4$ Stationary Solutions

We shall restrict ourselves to axisymmetric, stationary, asymptotically flat solutions. We consider one example over a black hole solution and we compute the charge with the mentioned technique.

A stationary solutions in an extended, ungauged $D = 4$ supergravity, whose bosonic sector consists in n_s scalar fields $\phi^s(x)$, n_v vector fields A^Λ_μ ($\Lambda =$

$1, \ldots, n_v$) and the graviton $g_{\mu\nu}$. The most general Lagrangian has the form

$$\begin{aligned}\frac{1}{e}\mathcal{L}_4 =& \frac{R}{2} - \frac{1}{2}G_{rs}(\phi)\partial_\mu\phi^r\partial^\mu\phi^s \\ &+ \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\mu\nu\sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\end{aligned} \tag{5.2.1}$$

where $e = e_4 = \sqrt{g_{\mu\nu}}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The four-dimensional scalar fields ϕ^s s parametrize a homogeneous, symmetric scalar manifold of the form:

$$\mathscr{M}_{scal} = \frac{G_{(4)}}{H_{(4)}}. \tag{5.2.2}$$

This makes that the fields are described by a non-linear σ-model action, that is is clearly invariant under the action of global space-time independent isometries of the scalar manifold. Indeed, if $G_{(4)}$ is the semisimple isometry group and $H_{(4)}$ its maximal compact subgroup of $\mathscr{M}_{scal}$, a generic element of it will map the scalar fields $\phi = (\phi^s)$ in new ones, wich in general are non-linear functions $\phi'^s = \phi'^s(\phi^t)$.

$$\forall g \in G_{(4)} : \longrightarrow g \star \phi = \phi'(\phi) : \quad G_{s't'}(\phi'(\phi))\frac{\partial\phi'^{s'z}}{\partial\phi^s}\frac{\partial\phi'^{t'}}{\partial\phi^t} = G_{st}(\phi). \tag{5.2.3}$$

The group $G_{(4)}$ can be promoted to a global symmetry group of the field equations and Bianchi identities, that means an on-shell symmetry, provided its (non-linear) action on the scalar fields (5.2.3) is combined with an electric-magnetic duality transformation on the vector field strengths and their magnetic duals, obtaining a complete action σ-model type on-shell invariant.

In the case of $D = 4$, which is what we are interested, we start from a stationary metric which has the general form:

$$ds^2 = -e^{2U}\left(dt + \omega_\varphi d\varphi\right)^2 + e^{-2U}ds_3^2, \tag{5.2.4}$$

where $ds_3^2 = g^{(3)}_{ij}dx^i dx^j$, the index $i, j = 1, 2, 3$ label the spatial coordinates $x_i = (r, \theta, \phi)$ and U, ω_ϕ, g_{ij} are all functions of the coordinates (r, θ). The two Killing vectors of the above metric are $\xi = \partial_t$ and $\psi = \partial_\varphi$.

One can perform a formal reduction to three dimensions along the time direction and dualize the vector fields of the theory to scalar fields, using the prescription of the seminal work [96]. Following this procedure, one ends up with an effective description in an Euclidean $D = 3$ model describing gravity coupled to $n = 2 + n_s + 2n_v$ scalar fields $\Psi^I(r, \theta)$.

We start noting that the 4-dimensional metric can be rewrite as

$$g_{\mu\nu} = \begin{pmatrix} e^{2U} & e^{2U}\omega_i \\ e^{2U}\omega_j & e^{-2U}g^{(3)}_{ij} + e^{2U}\omega_i\omega_j \end{pmatrix}, \tag{5.2.5}$$

If we want to diagonalize this matrix for do not have cross terms we can put

$$\begin{aligned}\mathbb{V}^0 =& e^U V^0 = e^U\left(dt + \omega\right) = e^U\left(dt + \omega_i dx^i\right), \\ \mathbb{V}^i =& e^{-U}V^i_k dx^k = \mathbb{V}^i{}_k dx^k,\end{aligned}$$

where V^0 and V^i are the vielbeins, then the metric is diagonal,

$$ds^2 = \left(\mathbb{V}^0\right)^2 - \left(\mathbb{V}^i\right)\left(\mathbb{V}^k\right). \tag{5.2.6}$$

Under these coordinates its easy to see that the theory is invariant under a coordinate transformation in time, $t + \xi(x^i)$, obviously $\mathbb{V}'^i = \mathbb{V}^i$ and

$$\begin{aligned}\mathbb{V}^0 = e^U V^0 \implies \mathbb{V}'^0 =& e^U (V^0)' = e^U (dt + \omega_i dx^i)' \\ \mathbb{V}'^0 =& e^U (d(t + \xi(x^i)) + \omega'_i dx^i) \\ \mathbb{V}'^0 =& e^U (dt + \partial_i \xi dx^i + \omega_i dx^i - \partial_i \xi dx^i), \\ \mathbb{V}'^0 =& e^U (dt + \omega_i dx^i).\end{aligned}$$

To proceed in the dimensional reduction we must write the potential in a different way. Writing

$$A = A_\mu dx^\mu = A_0 dt + A_\varphi d\varphi.$$

And we consider the next quantyties

$$V^0 =(dt + \omega), \tag{5.2.7}$$

$$A_3 =A_\varphi d\varphi - A_0 \omega, \tag{5.2.8}$$

$$A =A_0 V^0 + A_3. \tag{5.2.9}$$

With the respective strengths tensors in the forms

$$F =dA = dA_0 V^0 + \tilde{F}_3, \tag{5.2.10}$$

$$\tilde{F}_3 =\mathring{F} + A_0 F^0 = \frac{1}{2}\tilde{F}_{ij} dx^i dx^j, \tag{5.2.11}$$

$$\mathring{F} =d\tilde{A}_3. \tag{5.2.12}$$

Then, we have the first kinect term in (5.2.1) as

$$\begin{aligned}\frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\hat{F}^{\Sigma}_{\mu\nu}\hat{F}^{\Lambda\mu\nu} =& \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} g^{\mu\rho} g^{\nu\sigma} \hat{F}^{\Lambda}_{\mu\nu}\hat{F}^{\Sigma}_{\rho\sigma} \\ =& \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} g^{il} g^{jk} F^{\Lambda}{}_{ij} F^{\Sigma}_{lk} + \frac{1}{2}\mathcal{I}_{\Lambda\Sigma} F^{\Lambda}{}_{0i} F^{\Sigma}{}_{0j} g^{00} g^{ij} \\ =& \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} e^{2U} e^{2U} g^{(3)il} g^{(3)jl} F^{\Lambda}_{ij} F^{\Sigma}_{lk} + \frac{1}{2}\mathcal{I}_{\Lambda\Sigma} e^{-2U} e^{2U} g^{(3)ij} F^{\Lambda}_{0i} F^{\Sigma}_{0j} \\ =& \frac{1}{4} e^{4U} \mathcal{I}_{\Lambda\Sigma} F^{\Lambda}_{ij} F^{\Sigma lk} + \frac{1}{2}(\partial_i A^{\Lambda}{}_0)\mathcal{I}_{\Lambda\Sigma}(\partial^i A^{\Sigma}_0)\end{aligned}$$

And the second kinect term as

$$\begin{aligned}\frac{1}{4}\mathcal{R}_{\Lambda\Sigma}\, F^{\Lambda}_{\mu\nu}{}^{*} F^{\Sigma\mu\nu} =& \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma}, \\ =& \frac{e_3 e^{2U}}{4}\mathcal{R}_{\Lambda\Sigma}\, F^{\Lambda}_{ij}\left(\frac{1}{2}\frac{e^{2U}}{e_3}\epsilon^{ijk} F_{k0}\right) \\ &+ \frac{1}{4}\mathcal{R}_{\Lambda\Sigma} F_{0i}\epsilon^{ijk}\left(\frac{1}{2}\frac{-e^{2U}}{e_3}\epsilon^{ijk} F_{jk}\right) \\ =& \frac{1}{8} e^{4U}\mathcal{R}_{\Lambda\Sigma}\epsilon^{ijk} F_{ij} F_{k0} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}\epsilon^{ijk} F_{0i} F_{jk}.\end{aligned}$$

Also, we can write the curvature in function of 3-dimensional quantities

$$e_4 R_4 = e_3 \left(R_3 - 2\,(\partial U)^2 + \frac{e^{4U}}{4}(F^0) \right) \tag{5.2.13}$$

being $e_4 = e_3 e^{-2U}$, $e_4 = \sqrt{|g_{\mu\nu}|}$, $e_3 = \sqrt{|g^3_{ij}|}$.

The action (now in three dimension) looks as

$$S_{(3)} = \int e_3 \left[\frac{R_3}{2} - \partial_i U \partial^i U + \frac{e^{4U}}{8} F^0 F_0 - \frac{\mathcal{G}_{rs}}{2} \partial_i \phi^r \partial^i \phi^s + \frac{e^{2U}}{4} F^T_{ij} \mathcal{I}_{\Lambda\Sigma} F^{ij} \right.$$
$$\left. - \frac{e^{-2U}}{2} \partial_i A_0^T \mathcal{I}_{\Lambda\Sigma} \partial^i A_0 + \frac{1}{2e_3} \epsilon^{ijk} F^T_{ij} \mathcal{R}_{\Lambda\Sigma} \partial_k A_0 \right] \tag{5.2.14}$$

We apply the Lagrange multipliers to keep the invariance under duality

$$\mathcal{L}'_3 = \mathcal{L}_3 + \frac{1}{2}\epsilon^{ijk} \left[-\left(\partial_i A^\Lambda_j - \partial_j A^\Lambda_i\right) \partial_k A_\Lambda + \frac{1}{2} F^0_{ij} \partial_k \tilde{a} \right] \tag{5.2.15}$$

Instead of using the definitions of F^Λ_{ij} and $\mathring{F}_{ij} = \left(\partial_i A^\Lambda_j - \partial_j A^\Lambda_i\right)$, we can treat them as independent fields

$$\frac{\delta \mathcal{L}_3}{\delta \mathring{F}_{ij}} = 0 \qquad \Longrightarrow \tilde{F}^{ij} = \frac{e^{-2U}}{e_3} \epsilon^{ijk} I^{-1}_{\Lambda\Sigma} \left(\partial_k A - R_{\Lambda\Sigma} \partial_k A_0\right), \tag{5.2.16}$$

$$\frac{\delta \mathcal{L}_3}{\delta F^0_{ij}} = 0 \qquad \Longrightarrow F^{0ij} = -\frac{e^{-4U}}{e_3} \left(\partial_k a + \mathcal{Z}^T \mathbb{C}_{MN} \partial_k \mathcal{Z}\right). \tag{5.2.17}$$

defining

$$\chi_k = \partial_k a + \mathcal{Z}^M \mathbb{C}_{MN} \partial_k \mathcal{Z}^N \tag{5.2.18}$$

$$\tilde{a} = a - A_\Lambda A^\Lambda_0 \tag{5.2.19}$$

where

$$\mathcal{Z} = \begin{pmatrix} A^\Lambda_0 \\ A_\Lambda \end{pmatrix} \tag{5.2.20}$$

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{5.2.21}$$

and replacing in the Lagrangian we obtain

$$\frac{1}{e_3}\frac{R_3}{2} - \partial_i U \partial^i U - \frac{1}{2}\mathcal{G}_{rs} \partial_i \phi^r \partial^i \phi^s - \frac{1}{2} e^{-2U} \partial_k \mathcal{Z}^T \mathcal{M} \partial^k \mathcal{Z} - \frac{1}{4} e^{-4U} \chi^k \chi_k \tag{5.2.22}$$

Then, the Lagrangian is writen in a σ-model theory type:

$$\frac{1}{e_3}\mathcal{L}_3 = \frac{1}{2} R_3 - \frac{1}{2} \hat{\mathcal{G}}_{IJ} d\Phi^I \Phi^J \tag{5.2.23}$$

where

$$\frac{1}{2}\hat{\mathcal{G}}_{IJ} d\Phi^I \Phi^J = (dU)^2 + \frac{1}{2}\mathcal{G} d\phi^r d\phi^s + \frac{1}{4} e^{-4U} \chi^2 + \frac{1}{2} e^{-2U} d\mathcal{Z}^T \mathcal{M} d\mathcal{Z}. \tag{5.2.24}$$

Then, one ends up with an effective description in an Euclidean $D = 3$ model (which is related to the $D = 4$ one by a rescaling) describing gravity coupled to $n = 2 + n_s + 2n_v$ scalar fields $\Phi^I(r, \theta)$. After the dualization, all the propagating degrees of freedom are reduced to scalars by 3D Hodge-dualization, and, in particular, we have:

- n_s four-dimensional scalars ψ^s,
- the warp function U,
- $2n_v$ scalars $Z^M = \left\{A^\Lambda, A_\Lambda\right\}$ from the dimensional reduction of the four-dimensional vectors fields,
- the scalar a from the dualization of the Kaluza-Klein vector ω_ϕ.

5.3 Sigma-Model for $D = 3$

For a space-like Killing vector the metric on $\mathscr{M}_{scal}$ is positive definite, while for a time-like Killing vector which correspond to stationary solutions the metric is indefinite with $2n_v$ negative terms due to the fields $\mathcal{Z}$ originating from the n_v vector fields in the four-dimensional theory.

Then, if we focus in the stationary the target space $\mathscr{M}^{(3)}_{scal}$ is a pseudo-Riemannian scalar manifold which contain $\mathscr{M}^{(4)}_{scal}$ of the form

$$\mathscr{M}^{(4)}_{scal} \subset \mathscr{M}^{(3)}_{scal} = \frac{G_{(3)}}{H^*_{(3)}}. \tag{5.3.1}$$

The isometry group $G_{(3)}$ of the target space is the global symmetry group of the (5.2.23) Lagrangian $\mathcal{L}_3$, which is semisimple, non-compact Lie group while $H^*_{(4)}$ is a non-compact real form of $H_{(3)}$, the semisimple maximal compact subgroup of $G_{(3)}$.

We shall use for the scalar manifold the solvable Lie algebra parametrization, identifying the scalar fields Φ^I with parameters of a suitable solvable Lie algebra [104]. Indeed, the three-dimensional scalars Φ^I define a local *solvable parametrization* of the coset, i.e. the corresponding *physical patch* $\mathscr{U}$ is isometric to a solvable Lie group generated by a solvable Lie algebra $\mathscr{S}$ (see section (4.3.2):

$$\mathscr{M}^{(3)}_{scal} \supset \mathscr{U} \equiv e^{\mathscr{S}}. \tag{5.3.2}$$

The coset representative is chosen defining the following exponential map:

$$L(\Phi) = e^{-aT_\bullet} e^{\sqrt{2}\mathcal{Z}^M T_M} e^{\phi^r T_r} e^{2UH_0}. \tag{5.3.3}$$

where the generators $\{H_0, T_\bullet, T_s, T_M\}$ are the solvable generators and satisfy the following commutation relations: respects the following commutation relations

$$[T_\circ, T_M] = \frac{1}{2} T_M, \qquad [T_\circ, T_\bullet] = T_\bullet, \tag{5.3.4}$$

$$[T_M, T_N] = C_{MN} T_\bullet \qquad [T_\circ, T_r] = [T_\bullet, T_r] = 0, \tag{5.3.5}$$

$$[T_r, T_M] = (T_r)^N{}_M T_N, \qquad [T_s, T_r] = -\,(T_{sr})^{s'}\, T_{s'}. \tag{5.3.6}$$

The solvable Lie algebra $\mathscr{S}$ is defined by the Iwasawa decomposition [55] of the Lie algebra $\mathfrak{g}_3$ of $G_{(3)}$, with respect to its maximal compact subalgebra H_3. The generators T_M transform under the adjoint action of $G_{(4)} \subset G_{(3)}$ in the symplectic duality representation of the electric-magnetic charges.

The Lie algebra of $H^*_{(3)}$ is denoted by $\mathfrak{H}^*_3$ and is a subalgebra of $\mathfrak{g}_3$, the Lie algebra of $G_{(3)}$ All the above formulas are referred to a matrix representation in which $\mathfrak{H}^*_3$ and $\mathfrak{H}^*_3$ are defined by a *pseudo-Cartan involution* (see (4.3.1)). This involutive automorphism ς, acting on the algebra $\mathfrak{g}_3$ of $G_{(3)}$, leaves the algebra $\mathfrak{H}^*_3$ invariant. The action of ς on a general matrix A is

$$\varsigma(A) = -\eta A^\dagger \eta \tag{5.3.7}$$

where η is a suitable $H^*_{(3)}$-invariant metric, where $\eta = \eta^\dagger$ and $\eta^2 = \mathbf{1}$.

Analogous to section (4.3.1) The pseudo-Cartan ς-involution induces a (pseudo)-Cartan decomposition of $\mathfrak{g}_3$ of the form

$$\mathfrak{g} = \mathfrak{H}^*_3 \oplus \mathfrak{K}^*_3, \tag{5.3.8}$$

where in this case we have

$$\varsigma: \quad \varsigma(\mathfrak{H}^*_3) = \mathfrak{H}^*_3, \quad \varsigma(\mathfrak{K}^*_3) = -\mathfrak{H}^*_3. \tag{5.3.9}$$

The spaces follow the structure

$$[\mathfrak{H}^*_3, \mathfrak{H}^*_3] \subseteq \mathfrak{H}^*_3, \tag{5.3.10}$$

$$[\mathfrak{H}^*_3, \mathfrak{K}^*_3] \subseteq \mathfrak{K}^*_3, \tag{5.3.11}$$

$$[\mathfrak{K}^*_3, \mathfrak{K}^*_3] \subseteq \mathfrak{H}^*_3. \tag{5.3.12}$$

We see that $H^*_{(3)}$ has a linear adjoint action in the space $\mathfrak{K}^*_{(3)}$, which is thus the carrier of an $H^*_{(3)}$-representation.

5.4 Conserved currrent for $D = 3$

The σ-model action (4.4.17) reads

$$S = -\frac{\alpha}{8}\int dx \sqrt{|g^{(3)}|}\mathrm{Tr}\left(\mathcal{M}^{-1}\partial_i\mathcal{M}\mathcal{M}^{-1}\partial^i\mathcal{M}\right) \tag{5.4.1}$$

where $\alpha = \frac{1}{2\mathrm{Tr}(T_0T_0)}$. From the Lagrangian

$$\mathcal{L}_\sigma = A\mathrm{Tr}\left(\mathcal{M}^{-1}\partial_i\mathcal{M}\mathcal{M}^{-1}\partial^i\mathcal{M}\right), \tag{5.4.2}$$

From the Euler-Lagrange equations

$$\frac{\delta\left(\mathcal{L}\sqrt{g^{(3)}}\right)}{\delta\mathcal{M}} = -2A\sqrt{g^{(3)}}\mathcal{M}^{pm}\mathcal{M}^{nq}\partial_i\mathcal{M}_{qq'}\mathcal{M}^{q'q''}\partial^i\mathcal{M}^{q''p} \tag{5.4.3}$$

$$= -2A\left(\mathcal{M}^{-1}\partial_i\mathcal{M}\mathcal{M}^{-1}\partial^i\mathcal{M}\mathcal{M}^{-1}\right)^{nm}\sqrt{g^{(3)}}, \tag{5.4.4}$$

$$\frac{\delta\left(\mathcal{L}\sqrt{g^{(3)}}\right)}{\delta\partial_i\mathcal{M}} = 2A\left(\mathcal{M}^{-1}\partial^i\mathcal{M}\mathcal{M}^{-1}\right)^{nm}\sqrt{g^{(3)}}. \tag{5.4.5}$$

$$\begin{aligned}0 = &-2A\sqrt{g^{(3)}}\mathcal{M}^{-1}\partial_i\mathcal{M}\mathcal{M}^{-1}\partial^i\mathcal{M}\mathcal{M}^{-1} - 2A\partial^i\left(\sqrt{g^{(3)}}\mathcal{M}\partial^i\mathcal{M}\mathcal{M}^{-1}\right)\\ &-2A\partial_i\left(\sqrt{g^{(3)}}\mathcal{M}^{-1}\partial^i\mathcal{M}\right)\mathcal{M}^{-1}\end{aligned}$$

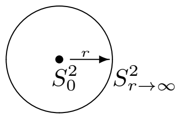

Figure 5.1: The integration in the boundary in infinity is equivalent to the integration over the boundary close to the solution. Source: self made.

We find

$$\Longrightarrow \quad \partial_i \left(J^i \sqrt{g^{(3)}} \right) = 0 \tag{5.4.6}$$

where

$$J^i = \frac{1}{2} \left(\mathcal{M}^{-1} \partial^i \mathcal{M} \right). \tag{5.4.7}$$

For duality we can write this one form like a two form

$$J^i = \frac{1}{\sqrt{g^{(3)}}} \epsilon^{ijk} J^{(2)}_{jk} \quad \Longrightarrow \quad \partial_i J^{(2)}_{jk} \epsilon^{ijk} = 0 \tag{5.4.8}$$

where

$$J^{(2)} = \frac{1}{2} J^{(2)}_{jk} dx^j dx^k \quad \Longrightarrow \quad dJ^{(2)} = 0. \tag{5.4.9}$$

We see that this conserved currrent has the property of

$$0 = \int_{M_3} dJ^{(2)} = \int_{\partial M_3} J^{(2)} \tag{5.4.10}$$

But the boundary is $\partial M = S^2_r - S^2_0$ then

$$\int_{S^2_r} J^{(2)} = \int_{S^2_0} J^{(2)}, \tag{5.4.11}$$

that means that is independent of the radius and stationary axisymmetric solutions can be described by n functions $\Phi^I(r,\theta)$, solutions to the sigma model equations, and characterized by a unique "initial point" $\Phi_0 \equiv \Phi^I_0$ at radial infinity

$$\Psi_0 = \lim_{r \longrightarrow \infty} \Psi^I(r,\theta). \tag{5.4.12}$$

and an "initial velocity" Q, at radial infinity, in the tangent space $T_{\Phi_0}(\mathscr{M}_{scal})$. Considering this, the $\mathfrak{g}_3$-valued Noether-charge matrix Q, in terms of the currents, reads:

$$Q = \frac{1}{4\pi} \int_{S^2} J_{A\theta\varphi} d\theta d\varphi,$$

Considering $J^{(2)}_{\theta\varphi} = \sqrt{g_3} \epsilon_{\theta\varphi r} J^r$

$$\begin{aligned} Q &= \frac{1}{4\pi} \int_{S^2} J_{A\theta\varphi} d\theta d\varphi, \\ Q &= \frac{1}{4\pi} \int_{S^2} \sqrt{g^3} J^{r(2)} d\theta d\varphi \end{aligned} \tag{5.4.13}$$

From it we may derive the set of Nöther currents J_{Am} and we can find the *constants of motion* as mass m using $T_A = T_0$, the NUT-charge n_{nut} associate to the $T_A = T_\bullet$, the scalar charges Σ_r where $T_A = T_r$, and electric-magnetic charges $\Gamma = \begin{pmatrix} e \\ m \end{pmatrix}$ when $T_A = T_M$, are they obtained from the components of the charges Q of the Nöether currents J_{Am} [103]:

$$J_{Am} \equiv k\mathrm{Tr}\left(T_A^\dagger J_m\right), \tag{5.4.14}$$

$$Q_A = k\mathrm{Tr}(QT_A^\dagger) = \frac{1}{4\pi}\int_{S^2} \sqrt{g} g^{rr} J_r^{(2)} d\theta d\varphi \tag{5.4.15}$$

where $T_A = \{T_\circ, T_\bullet, T_M, T_r\}$, and then we have

$$J_{\bullet m}^{(1)} = \frac{k}{2}\mathrm{Tr}\left(T_\bullet^\dagger \mathcal{M}^{-1}\partial_m \mathcal{M}\right) = -\frac{1}{2}e^{-2U}\left(\partial_m a + \mathcal{Z}^T \mathbb{C}\partial_m \mathcal{Z}\right), \tag{5.4.16}$$

$$J_{\circ m}^{(1)} = \frac{k}{2}\mathrm{Tr}\left(T_\circ^\dagger \mathcal{M}^{-1}\partial_m \mathcal{M}\right) = \partial_m U + \frac{1}{2}e^{-2U}\mathcal{Z}^T \mathcal{M}\partial_m \mathcal{Z} - aJ_{\bullet m}, \tag{5.4.17}$$

$$J_{Mm}^{(1)} = \frac{k}{2}\mathrm{Tr}\left(T_M^\dagger \mathcal{M}^{-1}\partial_m \mathcal{M}\right) = \frac{1}{\sqrt{2}}e^{-2U}\mathcal{M}_{(4)MN}\partial_m \mathcal{Z}^N + \sqrt{2}\mathbb{C}\mathcal{Z}J_{\bullet m}, \tag{5.4.18}$$

$$J_{sm}^{(1)} = \frac{1}{\sqrt{2}}\mathbb{L}_{4s}{}^{s} V_{4s}{}^{s'}\partial_m \phi^{s''} + e^{-2U}\mathcal{Z}^T T_s \mathcal{M}_{(4)}\partial_m \mathcal{Z} - T_{sMN}\mathcal{Z}^M \mathcal{Z}^N J_{\bullet m}. \tag{5.4.19}$$

where $\mathbb{L}_{4s}^{s'}$ is the coset representative of the symmetric scalar manifold in four-dimensions in the solvable parametrization, as a matrix in the adjoint representation of the solvable group, $V_{4s}^{s'}$ is the vielbein of the same manifold and the hat denotes rigid indices.

The other conserved quantity characterizing the axisymmetric solution is the angular momentum M_φ along the rotation axis. Usually it is derived from the standard Komar-integral, but it is useful to describe the global rotation of the solution by means of a new $\mathfrak{g}_3$-valued matrix Q_φ, firstly defined in[[103]:

$$Q_\varphi = -\frac{3}{4\pi}\int_{S^2}^{\infty} \varphi_{[i}\partial_{j]}dx^i \wedge dx^j = -\frac{3}{4\pi}\int_{S^2}^{\infty} g_{\varphi\varphi}^{(3)} J_\theta d\theta d\varphi. \tag{5.4.20}$$

The conserved quantities are then obtained as the flux of the currents across the 2-sphere at infinity, according to eq. (5.4.14) as mentioned before (see figure 5.1):

$$n_{nut} = -k\mathrm{Tr}\left(T_\bullet^\dagger Q\right), \qquad M_{ADM} = k\mathrm{Tr}\left(T_0^\dagger Q\right), \qquad \Sigma_s = k\mathrm{Tr}\left(T_s^\dagger Q\right),$$

$$\Gamma^M = \sqrt{2}k\mathbb{C}^{MN}\mathrm{Tr}\left(T_N^\dagger Q\right), \qquad \mathscr{J}_\varphi = k\mathrm{Tr}\left(T_\bullet^\dagger Q_\varphi\right). \tag{5.4.21}$$

Being $G_{(3)}$ the global symmetry group of the effective three-dimensional model, a generic element g of it maps a solution $\Phi^I(r,\theta)$ into an other solution $\Psi'^I(r,\theta)$ according to

$$\forall g \in G_{(3)} : \quad \mathcal{M}_3(\Psi^I) \xrightarrow{g} \mathcal{M}_3(\Psi'^I) = g\mathcal{M}_3(\Psi^I)g^\dagger. \tag{5.4.22}$$

That means

$$\forall g \in G_{(3)}: \quad Q \xrightarrow{g} Q' = (g^{-1})^\dagger Q g^\dagger, \quad Q_\varphi \xrightarrow{g} Q'_\varphi = (g^{-1})^\dagger Q_\varphi g^\dagger. \tag{5.4.23}$$

Equations (5.4.21) allow to compute the angular momentum of the transformed solution without having to explicitly derive the latter from (5.4.22), so that we can avoid to compute the corresponding Komar integral on it. This is one of the main advantages of working with Q_Ψ. The presence of a non-vanishing Q_Ψ is a characteristic of the $G_{(3)}$-orbits of rotating solutions and therefore one cannot generate rotation on a static $D = 4$ solution using $G_{(3)}$.

We can map point at radial infinity (Ψ_0^I) into the origin of the manifold by means of a transformation in $G_{(3)}/H_{(3)}$. This choice clearly breaks $G_{(3)}$ to the isotropy group $H_{(3)}$ and, as a consequence of this, the two matrices Q and Q_φ always lie in the coset space $\mathfrak{K}_3^*$.

5.5 Kern-Newmann black hole

Let put this technique in the case of Kerr-Newmann black hole. The components of the metric (5.2.4) are

$$g_{tt} = -\,e^{2U} = -\frac{\tilde{\Delta}}{\rho^2}, \qquad g_{rr} = \frac{\rho^2}{\Delta}, \tag{5.5.1}$$

$$g_{\theta\theta} = \rho^2, \qquad g_{\varphi\varphi} = \frac{\Delta\rho^2}{\tilde{\Delta}} \tag{5.5.2}$$

with the 3-dimensional metric like

$$g_{ij}^{(3)} = \begin{pmatrix} \frac{\tilde{\Delta}}{\Delta} & 0 & \\ 0 & \tilde{\Delta} & 0 \\ 0 & 0 & \Delta \sin^2\theta \end{pmatrix} \tag{5.5.3}$$

and

$$\omega = \omega_\varphi d\varphi = \frac{a(\tilde{\Delta} - \rho^2)}{\tilde{\Delta}} \sin^2\theta d\varphi. \tag{5.5.4}$$

Then, we must consider

$$A_0 = -\frac{1}{\rho^2}\left(Qr - aP\cos\theta\right), \tag{5.5.5}$$

$$A_\varphi = \frac{1}{\rho^2}\left(aQr\sin^2\theta - (r^2 + a^2)P\cos\theta\right) \tag{5.5.6}$$

Following (5.2.8), (5.2.10), (5.2.12) we have

$$\tilde{A}_3 = \frac{(Qra\sin^2\theta - \Delta P\cos\theta)}{\tilde{\Delta}} d\varphi, \tag{5.5.7}$$

$$\tilde{F}_{r\varphi} = \frac{a(2aPr\cos\theta - Q(r^2 - a^2\cos^2\theta))}{\rho^2\tilde{\Delta}} \sin^2\theta, \tag{5.5.8}$$

$$\tilde{F}_{\theta\varphi} = \frac{\Delta\left(2aQr\cos\theta + P\left(r - a^2\cos^2\theta\right)\right)\sin\theta}{\rho^2\tilde{\Delta}}. \tag{5.5.9}$$

Using (5.2.16), (5.2.17) and considering $\mathcal{I} = -1$ with $\mathcal{R} = 0$

$$\partial_r A = - e^{2U} e_3 g^{\theta\theta} g^{\varphi\varphi} \tilde{F}_{\theta\varphi}, \tag{5.5.10}$$

$$\partial_\theta A = e^{2U} e_3 g^{rr} g^{\varphi\varphi} \tilde{F}_{r\varphi} \tag{5.5.11}$$

$$\implies A = \frac{(Pr + aQ\cos\theta)}{\rho^2} + C. \tag{5.5.12}$$

therefore

$$\mathcal{Z}_M = \begin{pmatrix} -\frac{1}{\rho^2}\left(Qr - aP\cos\theta\right) \\ \frac{1}{\rho^2}\left(Pr + aQ\cos\theta\right) + C. \end{pmatrix} \tag{5.5.13}$$

where C is a integral constant. To obteind a we use (5.2.17)

$$a = \frac{2aM}{\rho^2}\cos\theta + K \tag{5.5.14}$$

where K is an integral constant.

5.5.1 Coset geometry for KN solution

We consider the scalar manifold $\mathscr{M} = SU(2,1)/U(1,1)$ with $\eta = (-1, 1, 19$. Let $g \in U(2,1)$ a 3×3 matrix such that $g = a_i g_i$ where g are the generators of $SU(2,1)$, then

$$g^\dagger \eta + \eta g = 0. \tag{5.5.15}$$

The scalars $\Phi_a = \{U, a, \phi, r, \mathcal{Z}\}$ correspond to a local solvable parametrization, i.e. the corresponding patch, to be dubbed physical patch $\mathscr{U}$ (see (5.3.2)), is isometric to a solvable Lie group generated by a solvable Lie algebra (see (4.3.2)) defined by the Iwasawa decomposition of the Lie algebra of $G_{(3)}$ with respect to its maximal compact subalgebra H. The generators of the solvable decomposition are

$$H_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad T_\bullet = \begin{pmatrix} -\frac{i}{2} & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ -\frac{i}{2} & 0 & \frac{i}{2} \end{pmatrix}$$
$$T_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \tag{5.5.16}$$

then we can built the parametrization of $SU(2,1)/U(1,1)$ correspond to the following exponential map:

$\mathbb{L} =$

$$\begin{pmatrix} \frac{1}{4}e^{-U}\left(2 + 2ia + 2e^{2U} + \mathcal{Z}_1^2 + \mathcal{Z}_1^2\right) & -\frac{1}{\sqrt{2}}(\mathcal{Z}_1 + i\mathcal{Z}_2) & \frac{1}{4}e^{-U}\left(-2 - 2ia + 2e^{2U} - \mathcal{Z}_1^2 - \mathcal{Z}_1^2\right) \\ -\frac{1}{\sqrt{2}}e^{-U}(\mathcal{Z}_1 - i\mathcal{Z}_2) & 1 & \frac{1}{\sqrt{2}}e^{-U}(\mathcal{Z}_1 - i\mathcal{Z}_2) \\ \frac{1}{4}e^{-U}\left(-2 + 2ia + 2e^{2U} + \mathcal{Z}_1^2 + \mathcal{Z}_1^2\right) & -\frac{1}{\sqrt{2}}(\mathcal{Z}_1 + i\mathcal{Z}_2) & \frac{1}{4}e^{-U}\left(2 - 2ia + 2e^{2U} - \mathcal{Z}_1^2 - \mathcal{Z}_1^2\right) \end{pmatrix} \tag{5.5.17}$$

With this, following section (5.4) we can obtain the currents J_i and the charges. In the case when we consider J_r we have

$$Q = \begin{pmatrix} 0 & 0 & 2M \\ 0 & 0 & -i\sqrt{2}(P - iQ) \\ 2M & \sqrt{2}(-iP + Q) & 0 \end{pmatrix} \tag{5.5.18}$$

5.6 AdS metric

Analogously to the case above we consider the AdS metric

$$ds^2 = -\left(1+\frac{r^2}{b^2}\right)dt^2 + \frac{1}{\sqrt{(1+rb^{-2})}}dr + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2. \qquad (5.6.1)$$

where $b = \sqrt{\frac{3}{\Lambda}}$. The field strengths are *off* and only scalars after the reduction and dualization is

$$U = \frac{1}{2}\text{Log}\left[1+\frac{r^2}{\sqrt{-\frac{3}{\Lambda}}}\right]. \qquad (5.6.2)$$

Then, the non-vanish current is

$$J_\circ = \left(\frac{2r}{r^2+\sqrt{-\frac{3}{\Lambda}}}, 0\right). \qquad (5.6.3)$$

5.6.1 Coset geometry for AdS solution

We choose the group $G_{2(2)}$ which has as subspace non-compact $H_{n.c.} = SO(4) \simeq SU(2)\times SU(2)$. From equation (4.3.11) we have basically that $h = a^i h_i$ where $h_i \in H_{n.c.}$ and $g = b^i g_i$ where $g_i \in \mathfrak{g}$, such that

$$[h, g] = (h)_m^{\ n} g_n, \qquad (5.6.4)$$

is a secular equation to obtain an adjoint representation. We find the generators $H_{n.c.}$, noting that they are symmetric* and that they commute between them. The Iwasawa parametrization is the most suitable for the computation of the metric on $G_{2(2)}/SO(4)$. We know that the Cartan subalgebra of is generated by $h_1 := C_{11}$ and $h_2 := C_5$ in [105]. The roots of G_2 can thus been computed by diagonalizing the adjoint action of H_i. We obtain

$$Solv = H_{n.c.} \oplus \{E_\alpha\}, \qquad (5.6.5)$$

*In the that they are real, if not they are hermetic which correspond to the *Borel algebra*. The respective matrices are

$$T_\circ = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}, \tag{5.6.6}$$

$$H_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}, \tag{5.6.7}$$

$$T_\bullet = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.6.8}$$

$$T_i = \left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right), \tag{5.6.9}$$

$$T_M = \left(\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right. \tag{5.6.10}$$

$$\left. , \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{5.6.11}$$

Then, as we see in section 5.4, the only charge different from zero is

$$M_{ADM} = -\frac{2r}{r^2 + \sqrt{3}\sqrt{-\frac{1}{\Lambda}}} \times \sqrt{(-9+3r^2\Lambda)^{-1}\left(9 + \frac{9\sqrt{3}r^2}{\sqrt{-\frac{1}{\Lambda}}} - 9r^4\Lambda + \sqrt{3}r^6\sqrt{-\frac{1}{\Lambda}}\Lambda^2\right)}\,\phi\cos(\theta) \tag{5.6.12}$$

Chapter 6

Conclusion and Comments

To learn is not to know; there are the learners and the learned. Memory makes the one, philosophy the others.

Alexandre Dumas, The Count of Monte Cristo

First part

In the first part of this thesis, we have presented, using a geometrical formalism, a method to find the semigroups with which it is possible to perform an S-expansion between two known algebras. In addition, the concept of zero-reduction towards an ideal-reduction has been extended, reproducing, in this way, a generalized IW-contraction. Numerous examples are shown to corroborate the method, and, at the same time, an interesting application in supergravity to obtain a Hidden Maxwell Superalgebra from an AdS Lorentz algebra in 4 dimensions.

In chapter 1 we have reviewed some concepts of the theory of Lie algebras and the main aspects of the S-expansion procedure. Precisely, in section 1.3 we shown how S-expansion procedure affects the geometry of a Lie group it was found how the magnitude of a vector and the angle change between two vectors. Also, it was found that the kind of algebra, after applying N S-Expansion is a non-simple Lie algebra. Then, in section 1.4, when is considered resonance and reduction, we developed an analytic method, able to give us the multiplication table(s) of the set(s) involved in an S-expansion process for reaching a target Lie (super)algebra from a starting one, after having properly chosen the partitions over subspaces of the considered (super)algebras.

Furthermore, in our example 1.5.2, we have shown in an interesting way that the particular subalgebra (where $\tilde{P}_a$ and $\tilde{Q}$ commute) of the hidden superalgebra underlying $D = 11$ supergravity is linked to $osp(32/1)$ by the semigroup $S_E^{(2)}$, only if this coincides with the M-algebra described in [20]. We conclude our observations saying that, previously in [40] and later in [26], the authors also found a singular solution, which, in our notation, corresponds to consider the singular limit $\tilde{Q}' \to 0$ in the target superalgebra. In this case, the authomorphism group of the FDA is enlarged to $Sp(32)$, and the whole procedure resembles an Inönü-Wigner contraction. Thus, we can clearly see the existence of a strong link between the mentioned superalgebras, given by the S-expansion (0_S-resonant-reduction) and the Inönü-Wigner contraction procedures. However, we will not treat the case involving the Inönü-Wigner contraction in our work, and we leave these considerations for the future.

In chapter 2, we have shown how to reproduce a generalized Inönü-Wigner contraction by considering a new prescription for the S-expansion procedure, which involves an infinite abelian semigroup $S_E^{(\infty)}$ and the removal of an ideal subalgebra. This infinite S-expansion procedure with subsequent ideal subtraction, represents an extension and generalization of the finite one, allowing a deeper view on the maps linking different algebras.

With all this "tools" we can use of the main advantage of S-expansion. We refer to the use of theorem VII.2 of the refence [20] which serves to construct the topological invariant of a respective S-expanded algebra from another of which the bilinear form is known, then is an extension and generalization of the mentioned theorem. This procedure allows to develop the dynamics and construct the Lagrangians of several theories. In this context, in particular, the construction of Chern-Simons forms becomes more accessible, and it would be particularly interesting to develop them in (super)gravity theories in higher dimensions, by following our approach.

Also, the work reproduces the results already presented in the literature, concerning to the generalized Inönü-Wigner contraction, and also gives some new features. Moreover, it gives a connection between the contraction processes and the expansion methods introduced in [17], which was an open question already mentioned in [12].

In chapter 3, it is shown one of the interesting applications mentioned above. We refer to obtain a particular hidden Maxwell superalgebra underlying supergravity in four dimensions. Thus, we have written the hidden Maxwell superalgebra in the Maurer-Cartan formalism, and then, we have considered the parametrization of the 3-form $A^{(3)}$ in terms of 1-forms, in order to show the way in which the trivial boundary contribution in four dimensions, $dA^{(3)}$, can be naturally extended by considering particular contributions to the structure of the extra fermionic generator, appearing in the hidden Maxwell superalgebra. These extensions contain terms which involve the cosmological constant. Interestingly, the presence of these terms strictly depends on the form of the extra fermionic generators, appearing in this hidden extension of $D = 4$ supergravity.

Future work

A future work of that is discussed in the section 1.4, includes the study of the particular cases in which the number of subspaces partitions of the target (super)algebra is different from the number of subspaces partition of the starting one, and the extension to the case of infinite algebras and semigroups, to made the connection with the *ideal-reduction* proposed in chapter 2.

In the generalization of IW-contraction as an infinite S-expansion procedure, we have restricted to the cases involving an infinite semigroup $S_E^{(\infty)}$ related to the set $(\mathbb{N}, +)$. We leave a possible extension to the set $(\mathbb{Z}, +)$ to future works. This further analysis would be interesting, since it would produce a S-expansion involving a complete group, rather than a semigroup. Moreover, if we consider ti use Grassmann parameters, as we saw in section 1.5.1, we can consider the relation of this technique with the *bosonization* [106], which has an exponential function in the procedure that gives the possibility of relating the presented technique due to the use of the group $\mathbb{Z}$.

About the Hidden Maxwell Superalgebra underlying $D = 4$ supergravity, it would be interesting to write the Lagrangian in four dimensions and to consider non-trivial boundary terms, by looking at the new structure of $dA^{(3)}$. Another interesting development of the present work would be the study of Chern-Simons theories and Lagrangians in even dimensions, such as the Born-Infeld case for four dimensional case. In particular we would like to see the relation of dA^3 and the generation of a topological invariant, proposed in [107].

Second part

In the second part, we have synthesized the general features of ungauged supergravities, including their global symmetry group G. We have shown two examples, in extended supergravites, where we use a Lagrangian with an asintotic solution, where one of them has a cosmological constant. The solution generating technique is useful in scenarios where the theory is not-gauged, since we need solve the Einstein, scalars and vector equations in the same time. With a gauged theory we need to study how uncoupling the equations to be resolved. At first we work, in the unfinished research,with the AdS metric with the global group $G_{2(2)}$, where we found the only charge. Bearing in mind that a solution with cosmological constant is always gauged for extended supegravities [36], it is planned to investigate the applicability of the solutions generating technique on gauged theories at this level.

The gauging procedure consists in promoting a suitable global symmetry group G of the Lagrangian to a local symmetry group gauged by the vector fields of the theory. Besides the introduction of minimal couplings, changes in the Lagrangian and the supersymmetry transformation rules are required in order to the resulting model has the same supersymmetries as the original ungauged one.

Future work

The work to be done is to solve the field equations of a Lagrangian with constant potential, to solve the gauged equations, using the solution-generating-technique. After this, evolve into more complicated potentials.

Another task is to approach the dimensional reduction and dualization to link higher dimensional theories with complicated topologies and solve them at the 3-dimensional level.

Appendix

-Vuoi un caffè?
-Si per favore.
-Andiamo.

A lot of people in the world

Appendix A

Calculations to found the semigroup

A.1 Resonance and reduction for moving from the BTI to the BTII

First of all, we solve the system (1.4.7), that in this case reads

$$\begin{cases} 1 = 1 \cdot (\Delta_0)\,, \\ 2 = 1 \cdot (\Delta_1)\,, \\ \tilde{P} = \Delta_0 + \Delta_1 + 1, \end{cases} \tag{A.1.1}$$

since

$$dim(\tilde{V}_0) = 1, \quad dim(V_0) = 1, \tag{A.1.2}$$

$$dim(\tilde{V}_1) = 2, \quad dim(V_1) = 1, \tag{A.1.3}$$

neglecting the zero element of the subspaces V_0 and $\tilde{V}_0$. Here we have denoted, as usual, with Δ_A, $A = 0, 1$, the cardinality of the subsets S_{Δ_A} associated with the subspace A, *i.e.* the number of elements in S_{Δ_A}. Solving the system above, we get the unique solution

$$\tilde{P} = 4, \quad \Delta_0 = 1, \quad \Delta_1 = 2. \tag{A.1.4}$$

Now, since $\tilde{S} = \{S_{1_0}\} \sqcup \{S_{2_1}\} \cup \{\lambda_{0_S}\}$, where λ_{0_S} is the zero element of the set $\tilde{S}$, we can write the following subset decomposition structure of the set $\tilde{S}$:

$$\begin{aligned} S_{1_0} =& \{\lambda_a\}, \\ S_{2_1} =& \{\lambda_b,\ \lambda_c\}. \end{aligned} \tag{A.1.5}$$

Here the index a, b, c identify general elements of the set $\tilde{S}$, and they are not running index. We do not yet identify them with numbers, because there still exists the possibility of having the same element in different subsets.

The next step consists in finding the multiplication rules between the elements of each subset in (A.1.5). We write the relations (1.4.9), which, in this case, read

$$
\begin{aligned}
&[(S_{1_0} \otimes V_0) \oplus (\{\lambda_{0_S}\} \otimes V_0)\,, (S_{1_0} \otimes V_0) \oplus (\{\lambda_{0_S}\} \otimes V_0)] = \\
&= \left(K^{(1_0)}_{(1_0)(1_0)} \, (C)^0_{00}\right) (S_{1_0} \otimes V_0) \oplus (\{\lambda_{0_S}\} \otimes V_0)\,, \\
&= \left(K^{(2_1)}_{(2_1)(1_0)} \, (C)^1_{10}\right) (S_{2_1} \otimes V_1) \oplus (\{\lambda_{0_S}\} \otimes V_1)\,, \\
&[(S_{2_1} \otimes V_1) \oplus (\{\lambda_{0_S}\} \otimes V_1)\,, (S_{2_1} \otimes V_1) \oplus (\{\lambda_{0_S}\} \otimes V_1)] = \\
&= \left(K^{(1_0)}_{(2_1)(2_1)} \, (C)^0_{11}\right) (S_{1_0} \otimes V_0) \oplus (\{\lambda_{0_S}\} \otimes V_0)\,.
\end{aligned}
\tag{A.1.6}
$$

These relations (which can also be rewritten in a simpler form, such as the one in (1.4.13)) give us a first view on the possibilities allowed by the multiplication table of the set $\tilde{S}$. In fact, we can now write

$$S_{1_0} \cdot S_{1_0} \subset S_{1_0} \cup \{\lambda_{0_S}\}, \tag{A.1.7}$$

$$S_{2_1} \cdot S_{1_0} \subset S_{2_1} \cup \{\lambda_{0_S}\}, \tag{A.1.8}$$

$$S_{2_1} \cdot S_{2_1} \subset S_{1_0} \cup \{\lambda_{0_S}\}, \tag{A.1.9}$$

where we have taken into account the presence of the zero element λ_{0_S} of the set $\tilde{S}$. Thus, we are now able to write the possible multiplication rules between the elements of the set $\tilde{S}$, namely

$$\lambda_a \lambda_a = \lambda_{a,0_S}\,, \tag{A.1.10}$$

$$\lambda_{b,c} \lambda_a = \lambda_{b,c,0_S}\,, \tag{A.1.11}$$

$$\lambda_{b,c} \lambda_{b,c} = \lambda_{a,0_S}\,, \tag{A.1.12}$$

where we have already taken into account the triviality of the multiplications rules of λ_{0_S}.

We have exhausted the information coming from the initial algebra, thus we now use the information coming from the target one, in order to build up the complete multiplication table of the set $\tilde{S}$.

We proceed by writing the relations between the S-expanded generators of the initial BTI algebra and the generators of the target BTII one, according to the usual S-expansion procedure described [20]. According to the identification criterion presented in Subsection 1.4.1, we can perform the identification

$$\lambda_a X_2 = Y_3 \tag{A.1.13}$$

$$\lambda_b X_1 = Y_1 \tag{A.1.14}$$

$$\lambda_c X_1 = Y_2. \tag{A.1.15}$$

We now write the commutators of the target BTII algebra in terms of the commutators between the S-expanded, resonant-reduced generators of the BTI one:

$$
\begin{aligned}
[Y_2, Y_3] &= Y_1, \\
[\lambda_c X_1, \lambda_a X_2] &= \lambda_b X_1, \\
\lambda_c \lambda_a \, [X_1, X_2] &= \lambda_b X_1.
\end{aligned}
\tag{A.1.16}
$$

Since for the BTI algebra we have $[X_1, X_2] = X_1$, from equation (A.1.16) we obtain

$$\lambda_c \lambda_a = \lambda_b. \tag{A.1.17}$$

This simple analysis can be performed in order to find the correct multiplication rules between the elements of the set $\tilde{S}$, thus we proceed in this way, computing the other commutators and factorizing the product between the elements of $\tilde{S}$, in order to end up with the complete multiplication table.

Let us observe that the commutator

$$[Y_1, Y_2] = 0 \tag{A.1.18}$$

does not give us any further information about the multiplication rule between λ_b and λ_c, due to the fact that, when we write it in terms of the commutator between S-expanded generators and we factorize the product between the elements of $\tilde{S}$, we are left with $\lambda_b \lambda_c [X_1, X_1] = 0$, which reproduces a trivial identity, since $[X_1, X_1] = 0$ in the BTI algebra. On the other hand, from the study of the last commutator we have to consider, we get

$$\begin{aligned} [Y_1, Y_3] &= 0, \\ [\lambda_b X_1, \lambda_a X_2] &= 0, \\ \lambda_b \lambda_a [X_1, X_2] &= 0. \end{aligned} \tag{A.1.19}$$

Since, from the initial BTI algebra, we know that $[X_1, X_2] = X_1 \neq 0$, from equation (A.1.19) we see clearly that the the zero element λ_{0_S} is naturally involved in the procedure, and we can finally write

$$\lambda_b \lambda_a = \lambda_{0_S}. \tag{A.1.20}$$

A.2 Only resonance for moving from the BTI to the BTII

If we search for a table in a case of using only resonance, we must to resolve again the equation (1.4.7) which, in this case, is solved by

$$\tilde{P} = 3, \quad \Delta_0 = 1, \quad \Delta_1 = 2. \tag{A.2.1}$$

Then, performing the usual procedure (see Section 1.4) as before

$$S_{1_0} \cdot S_{1_0} \subset S_{1_0} \cup, \tag{A.2.2}$$
$$S_{2_1} \cdot S_{1_0} \subset S_{2_1} \cup, \tag{A.2.3}$$
$$S_{2_1} \cdot S_{2_1} \subset S_{1_0} \cup, \tag{A.2.4}$$

where we have not taken into account the presence of the zero element λ_{0_S} of the set $\tilde{S}$. Thus, we are now able to write the possible multiplication rules between the elements of the set $\tilde{S}$, namely

$$\lambda_a \lambda_a = \lambda_a, \tag{A.2.5}$$
$$\lambda_{b,c} \lambda_a = \lambda_{b,c}, \tag{A.2.6}$$
$$\lambda_{b,c} \lambda_{b,c} = \lambda_a, \tag{A.2.7}$$

One again, we have exhausted the information coming from the initial algebra, thus we now use the information coming from the target one, in order to build up the complete multiplication table of the set $\tilde{S}$.

We proceed by writing the relations between the S-expanded generators of the initial BTI algebra and the generators of the target BTII one, according to the usual S-expansion procedure described [20]. According to the identification criterion presented in Subsection 1.4.1, we can perform the identification

$$\lambda_a X_2 = Y_3 \tag{A.2.8}$$
$$\lambda_b X_1 = Y_1 \tag{A.2.9}$$
$$\lambda_c X_1 = Y_2 . \tag{A.2.10}$$

we can reach the multiplication rules between the elements of the set $\tilde{S}$, after having faced the particular situation in which

$$[Y_1, Y_3] = 0, \tag{A.2.11}$$
$$[\lambda_b X_1, \lambda_a X_3] = 0, \tag{A.2.12}$$
$$\lambda_b \lambda_a [X_1, X_2] = 0, \tag{A.2.13}$$

where $[X_1, X_2] = X_1 \neq 0$. As we can see in equation (A.2.11), the generators Y_1 and Y_3 of the target algebra must commute, while the generators X_1 and X_2 of the starting algebra do not commute; so, the only way for reaching a consistent multiplication rule between the elements λ_a and λ_b consists in adding a zero element in the set $\tilde{S}$ involved in the process, such that

$$\lambda_b \lambda_a = \lambda_{0_S} . \tag{A.2.14}$$

The inclusion of the zero element is consistent, since this modification just affects the variable $\tilde{P}$ in the system (1.4.7), which increases of $+1$ (namely, $\tilde{P} = 4$).

A.3 Detailed calculations for reaching the hidden superalgebra underlying D = 11 supergravity, starting from the supersymmetric Lie algebra os p(32/1)

We can thus proceed, giving the internal decomposition of the target superalgebra (we first consider the case in which $\left[\tilde{Q}, \tilde{P}_a\right] \neq 0$). For the target superalgebra underlying $D = 11$ supergravity, we can write

$$\dim\left(\tilde{V}_0\right) = \underbrace{110}_{\tilde{J}_{ab},\, \tilde{Z}_{ab}} , \tag{A.3.1}$$
$$\dim\left(\tilde{V}_1\right) = \underbrace{64}_{\tilde{Q}_\alpha,\, \tilde{Q}'_\alpha} , \tag{A.3.2}$$
$$\dim\left(\tilde{V}_2\right) = \underbrace{11}_{\tilde{P}_a} + \underbrace{462}_{\tilde{Z}_{a_1\cdots a_5}} = 473. \tag{A.3.3}$$

where we have clearly set $\tilde{V}_0 = \{0\} \cup \{\tilde{J}_{ab}, \tilde{Z}_{ab}\}$, $\tilde{V}_1 = \{\tilde{Q}, \tilde{Q}'\}$, and $\tilde{V}_2 = \{\tilde{P}_a, \tilde{Z}_{a_1 \ldots a_5}\}$. The subspaces partition for the target superalgebra satisfies the following relations

$$\begin{aligned}
\left[\tilde{V}_0, \tilde{V}_0\right] &\subset \tilde{V}_0, & \text{(A.3.4)}\\
\left[\tilde{V}_0, \tilde{V}_1\right] &\subset \tilde{V}_0 \oplus \tilde{V}_1, & \text{(A.3.5)}\\
\left[\tilde{V}_0, \tilde{V}_2\right] &\subset \tilde{V}_0 \oplus \tilde{V}_2, & \text{(A.3.6)}\\
\left[\tilde{V}_1, \tilde{V}_1\right] &\subset \tilde{V}_0 \oplus \tilde{V}_2, & \text{(A.3.7)}\\
\left[\tilde{V}_1, \tilde{V}_2\right] &\subset \tilde{V}_0 \oplus \tilde{V}_1, & \text{(A.3.8)}\\
\left[\tilde{V}_2, \tilde{V}_2\right] &\subset \tilde{V}_0 \oplus \tilde{V}_2, & \text{(A.3.9)}
\end{aligned}$$

analogously to what we have done for $osp(32/1)$. We can now move to the study of the usual system (1.4.7), which, in this case, reads

$$\begin{cases}
110 = 55(\tilde{P} - 1 - \Delta_1 - \Delta_2), \\
64 = 32(\tilde{P} - 1 - \Delta_0 - \Delta_2), \\
473 = 473(\tilde{P} - 1 - \Delta_0 - \Delta_1), \\
\tilde{P} = \Delta_0 + \Delta_1 + \Delta_2 + 1,
\end{cases} \tag{A.3.10}$$

where Δ_0, Δ_1, Δ_2 respectively denote the cardinality of the subsets related to the subspaces V_0, V_1, and V_2. This system admits the unique solution

$$\tilde{P} = 6, \quad \Delta_0 = 2, \quad \Delta_1 = 2, \quad \Delta_2 = 1. \tag{A.3.11}$$

Thus, we are now able to write the following subset decomposition of the set $\tilde{S}$ involved in the process:

$$\begin{aligned}
S_{2_0} &= \{\lambda_a, \lambda_b\}, & \text{(A.3.12)}\\
S_{2_1} &= \{\lambda_c, \lambda_d\}, & \text{(A.3.13)}\\
S_{1_2} &= \{\lambda_e\}. & \text{(A.3.14)}
\end{aligned}$$

Thus, one can now write the usual relations (1.4.9) (or their simpler form, given by (1.4.13)) for the case under analysis, and find the following product structure for the subset decomposition of the set $\tilde{S}$

$$\begin{aligned}
S_{2_0} \cdot S_{2_0} &\subset S_{2_0} \cup \{\lambda_{0_S}\}, & \text{(A.3.15)}\\
S_{2_0} \cdot S_{2_1} &\subset S_{2_1} \cup \{\lambda_{0_S}\}, & \text{(A.3.16)}\\
S_{2_0} \cdot S_{1_2} &\subset S_{1_2} \cup \{\lambda_{0_S}\}, & \text{(A.3.17)}\\
S_{2_1} \cdot S_{2_1} &\subset (S_{2_0} \cap S_{1_2}) \cup \{\lambda_{0_S}\}, & \text{(A.3.18)}\\
S_{2_1} \cdot S_{1_2} &\subset S_{1_2} \cup \{\lambda_{0_S}\}, & \text{(A.3.19)}\\
S_{1_2} \cdot S_{1_2} &\subset (S_{2_0} \cap S_{1_2}) \cup \{\lambda_{0_S}\}, & \text{(A.3.20)}
\end{aligned}$$

where we have explicitly taken into account the presence of the zero element λ_{0_S}. This allows to reach the multiplication rules

$$
\begin{aligned}
\lambda_{a,b}\lambda_{a,b} &= \lambda_{a,b,0_S}, && \text{(A.3.21)}\\
\lambda_{a,b}\lambda_{c,d} &= \lambda_{c,d,0_S}, && \text{(A.3.22)}\\
\lambda_{a,b}\lambda_{e} &= \lambda_{e,0_S}, && \text{(A.3.23)}\\
\lambda_{c,d}\lambda_{c,d} &= \lambda_{a,b,e,0_S}, && \text{(A.3.24)}\\
\lambda_{c,d}\lambda_{e} &= \lambda_{e,0_S}, && \text{(A.3.25)}\\
\lambda_{e}\lambda_{e} &= \lambda_{a,b,e,0_S}, && \text{(A.3.26)}
\end{aligned}
$$

where we have already taken into account the triviality of the multiplications rules of λ_{0_S}.

We can now fix the degeneracy appearing in the above multiplication rules, by analyzing the information coming from the target superalgebra. According to the usual S-expansion procedure (see Ref. [20]), we have to write the commutation relations between the generators of the target superalgebra in terms of the commutation relations between the generators of the S-expanded $osp(32/1)$. After having performed the identification (according to the identification criterion presented in Subsection 1.4.1)

$$
\begin{aligned}
\lambda_a J_{ab} &= \tilde{J}_{ab}, & \lambda_b J_{ab} &= \tilde{Z}_{ab}, & \lambda_c Q &= \tilde{Q}, && \text{(A.3.27)}\\
\lambda_d Q &= \tilde{Q}', & \lambda_e P_a &= \tilde{P}_a, & \lambda_e Z_{a_1 \ldots a_5} &= \tilde{Z}_{a_1 \ldots a_5}, && \text{(A.3.28)}
\end{aligned}
$$

we are able to write the commutation relations of the superalgebra underlying $D = 11$ supergravity in terms of the commutation relations of the S-expanded generators of $osp(32/1)$. In the following, we will just consider the structure of the commutation relations, since the explicit values of the coefficients are not relevant to our analysis. For performing this calculation, we consider the case in which $\left[\tilde{Q}, \tilde{P}_a\right] \neq 0$.

Thus, taking into account the commutation relations for the initial algebra

$osp(32/1)$, we have:

$$\{\tilde{Q}', \tilde{Q}'\} = \{\lambda_d Q, \lambda_d Q\} = \lambda_d \lambda_d \{Q, Q\} = 0, \quad \text{(A.3.29)}$$
$$\rightarrow \lambda_d \lambda_d = \lambda_{0_S},$$

$$\{\tilde{Q}, \tilde{Q}'\} = \{\lambda_c Q, \lambda_d Q\} = \lambda_c \lambda_d \{Q, Q\} = 0, \quad \text{(A.3.30)}$$
$$\rightarrow \lambda_c \lambda_d = \lambda_{0_S},$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \left[\lambda_e P_a, \lambda_e P_b\right] = \lambda_e \lambda_e \left[P_a, P_b\right] = 0, \quad \text{(A.3.31)}$$
$$\rightarrow \lambda_e \lambda_e = \lambda_{0_S},$$

$$\left[\tilde{P}_a, \tilde{J}_{bc}\right] = \left[\lambda_e P_a, \lambda_a J_{bc}\right] = \lambda_e \lambda_a \left[P_a, J_{bc}\right] \propto \lambda_e \delta^{ab}_{ec} P^e, \quad \text{(A.3.32)}$$
$$\rightarrow \lambda_e \lambda_a = \lambda_e,$$

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \left[\lambda_a J_{ab}, \lambda_a J_{cd}\right] = \lambda_a \lambda_a \left[J_{ab}, J_{cd}\right] \propto \lambda_a \delta^{abf}_{ecd} \ J^e{}_f, \quad \text{(A.3.33)}$$
$$\rightarrow \lambda_a \lambda_a = \lambda_a,$$

$$\left[\tilde{P}_a, \tilde{Q}'\right] = \left[\lambda_e P_a, \lambda_d Q\right] = \lambda_e \lambda_d \left[P_a, Q\right] = 0, \quad \text{(A.3.34)}$$
$$\rightarrow \lambda_e \lambda_d = \lambda_{0_S},$$

$$\left[\tilde{Z}_{ab}, \tilde{Q}'\right] = \left[\lambda_b J_{ab}, \lambda_d Q\right] = \lambda_b \lambda_d \left[J_{ab}, Q\right] = 0, \quad \text{(A.3.35)}$$
$$\rightarrow \lambda_b \lambda_d = \lambda_{0_S},$$

$$\left[\tilde{P}_a, \tilde{Q}\right] = \left[\lambda_e P_a, \lambda_c Q\right] = \lambda_e \lambda_c \left[P_a, Q\right] \propto \lambda_d Q, \quad \text{(A.3.36)}$$
$$\rightarrow \lambda_e \lambda_c = \lambda_d,$$

$$\left[\tilde{Z}_{ab}, \tilde{Q}\right] = \left[\lambda_b J_{ab}, \lambda_c Q\right] = \lambda_b \lambda_c \left[J_{ab}, Q\right] \propto \lambda_d Q, \quad \text{(A.3.37)}$$
$$\rightarrow \quad \lambda_b \lambda_c = \lambda_d,$$

$$\left[\tilde{J}_{ab}, \tilde{Z}_{cd}\right] = \left[\lambda_a J_{ab}, \lambda_b J_{cd}\right] = \lambda_a \lambda_b \left[J_{ab}, Z_{cd}\right] \propto \lambda_b \delta^{abf}_{ecd} \ Z^e{}_f, \quad \text{(A.3.38)}$$
$$\rightarrow \lambda_a \lambda_b = \lambda_b,$$

$$\left[\tilde{J}_{ab}, \tilde{Q}\right] = \left[\lambda_a J_{ab}, \lambda_c Q\right] = \lambda_a \lambda_c \left[J_{ab}, Q\right] \propto \lambda_c Q, \quad \text{(A.3.39)}$$
$$\rightarrow \lambda_a \lambda_c = \lambda_c,$$

$$\left[\tilde{J}_{ab}, \tilde{Q}'\right] = \left[\lambda_a J_{ab}, \lambda_d Q\right] = \lambda_a \lambda_d \left[J_{ab}, Q\right] \propto \lambda_d Q, \quad \text{(A.3.40)}$$
$$\rightarrow \lambda_a \lambda_d = \lambda_d,$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \left[\lambda_b J_{ab}, \lambda_b J_{cd}\right] = \lambda_b \lambda_b \left[J_{ab}, Z_{cd}\right] = 0, \quad \text{(A.3.41)}$$
$$\rightarrow \quad \lambda_b \lambda_b = \lambda_{0_S},$$

$$\left[\tilde{Z}_{ab}, \tilde{P}_c\right] = \left[\lambda_b J_{ab}, \lambda_e P_c\right] = \lambda_b \lambda_e \left[J_{ab}, P_c\right] = 0, \quad \text{(A.3.42)}$$
$$\rightarrow \quad \lambda_b \lambda_e = \lambda_{0_S},$$

$$\{\tilde{Q}, \tilde{Q}\} = \{\lambda_c Q, \lambda_c Q\} = \lambda_c \lambda_c \{Q, Q\} \propto \lambda_e P_a + \lambda_b J_{ab} + \lambda_e Z_{abcde}, \quad \text{(A.3.43)}$$
$$\rightarrow \ \lambda_c \lambda_c = \lambda_b, \ \ \text{where we have set } \lambda_b = \lambda_e,$$

and the other commutation relations give us results that agree with the above ones.

We observe that, in equation (A.3.43), we must set

$$\lambda_b = \lambda_e, \tag{A.3.44}$$

in order to get consistent relations.

This procedure fixes the degeneracy of the multiplication rules between the elements of the subsets of $\tilde{S}$, and we are now able to write the following multiplication table

$$\begin{array}{c|ccccc} & \lambda_a & \lambda_b & \lambda_c & \lambda_d & \lambda_{0_S} \\ \hline \lambda_a & \lambda_a & \lambda_b & \lambda_c & \lambda_d & \lambda_{0_S} \\ \lambda_b & \lambda_b & \lambda_{0_S} & \lambda_d & \lambda_{0_S} & \lambda_{0_S} \\ \lambda_c & \lambda_c & \lambda_d & \lambda_b & \lambda_{0_S} & \lambda_{0_S} \\ \lambda_d & \lambda_d & \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} \\ \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} & \lambda_{0_S} \end{array} \tag{A.3.45}$$

Then, after having performed the identification

$$a \leftrightarrow 0, \quad b \leftrightarrow 2, \quad c \leftrightarrow 1, \quad d \leftrightarrow 3, \quad 0_S \leftrightarrow 4, \tag{A.3.46}$$

we can finally rewrite the table above as follows (in the usual order):

$$\begin{array}{c|ccccc} & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_3 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \end{array} \tag{A.3.47}$$

Appendix B

Calculations on Ideal-Subtraction

B.1 Calculations for reaching the non-standard Maxwell algebra

We consider the abelian semigroup $S_E^{(\infty)}$ and, following the notation adopted in Ref.s [20, 45] and the procedure developed in the present work, we show that it is possible to obtain an infinite-dimensional superalgebra as a S-expansion of $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$, using $S_E^{(\infty)}$.

We decompose the infinite semigroup $S_E^{(\infty)}$ as follows:

$$S_0 = \{\lambda_0, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \ldots\}, \tag{B.1.1}$$

$$S_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \ldots\}, \tag{B.1.2}$$

allowing resonance between the decomposition of the infinite semigroup $S_E^{(\infty)}$ and the partition over the subsets of the AdS superalgebra, which reads $V_0 = \{\tilde{J}_{ab}\}$, and $V_1 = \{\tilde{Q}_\alpha, \tilde{P}_a\}$.

If we now perform an identification and rename the generators as follows:

$$J^0_{ab} = \lambda_0 \tilde{J}_{ab}, \qquad \lambda_n \tilde{J}_{ab} \equiv \tilde{J}^n_{ab} = J^n_{ab}, \quad n \neq 0, 2, \tag{B.1.3}$$

$$Z^2_{ab} = \lambda_2 \tilde{J}_{ab}, \tag{B.1.4}$$

$$P^1_a = \lambda_1 \tilde{P}_a, \qquad \lambda_s \tilde{P}_a \equiv \tilde{P}^s_a = P^s_a, \quad s \neq 1, \tag{B.1.5}$$

$$Q^1_\alpha = \lambda_1 \tilde{Q}_\alpha, \qquad \lambda_s \tilde{Q}_\alpha = \tilde{Q}^s_\alpha = Q^s_\alpha, \quad s \neq 1, \tag{B.1.6}$$

we can then choose the set of generators on the right to belong to an ideal $\mathcal{I}$.

In order to make this clearer, we write the commutation relations between the expanded generators and we remark with **bold** characters those which we choose to belong to the ideal (those terms must be removed when we perform the ideal subtraction).

For the bosonic sector we have:

$$\left[J_{ab}^{0},J_{cd}^{0}\right]=\eta_{bc}J_{ad}^{0}-\eta_{ac}J_{bd}^{0}-\eta_{bd}J_{ac}^{0}+\eta_{ad}J_{bc}^{0}, \tag{B.1.7}$$
$$\left[J_{ab}^{0},Z_{cd}^{2}\right]=\eta_{bc}Z_{ad}^{2}-\eta_{ac}Z_{bd}^{2}-\eta_{bd}Z_{ac}^{2}+\eta_{ad}Z_{bc}^{2}, \tag{B.1.8}$$
$$\left[J_{ab}^{0},\mathbf{\tilde{J}_{cd}^{n}}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{J}_{ad}^{n}}-\eta_{\mathbf{ac}}\mathbf{\tilde{J}_{bd}^{n}}-\eta_{\mathbf{bd}}\mathbf{\tilde{J}_{ac}^{n}}+\eta_{\mathbf{ad}}\mathbf{\tilde{J}_{bc}^{n}}, \tag{B.1.9}$$
$$\left[\mathbf{\tilde{J}_{ab}^{n}},\mathbf{\tilde{J}_{cd}^{n}}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{J}_{ad}^{n+n}}-\eta_{\mathbf{ac}}\mathbf{\tilde{J}_{bd}^{n+n}}-\eta_{\mathbf{bd}}\mathbf{\tilde{J}_{ac}^{n+n}}+\eta_{\mathbf{ad}}\mathbf{\tilde{J}_{bc}^{n+n}}, \tag{B.1.10}$$
$$\left[J_{ab}^{0},P_{c}^{1}\right]=\eta_{bc}P_{a}^{1}-\eta_{ac}P_{b}^{1}, \tag{B.1.11}$$
$$\left[J_{ab}^{0},\mathbf{\tilde{P}_{c}^{s}}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{P}_{a}^{s}}-\eta_{\mathbf{ac}}\mathbf{\tilde{P}_{b}^{s}}, \tag{B.1.12}$$
$$\left[\mathbf{\tilde{J}_{ab}^{n}},P_{c}^{1}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{P}_{a}^{n+1}}-\eta_{\mathbf{ac}}\mathbf{\tilde{P}_{b}^{n+1}}, \tag{B.1.13}$$
$$\left[\mathbf{\tilde{J}_{ab}^{n}}\mathbf{\tilde{P}_{c}^{s}}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{P}_{a}^{n+s}}-\eta_{\mathbf{ac}}\mathbf{\tilde{P}_{b}^{n+s}}, \tag{B.1.14}$$
$$\left[P_{a}^{1},P_{b}^{1}\right]=Z_{ab}^{2}, \tag{B.1.15}$$
$$\left[\mathbf{\tilde{P}_{a}^{s}},\mathbf{\tilde{P}_{b}^{s}}\right]=\mathbf{\tilde{J}_{ab}^{s+s}}, \tag{B.1.16}$$
$$\left[\mathbf{\tilde{P}_{a}^{s}},P_{b}^{1}\right]=\mathbf{\tilde{J}_{ab}^{s+1}}, \tag{B.1.17}$$

while for the fermionic sector we have:

$$\left[\mathbf{\tilde{P}_{a}^{s}},\mathbf{\tilde{Q}_{\alpha}^{s}}\right]=\mathbf{\frac{1}{2}}\left(\mathbf{\Gamma_{a}\tilde{Q}^{s+s}}\right)_{\alpha}, \tag{B.1.18}$$
$$\left[\mathbf{\tilde{P}_{a}^{s}},Q_{\alpha}^{1}\right]=\mathbf{\frac{1}{2}}\left(\mathbf{\Gamma_{a}\tilde{Q}^{s+1}}\right)_{\alpha}, \tag{B.1.19}$$
$$\left[P_{a}^{1},\mathbf{\tilde{Q}_{\alpha}^{s}}\right]=\mathbf{\frac{1}{2}}\left(\mathbf{\Gamma_{a}\tilde{Q}^{s+1}}\right)_{\alpha}, \tag{B.1.20}$$
$$\left[P_{a}^{1},Q_{\alpha}^{1}\right]=\mathbf{\frac{1}{2}}\left(\mathbf{\Gamma_{a}\tilde{Q}_{\alpha}^{2}}\right)_{\alpha}, \tag{B.1.21}$$
$$\left[\mathbf{\tilde{J}_{ab}^{n}},\mathbf{\tilde{Q}_{\alpha}^{s}}\right]=\mathbf{\frac{1}{2}}\left(\mathbf{\Gamma_{ab}\tilde{Q}^{s+n}}\right)_{\alpha}, \tag{B.1.22}$$
$$\left[J_{ab}^{0},Q_{\alpha}^{1}\right]=\frac{1}{2}\left(\Gamma_{ab}Q^{1}\right)_{\alpha}, \tag{B.1.23}$$
$$\left\{\mathbf{\tilde{Q}_{\alpha}^{s}},\mathbf{\tilde{Q}_{\beta}^{s}}\right\}=-\mathbf{\frac{1}{2}}\left[\left(\mathbf{\Gamma^{ab}C}\right)_{\alpha\beta}\mathbf{\tilde{J}_{ab}^{s+s}}-\mathbf{2}\left(\mathbf{\Gamma^{a}C}\right)_{\alpha\beta}\mathbf{\tilde{P}_{a}^{s+s}}\right], \tag{B.1.24}$$
$$\left\{Q_{\alpha}^{1},\mathbf{\tilde{Q}_{\beta}^{s}}\right\}=-\mathbf{\frac{1}{2}}\left[\left(\mathbf{\Gamma^{ab}C}\right)_{\alpha\beta}\mathbf{\tilde{J}_{ab}^{s+1}}-\mathbf{2}\left(\mathbf{\Gamma^{a}C}\right)_{\alpha\beta}\mathbf{\tilde{P}_{a}^{s+1}}\right], \tag{B.1.25}$$
$$\left\{Q_{\alpha}^{1},Q_{\beta}^{1}\right\}=-\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}Z_{ab}^{2}-2(\mathbf{\Gamma^{a}C})_{\alpha\beta}\mathbf{\tilde{P}_{a}^{2}}\right], \tag{B.1.26}$$
$$\left[Z_{ab}^{2},Z_{cd}^{2}\right]=\eta_{\mathbf{bc}}\mathbf{\tilde{J}_{ad}^{4}}-\eta_{\mathbf{ac}}\mathbf{\tilde{J}_{bd}^{4}}-\eta_{\mathbf{bd}}\mathbf{\tilde{J}_{ac}^{4}}+\eta_{\mathbf{ad}}\mathbf{\tilde{J}_{bc}^{4}}, \tag{B.1.27}$$
$$\left[Z_{ab}^{2},Q_{\alpha}^{1}\right]=-\frac{1}{2}\left(\mathbf{\Gamma_{ab}\tilde{Q}^{3}}\right)_{\alpha}, \tag{B.1.28}$$

where n and s can also assume the same value, but the conditions $n \neq 0, 2$ and $s \neq 1$ must hold. The subtraction of the ideal correspond to erase the parts in **bold** character. After having performed the ideal subtraction, we end up with the non-standard Maxwell superalgebra described in Subsection 2.6.6.

B.2 Calculations for reaching the super-PP-wave algebra

We choose the following ideal:

$$\mathcal{I} = (\{\lambda_{2n}\}_{n=1}^{\infty} \times V_0) \oplus (\{\lambda_{2n+1}\}_{n=1}^{\infty} \times V_1) \oplus (\{\lambda_{2n}\}_{n=2}^{\infty} \times V_2)\,. \qquad \text{(B.2.1)}$$

We then subtract this ideal and what finally remains is given by

$$[\lambda_0 P_-, \lambda_1 Q_+] = -\frac{3}{2\sqrt{2}}\lambda_1 Q_+ I, \qquad \text{(B.2.2)}$$

$$[\lambda_0 P_-, \lambda_0 Q_-] = -\frac{1}{2\sqrt{2}}\lambda_0 Q_- I \qquad \text{(B.2.3)}$$

$$[\lambda_1 P_i, \lambda_0 Q_-] = \frac{1}{\sqrt{2}}\lambda_1 Q_+ \Gamma^- I \Gamma_i, \qquad \text{(B.2.4)}$$

$$[\lambda_1 P_{i'}, \lambda_0 Q_-] = \frac{1}{2\sqrt{2}}\lambda_1 Q_+ \Gamma^- I \Gamma_{i'}, \qquad \text{(B.2.5)}$$

$$[\lambda_1 P^*_m, \lambda_0 Q_-] = \frac{1}{2\sqrt{2}}\lambda_1 Q_+ \Gamma_m \Gamma^-, \qquad \text{(B.2.6)}$$

$$[\lambda_0 J_{mn}, \lambda_0 Q_\pm] = \frac{1}{2}\lambda_0 Q_\pm \Gamma_{mn}, \qquad \text{(B.2.7)}$$

$$\{\lambda_1 Q_+, \lambda_1 Q_+\} = -2\mathcal{C}\Gamma^+ \lambda_1 P_+, \qquad \text{(B.2.8)}$$

$$\{\lambda_0 Q_-, \lambda_0 Q_-\} = -2\mathcal{C}\Gamma^- \lambda_0 P_- - \sqrt{2}\mathcal{C}\Gamma^- I \Gamma^{ij} \lambda_0 J_{ij} + \frac{1}{\sqrt{2}}\mathcal{C}\Gamma^- I \Gamma^{i'j'} \lambda_0 J_{i'j'}, \qquad \text{(B.2.9)}$$

$$\{\lambda_1 Q_+, \lambda_0 Q_-\} = \left(-2\mathcal{C}\Gamma^m \lambda_1 P_m - 4\mathcal{C}I\Gamma^i \lambda_1 P^*_i - 2\mathcal{C}I\Gamma^{i'} \lambda_1 P^*_{i'}\right)\lambda_0 \mathcal{P}_-. \qquad \text{(B.2.10)}$$

Then, we only have to properly rename the generators in order to find (2.6.101).

Appendix C

Dirac-Schwinger-Zwanziger quantization

Electromagnetism without monopoles is $\partial_\mu\ {}^*F^{\mu\nu} = 0$, that means $F\mu\nu = \partial_\mu A_\nu - \partial_\nu a_\mu$, we A_μ is defined in all the space-time. Also we have the invarianze

$$A'_\mu = A_\mu + \partial_\mu \alpha \tag{C.0.1}$$

that satisfies $F\mu\nu$.

Also, a particle of mass m satisfies the Schrodinger equations which is invariant under (C.0.1) if ψ transform as $\psi' = e^{-\frac{iq\alpha}{\hbar}}\psi$. But if we consider the existence of magnetic monopoles,

$$\vec{B}_m = \frac{g}{4\pi r^3}\vec{r} \implies \vec{\nabla}\cdot\vec{B}_m = g\delta^3(\vec{r}) \tag{C.0.2}$$

Since the divergence of B is different from zero we cannot have $\vec{A}$ regular in all the space and satisfies the same equations as F. Neverless we can use the ambiguity of A transformation and we use different potentials in different regions of the space,

$$A_N = \frac{g}{4\pi i}\frac{(1-\cos\theta}{\sin\theta}\hat{e}_\theta, \tag{C.0.3}$$

$$A_S = -\frac{g}{4\pi i}\frac{(1+\cos\theta)}{\sin\theta}\hat{e}_\theta. \tag{C.0.4}$$

Then, we have

$$\nabla\times A_S = B_m.$$

B_m can be describe with A_N or A_S. In every place we must choose a function ψ_N or ψ_S, which differ by a phase. We know that the function have to be simple-evaluate from quantum mechanichs, but from the transformation of ψ' we know that both are simulstanously simple-evaluate if

$$e^{-\frac{iq\alpha(\phi)}{\hbar}} = e^{-\frac{iq\alpha(\phi+2\pi)}{\hbar}},$$

taking $\alpha(\phi) = -\frac{g}{2\pi}\phi$ we see that $qg = 2\pi n\hbar,\ n\in\mathbb{Z}$, and if we consider more than one monopole; $q_i g_j = 2\pi n_{ij}\hbar, \quad n_{ij}\in\mathbb{Z}$. Then, if we consider also dyons we have

$$q_i g_j - q_j g_i = 2\pi n_{ij}\hbar. \tag{C.0.5}$$

Which is the condition of Dirac-Schiwinger-Zwanzinger.

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