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About non-symplectic automorphisms of composite order of K3 surfaces

Tesis para optar al grado de Doctor en Matemática

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A mi familia.

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Introduction

An automorphism of finite order $n \geq 2$ of a complex projective K3 surface is called *non-symplectic* if its action on the vector space of holomorphic 2-forms is non-trivial, *purely non-symplectic* if such action has order n. By [Nik79a, Theorem 0.1] the rank of the transcendental lattice of a K3 surface carrying a purely non-symplectic automorphism of order n is divisible by the Euler's function of n. This implies that $\varphi(n) \leq 21$, and all positive integers $n \neq 60$ with such property occur as orders of purely non-symplectic automorphisms by [MO98, Main Theorem 3]. A classification of purely non-symplectic automorphisms is known for all prime orders [Nik79a, OZ98, OZ11, Vor83, OZ00, Kon92, AS08, AST11], when $\varphi(n) = 20$ [MO98], when the automorphism acts trivially on the Néron-Severi lattice and $\varphi(n)$ equals the rank of the transcendental lattice [Vor83, Kon92, OZ00, Sch10], for orders $6, 16$ [Dil12, ATST16] and $4, 8$ [AS15, ATS18] (the latter contain partial classifications). In case the automorphism has prime order its invariant lattice in $H^2(X, \mathbb{Z})$ is a p-elementary lattice. This makes the classification of such automorphisms easier, by means of lattice theory, since p-elementary lattices are classified. On the other hand, the classification of automorphisms of composite order is more subtle and requires the use of geometric arguments.

The aim of this thesis is to give a contribution to the classification of non-symplectic automorphisms of composite order. By *classification* we intend the following: determine the possible topological structures of the fixed locus and, according to this, the eigenspace decomposition of the induced automorphism of $H^2(X, \mathbb{C})$.

The first part of this thesis consists in the classification of automorphisms of order 9 of $K3$ surfaces. By [Muk88] an automorphism of order nine is non-symplectic. Moreover, we prove that it is necessarily purely non-symplectic and we prove the following result (see Theorem 3.2.1 and Table 3.2).

Theorem 1. *Let X be a complex K3 surface and σ be an automorphism of X of order nine. Then*

(i) *σ is purely non-symplectic, i.e. $\tau = \sigma^3$ is non-symplectic;*

(ii) *the topological structure of $\mathrm{Fix}(\sigma)$, $\mathrm{Fix}(\tau)$, the dimensions of the eigenspaces of σ^* and τ^* in $H^2(X, \mathbb{C})$ and the invariant lattice of τ^* in $H^2(X, \mathbb{Z})$ are described in the following Table.*

τ			σ		
Case	(N, k, g)	$S(\tau)$	*Case*	$(N_\sigma, k_\sigma, g_\sigma)$	(d_1, d_3)
A	*(1,0,3)*	$U(3) \oplus A_2$	*A1*	$(6, 0, -)$	*(4,0)*
			A2	$(3, 0, -)$	*(2,1)*
B	*(1,1,4)*	$U \oplus A_2$	*B*	$(6, 0, -)$	*(4,0)*
C	*(4,0,0)*	$U(3) \oplus A_2^4$	*C*	$(3, 0, -)$	*(4,3)*
D	*(4,1,1)*	$U \oplus A_2^4$	*D1*	$(7, 0, 0)$	*(8,1)*
			D2	$(3, 0, 1)$	*(4,3)*
			D3	$(6, 0, -)$	*(6,2)*
			D4	$(3, 0, -)$	*(4,3)*
E	*(4,2,2)*	$U \oplus E_6 \oplus A_2$	*E*	$(10, 0, 0)$	*(10,0)*
F	*(4,3,3)*	$U \oplus E_8$	*F*	$(10, 0, 0)$	*(10,0)*
G	*(7,3,0)*	$U \oplus E_6^2 \oplus A_2$	*G1*	$(10, 0, 0)$	*(12,2)*
			G2	$(3, 0, -)$	*(6,5)*
H	*(7,4,1)*	$U \oplus E_6 \oplus E_8$	*H*	$(14, 1, 0)$	*(16,0)*

The integers N, k, g are the number of isolated fixed points, the number of smooth connected fixed curves and the maximal genus of a fixed curve of τ respectively; the

> *integers $N_\sigma, k_\sigma, g_\sigma$ denote the same numbers for σ; $S(\tau)$ is the fixed lattice of τ^* in $H^2(X,\mathbb{Z})$ and d_1, d_3 are the dimensions of the eigenspaces of σ^* in $H^2(X,\mathbb{C})$ relative to the eigenvalues 1 and ζ_3 (a primitive 3rd root of unity) respectively.*

Moreover, all configurations described in the Table exist.

This result relies on the classification for order 3 [AS08] and generalizes [Tak10], where the author classified non-symplectic automorphisms of a 3-power order which act trivially on the Néron-Severi group. As a consequence of Theorem 1, we show that the moduli space of K3 surfaces with a non-symplectic automorphism of order 9 has three irreducible components of maximal dimension 2, whose members can be described as smooth quartic surfaces or elliptic surfaces, up to isomorphism. These results are contained in the paper [ACV20].

In the second part of this thesis we classify $K3$ surfaces with the action of a purely non-symplectic automorphism whose moduli space is 1-dimensional. This holds when $\varphi(n)$ is either 8 or 10, i.e. when $n = 11, 15, 16, 20, 22, 24$ or 30. More precisely, for any such order we classify the general members (X, σ) of the one-dimensional components of the moduli space of K3 surfaces with a purely non-symplectic automorphisms of order n. Observe that the case $n = 11$ has been classified in [OZ11, AST11] and the case $n = 16$ in [ATST16]. The main result is the following, where V^σ is the ζ_n-eigenspace of σ^* in $H^2(X,\mathbb{C})$.

Theorem 2. *Let X be a complex K3 surface and σ be a purely non-symplectic automorphism of X of order $n \geq 2$ such that $\varphi(n) \in \{8, 10\}$ and $\dim V^\sigma = 2$. Then the fixed locus of σ and of some of its powers σ_i, the vector d containing the dimensions of the eigenspaces of σ^* in $H^2(X,\mathbb{C})$ and the Néron-Severi lattice of a general K3 surface with such property are described in Table 4.1. Moreover, all configurations described in the Table exist.*

Since T_X has the structure of a $\mathbb{Z}[\zeta_n]$-module and $\varphi(n) = 8$ or 10, then $\operatorname{rk}\mathrm{NS(X)} \geq 22 - 2\varphi(n)$. The generality assumption in the statement of Theorem 2 means that the Néron-Severi lattice of X has the minimal rank. For orders $n = 15, 20, 22$ and 30 we also prove uniqueness results, i.e. we prove that there is a unique irreducible family of K3 surfaces carrying a purely non-symplectic automorphism of order n once the topological structure of the fixed locus of σ is given (see Propositions 4.4.3, 4.5.4, 4.6.5, 4.8.3).

Finally, for orders $n = 15$ and 22 we provide a full classification of K3 surfaces with a purely non-symplectic automorphism of order n (not only for the general ones in the components of maximal dimension). We recall the main results here.

Theorem 3. *Let σ_{15} be a purely non-symplectic automorphism of order 15 on a K3 surface X. The fixed locus of σ_{15} consists of a set of points or the union of a smooth rational curve and a set of points. All possibilities for the fixed locus of σ_{15} and of its powers are described in one of the rows of the following table, where N_{15} is the number of isolated fixed points of σ_{15}, g_i and k_i are the maximal genus and the number of fixed smooth rational curves of $\sigma_i = \sigma^{\frac{15}{i}}$ respectively, and $a_{j,i}$ is the number of fixed points of type $A_{j,i}$ of σ_i, for $i = 3, 5$.*

	N_{15}	k_{15}	$a_{1,5}$	$a_{2,5}$	g_5	k_5	$a_{1,3}$	g_3	k_3
A1	*5*	*0*	*1*	*0*	*2*	*0*	*2*	*2*	*0*
B4	*7*	*0*	*3*	*1*	*1*	*0*	*1*	*4*	*1*
D1	*10*	*1*	*5*	*2*	*1*	*1*	*6*	*0*	*2*
F3	*9*	*1*	*7*	*3*	*0*	*1*	*4*	*2*	*2*
F7	*12*	*1*	*7*	*3*	*0*	*1*	*5*	*2*	*3*
F8	*5*	*0*	*7*	*3*	*0*	*1*	*2*	*2*	*0*

Moreover, all cases in the table exist.

Theorem 4. *Let σ_{22} be a purely non-symplectic automorphism of order 22 on a K3 surface X. Then the fixed locus of σ_{22} and of its powers are described in one of the rows of the following table.*

	$\mathrm{Fix}(\sigma_{22})$	$\mathrm{Fix}(\sigma_{11})$	$\mathrm{Fix}(\sigma_2)$
$A1$	$\{p_1,\dots,p_6\}$	$C_1 \sqcup \{p_5, p_6\}$	$C_{10} \sqcup R$
$B1$	$R_1 \sqcup \{p_1,\dots,p_{11}\}$	$R_1 \sqcup \{p_1,\dots,p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_5$
$B2$	$R_1 \sqcup \{p_1,\dots,p_9\}$	$R_1 \sqcup \{p_1,\dots,p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_4$
$B3$	$\{p_1,\dots,p_5\}$	$R_1 \sqcup \{p_1,\dots,p_{11}\}$	$C_5 \sqcup R_2,$

where $g(C_i) = i$ *for* $i = 1, 5, 10$ *and* $g(R_j) = 0$ *for* $j = 1,\dots,5$. *Moreover, all cases exist.*

These results are contained in the paper [ACV].

We now describe briefly the content of each Chapter.

Chapter 1 is about the preliminaries definitions and theorems. We give a brief description for lattices, surfaces and elliptic fibrations.

Chapter 2 consists of studying the theory of $K3$ surfaces and their automorphisms, focusing in non-symplectic automorphisms, giving a description of their fixed locus. We also present some of the results known in the literature so far on non-symplectic automorphisms, and that will help us in the development of Chapters 3 and 4.

In Chapter 3 we prove Theorem 5, and we give examples for each case in the classification.

In Chapter 4 we prove Theorems 6, 7 and 8, with their respective examples.

Introducción

Un automorfismo de orden finito $n \geq 2$ de una superficie $K3$ proyectiva compleja es llamado *no-simpléctico* si su acción en el espacio vectorial de las 2-formas holomorfas es no trivial, y es *puramente no-simpléctico* si tal acción tiene orden n. De [Nik79a, Theorem 0.1] el rango del reticulado trascendental de una superficie $K3$ con un automorfismo puramente no-simpléctico de orden n es divisible por la función de Euler de n. Esto implica que $\varphi(n) \leq 21$ y todos los enteros positivos $n \neq 60$ con tal propiedad resultan ser los ordenes de automorfismos no-simplécticos [MO98, Main Theorem 3]. Se conoce una clasificación de automorfismos puramente no-simpléctios para todos los ordenes primos [Nik79a, OZ98, OZ11, Vor83, OZ00, Kon92, AS08, AST11], cuando $\varphi(n) = 20$ [MO98], cuando el automorfismo actúa en el reticulado de Néron-Severi y $\varphi(n)$ es igual al rango del reticulado trascendental [Vor83, Kon92, OZ00, Sch10], para los ordenes $6, 16$ [Dil12, ATST16] y $4, 8$ [AS15, ATS18] (los últimos contienen clasificaciones parciales). En caso de que el automorfismo tenga orden primo, su reticulado invariante en $H^2(X, \mathbb{Z})$ es un reticulado p-elemental. Esto hace que la clasificación de estos automorfismos sea más fácil, por medio de la teoría de reticulados, ya que los reticulados p-elementales están clasificados. Por otro lado, la clasificación de los automorfismos de orden compuesto es más sutil y se requiere del uso de argumentos geométricos.

El objetivo de esta tesis es es dar una contribución a la clasificación de automorfismos no-simplécticos de orden compuesto. Por *clasificación* entendemos lo siguiente: determinar las posibles estructuras topológicas del lugar fijo y de acuerdo a esto, la descomposición del espacio propio del automorfismo inducido de $H^2(X, \mathbb{C})$.

La primera parte de esta tesis consiste en la clasificación de los automorfismos de ordern 9 de una superficie $K3$. Según [Muk88] un automorfismo de order nueve es no-simpléctico. Más aún, probamos que necesariamente es puramente no-simpléctico y demostramos el siguiente resultado (ver Teorema 3.2.1 y la Tabla 3.2).

Teorema 5. *Sea X una superficie $K3$ compleja y σ un automorfismo de X de orden nueve. Entonces*

(i) *σ es puramente no-simpléctico, i.e. $\tau = \sigma^3$ es no-simpléctico;*

(ii) *la estructura topológica de* $\mathrm{Fix}(\sigma)$, $\mathrm{Fix}(\tau)$, *las dimensiones de los espacios propios de σ^* y τ^* in $H^2(X,\mathbb{C})$ y el reticulado invariante de τ^* en $H^2(X,\mathbb{Z})$ están descritos en la siguiente Tabla.*

τ			σ		
Casp	(N,k,g)	$S(\tau)$	*Caso*	$(N_\sigma,k_\sigma,g_\sigma)$	(d_1,d_3)
A	*(1,0,3)*	$U(3)\oplus A_2$	*A1*	$(6,0,-)$	*(4,0)*
			A2	$(3,0,-)$	*(2,1)*
B	*(1,1,4)*	$U\oplus A_2$	B	$(6,0,-)$	*(4,0)*
C	*(4,0,0)*	$U(3)\oplus A_2^4$	C	$(3,0,-)$	*(4,3)*
D	*(4,1,1)*	$U\oplus A_2^4$	*D1*	$(7,0,0)$	*(8,1)*
			D2	$(3,0,1)$	*(4,3)*
			D3	$(6,0,-)$	*(6,2)*
			D4	$(3,0,-)$	*(4,3)*
E	*(4,2,2)*	$U\oplus E_6\oplus A_2$	E	$(10,0,0)$	*(10,0)*
F	*(4,3,3)*	$U\oplus E_8$	F	$(10,0,0)$	*(10,0)*
G	*(7,3,0)*	$U\oplus E_6^2\oplus A_2$	*G1*	$(10,0,0)$	*(12,2)*
			G2	$(3,0,-)$	*(6,5)*
H	*(7,4,1)*	$U\oplus E_6\oplus E_8$	H	$(14,1,0)$	*(16,0)*

Los enteros N,k,g son el número de puntos fijos aislados, el número de curvas *suaves conexas y el género maximal de una curva fija de τ, respectivamente; los enteros $N_\sigma, k_\sigma, g_\sigma$ denotan los mismo números para σ; $S(\tau)$ es el reticulado fijo de τ^* en $H^2(X,\mathbb{Z})$ y d_1, d_3 son las dimensiones de los espacios propios de σ^* en $H^2(X,\mathbb{C})$ relativos a los valores propios 1 y ζ_3 (una raíz cúbica primitiva de la unidad) respetivamente.*

Más aún, todas las configuraciones descritas en la Tabla existen.

Este resultado se basa en la clasificación de orden 3 [AS08] y generaliza [Tak10], donde el autor clasificó los automorfismos de order potencia de 3 que actúan trivialmente sobre el grupo de Néron-Severi. Como consecuencia del Teorema 5, mostramos que el espacio de moduli de una superficie $K3$ con automorfismo no-simpléctico de orden 9 tiene tres componentes irreducibles de dimensión máxima 2, cuyos miembros pueden ser descritos como superficies cuárticas suaves o superficies elípticas, salvo isomorfismo. Estos resultados están contenidos en el artículo [ACV20].

En la segunda parte de esta tesis clasificamos las superficies $K3$ con la acción de un automorfismo puramente no-simpléctico cuyo espacio de moduli es 1-dimensional. Esto se tiene cuando $\varphi(n)$ es 8 o 10, i.e. cuando $n = 11, 15, 16, 20, 22, 24$ y 30. Más precisamente, para cualquier order, clasificamos el miembro general (X,σ) de la componente 1-dimensional del espacio de moduli de las superficies $K3$ con automorfismo puramente no-simpléctico de orden n. El caso para orden $n = 11$ fue clasificado en [OZ11, AST11] y el caso para $n = 16$ en [ATST16]. El resultado principal es el siguiente, donde V^σ es el espacio vectorial de ζ_n de σ^* en $H^2(X,\mathbb{C})$.

Teorema 6. *Sean X una superficie K3 compleja y σ un automorfismo puramente no-simpléctico de X de orden $n \geq 2$ tal que $\varphi(n) \in \{8, 10\}$ y $\dim V^\sigma = 2$. Entonces, el lugar fijo de σ y el de algunas de sus potencias σ_i, el vector d que contiene las dimensiones de los espacios propios de σ^* en $H^2(X,\mathbb{C})$ y el reticulado Néron-Severi de una superficie $K3$ general con tal propiedad están descritos en la Tabla 4.1. Más aún, todas las configuraciones descritas en la Tabla existen.*

Como T_X tiene la estructura de $\mathbb{Z}[\zeta_n]$-modulo y $\varphi(n) = 8$ o 10, entonces $\operatorname{rk}\mathrm{NS(X)} \geq$

$22 - 2\varphi(n)$. La generalidad en la suposión del Teorema 6 significa que el reticulado Néron Severi de X tiene rango minimal. También, para los ordernes $n = 15, 20, 22$ y 30 demostramos resultados de unicidad, es decir probamos que existe una única familia de superficies $K3$ con automorfismo puramente no-simpléctico de orden n, una vez que la estructura topológica del lugar fijo de σ sea dada (ver las Proposiciones 4.4.3, 4.5.4, 4.6.5, 4.8.3).

Por último, para los ordenes $n = 15$ y $n = 22$ damos una clasificación completa de las superficies $K3$ con automorfismo puramente no-simpléctico de orden n (no solo para las generales en la componente de dimensión maximal). Estos son los resultados principales.

Teorema 7. *Sea σ_{15} un automorfismo puramente no-simpléctico de de orden 15 sobre una superficie $K3$ X. El lugar fijo de σ_{15} consiste en un conjunto de puntos o en la unión de una curva racional suave y un conjunto de puntos. Todas las posibilidades para el lugar fijo de σ_{15} y sus potencias están descritas en una de las filas de la siguiente tabla, donde N_{15} es el número de puntos fijos aislados de σ_{15}, g_i y k_i son el género máximo y el número de curvas racionales suaves fijas de $\sigma_i = \sigma^{\frac{15}{i}}$ respectivamente, y $a_{j,i}$ es el número de puntos fijo de tipo $A_{j,i}$ de σ, para $i = 3, 5$.*

	N_{15}	k_{15}	$a_{1,5}$	$a_{2,5}$	g_5	k_5	$a_{1,3}$	g_3	k_3
A1	*5*	*0*	*1*	*0*	*2*	*0*	*2*	*2*	*0*
B4	*7*	*0*	*3*	*1*	*1*	*0*	*1*	*4*	*1*
D1	*10*	*1*	*5*	*2*	*1*	*1*	*6*	*0*	*2*
F3	*9*	*1*	*7*	*3*	*0*	*1*	*4*	*2*	*2*
F7	*12*	*1*	*7*	*3*	*0*	*1*	*5*	*2*	*3*
F8	*5*	*0*	*7*	*3*	*0*	*1*	*2*	*2*	*0*

Más aún, todos los casos en la tabla existen.

Teorema 8. *Sea σ_{22} un automorfismo puramente no-simpléctico de orden 22 sobre una superficie $K3$ X. Entonces el lugar fijo de σ_{22} y sus pontencias están descritos en una* *de las filas de la siguiente tabla.*

	$\mathrm{Fix}(\sigma_{22})$	$\mathrm{Fix}(\sigma_{11})$	$\mathrm{Fix}(\sigma_2)$
$A1$	$\{p_1, \dots, p_6\}$	$C_1 \sqcup \{p_5, p_6\}$	$C_{10} \sqcup R$
$B1$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_5$
$B2$	$R_1 \sqcup \{p_1, \dots, p_9\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_4$
$B3$	$\{p_1, \dots, p_5\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_2,$

donde $g(C_i) = i$ para $i = 1, 5, 10$ t $g(R_j) = 0$ for $j = 1, \dots, 5$. Más aún, todos los casos existen.

Estos resultados están contenidos en el artículo [ACV].

Ahora, describiremos brevemente el contendo de cada capítulo.

El Capítulo 1 trata de las definiciones y teoremas preliminares. Damos una breve descricpión para los reticulados, superficies y fibraciones elíticas.

El Capítulo 2 consiste en estudiar la teoría de las superficies $K3$ y sus automorfismos, enfocándonos en los automorfismos no-simplécticos, dando una descripción de sus lugares fijos. También presentamos algunos resultados conocidos hasta hora en la literatura para automorfismos no-simplécticos, y que nos ayudarán en el desarrollo de los Capítulos 3 y 4.

En el Capítulo 3 demostramos el Teorema 5, y damos ejemplos para cada caso en la clasificación.

En el Capítulo 4 demostramos los Teoremas 6, 7 and 8, con sus respectivos ejemplos.

Chapter 1

Preliminaries

1.1 Lattices

A general reference for this section are [BHPVdV04, §2, Ch. I] and [Dol83].

Definition 1.1.1.

(i) A **lattice** (L, q) is a finitely generated free $\mathbb{Z}$-module L endowed with a quadratic form $q : L \to \mathbb{Z}$. We will denote by $b : L \times L \to \mathbb{Z}$ the associated symmetric bilinear form.

(ii) The **rank of the lattice** L, denoted $\mathrm{rank}(L)$, is the dimension of the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$.

(iii) Given a basis $\{e_1, e_2, \ldots, e_n\}$ of L, the **Gram matrix** of the lattice associated to such basis is $E = (b(e_i, e_j))$.

(iv) The **determinant** of the lattice, denoted by $d(L)$, is the determinant of any of its Gram matrices (it is uniquely determined independently of the choice of the basis).

(v) If $d(L) \neq 0$ then we will say that (L, q) is **non-degenerate**.

(vi) If $d(L) = \pm 1$ then (L, q) is called **unimodular**.

(vii) If q takes values in $2\mathbb{Z}$, we will say that the lattice is **even**; otherwise it will be called **odd lattice**.

(viii) The **signature** of the lattice is the pair whose coordinates are the number of positive and negative eigenvalues $\mathrm{sign}(L) = (s_+, s_-)$ of the extension of the quadratic form to $L \otimes_{\mathbb{Z}} \mathbb{R}$.

(ix) If $\mathrm{sign}(L) = (r, 0)$ (or $(0, r)$) where $r = \mathrm{rank}(L)$, then the lattice is called **positive (negative) definite**, otherwise it is an **indefinite** lattice.

(x) The lattice L will be called **hyperbolic** if its signature is $(1, r-1)$, where $r = \mathrm{rank}(L)$.

(xi) The **dual of** L is $L^\vee = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$.

Let $b_{\mathbb{Q}} : L \otimes_{\mathbb{Z}} \mathbb{Q} \times L \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ be the $\mathbb{Q}$-linear extension of b. Observe that

$$\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b_{\mathbb{Q}}(v, x) \in \mathbb{Z}, \forall x \in L\}$$

and there is a natural inclusion $\phi : L \hookrightarrow L^\vee$ given by $\phi(x) = b(x, -)$.

Definition 1.1.2. Let (L_1, q_1) and (L_2, q_2) be two lattices.

(i) A **homomorphism of lattices** is a homomorphism $f : L_1 \to L_2$ of $\mathbb{Z}$-modules such that $q_2(f(x)) = q_1(x)$.

(ii) A bijective homomorphism of lattices is called **isometry of lattices**.

(iii) An injective homomorphism of lattices $i : L_1 \to L_2$ is called **primitive embedding** if $L_2/i(L_1)$ is a free abelian group.

(iv) A sublattice L_1 of L_2 is called **primitive** if the inclusion map $i : L_1 \to L_2$ is a primitive embedding.

For the following result see [BHPVdV04, Lemma 2.1, Lemma 2.2, Lemma 2.6]

Lemma 1.1.3. *Let L be a non-degenerate lattice then:*

(i) *The index as $\mathbb{Z}$-module of $\phi(L)$ in $L^\vee$ is $|d(L)|$.*

(ii) *If M is a submodule of L with* $\operatorname{rank}(M) = \operatorname{rank}(L)$, *then* $(L : M)^2 = d(M)d(L)^{-1}$, *where $(L : M)$ is the index of M in L.*

(iii) *If M is a submodule of L then* $\operatorname{rank}(M) + \operatorname{rank}(M^\perp) = \operatorname{rank}(L)$.

(iv) *If L is unimodular and $M \subset L$ is a primitive sublattice, then $|d(M)| = |d(M^\perp)|$. Moreover, if M is unimodular then $L = M \oplus M^\perp$.*

Definition 1.1.4. Let (L, q) be a lattice and $L^\vee$ its dual.

(i) The **discriminant group of** L is the finite abelian group $A_L := L^\vee/L$.

(ii) The quadratic form q induces a quadratic form $q_{A_L} : A_L \to \mathbb{Q}/2\mathbb{Z}$, called **discriminant quadratic form** of L.

Examples 1.1.5. The following are some well know lattices:

(i) The lattice $(\mathbb{Z}, (\pm 1))$ is an odd, unimodular lattice.

(ii) The lattice $U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ is a hyperbolic, even, rank two unimodular lattice of signature $(1, 1)$.

(iii) The lattices A_n, D_k, E_i for $n \geq 1, k \geq 4, i = 6, 7, 8$ are negative definite, even lattices associated to the Dynkin diagrams of the corresponding types. For each case in Table 1.1, we obtain the Gram matrix of the quadratic form q as follows: Let us denote by v_i a vertex of the graph, then in the entry $a_{i,j}$ of the matrix we write:

$$a_{i,j} = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } v_i \text{ meets } v_j \\ 0 & \text{if } v_i \text{ does not meet } v_j. \end{cases}$$

(iv) Given two lattices (L_1, q_1) and (L_2, q_2) we will denote by $L_1 \oplus L_2$ their direct sum in the standard way, endowed with the bilinear form $b_{L_1 \oplus L_2}((x_1, y_1), (x_2, y_2)) = b_{L_1}(x_1, x_2) + b_{L_2}(y_1, y_2)$. Moreover, we will denote by L^m the direct sum of m copies of a lattice (L, q) with itself.

(v) Given a lattice (L, q) we will denote by $L(m)$, $m \in \mathbb{Z}$ the lattice obtained multiplying by m a Gram matrix of (L, q).

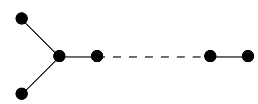
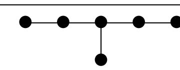
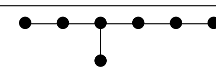
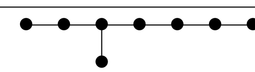
Lattice	rank(L)	$d(L)$	A_L	Dynkin diagram
A_n	n	$n+1$	$\mathbb{Z}/(n+1)\mathbb{Z}$	
D_k	k	4	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ if k is even $\mathbb{Z}/4\mathbb{Z}$ if k is odd	
E_6	6	3	$\mathbb{Z}/3\mathbb{Z}$	
E_7	7	2	$\mathbb{Z}/2\mathbb{Z}$	
E_8	8	1	$\{0\}$	

Table 1.1 Lattices of types A, D, E.

Theorem 1.1.6 ([Mil58])**.** *Let (L, q) be an indefinite unimodular lattice. If (L, q) is even, then $L \cong E_8(\pm 1)^m \oplus U^n$ for some $m, n \in \mathbb{Z}$. If (L, q) is odd, then $L \cong (1)^m \oplus (-1)^n$ for some $m, n \in \mathbb{Z}$.*

Definition 1.1.7. The lattice $\Lambda_{K3} = U^3 \oplus E_8^2$ is called $K3$ **lattice**.

The lattice Λ_{K3} is an even unimodular lattice of rank 22 and signature $\text{sign}(\Lambda_{K3}) = (3, 19)$.

Definition 1.1.8. Let p be a prime number. A lattice (L, q) is p**-elementary** if $A_L \simeq (\mathbb{Z}/p\mathbb{Z})^a$ for some non-negative integer a.

Theorem 1.1.9. (i) *An even, indefinite, p-elementary lattice (L, q) of rank r with $|d(L)| = p^a$, for $p \neq 2$ and $r \geq 2$ is uniquely determined, up to isometries, by the integer a.*

(ii) *For $p \neq 2$ a hyperbolic p-elementary even lattice L with $|d(L)| = p^a$ and $\text{rank}(L) = r$ exists if and only if the following conditions are satisfied: $a \leq r$, $r \equiv \pmod 2$ and*

$$\begin{cases} for \;\; a \equiv 0\,(mod\ 2), & r \equiv 2 \pmod 2 \\ for \;\; a \equiv 1\,(mod\ 2), & p \equiv (-1)^{r/2-1} \pmod 4 \end{cases}$$

Moreover $r > a > 0$, if $r \neq 2\,(mod\ 8)$.

(iii) *An even indefinite, 2-elementary lattice is determined by r, a and a third invariant $\delta \in \{0, 1\}$ see* [Nik79b].

1.2 Surfaces

In this thesis by surface we mean a 2-dimensional smooth complex projective algebraic variety.

Definition 1.2.1. Let X be a surface.

(i) A **divisor** on X is a formal finite linear combination $D = \sum_i a_i Y_i$, where $a_i \in \mathbb{Z}$ and $Y_i \subset X$ are irreducible hypersurfaces of X. The set of all divisors of X is denoted by $\text{Div}(X)$ and has the structure of a free abelian group.

(ii) A divisor is **effective** if $a_i \geq 0$, for all i.

(iii) The **support of** D is the union of all Y_i with $a_i \neq 0$, denoted as $\text{supp}(D)$.

(iv) A divisor is **prime** if it contains only one summand with coefficient 1.

(v) Given a non zero rational function f on X, its **principal divisor** is

$$\operatorname{div}(f) = \sum_{Y \subset X} n_Y(f) Y,$$

where Y is a hypersurface of X and $n_Y(f) \in \mathbb{Z}$ is the order of either a zero or a pole of f along Y (see [Har77, §6, Ch. II]). The set of all principal divisors of X is denoted by $\operatorname{PDiv}(X)$ and it is a subgrup of $\operatorname{Div}(X)$.

(vi) Two divisors D_1 and D_2 are **linearly equivalent** if $D_1 - D_2 \in \operatorname{PDiv}(X)$. In this case, we write $D_1 \sim D_2$.

(vii) The **Picard group of** X is the quotient

$$\operatorname{Pic}(X) := \operatorname{Div}(X)/\operatorname{PDiv}(X).$$

The elements of this group are called **classes of divisors** and denoted by $[D]$, $D \in \operatorname{Div}(X)$.

For any divisor D of X, there exists an open covering $\{U_i\}_{i \in I}$ of X such that the restriction of D to each U_i, is a principal divisor, i.e. for some meromorphic function f_i on U_i, we have:

$$D|_{U_i} = \sum_i a_i (Y_i \cap U_i) = \operatorname{div}(f_i).$$

Theorem 1.2.2. [Har77, Theorem 1.1, Ch.V] *Let X be a smooth projective surface. There is a unique intersection pairing* $\operatorname{Div}(X) \times \operatorname{Div}(X) \to \mathbb{Z}$, $(C, D) \to C \cdot D$ *such that:*

(i) *If C and D are non-singular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, is the number of points of $C \cap D$.*

(ii) *It is symmetric: $C \cdot D = D \cdot C$.*

(iii) *It is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D \ + \ C_2 \cdot D$.*

(iv) *It depends only on the linear equivalence classes: if* $C_1 \sim C_2$ *then* $C_1 \cdot D = C_2 \cdot D$.

From (iv) in Theorem 1.2.2, we can prove that $C' \cdot D' = C \cdot D$, for $C' \sim C$ and $D' \sim D$, thus the intersection pairing induces a bilinear map:

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}, \quad ([C],[D]) \mapsto C \cdot D.$$

The Picard group, modulo torsion, equipped with the quadratic form defined by the intersection pairing is called **Picard lattice of** X.

Definition 1.2.3. Let ω be a meromorphic 2-form on X and $\{U_i\}_{i\in I}$ an affine open covering of X such that $\omega|_{U_i} = f_{U_i} dz_{1i} \wedge dz_{2i}$. Then the principal divisors $\mathrm{div}(f_{U_i})$ glue together to define a divisor of X, called **canonical divisor of** X, denoted by K_X.

Remark 1.2.4. The class of K_X in $\mathrm{Pic}(X)$ is independent of the choice of the meromorphic 2-form ω: let ω and ω' be two meromorphic 2-forms on X and let $U, V \subset X$ be two open sets, such that:

$$\begin{aligned} \omega|_U &= \alpha_U dz_1 + dz_2 & \omega'|_U &= \beta_U dz_1 + dz_2 \\ \omega|_V &= \alpha_V dw_1 + dw_2 & \omega'|_V &= \beta_V dw_1 + dw_2, \end{aligned}$$

for α_U, β_U and α_V, β_V meromorphic functions on U and V, respectively. Let J_{UV} be the Jacobian of the coordinate change form U to V, then $\alpha_V = J\alpha_U$ and $\beta_V = J\beta_U$. Note that $\beta_V/\alpha_V = \beta_U/\alpha_U$ and this holds for any U, V, then $\{\mathrm{div}(\beta_V/\alpha_V)\}$ defines a principal divisor of X.

Proposition 1.2.5 (Adjunction formula). [Har77, Proposition 1.5, Ch. V].
Let X *be a surface and* C *be a smooth curve of genus* g *on* X*. Then:*

$$2g - 2 = (K_X + C) \cdot C.$$

For any divisor D on X, let $\mathcal{O}_X(D)$ be the sheaf such that $\mathcal{O}_X(D)(U)$ is the space of all meromorphic functions f on U such that $\mathrm{div}(f) + D_{|U}$ is an effective divisor. The

Euler characteristic of $\mathcal{O}_X(D)$ is

$$\chi(\mathcal{O}_X(D)) = \sum_{i=0}^{2}(-1)^i h^i(\mathcal{O}_X(D)), \tag{1.1}$$

where $h^i(\mathcal{O}_X(D))$ is the dimension of the sheaf cohomology group $H^i(X, \mathcal{O}_X(D))$ [Har77, Ch. 3]. The **Euler characteristic** of X is the Euler characteristic of $\mathcal{O}_X = \mathcal{O}_X(0)$.

Theorem 1.2.6 (Riemann-Roch). [Har77, Theorem 1.6, Ch.V] *Let X be a surface and D be a divisor of X, then:*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D - K_X) \cdot D + \chi(\mathcal{O}_X).$$

Another way to write $\chi(\mathcal{O}_X)$ in the above formula is $\dfrac{1}{12}(K_X^2 + e(X))$, where $e(X)$ denotes the topological Euler characteristic of X [Har77, Remark 1.6.1, Ch V].

Let $\mathbb{Z}_X$ be the sheaf of locally constant integer functions on X and $\mathcal{O}_X^* \subset \mathcal{O}_X$ be the subsheaf of non-vanishing holomorphic functions.

Proposition 1.2.7. [Har77, Appendix B] *Let X be a compact complex manifold. There is a short exact sequence of sheaves, called* **exponential sequence** *of X:*

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

where the map $\mathbb{Z}_X \to \mathcal{O}_X$ is the natural inclusion and the map exp *is given by $f \mapsto$* $\exp(2\pi i f)$.

Taking cohomology of the exponential sequence we obtain the exact sequence:

$$\begin{tikzcd}
H^1(X,\mathbb{Z}_X) \arrow[r] \arrow[d, "\wr"] & H^1(X,\mathcal{O}_X) \arrow[r, "\exp"] & H^1(X,\mathcal{O}_X^*) \arrow[r, "\tau"] \arrow[d, "\wr"] & H^2(X,\mathbb{Z}_X) \arrow[r] \arrow[d, "\wr"] & H^2(X,\mathcal{O}_X). \\
H^1(X,\mathbb{Z}) & & \mathrm{Pic}(X) & H^2(X,\mathbb{Z}) &
\end{tikzcd}$$

Moreover, the Picard group of X is isomorphic to $H^1(X, \mathcal{O}_X^*)$ (see [Har77, Corollary 6.16, Ch. II] and [Har77, Example 2.5, Ch. III]). By the exactness of the sequence the kernel of the map τ is isomorphic to the quotient of $H^1(X, \mathcal{O}_X)$ by the image of the $H^1(X, \mathbb{Z})$. This quotient is a projective complex torus and it is denoted by $\mathrm{Pic}(X)^0$.

Definition 1.2.8. The image of $H^1(X, \mathcal{O}_X^*)$ by τ in $H^2(X, \mathbb{Z})$ is the **Nerón-Severi group of** X, denoted by $\mathrm{NS}(X)$.

Hence we have the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X)^0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\tau} \mathrm{NS}(X) \longrightarrow 0.$$

Now, we recall some results from group cohomology theory.

Theorem 1.2.9 (Serre duality). [Har77, §7, Ch. III] *Let X be a smooth complex projective manifold of dimension n and let $D \in \mathrm{Div}(X)$. For every $0 \leq i \leq n$ there is a perfect pairing:*

$$H^i(X, \mathcal{O}_X(D)) \otimes H^{n-i}(X, \mathcal{O}_X(K_X - D)) \to H^n(X, \mathcal{O}_X(K_X)) \simeq \mathbb{C}.$$

In particular $h^i(X, D) = h^{n-i}(X, \mathcal{O}_X(K_X - D))$.

Let X be a complex manifold. The Dolbeault cohomology groups of X are denoted by $H^{p,q}(X)$ and their dimension is denoted by $h^{p,q}(X)$ [GH94, Ch. II and Ch. III].

Theorem 1.2.10. *Let X be complex manifold. Then $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$, where Ω_X^p is the sheaf of holomorphic p-forms of X.*

1.3 Elliptic fibrations

Let K be a field of characteristic $\neq 2, 3$.

Definition 1.3.1. An **elliptic curve** E defined over K is a smooth projective curve of genus one with a point O defined over K.

Proposition 1.3.2. [Mir89, Corollary II.2.3] *Let E be an elliptic curve over K. There are $a, b \in K$ such that E is defined by the equation*

$$y^2 = x^3 + ax + b. \tag{1.2}$$

The pair (a, b) is unique up to the action of K^* defined by $\lambda(a, b) = (\lambda^4 a, \lambda^6 b)$, $\lambda \in K^*$.

Definition 1.3.3.

(i) The equation 1.2 is called **Weierstrass equation** for E over K.

(ii) The **discriminant** of E is given by $\Delta = -16(4a^3 + 27b^2)$, and is well defined up to multiplication by 12th powers in K.

(iii) The **J-function** of (a, b) is defined by $J(a, b) = \dfrac{4a^3}{\Delta}$.

Lemma 1.3.4. [SS13, Lemma 2.3] *An elliptic curve E defined by a Weierstrass equation 1.2 is smooth if and only if $\Delta \neq 0$.*

Definition 1.3.5. Let X be a surface.

(i) An **elliptic fibration** on X over a curve C is a surjective morphism $\pi : X \to C$ such that all but finitely many are smooth curves of genus one. We will always assume π to be minimal, i.e. there are no (-1)-curves contained in its fibers.

(ii) The pair (X, π), where π is a minimal elliptic fibration, is an **elliptic surface** over C.

(iii) A **section** of an elliptic fibration π is a morphism $s : C \to X$ such that $\pi \circ s = \mathrm{id}_C$. We usually call section also the image $s = s(C) \subseteq X$.

The set of K-rational points of an elliptic curve (E, O) defined over K, denoted by $E(K)$, is a group with the point O as origin. By the Mordell-Weil theorem [Sil09, Ch. VIII] this is a finitely generated group. Similarly, the set of sections of an elliptic surface $\pi : X \to \mathbb{P}^1$ has the structure of a finitely generated abelian group after choosing the section s as the origin of the group law. This is due to the fact that π can be seen as an elliptic curve defined over $K(\mathbb{P}^1)$ and its sections can be identified with its rational points over $K(\mathbb{P}^1)$. The restriction of the group operation to a smooth fiber F of π is exactly the group law on F with origin in the intersection point of F with the section. The group of sections is called **Mordell-Weil group** of the pair (π, s).

The singular fibers of a minimal elliptic fibration are unions of rational curves which can have multiplicity bigger than one. The type of such fibers have been classified independently by Kodaira and Néron. We describe such classification in Table 1.2. In the first column we give the Kodaira symbol of the fibre and in parenthesis the associated affine Dynkin diagram. The irreducible components of the singular fibers are denoted by Θ_i and the numbers in blue are their respective multiplicities [Mir89, Theorem I.6.6].

Theorem 1.3.6. *The only possible fibers for a minimal elliptic fibration of a surface are those listed in Table 1.2.*

Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with a section $s : \mathbb{P}^1 \to X$. Then X is birational to a Weierstrass fibration, i.e. a surface which can be defined by an equation of the form (see [Mir89, §II.5]):

$$y^2 = x^3 + A(t)x + B(t), \tag{1.3}$$

where t denotes the complex coordinate in an affine chart of $\mathbb{P}^1$ and A, B define holomorphic functions on $\mathbb{P}^1$. Tate's algorithm (see [Mir89, Lecture IV, 3.]) allows to identify the Kodaira-Néron type of a singular fiber of an elliptic fibration in Weierstrass form. This is described in Table 1.3 ([Mir89, Table IV.3.1]), where α, β and δ are the orders of vanishing of A, B and Δ respectively, J is the value of the J-function and e is the Euler number of the fiber.

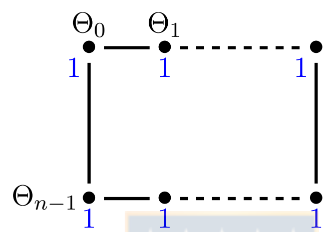
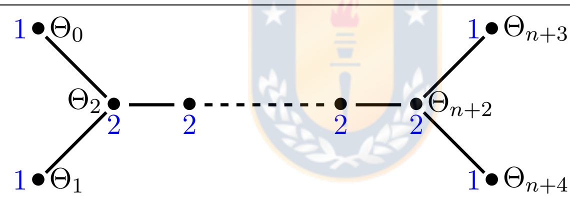
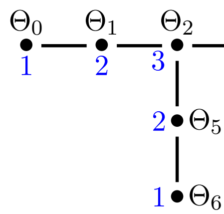
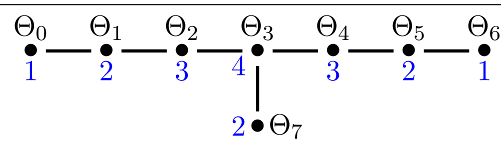
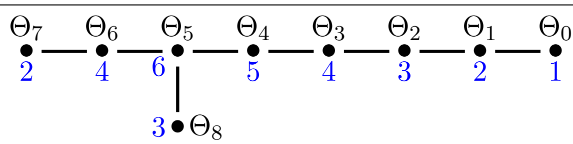
Fiber	Description
I_0	smooth elliptic curve
I_1	a rational curve Θ_0 with a node
$I_2 \ \ (\tilde{A}_1)$	two rational curves Θ_0, Θ_1 meeting at two points
II	a rational curve Θ_0 with a cusp
$III \ \ (\tilde{A}_1)$	two tangent rational curves Θ_1, Θ_2
$IV \ \ (\tilde{A}_2)$	$\Theta_1, \Theta_2, \Theta_3$ meeting at one point
$I_n \ \ (\tilde{A}_n)$	 $\Theta_i, \ i = 0, \dots, n-1$
$I_n^* \ \ (\tilde{D}_{n+4})$	 $\Theta_i, \ i = 0, 1, \dots, n+3, n+4$
$IV^* \ \ (\tilde{E}_6)$	 $\Theta_i, \ i = 0, 1, \dots, 6$
$III^* \ \ (\tilde{E}_7)$	 $\Theta_i, \ i = 0, 1, \dots, 7$
$II^* \ \ (\tilde{E}_8)$	 $\Theta_i, \ i = 0, 1, \dots, 8$

Table 1.2 Kodaira's table of singular elliptic fibers.

Type	α	β	δ	J	e
	0	0	0	$\neq 0, 1, \infty$	
I_0	$\alpha \geq 1$	0	0	0	0
	0	$\beta \geq 1$	0	1	
I_1	0	0	1	∞	1
I_n	0	0	n	∞	n
	2	3	6	$\neq 0, 1, \infty$	
I_0^*	$\alpha \geq 3$	3	6	0	6
	2	$\beta \geq 4$	6	1	
I_n^*	2	3	$n+6$	∞	$n+6$
II	$a \geq 1$	1	2	0	2
III	1	$\beta \geq 2$	3	1	3
IV	$\alpha \geq 2$	2	4	0	4
IV^*	$\alpha \geq 3$	4	8	0	8
III^*	3	$\beta \geq 5$	9	1	9
II^*	$\alpha \geq 4$	5	10	0	10

Table 1.3 Tate algorithm

Chapter 2

$K3$ surfaces and their automorphisms

2.1 $K3$ surfaces

As we said in section 1.2 by *surface* we mean a 2-dimensional smooth complex projective algebraic variety.

Definition 2.1.1. A surface X is called a $K3$ **surface** if

(i) it admits, up to scalar multiplication, a unique holomorphic 2-form ω_X that does not vanish at any point of X,

(ii) $H^1(X, \mathcal{O}_X) = \{0\}$.

The second singular complex cohomology group has the following Hodge decomposition [GH94, §7, Ch. 0]:

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Observe that $H^{2,0}(X)$, which is isomorphic to $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X)) = H^0(X, \mathcal{O}_X)$ by Theorem 1.2.10, is 1-dimensional and is generated by ω_X. Moreover $H^{0,2}(X) = \overline{H^{2,0}(X)}$.

Proposition 2.1.2. [BHPVdV04, Ch. VIII], [Mil58, §3] *Let X be a $K3$ surface. Then:*

(i) $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = \{0\}$.

(ii) *The group $H^2(X,\mathbb{Z})$ endowed with the quadratic form given by the cup product has a lattice structure isometric to the $K3$ lattice Λ_{K3} (Definition 1.1.7).*

The long cohomology sequence of the exponential sequence of a $K3$ surface X is

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow$$

$$\longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \longrightarrow H^3(X,\mathbb{Z}) \longrightarrow 0.$$

By Theorem 1.2.10 we have that $\dim H^2(X,\mathcal{O}_X) = \dim H^{0,2}(X) = 1$. By Proposition 2.1.2, $\dim H^1(X,\mathbb{C}) = \dim H^3(X,\mathbb{C}) = 0$ and $\dim H^2(X,\mathbb{C}) = 22$. Thus by Hodge decomposition we have:

Proposition 2.1.3 (Proposition (3.4), Ch.VIII, [BHPVdV04])**.**
For a $K3$ surface X we have

$$\begin{aligned}
&h^{0,1}(X) = h^{1,0}(X) = h^{2,1}(X) = h^{1,2}(X) = 0\\
&h^{0,2}(X) = h^{2,0}(X) = 1\\
&h^{1,1}(X) = 20.
\end{aligned}$$

Let us focus on this part of the long cohomology exponential sequence of X:

$$\begin{tikzcd}
H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{O}_X^*) \arrow[r, "\tau"] \arrow[d, "\wr"] & H^2(X,\mathbb{Z}) \arrow[r, "\tau"] & H^2(X,\mathcal{O}_X) \arrow[d, "\wr"] \\
\{0\} & \mathrm{Pic}(X) & & \mathbb{C}\overline{\omega}_X
\end{tikzcd}$$

The map τ is injective, so $\mathrm{Pic}(X)$ injects into $H^2(X,\mathbb{Z})$ and identifies with $\mathrm{NS}(X)$. The restriction of the quadratic form on $H^2(X,\mathbb{Z})$ induces a lattice structure on $\tau(\mathrm{Pic}(X))$, and thus on $\mathrm{Pic}(X)$, which can be proved to be the same as the one defined by the intersection product (see §1.2, Ch.1). The Picard lattice is an even lattice with $1 \leq \mathrm{rank}(\mathrm{Pic}(X)) \leq 20$ and signature $(1, \mathrm{rank}(\mathrm{Pic}(X)) - 1)$ [Huy16, §2.2, §3.3]. The rank of $\mathrm{Pic}(X)$ is called Picard number and it is denoted by $\rho(X)$.

We also have the following description of $\mathrm{NS}(X)$.

Theorem 2.1.4 (Theorem (2.13), Ch.IV, [BHPVdV04]). *Let X be a surface and let $i^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ be the natural inclusion. Then*

$$i^*\mathrm{NS}(X) = H^{1,1}(X) \cap i^*H^2(X,\mathbb{Z}).$$

In case X is a K3 surface Theorem 2.1.4 implies that

$$i^*\mathrm{NS}(X) = \omega_X^{\perp} \cap i^*H^2(X,\mathbb{Z}).$$

Definition 2.1.5. Let X be a K3 surface. The orthogonal to the lattice $NS(X)$ in $H^2(X,\mathbb{Z})$ is the **transcendental lattice** of X:

$$T_X = \mathrm{NS}(X)^{\perp_{H^2(X,\mathbb{Z})}}.$$

2.2 Automorphisms

We start stating a fundamental result in the theory of K3 surfaces, the Global Torelli theorem. A class $v \in \mathrm{NS}(X)$ is *ample* if $v^2 > 0$ and $v \cdot [D] > 0$ for every effective divisor D of X (see [Huy16, Proposition 1.5, Ch. 1]).

Definition 2.2.1. Let X and Y be two (projective) $K3$ surfaces and $\varphi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be an isometry of lattices.

(i) φ is a **Hodge isometry** if the $\mathbb{C}$-linear extension of φ preserves the Hodge decomposition;

(ii) φ is called **effective** if it maps an ample class of X to an ample class of Y.

Proposition 2.2.2. *Let X, Y be two $K3$ surfaces and $f : X \to Y$ be an isomorphism. Then $f^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an effective Hodge isometry. In particular an automorphism of a $K3$ surface X induces an effective Hodge isometry $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$.*

The following theorem is known as Global Torelli Theorem for $K3$ surfaces (see [Huy16, Theorem 5.3]).

Theorem 2.2.3. *Two $K3$ surfaces X and Y are isomorphic if and only if there is a Hodge isometry $\varphi : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. Moreover, if φ is an effective Hodge isometry then there exists a (unique) isomorphism $f : X \to Y$ such that $f^* = \varphi$.*

As we will see later, Theorem 2.2.3 allows to construct moduli spaces of polarized K3 surfaces with automorphisms.

Definition 2.2.4. Let X be a $K3$ surface with $H^{2,0}(X) = \mathbb{C}\omega_X$ and let $\sigma \in \mathrm{Aut}(X)$ with $\sigma^*(\omega_X) = \zeta\omega_X$, $\zeta \in \mathbb{C}^*$. We will say that σ is:

(i) **symplectic** if $\zeta = 1$.

(ii) **non-symplectic** if $\zeta \neq 1$.

(iii) **purely non-symplectic** if σ has finite order $n \geq 2$ and ζ is a primitive n-th root of unity.

Let us denote by ζ_n a primitive n-th root of unity, by μ_n the multiplicative group generated by ζ_n and by $\varphi(n)$ the cardinality of μ_n, i.e. the Euler function of n.

Let G be a subgroup of $\mathrm{Aut}(X)$ and $\alpha : G \to \mathbb{C}^\times$ be the character of the natural representation of G on the space $H^{2,0}(X) = \mathbb{C}\omega_X$. Then, there exists a positive integer $I(G)$ such that the following is an exact sequence (see [Nik79a, Theorem 0.1]):

$$1 \to \mathrm{Ker}(\alpha) \to G \xrightarrow{\alpha} \mu_{I(G)} \to 1, \tag{2.1}$$

where $\mu_{I(G)}$ is the multiplicative group of $I(G)$-th roots of unity.

Remark 2.2.5. Observe that an automorphism $\sigma \in \mathrm{Ker}(\alpha)$ if and only if σ is symplectic. Moreover, if $G = \langle \sigma \rangle$, then σ is purely non-symplectic if and only if $\mathrm{Ker}(\alpha) = \{\mathrm{id}\}$. In this case $G \cong \mu_{I(G)}$ is a finite cyclic group.

Theorem 2.2.6 (Lemma 1.1 [MO98])**.** *Let X be a $K3$ surface and $\sigma \in \mathrm{Aut}(X)$, such that $G = \langle \sigma \rangle$. Set $I(G) = I$, $\mathrm{rank}\ T_X = r$ and regard T_X as a $\mathbb{Z}[\langle \sigma \rangle]$-module via the natural action of σ^* on T_X. Let $\Phi_I(x) \in \mathbb{Z}[x]$ be the I-th cyclotomic polynomial. Then:*

(i) ([Nik79a, Theorem 0.1]) *the eigenvalues of $\sigma^*|T_X$ are the primitive I-th roots of unity. In particular $\mathrm{ord}(\sigma^*|T_X) = I$ and $I = 1$ if and only if $\sigma^*|T_X = id$.*

(ii) $\mathrm{Ann}(T_X) = \langle \Phi_I(\sigma) \rangle$, *and T_X is then naturally a torsion free $\mathbb{Z}[\langle \sigma \rangle]/\langle \Phi_I(\sigma) \rangle$-module, and*

(iii) *under the identification $\mathbb{Z}[\langle \sigma \rangle]/\langle \Phi_I(\sigma) \rangle = \mathbb{Z}[\zeta_I]$ through the correspondence $\sigma(\bmod \langle \Phi_I(\sigma) \rangle) \leftrightarrow \zeta_I$, there is an isomorphism $T_X \simeq \mathbb{Z}[\zeta_I]^{\oplus (\mathrm{rk}(T_X)/\varphi(I))}$ as $\mathbb{Z}[\zeta_I]$-modules.*

Observe that Theorem 2.2.6 implies that $\varphi(I(G))$ divides $\mathrm{rank}(T_X)$, so that $\varphi(I(G)) \leq \mathrm{rank}(T_X) \leq 21$. Table 2.1 contains all values that $I(G)$ and $\varphi(I(G))$ can take.

$\varphi(I(G))$	20	18	16	12	10	8	6	4	2	1
$I(G)$	66	54	60	42	22	30	18	12	6	2
	50	38	48	36	11	24	14	10	4	1
	44	27	40	28		20	9	8	3	
	33	19	34	26		16	7	5		
	25		32	21		15				
			17	13						

Table 2.1 The list of numbers n with $\varphi(I) \leq 21$

Theorem 2.2.7 (Main Theorem 3, [MO98])**.** *Among the candidates in Table 2.1, only 60 cannot be realised as $I(G)$ for some finite automorphism group G of a K3 surface. Moreover, for each positive integer $I \neq 60$ with $\varphi(I) \leq 20$ there exists a $K3$ surface X admitting an order n automorphism σ with $\langle \sigma \rangle \simeq \langle \alpha(\sigma) \rangle = \mu_I$.*

Nikulin and Mukai [Nik79a] and [Muk88] classified the finite groups G for which there exists a $K3$ surface X with an automorphism group isomorphic to G and $I(G) = 1$.

Definition 2.2.8. Let σ be an automorphism of a K3 surface. The **invariant lattice** of σ is given by:

$$S(\sigma) = \{x \in H^2(X, \mathbb{Z}) \mid \sigma^*(x) = x\}.$$

We will also denote $T(\sigma) = S(\sigma)^\perp$.

Proposition 2.2.9. *Let X be a $K3$ surface with a purely non-symplectic automorphism σ of finite order $n \geq 2$. Then* $\operatorname{rank} S(\sigma) > 0$ *and* $S(\sigma^i) \subseteq \mathrm{NS}(X)$ *for any* $i = 1, \ldots, n-1$.

Proof. By [Nik79a, Theorem 3.1] there is always an ample σ^*-invariant class on X, thus $\operatorname{rank} S(\sigma) > 0$. Now, let $x \in S(\sigma^i)$ with $i = 1, \ldots, n-1$, then we have:

$$\langle x, \omega_X \rangle = \langle (\sigma^i)^*(x), (\sigma^i)^*(\omega_X) \rangle = \langle x, \zeta_n^i \omega_X \rangle = \zeta_n^i \langle x, \omega_X \rangle,$$

where $\langle\ ,\ \rangle$ denotes the cup product in $H^2(X, \mathbb{C}) \supseteq H^2(X, \mathbb{Z})$. This implies $\langle x, \omega_X \rangle = 0$. By Theorem 2.1.4, we have that $x \in \mathrm{NS}(X)$. □

Since $S(\sigma) \subseteq \mathrm{NS}(X)$, then $T_X \subseteq T(\sigma)$. Recall that the action of σ on T_X and $T(\sigma)$ is by primitive roots of unity by Theorem 2.2.6.

We will now describe the structure of the **fixed locus** of an automorphism σ of a K3 surface X:

$$\mathrm{Fix}(\sigma) = \{x \in X : \sigma(x) = x\}.$$

The action of σ at each fixed point of it can be locally diagonalized as follows (see [Nik79a, §5]):

Lemma 2.2.10. *Let X be a $K3$ surface, σ be an automorphism of order n of X such that $\sigma^*(\omega_X) = \zeta_n^k \omega_X$, where $k \in \{1, \ldots, n\}$, and $p \in X$ be a fixed point for σ. Then σ can be locally linearized and diagonalized in a neighborhood of p so that it is given by a* *matrix of the form:*

$$A_{i,j} = \begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^j \end{pmatrix}, \textit{with } i+j \equiv k \pmod n,\ 1 \leq i \leq j \leq n.$$

This implies that the fixed locus of σ consists of a disjoint union of isolated points and non-singular curves. Moreover, since $\mathrm{Pic}(X)$ has hyperbolic signature, the fixed locus contains at most one curve of genus $g > 1$. Thus, if $\mathrm{Fix}(\sigma)$ contains a curve of genus $g \geq 2$ we have:

$$\mathrm{Fix}(\sigma) = \{p_1, p_2, \ldots, p_N\} \sqcup C_g \sqcup R_1 \sqcup R_2 \sqcup \cdots \sqcup R_k, \tag{2.2}$$

where p_i are isolated points, C_g and R_i are non-singular curves of genus g and 0, respectively.

In what follows we will denote by $a_{i,j}$ the total number of points of type $A_{i,j}$. We now recall two versions of Lefschetz fixed-point formula (see [AS68, Theorem 4.6]).

Theorem 2.2.11 (Holomorphic Lefschetz formula)**.** *Let σ be a finite order automorphism of a compact complex surface X. Then*

$$\sum_{i=0}^{2}(-1)^i \mathrm{tr}\left(\sigma^*\big|_{H^i(X,\mathcal{O}_X)}\right) = \sum_{p\in\mathrm{Fix}(\sigma)} \frac{1}{\det(I-\sigma^*\big|_{T_p})} + \sum_{C\subset\mathrm{Fix}(\sigma)} \left(\frac{1-g(C)}{1-\zeta} - \frac{\zeta\cdot C^2}{(1-\zeta)^2}\right),$$

where $g(C)$ is the genus of the curve C and ζ^{-1} is the eigenvalue of σ on the normal bundle of C.

In our notation and in the hypothesis of Lemma 2.2.10 with $k=1$, the previous formula is:

$$\sum_{i=0}^{2}(-1)^i \mathrm{tr}\left(\sigma^*\big|_{H^i(X,\mathcal{O}_X)}\right) = \sum_{i+j\equiv 0 \pmod n} \frac{a_{i,j}}{(1-\zeta_n^i)(1-\zeta_n^j)} + \alpha\frac{1+\zeta_n}{(1-\zeta_n)^2} \tag{2.3}$$

where $\alpha := \sum_{C\subset\mathrm{Fix}(\sigma)}(1-g(C))$.

Theorem 2.2.12 (Topological Lefschetz formula)**.** *Let σ be an order n automorphism of a surface X, then:*

$$\chi(\mathrm{Fix}(\sigma)) = \sum_i (-1)^i \mathrm{tr}\left(\sigma^*\Big|_{H^i(X,\mathbb{R})}\right).$$

Given an automorphism σ of order n of a K3 surface X, for any divisor m of n we define

$$H^2(X,\mathbb{C})_m = \{x \in H^2(X,\mathbb{C}) : \sigma^*(x) = \zeta_m x\}$$

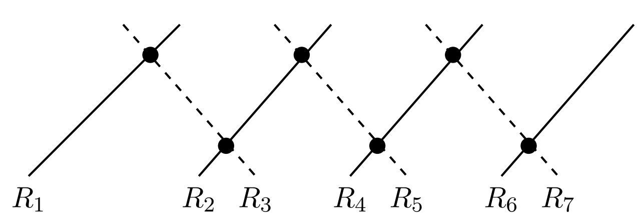
and denote by d_m its dimension (in particular d_1 is the rank of the invariant lattice). In this case, assuming the fixed locus of σ is as in (2.2), the topological Lefschetz fixed point formula is:

$$(2-2g) + 2k + N = \mathrm{tr}\left(\sigma^*\Big|_{H^2(X,\mathbb{R})}\right) + 2 = \sum_{m|n} d_m \left(\sum_{\gcd(r,m)=1} \zeta_m^r\right) + 2, \tag{2.4}$$

where we used the fact that σ^* is the identity on both $H^0(X,\mathbb{R}) \cong \mathbb{R}$ and $H^4(X,\mathbb{R}) \cong \mathbb{R}$ and $H^1(X,\mathbb{R}) = H^3(X,\mathbb{R}) = \{0\}$.

The following Lemma helps to identify the types of the fixed points (see [AS15, Lemma 4]).

Lemma 2.2.13. *Let $T = \sum_i R_i$ be a tree of smooth rational curves on a $K3$ surface X such that each R_i is invariant under the action of a purely non-symplectic automorphism σ of order n. Then, the points of intersection of the rational curves R_i are fixed by σ and the action at one fixed point determines the action on the whole tree.*



Remark 2.2.14. If $\sigma^*(\omega_X) = \zeta_n \omega_X$, then the local action at the intersection points of the curves R_i appear in the following order depending whether n is either even or odd:

If n is even $\quad \ldots\ A_{1,0},\ A_{1,0},\ A_{2,n-1}, \ldots A_{\frac{n}{2},\frac{n}{2}+1},\ A_{\frac{n}{2},\frac{n}{2}+1},\ \ldots$

If n is odd $\quad \ldots\ A_{1,0},\ A_{1,0},\ A_{2,n-1}, \ldots A_{\frac{n+1}{2},\frac{n+1}{2}},\ A_{\frac{n-1}{2},\frac{n+3}{2}},\ \ldots$

Finally, we recall a result which allows to prove, under certain conditions, that an elliptic fibration is left invariant by an automorphism.

Lemma 2.2.15. [AS15, Lemma 5] *Let X be a K3 surface with an automorphism σ and let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration. Let f be the class of the fiber. If σ^* fixes a class $x \in \operatorname{Pic}(X)$ such that $x^2 > 0$, then*

$$(f \cdot \sigma^*(f))x^2 \leq 2(x \cdot f)^2.$$

Moreover, if π has a section s with $x \cdot s = 0$, then

$$f \cdot \sigma^*(f) + 1 \leq \frac{2(x \cdot f)^2}{x^2}.$$

2.3 Classification of non-symplectic automorphisms

We briefly recall what is known about the classification of non-symplectic automorphisms of K3 surfaces.

A classification of purely non-symplectic automorphisms is known for all prime orders [Nik79a, OZ98, OZ11, Vor83, OZ00, Kon92, AS08, AST11], when $\varphi(n) = 20$ and when $\varphi(n) = 40$ [MO98], when the automorphism acts trivially on the Néron-Severi lattice and $\varphi(n)$ equals the rank of the transcendental lattice [Vor83, Kon92, OZ00, Sch10], for orders $6, 16$ [Dil12, ATST16], for 3-power order when the automorphism acts trivially on the Néron-Severi lattice [Tak10], and for order 4 [AS15] and 8 [ATS18] (the latter both contain partial classifications). In Table 2.1 we mark in green (orange) all orders of non-symplectic automorphisms whose classification is (partially) known.

If σ is a non-symplectic automorphism of prime order p of a K3 surface, then it is easy to prove that its invariant lattice $S(\sigma) \subset H^2(X, \mathbb{Z})$ is a p-elementary lattice (see

$\varphi(I(G))$	20	18	16	12	10	8	6	4	2	1
	66	54	48	42	22	30	18	12	6	2
	50	38	34	36	11	24	14	10	4	1
$I(G)$	44	27	40	28		20	9	8	3	
	33	19	32	26		16	7	5		
	25		17	21		15				
				13						

Table 2.2 Classification of non-symplectic automorphisms of K3 surfaces

Definition 1.1.8), which makes their classification easier. We now recall the main result.

Theorem 2.3.1. [AST11, Theorem 0.1] *Let L be a hyperbolic p-elementary lattice (p prime) of rank r with $d(L) = p^a$. Then L is isometric to the invariant lattice of a non-symplectic automorphism σ of order p on a $K3$ surface if and only if*

$$22 - r - (p-1)a \ \in 2(p-1)\mathbb{Z}_{\geq 0}.$$

Moreover, if σ is such automorphism, then its fixed locus $\mathrm{Fix}(\sigma)$ *is the disjoint union of smooth curves and isolated points and has the following form:*

$$\mathrm{Fix}(\sigma) = \begin{cases} \emptyset & \text{if} \quad L \cong U(2) \oplus E_8(2) \\ E_1 \sqcup E_2 & \text{if} \quad L \cong U \oplus E_8(2) \\ C_g \sqcup R_1 \sqcup \cdots \sqcup R_k \sqcup \{p_1, \ldots, p_N\} & \quad \text{otherwise,} \end{cases}$$

where E_i is a smooth elliptic curve, R_i is a smooth rational curve, p_i is an isolated fixed point and C_g is a smooth curve of genus

$$g = \frac{22 - r - (p-1)a}{2(p-1)}.$$

Moreover, N and k are given in the following table with the convention that $\mathrm{Fix}(\sigma)$ *contains no fixed curves if $k = -1$.*

	$p=2$	$p=3,5,7$	$p=11$	$p=13$	$p=17$	$p=19$
N	0	$\dfrac{-2+(p-2)r}{p-1}$	$\dfrac{2+9r}{10}$	9	7	5
k	$\dfrac{r-a}{2}$	$\dfrac{2+r-(p-1)a}{2(p-1)}$	$\dfrac{-2+r-10a}{20}$	1	0	0

Remark 2.3.2. We recall that by Theorem 1.1.9 an even indefinite p-elementary lattice L with $p \neq 2$ is uniquely determined, up to isomorphism, by its rank r and the integer a such that $\det(L) = p^a$. On the other hand, an even indefinite 2-elementary lattice L is uniquely determined by its rank r, the integer a such that $\det(L) = 2^a$ and an invariant $\delta \in \{0, 1\}$ defined in [Nik79b].

We now include several tables which describe the invariant lattice and its orthogonal complement for any given value of the topological invariants.

Figure 2.1 shows all values of (a, r, δ) which are realized for non-symplectic automorphisms of order 2 and the corresponding invariants (g, k) of the fixed locus.

Figure 2.2 shows all values of (a, m), where $2m = 22 - 2a$ which are realized for non-symplectic automorphisms of order 3 and the corresponding invariants (N, g, k) of the fixed locus. Moreover, Table 2.3 describes the lattice structure of the invariant lattice $S(\sigma)$ and its orthogonal complement $T(\sigma)$ for each choice of the triple (N, g, k).

Tables 2.4, 2.5 describe the fixed locus on non-symplectic automorphisms of prime order $p = 5, 11$ [AST11].

N	(k,g)	$T(\sigma)$	$S(\sigma)$
0	$(0,4)$	$U\oplus U(3)\oplus E_8\oplus E_8$	$U(3)$
	$(1,5)$	$U\oplus U\oplus E_8\oplus E_8$	U
1	$(0,3)$	$U\oplus U(3)\oplus E_6\oplus E_8$	$U(3)\oplus A_2$
	$(1,4)$	$U\oplus U\oplus E_6\oplus E_8$	$U\oplus A_2$
2	$(0,2)$	$U\oplus U(3)\oplus E_6\oplus E_6$	$U(3)\oplus A_2^2$
	$(1,3)$	$U\oplus U\oplus E_6\oplus E_6$	$U\oplus A_2^2$
3	$(0,-)$	$U\oplus U(3)\oplus A_2^5$	$U(3)\oplus E_6^*(3)$
	$(0,1)$	$U\oplus U\oplus A_2^5$	$U(3)\oplus A_2^3$
	$(1,2)$	$U\oplus U(3)\oplus A_2\oplus E_8$	$U\oplus A_2^3$
	$(2,3)$	$U\oplus U\oplus A_2\oplus E_8$	$U\oplus E_6$
4	$(0,0)$	$U\oplus U(3)\oplus A_2^4$	$U(3)\oplus A_2^4$
	$(1,1)$	$U\oplus U\oplus A_2^4$	$U\oplus A_2^4$
	$(2,2)$	$U\oplus U(3)\oplus E_8$	$U\oplus E_6\oplus A_2$
	$(3,3)$	$U\oplus U\oplus E_8$	$U\oplus E_8$
5	$(1,0)$	$U\oplus U(3)\oplus A_2^3$	$U\oplus A_2^5$
	$(2,1)$	$U\oplus U(3)\oplus E_6$	$U\oplus A_2^2\oplus E_6$
	$(3,2)$	$U\oplus U\oplus E_6$	$U\oplus E_8\oplus A_2$
6	$(2,0)$	$U\oplus U(3)\oplus A_2^2$	$U\oplus E_6\oplus A_2^3$
	$(3,1)$	$U\oplus U\oplus A_2^2$	$U\oplus E_6^2$
7	$(3,0)$	$U\oplus U(3)\oplus A_2$	$U\oplus E_6\oplus E_6\oplus A_2$
	$(4,1)$	$U\oplus U\oplus A_2$	$U\oplus E_6\oplus E_8$
8	$(4,0)$	$U\oplus U(3)$	$U\oplus E_6\oplus E_8\oplus A_2$
	$(5,1)$	$U\oplus U$	$U\oplus E_8\oplus E_8$
9	$(5,0)$	$A_2(-1)$	$U\oplus E_8\oplus E_8\oplus A_2$

Table 2.3 Order 3

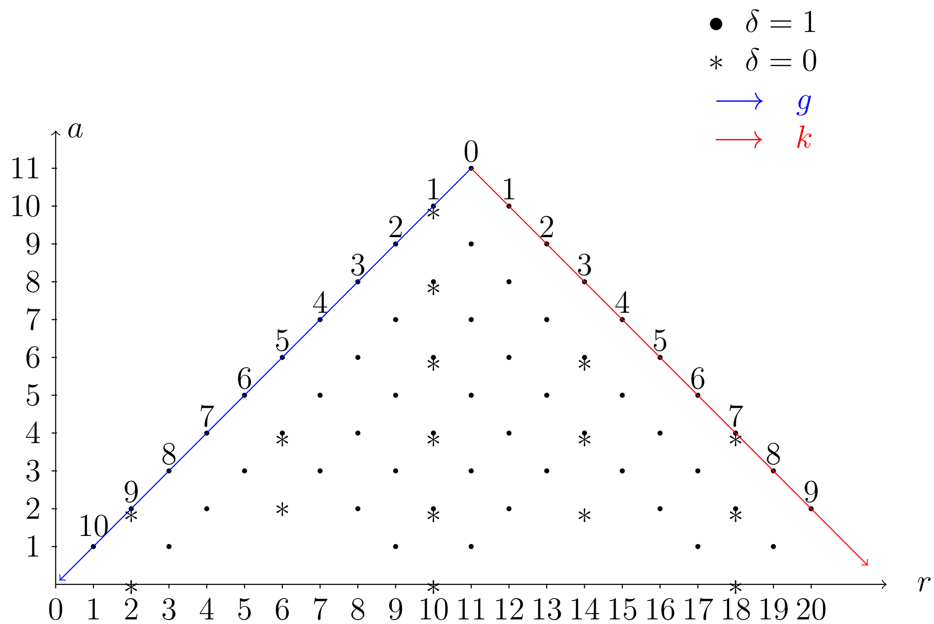


Figure 2.1 Order 2

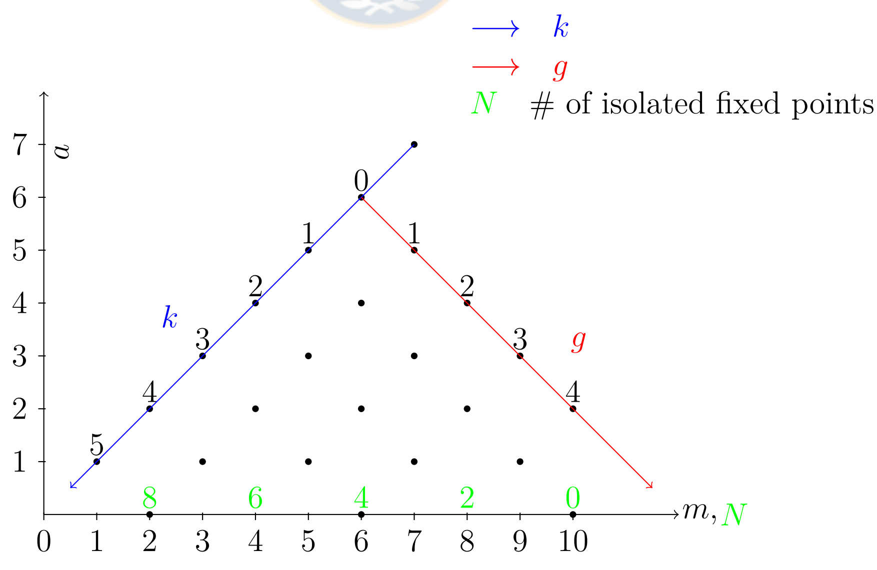


Figure 2.2 Order 3

(N,k,g)	$T(\sigma)$	$S(\sigma)$
$(1,0,2)$	$H_5 \oplus U \oplus E_8^2$	H_5
$(4,0,1)$	$H_5 \oplus U \oplus E_8 \oplus A_4$	$H_5 \oplus A_4$
$(4,0,-)$	$H_5 \oplus U(5) \oplus E_8 \oplus A_4$	$H_5 \oplus A_4^*(5)$
$(7,1,1)$	$U \oplus H_5 \oplus E_8$	$H_5 \oplus E_8$
$(7,0,0)$	$U \oplus H_5 \oplus A_4^2$	$H_5 \oplus A_4^2$
$(10,1,0)$	$U \oplus H_5 \oplus A_4$	$H_5 \oplus A_4 \oplus E_8$
$(13,2,0)$	$U \oplus H_5$	$H_5 \oplus E_8^2$

Table 2.4 Order 5

Case	(N,k,g)	$T(\sigma)$	$S(\sigma)$
A	$(2,0,1)$	$U^2 \oplus E_8^2$	U
	$(2,0,-)$	$U \oplus U(11) \oplus E_8^2$	$U(11)$
B	$(11,0,0)$	$K_{11}(-1) \oplus E_8$	$U \oplus A_{10}$

Table 2.5 Order 11

Chapter 3

Automorphisms of order 9 of $K3$ surfaces

This chapter contains a complete classification of automorphisms of order 9 of K3 surfaces and is based on the article [ACV20].

3.1 Preliminary results

Let σ be an automorphism of order nine of a $K3$ surface X. By [Nik79a] σ is non-symplectic. We will start proving that σ^3 is non-symplectic as well. In this section we will denote by ζ a 9th root of unity.

Lemma 3.1.1. *Let X be a $K3$ surface. If σ is an order 9 non-symplectic automorphism of X, then σ^3 is non-symplectic.*

Proof. Assume that σ^3 is symplectic i.e. $(\sigma^3)^*(\omega_X) = \omega_X$, where ω_X denotes a generator of $H^{2,0}(X)$. Since σ^3 has only isolated fixed points [Nik79a, §5], the same is true for σ. Assume p to be a fixed point of σ. By Lemma 2.2.10 σ can be locally linearized and diagonalized to be of the form

$$A_{i,j} = \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix}, \qquad \text{with } i + j \equiv 3 \pmod 9,$$

thus the possible types are $A_{1,2}, A_{4,8}$ and $A_{5,7}$. Let $a_{1,2}, a_{4,8}, a_{5,7}$ be the number of points of types $A_{1,2}, A_{4,8}$ and $A_{5,7}$ respectively. By the holomorphic Lefschetz formula [AS68, Theorem 4.6] we have

$$1+\zeta^{-3} = \frac{a_{1,2}}{(1-\zeta)(1-\zeta^2)} + \frac{a_{4,8}}{(1-\zeta^4)(1-\zeta^8)} + \frac{a_{5,7}}{(1-\zeta^5)(1-\zeta^7)},$$

which is inconsistent. Observe that the same formula shows that the fixed locus of σ is not empty. Therefore, σ^3 is non-symplectic. □

In [AS08, Theorem 2.2] the authors classified order 3 non-symplectic automorphisms τ of $K3$ surfaces relating the structure of their fixed locus to their action in cohomology (see Theorem 2.3.1 and Figure 2.2). In particular they proved that

$$\mathrm{Fix}(\tau) = \{p_1, \dots, p_N\} \sqcup C_g \sqcup R_1 \sqcup \cdots \sqcup R_k,$$

where the R_i's are smooth curves of genus 0, C_g is a smooth curve of genus g and moreover $N = 10 - m$, where $2m = \operatorname{rank} T(\tau) = 22 - \operatorname{rank} S(\tau)$.

If σ is an order 9 automorphism, then the eigenvalues of σ^* in $T(\sigma^3) \otimes_{\mathbb{Z}} \mathbb{C} \subseteq H^2(X, \mathbb{C})$ are the primitive 9th roots of unity, thus $2m$ is divisible by $\varphi(9) = 6$. This implies that $m \in \{3, 6, 9\}$, thus by Figure 2.2 and Table 2.3 the fixed locus of σ^3 is described in Table 3.1.

Case	(N, k, g)	m
A	(1,0,3)	9
B	(1,1,4)	9
C	(4,0,0)	6
D	(4,1,1)	6
E	(4,2,2)	6
F	(4,3,3)	6
G	(7,3,0)	3
H	(7,4,1)	3

Table 3.1 Fixed locus of σ^3

We will denote by N_σ the number of isolated points in $\mathrm{Fix}(\sigma)$, by g_σ the maximal genus of a curve in it (if any) and by k_σ the number of smooth rational curves in it. By Lemma 2.2.10 the fixed points of σ are of one of the following types:

$$A_{1,0},\ A_{2,8},\ A_{3,7},\ A_{4,6},\ A_{5,5}.$$

To simplify notation we define $A_i := A_{i+1,9-i}$ for $i = 0,1,2,3,4$. The points of type A_0 are those which are contained in a curve fixed by σ, those of type A_2 and A_3 are contained in a fixed curve by σ^3 only and the points of the remaining types are isolated for both σ and σ^3. We will denote by a_i the number of fixed points of type A_i.

Given $j \in \{1,3,9\}$ let

$$H^2(X,\mathbb{C})_j = \{x \in H^2(X,\mathbb{C}) : \sigma^* x = \zeta_j x\},$$

where ζ_j is a primitive jth root of unity, and let d_j be its dimension. In particular d_1 is equal to the rank of the invariant lattice.

Lemma 3.1.2. *Let $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma)} (1 - g(C))$. The following relations hold:*

$$\begin{cases} a_1 + a_4 = 3\alpha + 1 \\ a_2 = 2\alpha + 1 \\ a_3 + 3a_1 = 8\alpha + 4. \end{cases} \tag{Eq. 1.4}$$

In particular $\alpha \geq 0$. Moreover $N_\sigma + 2\alpha = 2 + d_1 - d_3$.

Proof. By the holomorphic Lefschetz formula 2.3 we have that:

$$1 + \bar{\zeta} = \sum_{i=1}^{4} \frac{a_i}{(1-\zeta^{i+1})(1-\zeta^{9-i})} + \alpha \frac{1+\zeta}{(1-\zeta)^2}.$$

This gives the first three equalities. The last equality follows from the topological Lefschetz formula 2.4 since $\chi(\mathrm{Fix}(\sigma)) = N_\sigma + 2\alpha$, $\mathrm{tr}(\sigma^*|_{H^0(X,\mathbb{R})}) = \mathrm{tr}(\sigma^*|_{H^4(X,\mathbb{R})}) = 1$

and

$$\mathrm{tr}(\sigma^*|_{H^2(X,\mathbb{R})}) = d_1 + d_3(\zeta^3+\zeta^6) + d_9(\zeta+\zeta^2+\zeta^4+\zeta^5+\zeta^7+\zeta^8) = d_1 - d_3.$$

□

Remark 3.1.3. Observe that, by Lemma 2.2.13, the action of σ on a tree of smooth rational curves is as in Figure 3.1, where double curves are in $\mathrm{Fix}(\sigma)$.

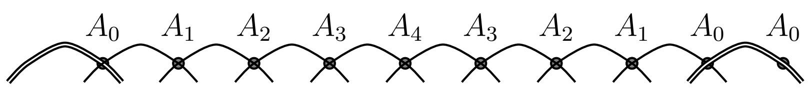


Figure 3.1 The action of σ on a tree of rational curves

We start proving some preliminary results.

Proposition 3.1.4. *The genus of a smooth curve in the fixed locus of an order nine non-symplectic automorphism σ of a K3 surface is either* 0 *or* 1*. Moreover, if σ^3 has invariants $(N,k,g)=(4,3,3)$ then the fixed locus of σ contains a curve of genus* 0.

Proof. Let σ be an automorphism as in the statement. By Table 3.1 the genus of a curve fixed by σ^3 is at most 4. Moreover, by Lemma 3.1.2 we have that $\alpha = (1-g_\sigma)+k_\sigma-1 \geq 0$, thus when the invariants of σ^3 are either $(N,k,g)=(1,0,3)$ or $(1,1,4)$, the automorphism σ can not fix the curve of positive genus.

Assume that σ fixes a smooth curve C of genus two, i.e. σ^3 has invariants $(N,k,g)=(4,2,2)$. The linear system $|C|$ is base point free and defines a degree two morphism $\pi : X \to \mathbb{P}^2$ which can be factorized as the composition of a birational morphism $\theta : X \to X'$ contracting all smooth rational curves orthogonal to C and a double cover $u : X' \to \mathbb{P}^2$ branched along a reduced plane sextic S [SD74]. Since X' has rational double points, then S has simple singularities, in particular either double or triple points [BHPVdV04, III, §7]. Since $|C|$ is invariant for σ, then σ induces an order nine automorphism $\bar{\sigma}$ of $\mathbb{P}^2$ which preserves the sextic S. Moreover, since σ fixes C pointwise, $\bar{\sigma}$ fixes pointwise the line $\pi(C)$ in $\mathbb{P}^2$. Thus, up to a coordinate change we can assume that $\bar{\sigma}(x_0, x_1, x_2) = (\zeta x_0, x_1, x_2)$, where ζ is a primitive 9th root of unity. Since there are no reduced plane sextics with simple singularities which are invariant for $\bar{\sigma}$, this case does not occur.

We now consider the case when σ^3 has invariants $(N, k, g) = (4, 3, 3)$. By Table 2.3 the invariant lattice $S(\sigma^3)$ of σ^3 is isomorphic to $U \oplus E_8$. Since $S(\sigma^3)$ is unimodular we have that $\mathrm{Pic}(X) = S(\sigma^3) \oplus M$, where M is a negative definite lattice. This implies that X has a σ^3-invariant elliptic fibration $\pi : X \to \mathbb{P}^1$ with a section E having a reducible fiber F_0 of type II^* (see the proof of [AS15, Proposition 3]) and [Kon89, Lemma 3.1]). The genus three curve C fixed by σ^3 clearly intersects all fibers of π, thus σ^3 preserves each fiber. This implies that E is fixed by σ^3 and C intersects the general fiber of π in two points by the Riemann-Hurwitz formula. By [AS08, Lemma 5], since π has a section E, C is invariant for σ, $C^2 = 4$ and $C \cdot E = 0$, we obtain that $f \cdot \sigma^*(f) \leq 1$, where f denotes the class of a fiber of π. This implies that $\sigma^*(f) = f$, i.e. π is invariant for σ. In particular σ preserves the section E. Moreover σ acts with order 3 on the basis of π, since a smooth genus one curve does not have automorphisms of order 9 fixing points. In particular σ does not fix C pointwise. The fiber F_0 is also invariant for σ (since otherwise π would have three fibers of type II^*, which is impossible for a K3 surface), in particular σ must fix the component of multiplicity 6 of F_0, since it has genus zero and contains at least three fixed points. □

We now study the case when σ^3 fixes a smooth curve of genus one.

Proposition 3.1.5. *Let σ be an automorphism of order 9 of a K3 surface X such that σ^3 fixes pointwise an elliptic curve E. Then $|E|$ defines a σ-invariant elliptic fibration $\pi : X \to \mathbb{P}^1$ such that σ induces an order nine automorphism of $\mathbb{P}^1$ with two fixed points corresponding to the fiber E and to a singular fiber E'. Moreover one of the following holds:*

(i) *E' is of type I_0^* and* $\mathrm{Fix}(\sigma^3) = E \cup R \cup \{p_1, \dots, p_4\}$, *where R is the central component of E' and the p_i's belong each to one of the four branches of E'. Moreover there are the following possibilities for* $\mathrm{Fix}(\sigma)$*:*

D1. $\mathrm{Fix}(\sigma) = R \cup \{p_1, \dots, p_4, q_1, q_2, q_3\}$ *where* $q_1, q_2, q_3 \in E$*,*

D2. $\mathrm{Fix}(\sigma) = E \cup \{q_4, q_5, p_4\}$*, where* $q_4, q_5 \in R$*,*

D3. $\mathrm{Fix}(\sigma) = \{q_1, \dots, q_5, p_4\}$*, where* $q_1, q_2, q_3 \in E$ *and* $q_4, q_5 \in R$*,*

D4. $\mathrm{Fix}(\sigma) = \{q_4, q_5, p_4\}$*, where* $q_4, q_5 \in R$*.*

(ii) *E' is of type I_9^*, $\mathrm{Fix}(\sigma^3) = E \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup \{q_1, \dots, q_7\}$ and $\mathrm{Fix}(\sigma) = R_1 \cup R_4 \cup \{p_1, \dots, p_{14}\}$, where the R_i are components of the fiber E', $p_1, \dots, p_{11} \in E'$ and $p_{12}, p_{13}, p_{14} \in E$.*

Proof. Observe that $\sigma(E) = E$, since E is the unique elliptic curve in $\mathrm{Fix}(\sigma^3)$. This implies that the elliptic fibration π defined by $|E|$ is σ-invariant. The action induced by σ^3 on the basis of the fibration is not trivial, since otherwise the action of σ^3 at a point of E would be the identity on the tangent space, contradicting the fact that σ^3 is non-symplectic. Thus σ induces an order nine automorphism on the basis of the fibration. The two fixed points of such action correspond to the fiber E and to another fiber E' which must contain all smooth rational curves and points fixed by σ^3. By Table 3.1 there are two possible cases for $\mathrm{Fix}(\sigma^3)$: either $(N, k, g) = (4, 1, 1)$ or $(7, 4, 1)$.

In the first case the fiber E' is of type I_0^*. All other possible types for the fiber can be ruled out using the fact that σ^3 has only one fixed curve, four isolated points and using Lemma 2.2.13, which implies that in any tree of smooth rational curves, two fixed curves by σ^3 have distance three (in the intersection graph). Thus the central component of E' is fixed by σ^3 and the remaining components contain one isolated fixed point each. The automorphism σ can either fix E, act on it with three fixed points, or without fixed points, by Riemann-Hurwitz formula. Moreover, it either fixes the central component of E' and four isolated points, or it acts with order three on the central component and has three isolated fixed points on E'. If σ fixes the central component of E', then $\alpha = 1$, $a_3 = 3$ by Lemma 3.1.2 and E must contain three fixed points. Thus the only possible cases for the action of σ are the four cases in the statement.

In the second case, using again Lemma 2.2.13, we find that E' is of type I_9^* with the configurations in Figure 3.2, where each vertex represents a curve and double circles represent the curves $R_1, \dots, R_4$ pointwise fixed by σ^3. Moreover, E' contains 7 isolated fixed points, where non-fixed components meet.

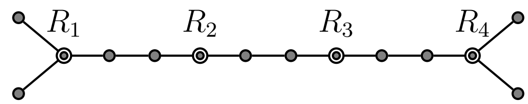


Figure 3.2 Configuration I_9^*

Since the intersection graph of E' has no order 3 symmetry, the automorphism σ preserves each component of E'. Since the curves R_1, R_4 contain three fixed points each, then σ must fix them pointwise. Moreover, it fixes 11 isolated points in E' (where non-fixed components meet), 4 of them on R_2 and R_3. By Lemma 3.1.2, since $a_2 = 2\alpha + 1 = 5$, σ also has fixed points in E. By Riemann-Hurwitz formula, σ has 3 fixed points in E. □

We now prove a remark about order three automorphisms of hyperelliptic curves.

Lemma 3.1.6. *Let C be a hyperelliptic curve of genus $g \geq 2$ and let f be an automorphism of C of order 3, then f has $2, 3$ or 4 fixed points.*

Proof. By [Har77, Proposition 5.3. Ch. IV] C has a unique hyperelliptic involution i. This implies that any automorphisms of C commutes with i. Let $\pi : C \to \mathbb{P}^1$ be the quotient by the hyperelliptic involution i. Since i commutes with f, then f induces an automorphism $\bar{f}$ of order 3 on $\mathbb{P}^1$. By Riemann-Hurwitz formula $\bar{f}$ fixes 2 points $q_1, q_2 \in \mathbb{P}^1$. Thus, since $\deg(\pi)$ and $\deg(f)$ are coprime, the fixed locus of f is equal to $\pi^{-1}(\{q_1, q_2\})$. □

3.2 Classification theorem

In this section we state and prove the main result of this Chapter. In the following σ denotes a non-symplectic automorphism of a K3 surface X of order nine, and we use the following notation for its fixed locus:

$$\mathrm{Fix}(\sigma) = \{p_1, p_2, \dots, p_{N_\sigma}\} \sqcup C_{g_\sigma} \sqcup R_1 \sqcup \dots \sqcup R_{k_\sigma},$$

where $p_1, \dots, p_{N_\sigma}$ are isolated fixed points, C_{g_σ} is a smooth curve of genus g_σ and $R_1, \dots, R_{k_\sigma}$ are smooth rational curves. Moreover, d_1 and d_3 will denote the dimensions of the eigenspaces of σ^* relative to the eigenvalues 1 and ζ_3 (see Section 3.1).

Theorem 3.2.1. *Let X be a complex K3 surface, σ be a non-symplectic automorphism of X of order nine. Then*

(i) *σ is purely non-symplectic, i.e. $\tau = \sigma^3$ is non-symplectic;*

(ii) *the topological structure of* $\mathrm{Fix}(\sigma)$, $\mathrm{Fix}(\tau)$, *the invariant lattice $S(\tau)$ of τ^* in $H^2(X,\mathbb{Z})$ and the dimensions of the eigenspaces of σ^* in $H^2(X,\mathbb{C})$ are described in Table 3.2.*

Moreover, all configurations described in Table 3.2 exist.

	σ^3			σ	
Case	(N,k,g)	$S(\tau)$	Case	$(N_\sigma, k_\sigma, g_\sigma)$	(d_1, d_3)
A	(1,0,3)	$U(3) \oplus A_2$	A1	$(6,0,-)$	(4,0)
			A2	$(3,0,-)$	(2,1)
B	(1,1,4)	$U \oplus A_2$	B	$(6,0,-)$	(4,0)
C	(4,0,0)	$U(3) \oplus A_2^4$	C	$(3,0,-)$	(4,3)
D	(4,1,1)	$U \oplus A_2^4$	D1	$(7,0,0)$	(8,1)
			D2	$(3,0,1)$	(4,3)
			D3	$(6,0,-)$	(6,2)
			D4	$(3,0,-)$	(4,3)
E	(4,2,2)	$U \oplus E_6 \oplus A_2$	E	$(10,0,0)$	(10,0)
F	(4,3,3)	$U \oplus E_8$	F	$(10,0,0)$	(10,0)
G	(7,3,0)	$U \oplus E_6^2 \oplus A_2$	G1	$(10,0,0)$	(12,2)
			G2	$(3,0,-)$	(6,5)
H	(7,4,1)	$U \oplus E_6 \oplus E_8$	H	$(14,1,0)$	(16,0)

Table 3.2 Classification of automorphisms of order 9 of K3 surfaces

Proof. Part (i) follows from Lemma 3.1. We now analyse the cases for $\mathrm{Fix}(\sigma^3)$ as they appear in Table 3.1 and deduce the possibilities for $\mathrm{Fix}(\sigma)$ according to Lemma 3.1.2.

A. In this case $\mathrm{Fix}(\sigma^3) = C_3 \sqcup \{p\}$. By Proposition 3.1.4, C_3 is not contained in $\mathrm{Fix}(\sigma)$, thus $\alpha = 0$ and we obtain two possible cases:

$$(a_1, a_2, a_3, a_4) = (0, 1, 4, 1) \text{ (case A1) and}$$

$$(a_1, a_2, a_3, a_4) = (1, 1, 1, 0) \text{ (case A2).}$$

B. In this case $\mathrm{Fix}(\sigma^3) = C_4 \sqcup R \sqcup \{p\}$. By [AS08, Corollary 4.3] the curve C_4 is hyperelliptic. Since $a_1 + a_4 \leq n = 1$, then $\alpha = 0$, i.e. σ has no fixed curves. Moreover $a_3 + 3a_1 = 4$. The case $a_3 = 1$ does not occur since $\mathrm{Fix}(\sigma)$ would contain only three isolated points, two of them on R by the Riemann-Hurwitz formula, and no one on C_4, contradicting Lemma 3.1.6. Thus the only possible case is

$$(a_1, a_2, a_3, a_4) = (0, 1, 4, 1).$$

C. In this case $\mathrm{Fix}(\sigma^3) = R \sqcup \{p_1, \ldots, p_4\}$. Thus α is either 0 or 1. The latter case can not occur since it would give $a_2 = 3$, while R contains exactly two fixed points. Thus $\alpha = 0$ and the only solution of (Eq. 1.4) in Lemma 3.1.2 is

$$(a_1, a_2, a_3, a_4) = (1, 1, 1, 0).$$

D. In this case $\mathrm{Fix}(\sigma^3) = C_1 \sqcup R \sqcup \{p_1, \ldots, p_4\}$. By Proposition 3.1.5 there are four possible cases D1, D2, D3, D4 for the fixed locus of σ. The number of isolated fixed points of each type can be computed as before by means of Lemma 3.1.2.

E. In this case $\mathrm{Fix}(\sigma^3) = C_2 \sqcup R_1 \sqcup R_2 \sqcup \{p_1, \ldots, p_4\}$. By Proposition 3.1.4 the curve C_2 is not fixed by σ. Moreover, by the Riemann-Hurwitz formula and Lemma 3.1.6, C_2 contains exactly 4 fixed points for σ. Since $a_1 + a_4 \leq n = 4$, then α is either 0 or 1. If $\alpha = 0$, then R_1, R_2 should contain two isolated fixed points each. By the Riemann-Hurwitz formula for C_2 and Lemma 3.1.6, it should be $a_2 + a_3 = 8$, and there is no solution of (Eq. 1.4) satisfying these conditions. Thus we can assume

$\alpha = 1$ and $R_1 \subseteq \mathrm{Fix}(\sigma)$. Since $a_2 + a_3 = 6$ the only solution of (Eq. 1.4) is

$$(a_1, a_2, a_3, a_4) = (3, 3, 3, 1).$$

F. In this case $\mathrm{Fix}(\sigma^3) = C_3 \sqcup R_1 \sqcup R_2 \sqcup R_3 \sqcup \{p_1, \ldots, p_4\}$. By Proposition 3.1.4, C_3 is not fixed by σ and $\alpha \geq 1$. Moreover C_3 is hyperelliptic by [AS08, Corollary 4.3], thus by the Riemann-Hurwitz formula and Lemma 3.1.6 it contains exactly 2 fixed points. The cases $\alpha = 2$ or 3 are not possible since in both cases a_2 would be bigger than the number of fixed points on the curves C_3, R_i. If $\alpha = 1$, then $a_2 + a_3 = 6$ and the only possible solution of (Eq. 1.4) is

$$(a_1, a_2, a_3, a_4) = (3, 3, 3, 1).$$

G. In this case $\mathrm{Fix}(\sigma^3) = R_1 \sqcup R_2 \sqcup R_3 \sqcup R_4 \sqcup \{p_1, \ldots, p_7\}$. Observe that $\alpha \in \{0, 1, 2, 3, 4\}$ counts how many curves among the R_i's are fixed by σ. We first assume that all the R_i's are preserved by σ, thus $a_2 + a_3 = 2(4 - \alpha)$. Under this condition, the system (Eq. 1.4) has no solution whenever $\alpha = 0, 2, 3, 4$. If $\alpha = 1$ the only solution of (Eq. 1.4) is

$$(a_1, a_2, a_3, a_4) = (3, 3, 3, 1) \text{ (case G1)}.$$

Now assume that three of the rational curves R_i are permuted by σ. The case $\alpha = 1$ is incompatible with (Eq. 1.4), while if $\alpha = 0$ the only solution is

$$(a_1, a_2, a_3, a_4) = (1, 1, 1, 0) \text{ (case G2)}.$$

H. In this case $\mathrm{Fix}(\sigma^3) = C_1 \sqcup R_1 \sqcup \cdots \sqcup R_4 \sqcup \{p_1, \ldots, p_7\}$. By Proposition 3.1.5 σ fixes two curves of genus zero and has 14 isolated points, 3 of them on C_1 and the others equally distributed on the two rational curves fixed by σ^3. Thus the only solution is

$$(a_1, a_2, a_3, a_4) = (6, 5, 2, 1).$$

In each case the values of d_1 and d_3 can be computed by means of Lemma 3.1.2 and using the fact that $d_1 + 2d_3 = 22 - 2m$. The existence part of the statement will be proved in the following section. □

3.3 Examples

We now provide examples for all cases in Table 3.2. As before, ζ denotes a primitive 9th root of unity.

Example 3.3.1 (Case A1)**.** Let $F_1, F_4 \in \mathbb{C}[x_0, x_1]$ be general homogeneous polynomials of degree 1 and 4 respectively. The following is a smooth $K3$ surface:

$$X = \{F_4(x_0, x_1) + F_1(x_0, x_1)x_2^3 + x_2x_3^3 = 0\} \subset \mathbb{P}^3$$

with the automorphism $\sigma(x_0, x_1, x_2, x_3) = (x_0, x_1, \zeta^6 x_2, \zeta^4 x_3)$. The fixed locus of σ^3 is the union of the curve $C = X \cap \{x_3 = 0\}$ of genus 3 and the point $p_1 = (0, 0, 0, 1)$. Thus $(N, k, g) = (1, 0, 3)$. The fixed locus of σ contains exactly 6 points, five of them on the curve C (the four roots of $F_4(x_0, x_1)$ and the point $q_1 = (0, 0, 1, 0)$), thus we are in **case A1**.

Example 3.3.2 (Case A2 and its descendents)**.** For a general choice of the coefficients the following is a $K3$ surface:

$$X = \{ax_0^2x_1x_2 + bx_1^2x_2^2 + cx_2^3x_0 + dx_1^3x_0 + fx_0^4 + x_2x_3^3 = 0\} \subseteq \mathbb{P}^3$$

which carries the automorphism $\sigma(x_0, x_1, x_2, x_3) = (x_0, \zeta^3 x_1, \zeta^6 x_2, \zeta x_3)$. The fixed locus of σ^3 is the union of the curve $C = X \cap \{x_3 = 0\}$ of genus 3 and the point $p_1 = (0, 0, 0, 1)$, thus $(N, k, g) = (1, 0, 3)$. The fixed locus of σ contains p_1 and the points $p_2 = (0, 1, 0, 0), p_3 = (0, 0, 1, 0)$. Thus we are in **case A2**.

For special values of the coefficients we obtain other examples. In each case we assume that the coefficients are general with the given assumption.

(i) if $b = 0$, the curve C is the union of a smooth cubic E and a line L and the surface X has three A_2 singularities in $L \cap E$. The minimal resolution $\pi : \tilde{X} \to X$ is a K3 surface and σ lifts to an automorphism $\tilde{\sigma}$ on $\tilde{S}$. Each exceptional divisor over the three singular points contains a fixed point for $\widetilde{\sigma^3}$ and $\tilde{p}_1 = \pi^{-1}(p_1) \in \mathrm{Fix}(\widetilde{\sigma^3})$. Thus the invariants for $\widetilde{\sigma^3}$ are $(N, k, g) = (4, 1, 1)$. Since the three exceptional divisors of type A_2 are permuted by $\tilde{\sigma}$, $\mathrm{Fix}(\tilde{\sigma})$ only contains the preimages $\tilde{p}_i$ of the points p_i, $i = 1, 2, 3$, thus this is an example of **case D4**.

(ii) if $a = -d - 2f, b = f - d, c = d$ the curve C acquires three nodes and the surface S has three A_2 singularities. Thus $\widetilde{\sigma^3}$ fixes three points, one in each of the exceptional divisors, the point $\tilde{p}_1$ and the proper transform of C, which has genus 0. Thus the invariants for σ^3 are $(N, k, g) = (4, 0, 0)$ and this is an example of **case C**.

(iii) if $c = 0$ the curve C acquires a tacnode at $p_3 = (0, 0, 1, 0)$, which gives a singularity of type E_6 of the surface. In this case $\mathrm{Fix}(\widetilde{\sigma^3})$ contains the proper transform of C, which has genus 1, a rational curve and three points in the exceptional divisor of type E_6, and $\tilde{p}_1$. Thus $(N, k, g) = (4, 1, 1)$. The automorphism $\tilde{\sigma}$ preserves the exceptional divisor of type E_6 and thus fixes its central component. By Theorem 3.2.1 this is an example of **case D1**.

Example 3.3.3 (Case B and its descendents)**.** Consider the elliptic surface with Weierstrass equation

$$y^2 = x^3 + t(t^3 - a)(t^3 - b)(t^3 - c), \text{ for } a, b, c \in \mathbb{C},$$

which carries the automorphism $\sigma(x, y, t) = (\zeta^4 x, \zeta^6 y, \zeta^3 t)$. For general $a, b, c \in \mathbb{C}$ this defines a K3 surface. More precisely, if a, b, c are distinct and non-zero, then by [Mir89, Table IV.3.1] the elliptic fibration has a singular fiber of type II for $t = 0$, of type IV for $t = \infty$ and nine fibers of type II over the zeroes of $P(t) = (t^3 - a)(t^3 - b)(t^3 - c)$. The automorphism σ^3 preserves each fiber of the elliptic fibration. The fixed locus of σ^3 clearly contains the curve $C = \{x = y^2 - tP(t) = 0\}$, which is hyperelliptic of genus

4 and is a 2-section of the fibration, and the section at infinity S_∞. Moreover, since neither C or S_∞ passes through the center p of the fiber of type IV, then p is an isolated fixed point of σ^3. Thus this gives a family of examples of **case B**. Observe that σ only preserves the fibers over $t = 0$ and $t = \infty$. This implies that its fixed locus consists of 4 points in the fiber over $t = \infty$ (the center p and the points where S_∞ and C intersect) and two points on the fiber over $t = 0$ (where S_∞ and C intersect).

For special values of $a, b, c \in \mathbb{C}$ we obtain other examples:

(i) $a = 0$: the fibration has one fiber of type IV^* over $t = 0$, IV over $t = \infty$ and six fibers of type II over the zeros of $(t^3 - b)(t^3 - c)$. The fixed locus of σ^3 in this case contains the curve C, which has genus two, the section at infinity S_∞, the central component and three isolated points of the fiber of type IV^* and the center of the fiber of type IV. Thus this gives a family of examples of **case E**.

(ii) $b = c$: the fibration has four fibers of type II over $t = 0$ and the zeroes of $t^3 - a$, and four fibers of type IV over $t = \infty$ and the zeroes of $t^3 - b$. The fixed locus of σ^3 contains the curve C, which has genus one, the section at infinity S_∞ and the four centers of the fibers of type IV. Thus we are in case D. The fixed locus of σ contains no curves and six isolated points, four on the fiber over $t = \infty$ (the center and the intersection points with C and S_∞) and two points on the fiber of type II (where C and S_∞ intersect). This gives a family of examples of **case D3**.

(iii) $c = \infty$: in this case the equation of the fibration is $y^2 = x^3 + t(t^3 - a)(t^3 - b)$, so that there are 7 fibers of type II, over $t = 0$ and over the zeroes of $(t^3 - a)(t^3 - b)$, and one fiber of type II^* over $t = \infty$. The fixed locus of σ^3 contains the curve C, which has genus 3, the section at infinity and two rational curves in the fiber of type II^*. Moreover it fixes 4 points in the fiber of type II^*. Thus we obtain a family of examples of **case F**.

(iv) $a = 0, b = c$: the fibration has a fiber of type IV^* over $t = 0$, of type IV over $t = \infty$ and three fibers of type IV over the zeroes of $t^3 - b$. Observe that in this case C splits in the union of two sections of the fibration:

$$C_1 = \{x = y - t^2(t^3 - b) = 0\}, \quad C_2 = \{x = y + t^2(t^3 - b) = 0\}.$$

The fixed locus of σ^3 contains the curves C_1, C_2, the section at infinity S_∞ and the central component of the fiber of type IV^*. Moreover, σ^3 fixes the centers of the four fibers of type IV. Observe that σ must preserve each component of the fiber over $t = 0$, so that it fixes its central component. Thus we obtain a family of examples of **case G1**.

(v) $a = 0, c = \infty$: in this case the equation of the fibration is $y^2 = x^3 + t^4(t^3 - b)$, so that there is a fiber of type IV^* over $t = 0$, of type II^* over $t = \infty$ and three fibers of type II over the zeroes of $t^3 - b$. The fixed locus of σ^3 contains the curve C, which has genus 1, the section at infinity, two rational curves in the fiber of type II^* and one rational curve in the fiber of type IV^*. Moreover it has 4 fixed points in the fiber of type II^* and 3 fixed points in the fiber of type IV^*. Thus we obtain a family of examples of **case H**.

Example 3.3.4 (Case G2)**.** Consider the elliptic K3 surface X with Weierstrass equation $y^2 = t^4(t^3 - b)^2$ (the case (iii) in Example 3.3.3). An explicit computation shows that, fixed S_∞ as zero section, C_1 is a 3-torsion section and $C_2 = C_1 \oplus C_1$, where $\oplus$ denotes the sum in the group law of the elliptic curve over $\mathbb{C}(t)$ associated to the fibration. The translation by the section C_1 defines a symplectic automorphism φ of order three on X. A computation with Magma [BCP97] available at `https://bit.ly/2FwQ0ZU` gives that

$$\sigma'(x, y, t) = (\alpha(x, y, t), \beta(x, y, t), t),$$

where

$$\alpha(x, y, t) = -2\frac{(t^5 + t^2)}{x^2}(y - (t^5 + t^2))$$

$$\beta(x, y, t) = -4\frac{(t^5 + t^2)^2}{x^3}(y - (t^5 + t^2)) + (t^5 + t^2).$$

Since σ' commutes with σ, as can be checked directly, then $\sigma' := \varphi \circ \sigma$ is a non-symplectic automorphism of order 9 on X such that $(\sigma')^3 = \sigma^3$. The sections S_∞, C_1, C_2 are permuted by σ'. This implies that the three branches of the fiber of type IV^* are also permuted by σ while the central component, which is σ'-invariant, contains two isolated fixed points of σ'. Finally, σ' acts on the fiber of type IV permuting the three components and fixing the center. Thus σ' fixes exactly three isolated points, giving an example of **case G2**.

Example 3.3.5 (Case D2)**.** Consider the elliptic fibration

$$y^2 = x^3 + x + t^9 + c, \quad c \in \mathbb{C}$$

with the automorphism $\sigma(x, y, t) = (x, y, \zeta t)$. The fibration has a singular fiber of type I_0^* over $t = \infty$ and 9 fibers of type I_1 over the zeroes of $4 + 27(t^9 + c)^2 = 0$. Observe that σ^3 preserves the fibers over $t = 0$ and $t = \infty$, thus $\mathrm{Fix}(\sigma^3)$ contains a curve of genus 1 and the central curve of the fiber I_0^*. Moreover, the fiber of type I_0^* contains 4 isolated fixed points for σ^3, so that the invariants for σ^3 are $(N, k, g) = (4, 1, 1)$. The automorphism σ on the fiber over $t = 0$ acts as the identity, thus this corresponds to **case D2**.

Remark 3.3.6. Examples for members of the families B, E, F and for H have been given also by Taki in [Tak10]. The case A1 is missing in [Tak10, Theorem 1.3], as was also observed in [Bra19, Remark 6.7].

3.4 Moduli

In this section we will consider the families of K3 surfaces with a non-symplectic automorphism of order 9 of maximal dimension, i.e. those of type A_1, A_2 and B. We show that their moduli space is irreducible and give birational maps to moduli spaces of curves with automorphisms.

Proposition 3.4.1. *Let X be a K3 surface with a non-symplectic automorphism σ of order* 9*, such that* $\mathrm{Pic}(X) = S(\sigma^3)$*. If* $\mathrm{Fix}(\sigma)$ *is of*

(i) *type A1, then up to isomorphism $(X, \langle\sigma\rangle)$ is as in Example 3.3.1,*

(ii) *type A2, then up to isomorphism $(X, \langle\sigma\rangle)$ is as in Example 3.3.2,*

(iii) *type B, then up to isomorphism $(X, \langle\sigma\rangle)$ is as in Example 3.3.3.*

Proof. We first assume that σ^3 has invariants $(N, k, g) = (1, 1, 3)$. Let C be the smooth genus three curve in the fixed locus of σ^3. By [AS08, Proposition 4.9] the linear system $|C|$ defines an embedding $\varphi_C : X \to \mathbb{P}^3$. After identifying X with its image in $\mathbb{P}^3$ we can assume that

$$\sigma^3(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, \zeta^3 x_3).$$

Since C is invariant for σ, then σ is given by a projectivity of $\mathbb{P}^3$ which leaves invariant the hyperplane $x_3 = 0$ and the point $p = (0, 0, 0, 1)$. An order three automorphism of $H \cong \mathbb{P}^2$ is exactly one of the following up to a coordinate change and up to taking powers:

$$f_1(x_0, x_1, x_2) = (x_0, x_1, \zeta^3 x_2), \quad f_2(x_0, x_1, x_2) = (x_0, \zeta^3 x_1, \zeta^6 x_2).$$

This implies that, up to coordinate changes and up to taking powers, σ is one of the following:

$$\sigma_{1,k}(x_0, x_1, x_2, x_3) = (x_0, x_1, \zeta^3 x_2, \zeta^k x_3),\ \sigma_2(x_0, x_1, x_2, x_3) = (x_0, \zeta^3 x_1, \zeta^6 x_2, \zeta x_3),$$

where $k = 1, 2$. Analyzing each of these automorphisms we obtain that the only ones which allow a smooth homogeneous invariant polynomial of degree 4 are $\sigma_{1,2}$ and σ_2. In case the automorphism is $\sigma_{1,2}$ we find that the family of smooth invariant polynomials is the one in Example 3.3.1; in case it is σ_2 the family of smooth invariant polynomials is the one in Example 3.3.2.

We now assume that σ is of type B. By [AS08, Proposition 4.2] and its proof, X has an elliptic fibration $\pi : X \to \mathbb{P}^1$ with Weierstrass form

$$y^2 = x^3 + p(t),$$

where $\deg(p) = 10$ and in this model $\sigma^3(x, y, t) = (\zeta^3 x, y, t)$. The fibration has a reducible fiber of type IV over $t = \infty$ and 10 singular fibers of type II over the zeroes of p. An argument similar to the one in the proof of Proposition 3.1.4 shows that σ preserves this fibration, using [AS15, Lemma 5]. Observe that σ induces an order three automorphism on the basis of the fibration with two fixed points, one of them at $t = \infty$. Up to a coordinate change in t we can assume that σ acts as $t \mapsto \zeta^3 t$ on the basis of π. This implies that up to a constant $p(t) = t(t^3 - a)(t^3 - b)(t^3 - c)$ for distinct $a, b, c \in \mathbb{C}$, which gives the family in Example 3.3.3. □

Let (X, σ) be a K3 surface with a non-symplectic automorphism of order 9 such that $\sigma^*(\omega_X) = \zeta\omega_X$, fix an isometry $\varphi : H^2(X, \mathbb{Z}) \to L_{K3}$, let $\rho = \varphi\sigma^*\varphi^{-1}$, $\iota = \rho^3$, $M \subseteq L_{K3}$ be the fixed lattice of ι and N be its orthogonal complement. By [DK07, §11] the moduli space of (M, ρ)-polarized K3 surfaces is isomorphic to a complex ball quotient $\mathcal{D}/\Gamma$ where

$$\mathcal{D} = \{z \in \mathbb{P}(V) : \langle z, \bar{z}\rangle > 0\} \cong \mathbb{B}_d, \quad \Gamma = \{\gamma \in O(N) : \gamma \circ \rho = \rho \circ \gamma\}$$

with $V = \{z \in N \otimes \mathbb{C} : \rho(z) = \zeta z\}$ and $d = \dim(V) - 1 = \frac{m}{3} - 1$. The points in the moduli space which correspond to (M, ρ)-ample polarized K3 surfaces, i.e. such that ρ is induced by an automorphism of the surface, belong to an open subset defined as the quotient of the complement of a divisor Δ in $\mathcal{D}$ [DK07, Lemma 11.5].

Given a general pair (X, σ) as in Example 3.3.1, we will denote by $\mathcal{A}_1$ the corresponding moduli space of (M, ρ)-polarized K3 surfaces. Similarly we define $\mathcal{A}_2$ and $\mathcal{B}$ as the moduli spaces corresponding to the general members of the families of type $A2$ and B. By Theorem 3.2.1 and Proposition 3.4.1 we obtain the following.

Corollary 3.4.2. *The moduli space of $K3$ surfaces with a non-symplectic automorphism of order 9 has three irreducible components of maximal dimension 2: $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{B}$.*

Remark 3.4.3. The moduli space $\mathcal{A}_1$ is birational to the moduli space $\mathcal{M}_3^3$ of genus three curves C carrying a cyclic automorphism f of order 3 with $C/(f) \cong \mathbb{P}^1$. The isomorphism is given by the map

$$(X, \langle\sigma\rangle) \mapsto (C, \sigma_{|C}), \tag{3.1}$$

where C is the fixed curve of σ^3. Conversely, given a pair (C, f) as above, observe that C is not hyperelliptic by Lemma 3.1.6, since it contains 5 fixed points for f. Thus C can be embedded in $\mathbb{P}^2$ as a smooth plane quartic and f is induced by an order three automorphism of $\mathbb{P}^2$. Up to a change of coordinates we can assume

$$f(x_0, x_1, x_2) = (x_0, x_1, \zeta^3 x_2).$$

Let Y be the triple cover of $\mathbb{P}^2$ branched along C and $2L$, where $L = \{x_2 = 0\}$ is the line fixed by f. The normalization of Y is a K3 surface X with an order nine automorphism σ, obtained lifting f, which gives a pair $(X, \langle\sigma\rangle)$ in $\mathcal{A}_1$.

Similarly one can show that $\mathcal{A}_2$ is birational to the moduli space of non-hyperelliptic curves of genus three C with a cyclic automorphism f of order 3 with $g(C/(f)) = 1$.

Finally, the map (3.1) defines an isomorphism between $\mathcal{B}$ and the moduli space of hyperelliptic curves of genus 4 with a cyclic automorphism f of order 3. Observe that one such curve C can be defined by an equation of the form

$$y^2 - t(t^3 - as^3)(t^3 - bs^3)(t^3 - cs^3) = 0,$$

with distinct $a, b, c \in \mathbb{C}$, in $\mathbb{P}(1,1,5)$ and in these coordinates $f(s,t,y) = (s, \zeta^3 t, \zeta^6 y)$. The triple cover of $\mathbb{P}(1,1,5)$ branched along C is birational to a K3 surface with an order nine automorphism which gives a point in $\mathcal{B}$.

Chapter 4

Non-symplectic automorphisms with 1-dimensional moduli space

The content of this Chapter is based on the paper [ACV].

4.1 Motivation

Let X be a K3 surface and σ be a purely non-symplectic automorphism of X of order $n \geq 2$, i.e. $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $H^{2,0}(X) = \mathbb{C}\omega_X$ and ζ_n is a primitive nth root of unity. The period line $\mathbb{C}\omega_X$ belongs to the domain

$$\mathcal{D} = \{\mathbb{C}z \in \mathbb{P}(V^\sigma) : (z, z) = 0, (z, \bar{z}) > 0\},$$

where $V^\sigma = H^2(X, \mathbb{C})^{\sigma^*}_{\zeta_n}$ is the ζ_n-eigenspace of σ^* in $H^2(X, \mathbb{C})$. Observe that, for $n \geq 3$ and $z \in V^\sigma$ we have

$$(z, z) = (\sigma^* z, \sigma^* z) = \zeta_n^2 (z, z),$$

thus the condition $(z, z) = 0$ is not necessary. In particular $\dim(\mathcal{D})$, which can be interpreted as the number of moduli of (X, σ), is equal to $\dim(V^\sigma) - 1$ (see [DK07, §11]). On the other hand, by Theorem 2.2.6, T_X has the structure of a free $\mathbb{Z}[\zeta_n]$-module via the correspondence $\sigma^* \mapsto \zeta_n$, so that $\mathrm{rk}(T_X) = \dim(V^\sigma)\varphi(n)$. Since $\mathrm{rk}(T_X) \leq 21$, this implies that

$$\dim(\mathcal{D}) \leq \gamma(n) := \left\lfloor \frac{21}{\varphi(n)} \right\rfloor - 1.$$

In this Chapter we classify all pairs (X, σ), where X is a K3 surface and σ a non-symplectic automorphism of order n such that $\gamma(n) = 1$. This happens exactly for $n = 11, 22, 15, 30, 16, 20, 24$, since in these cases $\varphi(n) = 10$ or 8 (see Table 2.1). More precisely, for any such order we classify the general members (X, σ) of the one-dimensional components of the moduli space of K3 surfaces with a purely non-symplectic automorphisms of order n. Moreover, for $n = 15$ and 22 we provide a full classification of K3 surfaces with a purely non-symplectic automorphism of order n (not only for the general ones in the components of maximal dimension).

4.2 Classification theorem

We now formulate the classification Theorem. We will denote by σ_i a power of an automorphism σ of order i. The local action of σ in a neighborhood of one of its fixed points is of the form

$$A_{i,n} = \begin{pmatrix} \zeta_n^{i+1} & 0 \\ 0 & \zeta_n^{n-i} \end{pmatrix}, \quad i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

We will say that a fixed point is of type $A_{i,n}$ if it is given by the matrix $A_{i,n}$ and the number of fixed points of this type will be denoted by $a_{i,n}$. When $i = 0$ the fixed point belongs to a fixed curve, otherwise it is an isolated fixed point.

We recall the notation for the fixed locus of an automorphism σ and its powers: g_i, k_i, N_i denote the maximal genus of a fixed curve, the number of fixed curves and the number of isolated fixed points of σ_i respectively. Finally, we recall that the vector $d = (d_n, \ldots, d_k, \ldots, d_1)$ contains the dimensions of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$. More precisely, given any divisor k of n, d_k is the dimension of the eigenspace corresponding to a primitive k-th root of unity. In particular d_1 is the rank of the invariant lattice of

σ^* and we will fix $d_n = \dim(V^\sigma) = 2$.

Theorem 4.2.1. *Let X be a complex K3 surface and σ be a purely non-symplectic automorphism of X of order $n \geq 2$ such that $\varphi(n) \in \{8, 10\}$ and $\dim V^\sigma = 2$. Then the fixed locus of σ and of some of its powers σ_i, the vector d containing the dimensions of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$ and the Néron-Severi lattice of a general K3 surface with such property are described in Table 4.1.*

Remark 4.2.2. Since T_X has the structure of a $\mathbb{Z}[\zeta_n]$-module and $\varphi(n) = 8$ or 10, then $\operatorname{rk}\operatorname{NS}(\mathrm{X}) \geq 22 - 2\varphi(n)$. The generality assumption in the statement of Theorem 4.2.1 means that the Néron-Severi lattice of X has the minimal rank.

We recall a result contained in [GS13, Theorem 1.4 and Theorem 1.5].

Proposition 4.2.3. *Let X be a K3 surface with a non-symplectic automorphism σ of order n. If either $n = 5$, or $n = 11$ and the fixed locus of σ contains a curve, then X admits a non-symplectic automorphism τ of order $2n$ with $\tau^2 = \sigma$.*

Moreover, if $n = 11$ and the fixed locus of σ consists of only isolated fixed points, then X does not admit a non-symplectic automorphism of order 22.

We will now prove Theorem 4.2.1 studying each order separately.

4.3 Order 11

Non-symplectic automorphisms of order 11 have been classified in [OZ11] and [AST11, §7]. In particular the proof of Theorem 4.2.1 for order 11 follows from the following result.

Theorem 4.3.1. *Let X be a K3 surface with a non-symplectic automorphism σ of order 11 such that $\operatorname{rk} S(\sigma) = 2$ (or equivalently $\dim(V^\sigma) = 2$). Then two cases can occur:*

a) $\operatorname{Fix}(\sigma_{11}) = C_1 \sqcup \{p_1, p_2\}$ *and* $S(\sigma) \cong U$*,*

b) $\operatorname{Fix}(\sigma_{11}) = \{p_1, p_2\}$ *and* $S(\sigma) \cong U(11)$*,* *where C_1 is a smooth curve of genus one. In both cases $d = (2, 2)$. Moreover, up to isomorphism, (X, σ) belongs to the family in Example 4.3.2 in case a) and to the family in Example 4.3.3 in case b).*

Example 4.3.2. Given $a \in \mathbb{C}$, let X_{11a} be the elliptic fibration with Weierstrass equation

$$y^2 = x^3 + ax + (t^{11} - 1).$$

For general $a \in \mathbb{C}$ the fibration has one fiber of Kodaira type II over $t = \infty$ and 22 fibers of type I_1. Observe that X_{11a} carries the order 11 automorphism

$$\sigma_{11a}(x, y, t) = (x, y, \zeta_{11} t),$$

which fixes the smooth fiber over $t = 0$ and two points in the fiber over $t = \infty$.

Example 4.3.3. Consider the rational elliptic surface $\phi : Y \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + x + t.$$

The fibration has a fiber of type II^* over $t = \infty$ and two fibers of type I_1 over the zeroes of $\Delta = 4 + 27t^2$, thus it is extremal. Given $\alpha \in \mathbb{P}^1$ such that $\phi^{-1}(\alpha)$ is smooth, let $\phi_{\alpha,e} : Y_{\alpha,e} \to \mathbb{P}^1$ be the principal homogeneous space of ϕ associated to a non-trivial 11-torsion element e in $\phi^{-1}(\alpha)$. We recall that $\phi_{\alpha,e}$ has the same configuration of singular fibers as ϕ and it has a fiber $F = 11F_0$ of multiplicity 11 over α such that $(F_0)_{|F_0} = e \in \mathrm{Pic}^0(F_0)$. Let $\psi_{\alpha,e} : Z_{\alpha,e} \to \mathbb{P}^1$ be the degree 11 base change of $\phi_{\alpha,e}$ branched along $t = \infty$ and $t = \alpha$. A minimal resolution of $Z_{\alpha,e}$ is a K3 surface $X_{\alpha,e}$ carrying an elliptic fibration $\pi_{\alpha,e}$ induced by $\psi_{\alpha,e}$ which has 22 fibers of type I_1 over the two fibers of type I_1 of $\psi_{\alpha,e}$ and a fiber of type II over $t = \infty$ (see [Mir89, Table VI.4.1]). The covering automorphism of $Z_{\alpha,e} \to Y_{\alpha,e}$ induces an order 11 automorphism

σ_{11b} of $X_{\alpha,e}$.

$$\begin{tikzcd} X_{\alpha,e} \arrow[r] \arrow[d, "\pi_{\alpha,e}"] & Z_{\alpha,e} \arrow[r] \arrow[d, "\psi_{\alpha,e}"] & Y_{\alpha,e} \arrow[d, "\phi_{\alpha,e}"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \arrow[r, "11:1"] & \mathbb{P}^1. \end{tikzcd}$$

We will denote by (X_{11b}, σ_{11b}) the family of K3 surfaces with automorphism obtained with this construction. The automorphism σ_{11b} fixes exactly two points in the fiber of $\pi_{\alpha,e}$ of type II.

Observe that in both Examples the automorphism is non-symplectic, since there exist no symplectic automorphisms of K3 surfaces of order 11 [Muk88].

4.4 Order 22

In this section we will give the classification of purely non-symplectic automorphisms of order 22 with $\dim(V^\sigma) = 2$. The full classification will be given in Section 4.10.

Proposition 4.4.1. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{22} of order 22 such that $\dim(V^\sigma) = \dim H^2(X,\mathbb{C})^{\sigma^*}_{\zeta_{22}} = 2$. Then the fixed locus of $\sigma_{11} = \sigma_{22}^2$ and $\sigma_2 = \sigma_{22}^{11}$ are as follows:*

$\mathrm{Fix}(\sigma_{22})$	$\mathrm{Fix}(\sigma_{11})$	$\mathrm{Fix}(\sigma_2)$
$\{p_1, \dots, p_6\}$	$C_1 \sqcup \{p_5, p_6\}$	$C_{10} \sqcup R,$

where $g(C_1) = 1, g(C_{10}) = 10$ and $g(R) = 0$. Moreover $d = (2, 0, 0, 2)$ and $\mathrm{NS}(X) \cong U$ for a general K3 surface with such property.

Proof. Decomposing $H^2(X,\mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain:

$$\dim H^2(X,\mathbb{C}) = 22 = 10d_{22} + 10d_{11} + d_2 + d_1 = 20 + 10d_{11} + d_2 + d_1.$$

Thus $d_1 + d_2 = 2$ and $d_{11} = 0$, so either $d = (2, 0, 1, 1)$ or $(2, 0, 0, 2)$. Moreover, by the topological Lefschetz formulas we have

$$\begin{cases} \chi(\mathrm{Fix}(\sigma_{22})) & = d_{22} - d_{11} - d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_{11})) & = -d_{22} - d_{11} + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_2)) & = -10d_{22} + 10d_{11} - d_2 + d_1 + 2. \end{cases} \tag{4.1}$$

This implies that $\chi(\mathrm{Fix}(\sigma_{11})) = 2$.

By Proposition 4.2.3, if a K3 surface admits a non-symplectic automorphism of order 11 without fixed curves, it does not admit a non-symplectic automorphism of order 22. This result and Theorem 4.3.1 imply that $\mathrm{Fix}(\sigma_{11})$ is the union of a smooth genus one curve C and two points p, q of types $A_{1,11}$ and $A_{4,11}$. On the other hand the same equations give that $\chi(\mathrm{Fix}(\sigma_{22})) = 4$ if $d = (2,0,1,1)$ and $= 6$ if $d = (2,0,0,2)$. This implies that σ_{22} has 4 fixed points on C and either exchanges or fixes p and q. The holomorphic Lefschetz formula for σ_{22} implies that the first case does not occur (observe that the fixed points of σ_{22} on C are of type $A_{10,22}$). Finally $\chi(\mathrm{Fix}(\sigma_2)) = -16$. By Figure 2.1 this implies that the fixed locus of σ_2 is either a genus 9 curve or the union of a genus 10 curve and a rational curve. The first case is not possible since a curve of genus 9 has no order 11 automorphisms by the Riemann-Hurwitz formula.

Finally, observe that for a general $K3$ surface as in the statement $\mathrm{rk}\,\mathrm{NS}(X) = 22 - 2\varphi(22) = 2$ and $S(\sigma_{11}) \subseteq \mathrm{NS}(X)$ by Proposition 2.2.9, thus $\mathrm{NS}(X) = S(\sigma_{11}) \cong U$ by Theorem 4.3.1. □

Example 4.4.2. The elliptic K3 surface in Example 4.3.2

$$y^2 = x^3 + ax + (t^{11} - 1), \ a \in \mathbb{C}$$

has the order 22 automorphism

$$\sigma(t, x, y) = (\zeta_{11} t, x, -y),$$

which fixes four points in the smooth fiber over $t = 0$ and two points in the fiber of type II over $t = \infty$. The involution σ^{11} fixes the curve $y = 0$, which has genus 10, and the section at infinity. Since σ^{11} has fixed curves and since there exist no symplectic automorphism of a K3 surface of order 11, then σ is purely non-symplectic by [Muk88].

Proposition 4.4.3. *Let X be a K3 surface with a non-symplectic automorphism σ of order 22 such that $\dim(V^\sigma) = 2$. Then (X, σ) belongs to the family in Example 4.4.2 up to isomorphism.*

Proof. By Proposition 4.4.1 $\mathrm{Fix}(\sigma_{11})$ contains an elliptic curve C_1 and two points. Thus by Theorem 4.3.1 (X, σ_{11}) belongs to the family in Example 4.3.2 up to isomorphism, i.e. it carries an elliptic fibration $\pi : X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + ax + (t^{11} - 1),\ \ a \in \mathbb{C}$$

and $\sigma_{11}(t, x, y) = (\zeta_{11} t, x, y)$. The lattice generated by the class of a fiber and the class of a section of π is isometric to the lattice U and is fixed by the automorphism σ_{11}^*, thus it coincides with $S(\sigma_{11})$ by Proposition 4.3.1. Since σ_{22}^* preserves the lattice $S(\sigma_{11})$ and this contains a unique class of elliptic fibration and a unique class of smooth rational curve, then σ_{22}^* preserves both.

By Proposition 4.4.1 the fixed locus of the involution σ_2 is the disjoint union of a smooth curve C_{10} of genus 10 and a smooth rational curve R. The curve C_{10} is clearly transverse to the fibers of π, thus each fiber of π contains fixed points of σ_2. This implies that the action induced by σ_2 on $\mathbb{P}^1$ is the identity, i.e. each fiber of π is preserved by σ_2. Moreover, the unique section S of π must be pointwise fixed by σ_2, so that $R = S$. Since σ_2 is an involution which preserves each fiber of π and fixes S, then it is defined by $(t, x, y) \mapsto (t, x, -y)$. This shows that the action of $\sigma_{22} = \sigma_{11} \circ \sigma_2$ on π is the one described in Example 4.4.2, concluding the proof. $\square$

4.5 Order 15

In this section we will give the classification of purely non-symplectic automorphisms of order 15 with $\dim H^2(X, \mathbb{C})_{\zeta_{15}} = 2$. The full classification will be given in Section 4.11.

Proposition 4.5.1. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{15} of order 15 such that $\dim(V^\sigma) = 2$. Then the fixed loci of the powers of σ_{15} are as follows:*

	$\mathrm{Fix}(\sigma_{15})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_3)$
a)	$\{p_1, \ldots, p_5\}$	$C_2 \sqcup \{p_1\}$	$C_2' \sqcup \{p_2, p_3\}$
b)	$\{p_1, \ldots, p_7\}$	$C_1 \sqcup \{p_1, \ldots, p_4\}$	$C_4 \sqcup R \sqcup \{p_1\}$,

where $g(C_2) = g(C_2') = 2$, $g(C_1) = 1$, $g(C_4) = 4$ and $g(R) = 0$. Moreover, $d = (2, 1, 0, 2)$ in case a) and $d = (2, 0, 1, 4)$ in case b).

Finally $\mathrm{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ for a general K3 surface X in case a) and $\mathrm{NS}(X) \cong H_5 \oplus A_4$ for a general K3 surface X in case b).

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain:

$$22 = 8d_{15} + 4d_5 + 2d_3 + d_1 = 16 + 4d_5 + 2d_3 + d_1,$$

thus $d \in \{(2, 1, 0, 2), (2, 0, 2, 2), (2, 0, 1, 4), (2, 0, 0, 6)\}$. By the topological Lefschetz fixed point formulas:

$$\begin{cases} \chi(\mathrm{Fix}(\sigma_{15})) & = d_{15} - d_5 - d_3 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_5)) & = -2d_{15} - d_5 + 2d_3 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_3)) & = -4d_{15} + 4d_5 - d_3 + d_1 + 2. \end{cases} \tag{4.2}$$

We will show that $d = (2, 1, 0, 2)$ and $d = (2, 0, 1, 4)$ are the only possible cases.

Assume that $d = (2, 1, 0, 2)$. Thus $\chi(\mathrm{Fix}(\sigma_{15})) = 5$, $\chi(\mathrm{Fix}(\sigma_5)) = -1$ and $\chi(\mathrm{Fix}(\sigma_3)) = 0$. By Table 2.4 we have that $\mathrm{Fix}(\sigma_5)$ is the union of a curve C_2 of genus 2 and one point. Since $\chi(\mathrm{Fix}(\sigma_{15})) = 5$ the action of σ on C_2 has order 3 with 4 fixed points by the Riemann-Hurwitz formula. In particular $\mathrm{Fix}(\sigma_{15})$ is the union of 5 points. Finally by Table 2.3 $\mathrm{Fix}(\sigma_3)$ is either the union of a genus 2 curve and 2 points or contains a curve of genus three. The second case is not possible since there is no genus 3 curve with an order five automorphism by [Bro91, Table 5].

If $d \neq (2,1,0,2)$, then $\chi(\mathrm{Fix}(\sigma_5)) = 4$ and $\chi(\mathrm{Fix}(\sigma_3)) = -6, -3, 0$ if

$$d = (2,0,2,2), (2,0,1,4), (2,0,0,6)$$

respectively. By Table 2.4 $\mathrm{Fix}(\sigma_5)$ is either the union of an elliptic curve C_1 and 4 points, or the union of 4 points. Observe that C_1 can not be contained in $\mathrm{Fix}(\sigma_3)$ since by [AS08] this would imply $\chi(\mathrm{Fix}(\sigma_3)) \geq 3$. Thus, looking at the possible actions of σ on C_1 and the 4 points we find that $\chi(\mathrm{Fix}(\sigma_{15}))$ is either 1, 4 or 7.

If $d = (2,0,0,6)$, then $\chi(\mathrm{Fix}(\sigma_{15})) = 10$ by (4.2), giving a contradiction.

If $d = (2,0,1,4)$, then $\chi(\mathrm{Fix}(\sigma_{15})) = 7$ by (4.2). Thus $\mathrm{Fix}(\sigma_5)$ is the union of an elliptic curve C_1 and 4 points and $\mathrm{Fix}(\sigma_{15})$ consists of 7 points, 3 of them on C_1. Moreover $\chi(\mathrm{Fix}(\sigma_3)) = -3$, thus by Table 2.3 $\mathrm{Fix}(\sigma_3)$ is either the union of a genus 4 curve, a rational curve and one point, or it contains a curve of genus 3. The last case is not possible by [Bro91, Table 5].

If $d = (2,0,2,2)$, then $\chi(\mathrm{Fix}(\sigma)) = 4$ and $\chi(\mathrm{Fix}(\sigma_3)) = -6$ by (4.2). Observe that σ has 7 types of isolated fixed points. The fixed points of type $A_{1,15}, A_{4,15}, A_{7,15}$ are isolated fixed points for σ_3 too, while points of type $A_{2,15}, A_{3,15}, A_{5,15}, A_{6,15}$ lie on a curve fixed by σ_3. Thus it has to be

$$a_{1,15} + a_{4,15} + a_{7,15} \leq a_{1,3}, \quad a_{1,15} + a_{4,15} + a_{7,15} \equiv a_{1,3} \mod 5.$$

Moreover, points of type $A_{4,15}, A_{5,15}$ lie on a curve fixed by σ_5, while points of type $A_{1,15}, A_{2,15}, A_{3,15}, A_{6,15}, A_{7,15}$ are isolated fixed points for σ_5 too. Checking types one has

$$a_{1,15} + a_{3,15} + a_{6,15} \leq a_{1,5}, \quad a_{2,15} + a_{7,15} \leq a_{2,5}. \tag{4.3}$$

Since $\chi(\mathrm{Fix}(\sigma_3)) = -6$, then $a_{1,3} = 0$ by [AST11]. Applying holomorphic Lefschetz formula to σ with this condition and using the fact that $\alpha = 0$, we find that $a = (0,1,0,0,3,0,0)$. Since $a_{5,15} = 3$, then we find that $\mathrm{Fix}(\sigma_5)$ contains an elliptic curve and σ fixes three points on it.

For $\mathrm{Fix}(\sigma_3)$ there are two possibilities by Table 2.3: it is either a curve of genus four or the union of a genus five curve and a rational curve. We will now exclude both cases by geometric arguments.

Assume that the fixed locus of σ_3 is the union of a curve C_5 of genus 5 and a rational curve R. By [AS08, Proposition 4.2], a K3 surface with a non-symplectic automorphism of order 3 with this fixed locus admits an elliptic fibration π with Weierstrass equation

$$y^2 = x^3 + p_{12}(t), \text{ with } \sigma_3 : (x, y, t) \mapsto (\zeta_3 x, y, t),$$

where $p_{12}(t)$ is a polynomial of degree 12. Observe that the section at infinity of π is pointwise fixed by σ_3, thus it is the rational curve R. Moreover, C_5 is the curve defined by $x = 0$. Since σ_3 has no other smooth rational fixed curves, an easy analysis of the possible singular fibers shows that the elliptic fibration has no reducible singular fibers and has exactly 12 fibers of type II. By [AS15, Lemma 5], since $C_5^2 = 8$ and $C_5 \cdot f = 2$, where f denotes the class of a fiber of π, we obtain that $f \cdot \sigma^*(f) = 0$. Thus π is invariant for σ. Since the section at infinity is invariant but not fixed by σ_5, then σ_5 induces an automorphism $\bar{\sigma}_5$ of order 5 on the basis $\mathbb{P}^1$ of the fibration. This implies that the elliptic curve C_1 in $\mathrm{Fix}(\sigma_5)$ is a fiber of the elliptic fibration. On the other hand σ_5 must permute the $12II$ fibers, thus at least two of them should be preserved. This contradicts the fact that $\bar{\sigma}_5$ has two fixed points in $\mathbb{P}^1$.

Assume now that the fixed locus of σ_3 is a genus four curve. By Table 2.3 the invariant lattice of σ_3 is isometric to $U(3)$. Thus the quotient of the K3 surface by σ_3 is a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the branch locus is a curve of genus 4 of type $(3, 3)$. The automorphism σ_5 descends to an automorphism $\bar{\sigma}_5$ of $\mathbb{P}^1 \times \mathbb{P}^1$ which has a fixed curve, thus up to a coordinate change we can assume

$$\bar{\sigma}_5 : (x_0 : x_1), (y_0 : y_1) \mapsto (x_0 : x_1), (\xi_5 y_0 : y_1).$$

However there is no curve of type $(3, 3)$ which is invariant for this automorphism, giving a contradiction. Thus we have proved that the only admissible cases are those with

$d = (2, 1, 0, 2)$ and $d = (2, 0, 1, 4)$.

We now compute the Néron-Severi lattice of a general X in both cases. Observe that, since $d_{15} = 2$ and $\varphi(15) = 8$, the Néron-Severi lattice of X has rank $22 - 2 \cdot 8 = 6$. In the first case the invariant lattice $S(\sigma^5) = S(\sigma_3)$ has rank $d_1 + 4d_5 = 6$, thus $\mathrm{NS}(X) = S(\sigma_3) \cong U(3) \oplus A_2 \oplus A_2$, where the last isomorphism is by Table 2.3 . In the second case the invariant lattice $S(\sigma^3) = S(\sigma_5)$ has rank $d_1 + 2d_3 = 6$, thus we conclude as before that $\mathrm{NS}(X) = S(\sigma_5) \cong H_5 \oplus A_4$ by Table 2.4 . □

Example 4.5.2. Let X be a double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(3, 1, 1, 1)$:

$$y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3$$

with general $a \in \mathbb{C}$. Then X is a K3 surface carrying an order 15 automorphism

$$\sigma(x_0, x_1, x_2, y) = (x_0, \zeta_5 x_1, \zeta_3 x_2, y)$$

whose fixed locus is the union of 5 points, which project to the 3 fundamental points of $\mathbb{P}^2$. Observe that $\sigma_5 = \sigma^3$ fixes the genus two curve defined by $x_1 = 0$ and the point $(0, 1, 0, 0)$, while $\sigma_3 = \sigma^5$ fixes the genus two curve $x_2 = 0$ and the points $(0, 0, 1, \pm 1)$. Since both σ_3 and σ_5 fix curves, then none of them is symplectic by [Muk88], thus σ_{15} is purely non-symplectic. Thus is an example of case a) in Proposition 4.5.1.

Example 4.5.3. Consider the elliptic surface with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with general $a \in \mathbb{C}$. Then X is a K3 surface with the automorphism of order 15:

$$\sigma(t, x, y) = (\zeta_5 t, \zeta_3 x, y).$$

The elliptic fibration has one fiber of type IV over $t = \infty$ and 10 fibers of type II. The automorphism σ^5 of order 3 fixes the genus 4 curve defined by $x = 0$, the section at infinity and the center of the fiber of type IV. The automorphism σ^3 of order 5 fixes the smooth fiber over $t = 0$ and four points in the fiber of $t = \infty$. The automorphism σ fixes 3 points in the fiber over $t = 0$ and 4 points in the fiber over $t = \infty$. The automorphism is purely non-symplectic by the same reason of the previous Example. Thus this is an example of case b) in Proposition 4.5.1.

Proposition 4.5.4. *Let X be a K3 surface with a purely non-symplectic automorphism σ of order 15 such that $\dim(V^\sigma) = 2$. Then (X, σ) belongs to one of the families in Examples 4.5.2 and 4.5.3 up to isomorphism.*

Proof. Let X be a K3 surface with a purely non-symplectic automorphism $\sigma = \sigma_{15}$ of order 15. By Proposition 4.5.1, $\mathrm{Fix}(\sigma_{15})$ contains either 5 or 7 isolated fixed points.

In the first case, $\mathrm{Fix}(\sigma_{15})$ consists of 5 fixed points, $\mathrm{Fix}(\sigma_5)$ is the union of a curve C_1 of genus 2 and one point and $\mathrm{Fix}(\sigma_3)$ is the union of a genus 2 curve C_2' and 2 points. Let $\varphi : X \to \mathbb{P}^2$ be the morphism associated to the linear system $|C_2'|$, which is a degree two morphism branched along a plane sextic which possibly contracts the smooth rational curves disjoint from C_2' [SD74]. Since $[C_2']$ is fixed by σ^*, the automorphism σ descends to an automorphism $\bar{\sigma}$ of $\mathbb{P}^2$. Let $\bar{\sigma}_3 = \bar{\sigma}^5$ and $\bar{\sigma}_5 = \bar{\sigma}^3$. Up to a projectivity we can assume that $\bar{\sigma}$, and thus $\bar{\sigma}_3$ and $\bar{\sigma}_5$ are diagonal. Observe that both $\bar{\sigma}_3$ and $\bar{\sigma}_5$ must fix pointwise a line and a point in $\mathbb{P}^2$, since both σ_3 and σ_5 fix pointwise a curve. Moreover, by the previous description, the two lines must be distinct. Thus we can assume that

$$\bar{\sigma}_3(x_0 : x_1 : x_2) = (x_0 : x_1 : \zeta_3 x_2) \quad \bar{\sigma}_5(x_0 : x_1 : x_2) = (x_0 : \zeta_5 x_1 : x_2).$$

The branch sextic B of φ is invariant for both $\bar{\sigma}$. Observe that B can not contain a line fixed by either $\bar{\sigma}_3$ or $\bar{\sigma}_5$ since otherwise $\mathrm{Fix}(\sigma_3)$ and $\mathrm{Fix}(\sigma_5)$ would contain a smooth rational curve. This implies, up to rescaling the variables B is defined by an equation of the form:

$$x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3 = 0,$$

with $a \in \mathbb{C}$. Observe that this is a smooth curve. This implies that an equation for X is the one given in Example 4.5.2.

We now consider the second case, i.e. when $\mathrm{Fix}(\sigma_{15})$ consists of 7 points, $\mathrm{Fix}(\sigma_5)$ is the union of an elliptic curve C_1 and 4 points and $\mathrm{Fix}(\sigma_3)$ is the union of a genus 4 curve C_4, a rational curve R and one point. By [AS08, Prop. 4.2] X admits a jacobian elliptic fibration π with Weierstrass equation

$$y^2 = x^3 + p_{12}(t),$$

with $\sigma_3(x, y, t) = (\zeta_3 x, y, t)$, where $p_{12}(t)$ has degree 12 and exactly one double root. Observe that in these coordinates C_4 is defined by $x = 0$ and the section at infinity is the curve R. By [AS15, Lemma 5], since $C_4^2 = 6$ and C_4 intersects a general fiber at two points, we obtain that $f \cdot \sigma^*(f) \leq 1$, where f denotes the class of a fiber of π. This implies that $f \cdot \sigma^*(f) = 0$, and thus $f = \sigma^*(f)$, i.e. the elliptic fibration is invariant for σ. Since R is invariant for σ_5 but not pointwise fixed, the action of σ_5 on the basis of the fibration has order 5. This description implies that, up to a coordinate change, the fibration π has Weierstrass equation of the form

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$, as in Example 4.5.3. □

4.6 Order 30

We first recall the following result, as suggested us Alice Garbagnati in a private communication.

Lemma 4.6.1. *Let X be a K3 surface admitting a purely non-symplectic automorphism σ of order 15. Then X admits a purely non-symplectic automorphism ψ of order* 30 *with $\psi^2 = \sigma$.*

Proof. Let $\sigma_5 = \sigma^3$ and $\sigma_3 = \sigma^5$. By Proposition 4.2.3, X admits a non-symplectic automorphism τ of order 10 such that $\tau^2 = \sigma_5$. Let

$$1 \longrightarrow \ker(\alpha) \longrightarrow G \xrightarrow{\alpha} \mu_{I(G)} \longrightarrow 1$$

be the exact sequence 2.1 described in Section 2.2. If $G = \langle \tau, \sigma_3 \rangle$, the action of any non trivial element of G on ω_X is the multiplication by a 30th root of unity $\mu \neq 1$, since the orders of τ and σ_3 are coprime. Thus $I(G) = 30$, $\ker(\alpha)$ is trivial, $G \simeq \mu_{30}$ and the composition $\psi = \tau \circ \sigma_3$ is an automorphism of order 30 of X. $\square$

Proposition 4.6.2. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{30} of order 30 such that $\dim(V^\sigma) = 2$. Then there are two possibilities for the fixed locus of σ_{30} and its powers:*

	$\mathrm{Fix}(\sigma_{30})$	$\mathrm{Fix}(\sigma_{15})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_3)$	$\mathrm{Fix}(\sigma_2)$
a)	$\{p_1\}$	$\{p_1, \dots, p_5\}$	$C_2 \sqcup \{p_1\}$	$C_2' \sqcup \{p_2, p_3\}$	C_{10}
b)	$\{p_1, p_2, p_5\}$	$\{p_1, \dots, p_7\}$	$C_1 \sqcup \{p_1, \dots, p_4\}$	$C_4 \sqcup R \sqcup \{p_1\}$	$C_9 \sqcup R$

where C_g, C_g' has genus g and $g(R) = 0$. Moreover, $d = (2, 0, 1, 0, 0, 0, 1, 1)$ in case a) and $d = (2, 0, 0, 1, 0, 0, 1, 3)$ in case b).

Finally $\mathrm{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ for a general K3 surface X in case a) and $\mathrm{NS}(X) \cong H_5 \oplus A_4$ for a general K3 surface X in case b).

Proof. Let $\chi_i = \chi(\mathrm{Fix}(\sigma_i)), i = 30, 15, 5, 3, 2$.

First observe that given a one dimensional family of K3 surfaces admitting a purely non-symplectic automorphism of order 30, every element in the family admits a purely non-symplectic automorphism of order 15. Thus this corresponds to one of the two families in Proposition 4.5.1 and the vector $(\chi_{15}, \chi_5, \chi_3)$ is either $(5, -1, 0)$ in case a) or $(7, 4, 3)$ in case b).

Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain:

$$22 = 8d_{30} + 8d_{15} + 4d_{10} + 2d_6 + 4d_5 + 2d_3 + d_2 + d_1. \tag{4.4}$$

Assuming $d_{30} = 2$, this gives $d_{15} = 0$.

Using topological Lefschetz fixed point formulas we compute the topological Euler characteristic of the fixed loci of powers of σ by:

$$\begin{cases} \chi_{30} & = d_{10} + d_6 - d_5 - d_3 - d_2 + d_1 \\ \chi_{15} & = -d_{10} - d_6 - d_5 - d_3 + d_2 + d_1 + 4 \\ \chi_5 & = -d_{10} + 2d_6 - d_5 + 2d_3 + d_2 + d_1 - 2 \\ \chi_3 & = 4d_{10} - d_6 + 4d_5 - d_3 + d_2 + d_1 - 6 \\ \chi_2 & = -4d_{10} - 2d_6 + 4d_5 + 2d_3 - d_2 + d_1 - 14. \end{cases} \tag{4.5}$$

We first assume to be in case a) of Proposition 4.5.1, i.e. $\mathrm{Fix}(\sigma_5)$ is the union of a smooth curve C_2 of genus 2 and a point p_1, $\mathrm{Fix}(\sigma_3)$ is the union of a smooth curve C_2' of genus 2 and two isolated points and $\mathrm{Fix}(\sigma_{15})$ consists of 5 isolated points $p_1, \dots, p_5$. In particular $(\chi_{15}, \chi_5, \chi_3) = (5, -1, 0)$. Moreover, since the fixed locus of σ_{15} only contains isolated points, the same holds for σ_{30}. Thus $\chi_{30} \geq 0$.

By (4.4) and (4.5) we get the possibilities in Table 4.2. In particular χ_{30} is either 3 or 1, thus $\mathrm{Fix}(\sigma_{30})$ is either the union of p_1 and two of the p_i's with $i \geq 2$ (and the other two are exchanged) or $\mathrm{Fix}(\sigma_{30}) = \{p_1\}$ and σ_{30} has no fixed points on C_2.

By Proposition 4.5.4 and its proof the linear system associated to C_2 defines a double cover $\varphi : X \to \mathbb{P}^2$ which can be defined in $\mathbb{P}(1, 1, 1, 3)$ by an equation of the form

$$y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3,$$

where $a \in \mathbb{C}$ and in these coordinates $\sigma_{15}(x_0, x_1, x_2, y) = (x_0, \zeta_5 x_1, \zeta_3 x_2, y)$. Since σ_{30} preserves C_2, then it induces an automorphism $\bar{\sigma}_{30}$ of $\mathbb{P}^2$. The involution σ_2 either induces the identity or an involution of $\mathbb{P}^2$. The latter is not possible since the fixed locus of σ_2 would contain a curve of genus at most 2, while $\chi_2 \leq -8$ by Table 4.2. Thus σ_2 coincides with the (automorphism induced by) the covering involution of φ, which fixes a smooth genus 10 curve, so that $\chi_2 = -18$ and $\chi_{30} = 1$ by Table 4.2. Thus σ_{30}

fixes a unique point. Since C_2 is invariant for σ_{30}, then $\varphi(C_2)$ is a line which contains two fixed points for $\bar{\sigma}_{30}$. Since $\chi_{30} = 1$, their preimages by φ are four points exchanged in pairs by σ_2.

Assume now to be in case b) of Proposition 4.5.1, i.e. $\mathrm{Fix}(\sigma_3)$ is the union of a curve C_4 of genus 4, a rational curve R and a point, $\mathrm{Fix}(\sigma_5)$ is union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma_{15})$ is the union of 7 points (3 on C_1). In particular $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$. Moreover, $\chi_{30} \geq 0$ since $\mathrm{Fix}(\sigma_{15})$ only contains isolated points, and thus the same holds for σ_{30}. There are five possible vectors d such that $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$ (see Table 4.3).

By Proposition 4.5.4 X admits an elliptic fibration $\pi : X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$. By the same argument in the proof of Proposition 4.5.4, using [AS15, Lemma 5], one concludes that the elliptic fibration is invariant for σ_{30}. Since $\chi_2 \leq -8$, then σ_2 fixes a curve of genus > 1. Such curve is clearly transverse to all fibers of π, thus σ_2 induces the identity on the basis of the fibration. Moreover, σ_2 must fix the section at infinity R of the fibration, since it preserves R and each fiber of π. This implies that $\sigma_2(x, y, t) = (x, -y, t)$. In particular σ_2 fixes R and the curve defined by $y = 0$, which has genus 9, so that $\chi_2 = -14$. Moreover σ_{30} fixes three points: two points on R and the center of the fiber of type IV over $t = \infty$.

The Néron-Severi lattice of a general X in cases a) and b) is the same as Proposition 4.5.1. $\square$

Example 4.6.3. Consider the double cover X of $\mathbb{P}^2$ in Example 4.5.2:

$$y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3.$$

Then X carries the order 30 automorphism

$$\sigma(x_0, x_1, x_2, y) = (x_0, \zeta_5 x_1, \zeta_3 x_2, -y).$$

Observe that for general $a \in \mathbb{C}$ the fixed locus of σ_2 is the smooth plane sextic defined by curve $y = 0$, which has genus 10. Moreover, the fixed locus of σ_{30} consists of the point $(0, 1, 0, 0)$.

Example 4.6.4. The elliptic K3 surface in Example 4.5.3

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

carries the order 30 automorphism

$$\sigma(t, x, y) = (\zeta_5 t, \zeta_3 x, -y).$$

Observe that for general $a \in \mathbb{C}$ the fixed locus of σ_2 is the curve $y = 0$, which has genus 9. Moreover, as observed in the proof of Proposition 4.6.2, the fixed locus of σ_{30} consists of two points in the section at infinity (over $t = 0$ and $t = \infty$) and the center of the fiber of type IV over $t = \infty$.

The proof of Proposition 4.6.2 also implies the following result.

Proposition 4.6.5. *Let X be a K3 surface with a non-symplectic automorphism σ of order 30 such that $\dim(V^\sigma) = 2$. Then (X, σ) belongs to one of the families in Examples 4.6.3 and 4.6.4 up to isomorphism.*

4.7 Order 16

Purely non-symplectic automorphisms of order 16 on K3 surfaces have been classified in [ATST16]. The following result has the same statement as [ATST16, Theorem 4.1], but we provide a slightly different proof since we use the weaker hypothesis $\dim(V^\sigma) = 2$.

Theorem 4.7.1. *Let σ_{16} be a purely non-symplectic automorphism of order 16 of a K3 surface X and assume that $S(\sigma_2)$ has rank 6 (or equivalently $\dim(V^\sigma) = 2$). Then there* *are two possibilities for the fixed locus of σ_{16} and its powers:*

	$\mathrm{Fix}(\sigma_{16})$	$\mathrm{Fix}(\sigma_8)$	$\mathrm{Fix}(\sigma_4)$	$\mathrm{Fix}(\sigma_2)$
a)	$\{p_1,\dots,p_6\}\sqcup R$	$\{p_1,\dots,p_6\}\sqcup R$	$\{p_1,\dots,p_6\}\sqcup R$	$C_7\sqcup R\sqcup R'$
b)	$\{p_1,p_2,p_7,p_8\}$	$\{p_1,\dots,p_6\}\sqcup R$	$\{p_1,\dots,p_6\}\sqcup R$	$C_6\sqcup R$

where $g(C_6)=6$, $g(C_7)=7$ and $g(R)=g(R')=0$. Moreover, $d=(2,0,0,0,6)$ in case a) and $d=(2,0,0,2,4)$ in case b). Finally, $\mathrm{NS}(X)\cong U\oplus D_4$ for a general X in case a) and $\mathrm{NS}(X)\cong U(2)\oplus D_4$ for a general X in case b).

Proof. Decomposing $H^2(X,\mathbb{C})$ as the direct sum of the eigenspaces of σ_{16}^* we obtain:

$$22 = 8d_{16}+4d_8+2d_4+d_2+d_1. \tag{4.6}$$

Since $d_{16}=2$, this implies that d_8 is either 0 or 1 and gives the 14 possibilities for the vector d in Table 4.4.

Let N_i be the number of isolated fixed points of σ_i, $\chi_i=\chi(\mathrm{Fix}(\sigma_i))$ and

$$\alpha_i = \sum_{C\subset \mathrm{Fix}(\sigma_i)} (1-g(C))$$

for $i\in\{2,4,8,16\}$. By the topological Lefschetz fixed point formula we get

$$\begin{cases} \chi_{16} = & = -d_2+d_1+2 \\ \chi_8 = & = -d_4+d_2+d_1+2 \\ \chi_4 = & = -4d_8+2d_4+d_2+d_1+2 \\ \chi_2 = & = -8d_{16}+4d_8+2d_4+d_2+d_1+2. \end{cases} \tag{4.7}$$

Table 4.4 shows the values of $(\chi_{16},\chi_8,\chi_4,\chi_2)$ for each possible vector d.

Observe that $\chi_2=-8$. By [Nik79a], $N_2=0$ and $\mathrm{Fix}(\sigma_2)$ is the union of a curve of genus g and k rational curves with $(g,k)=(5,0),(6,1)$ or $(7,2)$.

Moreover, $\chi_4=0$ or 8. By [AS15, Proposition 1] we have that $N_4=2\alpha_4+4$. Since

$\chi_4 = 2\alpha_4 + N_4$ one has

$$\chi_4 = 4\alpha_4 + 4.$$

If $\chi_4 = 0$, then $\alpha_4 = -1$, but this is not possible since $\mathrm{Fix}(\sigma_4) \subseteq \mathrm{Fix}(\sigma_2)$ and it is not compatible with the aforementioned possibilities for $\mathrm{Fix}(\sigma_2)$. Thus $\chi_4 = 8$, $\alpha_4 = 1$ and $\mathrm{Fix}(\sigma_4)$ contains a rational curve (and no more curves) and 6 points. This implies that the case $(g, k) = (5, 0)$ is impossible.

Let us assume now that $\mathrm{Fix}(\sigma_2)$ is the union of a smooth curve C_6 of genus 6 and a rational curve R. By the previous analysis we know that σ_4 fixes pointwise R and has 6 isolated fixed points $p_1, \ldots, p_6$ on C_6.

By the Riemann-Hurwitz formula for σ_8 on C_6, we observe that either a) 2 of the p_i's are fixed and the other four are permuted in pairs by σ_8 or b) the points $p_1, \ldots, p_6$ are fixed points for σ_8. Observe that case a) is not possible since $\chi_4 = 8$ and $\chi_8 = 4$ does not appear in Table 4.4.

By the Riemann-Hurwitz formula for σ_{16} on C_6, we obtain that σ_{16} fixes 2 of the p_i's and exchanges the other four in pairs. Thus $(\chi_{16}, \chi_8, \chi_4, \chi_2) = (4, 8, 8, -8)$.

Observe that six of the fixed points of σ_8 lie on a curve fixed pointwise by σ_2 and not by σ_4, thus the local action of σ_8 at such points is either of type $A_{2,8}$ or $A_{3,8}$. By [ATS18, Proposition 2.2] we have that $6 = 2 + 4\alpha_8$, thus $\alpha_8 = 1$. This implies that $N_8 = 6$ and the curve R is pointwise fixed by σ_8. On the other hand, by [ATST16, Proposition 2], N_{16} is bigger or equal to $2\alpha_{16} + 1$. This implies that $\alpha_{16} = 0$, i.e. R is not pointwise fixed by σ_{16}.

Let us assume now that $\mathrm{Fix}(\sigma_2)$ is the union of a smooth curve C_7 of genus 7 and two rational curves R, R'. We already know that one rational curve is fixed by σ_4, say R. Thus σ_4 fixes 2 points q_1, q_2 on R' and 4 points $p_1, \ldots, p_4$ on C_7. This implies that the curves R and R' cannot be exchanged by σ_{16} nor by σ_8 and that $\chi_{16} \geq 4$ and $\chi_8 \geq 4$.

By the Riemann-Hurwitz formula for σ_8 on C_7, either the four p_i's are fixed by σ_8 or none of them is fixed by σ_8. This implies that either $\chi_8 = 4$ or $\chi_8 = 8$. Looking at Table 4.4, we find that we are left with three possibilities:

d_{16}	d_8	d_4	d_2	d_1	χ_{16}	χ_8	χ_4	χ_2
2	0	0	2	4	4	8	8	-8
2	0	0	1	5	6	8	8	-8
2	0	0	0	6	8	8	8	-8

In particular $\chi_8 = 8$ and $\{p_1, \ldots, p_4, q_1, q_2\} \subset \mathrm{Fix}(\sigma_8)$. Moreover, by [ATS18, Proposition 2.2] we obtain that $2 + 4\alpha_8 = 6$, thus $\alpha_8 = 1$. This implies that σ_8 fixes pointwise the curve R.

By the Riemann-Hurwitz formula for σ_{16} on C_7, either a) σ_{16} fixes the four p_i's and thus $\chi_{16} = 8$, or b) it does not fix any of them and $\chi_{16} = 4$. By [ATST16, Proposition 2, Remark 1.3] the cases $(N_{16}, \alpha_{16}) = (2, 1)$ and $(N_{16}, \alpha_{16}) = (8, 0)$ are impossible. Thus in case a) $\alpha_{16} = 1$ and $N_{16} = 4$, i.e. the fixed locus of σ_{16} contains $p_1, \ldots, p_4, q_1, q_2$ and the curve R On the other hand in case b) we have that $\alpha_{16} = 0$, i.e. σ_{16} fixes exactly q_1, q_2 and two points on R. We now show that this case can not appear. By [ATST16, Remark 1.3] if $N_{16} = 4$, then $n_{3,16} = n_{7,16} = 1$ and $n_{8,16} = 2$. Observe that the points of type $A_{8,16}$ lie on a curve fixed by σ_8, thus they must be the two points on R. This implies that the points of type $A_{3,16}$ and $A_{7,16}$ are q_1, q_2. However, two isolated fixed points of σ_{16} lying on an invariant smooth rational curve can not be of these types by the proof of [AS15, Lemma 4].

Observe that both in case a) and b) we have that $d_4 = d_8 = 0$ and $d_1 + d_2 = 6$. This implies that $S(\sigma_8) = S(\sigma_4) = S(\sigma_2)$ has rank 6. If X is general, than $\mathrm{rk}\,\mathrm{NS}(X) = 22 - 2\varphi(20) = 6$ and thus by Remark 2.2.9 $\mathrm{NS}(X) = S(\sigma_2)$. This implies that $\mathrm{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ in case a) and $\mathrm{NS}(X) = S(\sigma_2) \cong U(2) \oplus D_4$ in case b), see [ATST16]. □

For the following examples see [ATST16, Example 4.2].

Example 4.7.2. Consider the elliptic fibration defined by

$$y^2 = x^3 + t^2x + at^3 + t^{11},\ a \in \mathbb{C}$$

with the order 16 automorphism $\sigma(t,x,y) = (\zeta_8 t, \zeta_8 x, \zeta_{16}^3 y)$. The action of σ^* on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by ζ_{16}, thus σ is purely non-symplectic. The fibration has a fiber of type I_0^* over $t = 0$ and a fiber of type II over $t = \infty$. The automorphism σ fixes the central component of the fiber ${I_0}^*$, four points in the other components of the same fiber and two more fixed points in the fiber over $t = \infty$. Thus this is an example of case a).

Example 4.7.3. Consider the double cover Y of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1,1,1,3)$:

$$y^2 = x_0(x_0^4x_2 + x_1^5 + x_1x_2^4 + ax_1^3x_2^2) = 0,\ a \in \mathbb{C}.$$

Observe that for general $a \in \mathbb{C}$ the branch curve is the union of a smooth plane quintic C and a line L. The surface Y has the order 16 automorphism

$$\sigma(x_0, x_1, x_2, y) = (x_0, \zeta_8^7 x_1, \zeta_8^3 x_2, \zeta_{16}^3 y).$$

The surface Y has 5 singular points of type A_1 over the intersection points of C and L. Its minimal resolution X is a K3 surface and σ lifts to an automorphism $\tilde{\sigma}$ of X. The automorphism $\tilde{\sigma}$ has 4 fixed points: two of them over the points $(1,0,0,0)$ and $(0,1,0,0)$ and the other two in the exceptional divisor over $(0,0,1,0)$ (which is a singular point of Y). Thus this is an example of case b) in Theorem 4.7.1.

4.8 Order 20

Proposition 4.8.1. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{20} of order 20 such that $\dim(V^\sigma) = 2$. Then the fixed locus of σ_{20} and its powers are as follows:*

$\mathrm{Fix}(\sigma_{20})$	$\mathrm{Fix}(\sigma_{10})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_4)$	$\mathrm{Fix}(\sigma_2)$
$\{p_1, p_2, p_3\}$	$\{p_1, \ldots, p_7\}$	$C_2 \sqcup \{p_1\}$	$\{p_1, \ldots, p_6\} \sqcup R$	$C_6 \sqcup R$

where $g(C_i) = i$ *for* $i = 2, 6$ *and* $g(R) = 0$. *Moreover* $d = (2, 0, 1, 0, 0, 2)$ *and* $\mathrm{NS}(X) = S(\sigma_2)$ *for a general such K3 surface* X.

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain: $22 = 8d_{20} + 4d_{10} + 4d_5 + 2d_4 + d_2 + d_1$. Moreover by the topological Lefschetz fixed point formula we get

$$\begin{cases} \chi(\mathrm{Fix}(\sigma_{20})) & = d_{10} - d_5 - d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_{10})) & = 2d_{20} - d_{10} - d_5 - 2d_4 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_5)) & = -2d_{20} - d_{10} - d_5 + 2d_4 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_4)) & = -4d_{10} + 4d_5 - d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_2)) & = -8d_{20} + 4d_{10} + 4d_5 - 2d_4 + d_2 + d_1 + 2. \end{cases} \tag{4.8}$$

Looking at Table 4.5 one finds that $\chi(\mathrm{Fix}(\sigma_5))$ is either -1 or 4.

Assume first that $\chi(\mathrm{Fix}(\sigma_5)) = -1$, so that $\mathrm{Fix}(\sigma_5)$ is the union of a smooth curve C of genus two and one point p by Table 2.4 . In all these cases $\chi(\mathrm{Fix}(\sigma_{10})) = 7$, so that $\mathrm{Fix}(\sigma_{10})$ is the union of p and 6 points on C. By the Riemann-Hurwitz formula, this implies that σ_{20} has two fixed points on C, so that $\mathrm{Fix}(\sigma_{20})$ consists of three points. Moreover, in all these cases $\chi(\mathrm{Fix}(\sigma_2)) = -8$ by Table 4.5, so that $\mathrm{Fix}(\sigma_2)$ is either a) the union of a curve of genus 7 and two rational curves, or b) the union of a curve of genus six and a rational curve, or c) a genus 5 curve by [Nik79a]. The same theorem implies that the rank of the invariant lattice of σ_2 is equal to 6, thus $\mathrm{NS}(X) = S(\sigma_2)$. In all cases σ_{20} acts with order 10 on the curve of positive genus, since otherwise either σ_{10} or σ_4 should contain such curve in its fixed locus, contradicting the previous remarks for σ_{10} and [AS15, Theorem 0.1].

In case a), σ_{20} must fix exactly three points on the curve C of genus 7 and exchange the two rational curves. By the Riemann-Hurwitz formula this implies that σ_5 fixes the same points on C and σ_4 has exactly 8 fixed points on C and exchanges the two rational curves. This is not possible since by [AS15, Theorem 0.1] the number of fixed points n of σ_4 equals $2\alpha + 4$, where $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma_4)} (1 - g(C))$.

In case b), σ_{20} has exactly one fixed point on the genus six curve and two points on the rational curve R, while σ_5 has exactly five fixed points on the curve by the Riemann-Hurwitz formula. By the same formula σ_4 has 6 fixed points on C. By [AS15, Theorem 0.1] R is fixed by σ_4.

Case c) is impossible since, by the Riemann-Hurwitz formula, a curve of genus 5 can not have an order five automorphism with more than two fixed points (and σ_5 would have this property).

Assume now that $\chi(\mathrm{Fix}(\sigma_5)) = 4$. By Table 2.3 , $\mathrm{Fix}(\sigma_5)$ contains either four isolated points or an elliptic curve and four isolated points. In both cases $a_{1,5} = 3, a_{2,5} = 1$. Observe that points of type $A_{4,20}, A_{5,20}, A_{9,20}$ lie on a curve fixed by σ_5, while points of type $A_{i,20}, i \in \{1, 2, 3, 6, 7, 8\}$ are isolated points for σ_5. Since the action of σ_{20} on $\mathrm{Fix}(\sigma_5)$ has order 2 or 4, in both cases the point of type $A_{2,5}$ is fixed by σ_{20} and $a_{2,20} + a_{7,20} = 1$ and $a_{1,20} + a_{3,20} + a_{6,20} + a_{8,20}$ is either 1 or 3.

If $\mathrm{Fix}(\sigma_5)$ consists of four isolated points, $a_{4,20} + a_{5,20} + a_{9,20} = 0$ since there are no curves in $\mathrm{Fix}(\sigma_5)$. A Magma computation shows that the holomorphic Lefschetz formula has no solutions satisfying these conditions.

If $\mathrm{Fix}(\sigma_5)$ consists of four isolated points and an elliptic curve E, by the Riemann-Hurwitz formula E contains 0, 2 or 4 isolated points for σ_{20} thus $a_{4,20} + a_{5,20} + a_{9,20} \in \{0, 2, 4\}$. The holomorphic Lefschetz formula has no solutions with these restrictions.

Observe that, since $d_{20} = 2$ and $\varphi(20) = 8$, the Néron-Severi lattice of X has rank $22 - 2 \cdot 8 = 6$. Moreover $S(\sigma_2) \subseteq \mathrm{NS}(X)$ by Proposition 2.2.9. On the other hand, since the fixed locus of σ_2 is the union of a curve of genus 6 and a rational curve, then $\mathrm{rk}\, S(\sigma_2) = 6$, thus $S(\sigma_2) = \mathrm{NS}(X)$. □

Example 4.8.2. Consider the double cover Y of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_0(x_1^5 + x_2^5 + x_0^2x_2^3 + ax_0^4x_2) = 0, \ a \in \mathbb{C}.$$

Observe that the branch curve is the union of a smooth plane quintic C and a line L.

The surface Y has the order 20 automorphism

$$\sigma(x_0, x_1, x_2, y) = (-x_0, \zeta_5 x_1, x_2, iy).$$

The surface Y has 5 singular points of type A_1 over the intersection points of C and L. Its minimal resolution X is a K3 surface and σ lifts to an automorphism $\tilde{\sigma}$ of X. The automorphism $\tilde{\sigma}$ has 3 fixed points over the fundamental points of $\mathbb{P}^2$.

Proposition 4.8.3. *Let X be a K3 surface with a purely non-symplectic automorphism σ of order 20 such that $\dim(V^\sigma) = 2$. Then (X, σ) belongs to the family in Example 4.8.2 up to isomorphism.*

Proof. By Theorem 4.8.1 the fixed locus of σ_5 is the union of a curve C_2 of genus two and one point. The linear system $|C_2|$ defines a morphism $\varphi : X \to \mathbb{P}^2$ of degree two which contracts all smooth rational curves orthogonal to C_2. Since σ leaves C_2 invariant, then it induces an automorphism $\bar{\sigma}$ of $\mathbb{P}^2$. Up to a projectivity, we can assume $\bar{\sigma}$ to be diagonal.

Let $\bar{\sigma}_2 = \bar{\sigma}^{10}$ and assume it has order two. Thus its fixed locus is the union of a line and one point, so that $\mathrm{Fix}(\sigma_2)$ contains a fixed curve of genus at most 2, contradicting the fact that σ_2 fixes a curve of genus 6. Thus σ_2 coincides with the covering involution of φ.

Now consider the automorphism $\bar{\sigma}_4 = \bar{\sigma}^5$, whose order is equal to two. Its fixed locus contains a line, we can assume it to be $L = \{x_0 = 0\}$ up to projectivites. By Theorem 4.8.1 the line L must be a component of the branch curve B of φ.

Finally, let $\bar{\sigma}_5 = \bar{\sigma}^4$. Since σ_5 has a fixed curve, then $\bar{\sigma}_5$ must fix a line L' which is not equal to the line fixed by $\bar{\sigma}_4$, thus up to projectivities we can assume $L' = \{x_1 = 0\}$.

In these coordinates

$$\bar{\sigma}(x_0, x_1, x_2) = (-x_0, \zeta_5 x_1, x_2).$$

The branch curve B is reduced, invariant for $\bar{\sigma}$ and must contain the line L has a component. This implies that its equation is of the form:

$$x_0(x_2^5 + ax_0^2x_2^3 + bx_0^4x_2 + cx_1^5) = 0,$$

with $a, b, c \in \mathbb{C}$. In particular B has 5 singular points in the intersection of the line L and the quintic curve Q (i.e. φ contracts 5 disjoint smooth rational curves). All the above implies (X, σ) can be defined by an equation as in Example 4.8.2. $\square$

Remark 4.8.4. It follows from Proposition 4.8.3 that there are five disjoint smooth rational curves $R_1, \dots, R_5$ in X, each intersecting at one point the two fixed curves C_6 and R of σ_2. The classes of the curves $R, R_1, \dots, R_5$ all belong to the invariant lattice $S(\sigma_2)$. Observe that the classes of $2R + R_1 + R_2 + R_3 + R_4$, $2R + R_1 + R_2 + R_3 + R_5$, R, R_1, R_2, R_3, R_4 generate a lattice S isometric to $U(2) + D_4$. Since S is contained in $S(\sigma_2)$ and $\det(S(\sigma_2))$ is equal to $\det(S) = -2^4$ by Theorem 2.3.1, then $S = S(\sigma_2)$.

4.9 Order 24

Proposition 4.9.1. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{24} of order 24 such that $\dim(V^\sigma) = 2$. Then the fixed locus of σ_{24} and its powers are as follows:*

$\mathrm{Fix}(\sigma_{24})$	$\mathrm{Fix}(\sigma_{12})$	$\mathrm{Fix}(\sigma_6)$	$\mathrm{Fix}(\sigma_3)$	$\mathrm{Fix}(\sigma_2)$
$\{p_1, p_2, p_3, p_{12}, p_{13}\}$	$\{p_1, p_2, p_3, p_{12}, p_{13}\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_4 \sqcup R_1 \sqcup \{p_1\}$	$C_7 \sqcup R_1 \sqcup R_2,$

where $g(C_i) = i$ for $i = 4, 7$ and $g(R_1) = g(R_2) = 0$. Moreover $d = (2, 0, 1, 0, 0, 2)$ and $\mathrm{NS}(X) \cong U \oplus D_4$ for a general such K3 surface X.

Proof. We have $22 = 8d_{24} + 4d_{12} + 4d_8 + 2d_6 + 2d_4 + 2d_3 + d_2 + d_1$. Moreover by the topological Lefschetz fixed point formula we get the system of equations (4.9). Computing all possible values of the vector d one can see that $\chi(\mathrm{Fix}(\sigma_3)) \in \{0, -3, -6\}$. Assume $\chi(\mathrm{Fix}(\sigma_3)) = 0$. By Table 2.3 $\mathrm{Fix}(\sigma_3)$ is either the union of genus two curve and two isolated points or the union of a genus three curve, a smooth rational curve and two isolated points. Clearly $\mathrm{Fix}(\sigma_6) \subseteq \mathrm{Fix}(\sigma_3)$ and in this case $\chi(\mathrm{Fix}(\sigma_6)) = 16$ or 8. The first case is incompatible with the structure of the fixed locus of σ_3. If $\chi(\mathrm{Fix}(\sigma_6)) = 8$, then the fixed locus of σ_3 must be the union of a genus three curve C, a smooth rational curve R and two isolated points p, q. The automorphism σ_6 fixes 4 points on C and p, q. Moreover, it either fixes pointwise R or it has two isolated fixed points on it.

$$\begin{cases} \chi(\mathrm{Fix}(\sigma_{24})) & = d_6 - d_3 - d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_{12})) & = 2d_{12} - d_6 - 2d_4 - d_3 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_8)) & = -2d_6 + 2d_3 - d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_6)) & = 4d_{24} - 2d_{12} - 4d_8 - d_6 + 2d_4 - d_3 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_4)) & = -4d_{12} + 2d_6 - 2d_4 + 2d_3 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_3)) & = -4d_{24} - 2d_{12} + 4d_8 - d_6 + 2d_4 - d_3 + d_2 + d_1 + 2 \\ \chi(\mathrm{Fix}(\sigma_2)) & = -8d_{24} + 4d_{12} - 4d_8 + 2d_6 + 2d_4 + 2d_3 + d_2 + d_1 + 2 \end{cases} \tag{4.9}$$

Both cases are incompatible with [Dil12, Theorem 4.1] since the fixed points of σ_6 contained in the fixed curve of σ_3 are those of type $A_{2,6}$ (of type $\frac{1}{6}(3,4)$ in [Dil12]).

If $\chi(\mathrm{Fix}(\sigma_3)) = -6$ we have $\chi(\mathrm{Fix}(\sigma_6)) = 10$ and this can be seen to be incompatible with [Dil12, Theorem 4.1] with an argument similar to the previous one.

If $\chi(\mathrm{Fix}(\sigma_3)) = -3$ then by Table 2.3 $\mathrm{Fix}(\sigma_3)$ is either the union of a curve of genus 3 and one point, or the union of a curve of genus 4, a smooth rational curve and one point. In these cases we have $\chi(\mathrm{Fix}(\sigma_6)) = 13$, which excludes the first possibility for $\mathrm{Fix}(\sigma_3)$. Thus $\mathrm{Fix}(\sigma_3)$ is the union of a curve C of genus 4, a smooth rational curve R_1 and one point p. Using the Riemann-Hurwitz formula for σ_6 and the fact that $\chi(\mathrm{Fix}(\sigma_6)) = 13$ we obtain that σ_6 fixes p and 10 points on C. Moreover, by [Dil12, Theorem 4.1] the curve R_1 is pointwise fixed by σ_6. In this case one computes that $\chi(\mathrm{Fix}(\sigma_{12}))$ is either 5 or 1, but the second case is not possible since σ_{12} either fixes pointwise or has two fixed points on R_1. Thus σ_{12} fixes p, two points on C and it either fixes pointwise or has two fixed points on R_1. A computation using holomorphic Lefschetz formula shows that the first case does not occur. In this case one computes that $\chi(\mathrm{Fix}(\sigma_{24})) \in \{-1, 1, 3, 5, 7\}$. The only cases compatible with the structure of $\mathrm{Fix}(\sigma_{12})$ are $\chi(\mathrm{Fix}(\sigma_{24})) = 3$ or 5. The first case is impossible by the Riemann-Hurwitz formula.

Assuming $\chi(\mathrm{Fix}(\sigma_3)) = -3$, $\chi(\mathrm{Fix}(\sigma_6)) = 13$, $\chi(\mathrm{Fix}(\sigma_{12})) = 5$ and $\chi(\mathrm{Fix}(\sigma_{24})) = 5$ we find two possible vectors $d = (2, 0, 0, 0, 0, 1, 0, 4), (2, 0, 0, 1, 0, 0, 1, 3)$. For these cases $\chi(\mathrm{Fix}(\sigma_2)) = -8$ and $\chi(\mathrm{Fix}(\sigma_4)) = 8$. Moreover $\chi(\mathrm{Fix}(\sigma_8)) = 8$ in the first case and $= 2$ in the second case.

By [Nik79a] the fixed locus of σ_2 is either the union of a curve C_7 of genus 7 and two smooth rational curves (R_1 and R_2), or the union of a curve C_6 of genus 6 and R_1. The latter is not possible by the Riemann-Hurwitz formula applied to σ_6 restricted to C_6. Since $\chi(\mathrm{Fix}(\sigma_4)) = 8$, σ_4 must fix 4 points on C_7, two points on R_1 and it either fixes pointwisely R_2 or it has two fixed points on it. This implies that $\mathrm{Fix}(\sigma_8)$ contains isolated points and, at most, a smooth rational curve. Thus $\chi(\mathrm{Fix}(\sigma_8)) \geq \chi(\mathrm{Fix}(\sigma_{24})) = 5$, which excludes the case $d = (2, 0, 0, 1, 0, 0, 1, 3)$.

Finally, by Theorem 2.3.1 and Figure 2.1 the invariant lattice of σ_2 is isometric to $U \oplus D_4$. For a general K3 surface we have $\mathrm{rk}\,\mathrm{NS}(X) = 22 - 2\varphi(24) = 6$. Moreover $S(\sigma_2) \subseteq \mathrm{NS}(X)$ by Proposition 2.2.9, thus $S(\sigma_2) = \mathrm{NS}(X)$. $\square$

Example 4.9.2. Consider the elliptic surface with equation

$$y^2 = x^3 + t(t^4 - 1)(t^4 - a), \; a \in \mathbb{C}.$$

For general $a \in \mathbb{C}$ it is a K3 surface and carries the order 24 automorphism

$$\sigma_{24}(t, x, y) = (it, \zeta_{12}x, \zeta_8 y).$$

The action of σ^* on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by $\zeta_{12}\zeta_4\zeta_8^{-1}$, thus σ is purely non-symplectic. For general $a \in \mathbb{C}$ the elliptic fibration has a singular fiber F_∞ of type I_0^* over $t = \infty$ and 9 fibers of type II. The automorphism

σ_2 fixes the section at infinity R_1, the genus 7 curve defined by $y = 0$ and the central component R_2 of the fiber F_∞. The automorphism σ_3 fixes R_1, the curve of genus 4 defined by $x = 0$ and the intersection point p_1 between R_2 and the component of F_∞ intersecting R_1. Observe that the remaining three components of F_∞ are permuted by σ_3. The automorphism σ_6 fixes the 9 singular points $p_3, \dots, p_{11}$ of the fibers of type II, the point p_1 and the intersection point p_2 between the fiber F_∞ and the curve $x = 0$. Finally, the automorphisms σ_{12} and σ_{24} fix the singular point p_3 of the fiber F_0 of type II over $t = 0$, the intersection points of R_1 with the fibers F_0, F_∞, p_1 and p_2.

4.10 Classification for order 22

We now prove a complete classification theorem of purely non-symplectic automorphisms of order 22 on a K3 surface, according to their fixed locus.

Theorem 4.10.1. *Let σ_{22} be a purely non-symplectic automorphism of order 22 on a K3 surface X. Then the fixed locus of σ_{22} and of its powers $\sigma_{11} = \sigma_{22}^2$ and $\sigma_2 = \sigma_{22}^{11}$ are described in one of the rows of the following table.*

	$\mathrm{Fix}(\sigma_{22})$	$\mathrm{Fix}(\sigma_{11})$	$\mathrm{Fix}(\sigma_2)$	$(d_{22}, d_{11}, d_2, d_1)$
$A1$	$\{p_1, \dots, p_6\}$	$C_1 \sqcup \{p_5, p_6\}$	$C_{10} \sqcup R$	$(2, 0, 0, 2)$
$B1$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_5$	$(1, 0, 1, 11)$
$B2$	$R_1 \sqcup \{p_1, \dots, p_9\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_1 \sqcup \cdots \sqcup R_4$	$(1, 0, 2, 10)$
$B3$	$\{p_1, \dots, p_5\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$	$C_5 \sqcup R_2$	$(1, 0, 5, 7)$,

where $g(C_i) = i$ for $i = 1, 5, 10$ and $g(R_j) = 0$ for $j = 1, \dots, 5$. Moreover, all cases exist.

Proof. Let σ_{11} be the square of σ_{22}. According to Table 2.5 and Proposition 4.2.3 the fixed locus of σ_{11} is either a) the union of a smooth elliptic curve and 2 points, or b) the union of a rational curve and 11 points. In the first case $m := \frac{22 - \mathrm{rk}\, S(\sigma_{11})}{10}$ is 2, while in the second case $m = 1$.

Recall that fixed points of type $A_{10,22}$ lie on a curve in $\mathrm{Fix}(\sigma_{11})$, while points of type

$A_{i,22}, A_{10-i,22}$ correspond to isolated points for σ_{11} of type $A_{i,11}$, $i = 1, \dots, 4$. Lefschetz holomorphic formula together with the restrictions

$$a_{5,22} \leq a_{5,11}, \quad a_{i,22} + a_{10-i,22} \leq a_{i,11}, \; i = 1, 2, 3, 4$$

give the solutions in the following Table, where we compute $\chi_{22} = \chi(\mathrm{Fix}(\sigma_{22}))$ and $\chi_2 = \chi(\mathrm{Fix}(\sigma_2))$ by (4.1).

	$a_{1,22}$	$a_{2,22}$	$a_{3,22}$	$a_{4,22}$	$a_{5,22}$	$a_{6,22}$	$a_{7,22}$	$a_{8,22}$	$a_{9,22}$	$a_{10,22}$	α	χ_{22}	χ_2
A1	0	0	0	1	0	0	0	0	1	4	0	6	-16
B1	3	2	1	1	1	2	1	0	0	0	1	13	2
B2	3	2	2	1	1	0	0	0	0	0	1	11	0
B3	0	0	0	1	1	0	0	0	1	2	0	5	-6.

In case A1 we have $d_1 + d_2 = 2$ and $d_{11} + d_{22} = 2$ by Table 2.5. Since $\chi_{22} = 6$, then $(d_{22}, d_{11}, d_2, d_1) = (2, 0, 0, 1)$ by (4.1). The description of the fixed locus of σ_2 is thus obtained as in the proof of Proposition 4.4.1.

We now study the possibilities for $\mathrm{Fix}(\sigma_{22})$ when $\mathrm{Fix}(\sigma_{11})$ is the union of a rational curve and 11 points. By Figure 2.1 the fixed locus of the involution $\sigma_2 = \sigma_{22}^{11}$ is the union of a curve of genus g_2 and k_2 rational curves and $\chi(\mathrm{Fix}(\sigma_2)) = 2(1 - g_2 + k_2)$.

Thus in case B1 one has $(g_2, k_2) \in \{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)\}$. The only admissible case is $(g_2, k_2) = (5,5)$ since otherwise, recalling that isolated points of σ_{22} lie on fixed curves for σ_2, one gets a contradiction with the Riemann-Hurwitz formula.

As for case B2, one has $(g_2, k_2) \in \{(1,0), (2,1), (3,2), (4,3), (5,4), (6,5)\}$. The first four give a contradiction to the Riemann Hurwitz formula. Case $(g_2, k_2) = (6,5)$ is not admissible since by [FK92, Proposition V.2.14] a curve of genus 6 does not admit an automorphism of order 11 acting on it.

Similarly, in case B3 the possibilities are $(g_2, k_2) \in \{(4,0), (5,1), (6,2)\}$ and the only admissible one is $(g_2, k_2) = (5,1)$.

The vector $d = (d_{22}, d_{11}, d_2, d_1)$ is obtained in all cases my means of (4.1).

An example for case A1 has been given in Example 4.4.2. In the following we will provide examples for the cases B1, B2, B3. □

Example 4.10.2. (Case B1) Let X be the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t^7 x + t^5.$$

The fibration has one fiber of type II^* over $t = 0$, III over $t = \infty$ and 11 fibers of type I_1. The automorphism

$$\sigma_{22}(x, y, t) = (\zeta_{11}^{10} x, -\zeta_{11}^{4} y, \zeta_{11}^{6} t)$$

is purely non-symplectic of order 22 since its action on the two form $\frac{dx \wedge dt}{2y}$ is the multiplication by $-\zeta_{11}$. The automorphism σ_{22} preserves the fibers over $t = 0$ and $t = \infty$. In the fiber over $t = 0$, which is of type II^*, it must fix the component of multiplicity 6 and has 8 isolated fixed points in the other components. In the fiber over $t = \infty$ it fixes three isolated points. The involution σ_2 preserves each fiber of the elliptic fibration, thus it must fix R, three more components of the fiber over $t = 0$, the section at infinity and the 3-section $y = 0$, which has genus 5. This correspond to case B1.

Example 4.10.3. (Case B2) Let X be the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t^5 x + t^2.$$

The fibration has a fiber of type IV over $t = 0$, a fiber of type III^* over $t = \infty$ and 11 fibers of type I_1. The automorphism

$$\sigma_{22}(x, y, t) = (\zeta_{11}^{8} x, -\zeta_{11} y, \zeta_{11} t)$$

is purely non-symplectic of order 22 since its action on the two form $\frac{dx \wedge dt}{2y}$ is the multiplication by $-\zeta_{11}^{8}$. By [AST11, Example 7.4], σ_{11} has fixed locus $R \cup \{p_1, \ldots, p_{11}\}$, where R is the central component of the fiber of type III^*. The involution σ_2 maps (x, y, t) to $(x, -y, t)$, thus it preserves each fiber. This implies that it fixes R and two more rational components of the fiber of type III^*, as well as the section at infinity and the 3-section $y = 0$, whose genus is 5. This corresponds to case B2.

Example 4.10.4. (Case B3) We recall that the elliptic K3 surface defined by

$$y^2 = x^3 + ax + (t^{11} - 1), \ a \in \mathbb{C}^*$$

with the automorphism $\sigma_{22}(x, y, t) = (x, -y, \zeta_{11}t)$ is an example of case A (see Example 4.4.2). If a is such that $a^3 = -\frac{27}{4}$, thus the fibration admits a singular fiber of type II over $t = 0$, I_{11} over $t = \infty$ and 11 fibers of type I_1. The fixed locus of the automorphism σ_{11} is contained in the fibers over $t = 0$ and $t = \infty$. Since it fixes 11 isolated points and one rational curve, then it must fix one of the components of the fiber of type I_{11}, say R, has 9 fixed points in the other components of the same fibers and two more fixed points in the fiber of type II. The involution σ_2 fixes the section at infinity and the curve $y = 0$, which has genus 5. Moreover, σ_2 can not preserve each component of the fiber of type I_{11} by Lemma 2.2.13. Thus σ_2 acts on the fiber of type I_{11} as a reflection, without fixed components and with a unique invariant component. This corresponds to case B3.

4.11 Classification for order 15

In this section we prove a complete classification theorem of purely non-symplectic automorphisms of order 15 on a K3 surface, according to their fixed locus.

Theorem 4.11.1. *Let σ_{15} be a purely non-symplectic automorphism of order 15 on a K3 surface X. The fixed locus of σ_{15} consists of a set of points or the union of a smooth rational curve and a set of points. All possibilities for the fixed locus of σ_{15} and of its powers $\sigma_5 = \sigma_{15}^3$ and $\sigma_3 = \sigma_{15}^5$ are described in one of the rows of the following table, where N_{15} is the number of isolated fixed points of σ_{15}, g_i and $k_i + 1$ are the maximal genus and the number of fixed smooth curves of σ_i respectively, and $a_{j,i}$ is the number of fixed points of type $A_{j,i}$ of σ_i, for $i = 3, 5$.*

	N_{15}	k_{15}	$a_{1,5}$	$a_{2,5}$	g_5	k_5	$a_{1,3}$	g_3	k_3	(d_{15}, d_5, d_3, d_1)
A	5	0	1	0	2	0	2	2	0	$(2, 1, 0, 2)$
$B1$	7	0	3	1	1	0	1	4	1	$(2, 0, 1, 4)$
$B2$	7	0	3	1	1	0	6	0	2	$(1, 2, 0, 6)$
$D1$	10	1	5	2	1	1	6	0	2	$(1, 1, 0, 10)$
$F3$	9	1	7	3	0	1	4	2	2	$(1, 0, 2, 10)$
$F7$	12	1	7	3	0	1	5	2	3	$(1, 0, 1, 12)$
$F8$	5	0	7	3	0	1	2	2	0	$(1, 0, 4, 6)$

Moreover, all cases in the table exist.

Proof. According to Table 2.4 the fixed locus of $\sigma_5 = \sigma_{15}^3$ is the union of a smooth curve of genus g_5, k_5 rational curves, $a_{1,5}$ isolated points of type $A_{1,5}$ and $a_{2,5}$ isolated points of type $A_{2,5}$, where $g_5, k_5, a_{1,5}, a_{2,5}$ are as in one of the lines of the following Table, where $4m = 22 - \operatorname{rk} S(\sigma_5)$.

	$a_{1,5}$	$a_{2,5}$	g_5	k_5	m
A	1	0	2	0	5
B	3	1	1	0	4
C	3	1	-	-	4
D	5	2	1	1	3
E	5	2	0	0	3
F	7	3	0	1	2
G	9	4	0	2	1

We recall that $\alpha = \sum_{C\subset \mathrm{Fix}(\sigma_{15})}(1-g(C))$. In order to find all possibilities for $\mathrm{Fix}(\sigma_{15})$, we will look for a solution $a := (a_{1,15}, a_{2,15}, \ldots, a_{7,15}, \alpha)$ of the holomorphic Lefschetz formula compatible with the system of equations (4.2).

Remark 4.11.2. We recall that points of type $A_{4,15}, A_{5,15}$ lie on a curve fixed by σ_5 and not by σ_{15}. Thus if $a_{4,15} + a_{5,15} > 0$, there is at least a curve in $\mathrm{Fix}(\sigma_5)\backslash\mathrm{Fix}(\sigma_{15})$.

Remark 4.11.3. Observe that by [Bro91], a curve of genus 3 does not admit an automorphism of order 5. Thus if $\mathrm{Fix}(\sigma_3)$ contains a curve of genus 3, such curve is also fixed by σ_{15}.

We now analyze each line of the previous table separately.

- **Case A**: corresponds to $\chi(\mathrm{Fix}(\sigma_5)) = -1$. By equations (4.3) it follows that $a_{2,15} = a_{7,15} = 0$. The only solution of the holomorphic Lefschetz formula with this property is $a = (0, 0, 1, 2, 2, 0, 0, 0)$. In particular $\chi(\mathrm{Fix}(\sigma_{15})) = 5$. It follows from equations (4.2) that $d = (2, 1, 0, 2)$. The proof thus follows as in the proof of Proposition 4.5.1.

- **Case B**: corresponds to $\chi(\mathrm{Fix}(\sigma_5)) = 4$, i.e. $\mathrm{Fix}(\sigma_5)$ is the disjoint union of a smooth curve of genus one and 4 points. By [AST11, Example 5.6] X has an elliptic fibration $\pi : X \to \mathbb{P}^1$ which can be defined by a Weierstrass equation of the form
$$y^2 = x^3 + (t^5 + \alpha)x + (t^{10} + \beta t^5 + \gamma),\ \alpha, \beta, \gamma \in \mathbb{C},$$
where $\sigma_5(x, y, t) = (x, y, \zeta_5 t)$. The automorphism σ_5 fixes pointwise the smooth fiber F_0 over $t = 0$ and leaves invariant the fiber F_∞ over $t = \infty$, which contains 4 fixed points. This property and the fact that $24 - e(F_\infty)$ must be divisible by 5, imply that F_∞ is of Kodaira type IV, i.e. the union of three smooth rational curves intersecting transversally at one point. Observe that the elliptic fibration π is invariant for σ_3, since the smooth fiber over $t = 0$ is invariant for σ_3, and thus the same holds for the associated linear system. Moreover σ_3 must preserve all fibers of π, since otherwise 15 should divide $24 - e(F_\infty) = 20$, a contradiction. The remaining singular fibers of π, considering the fact that they are preserved by σ_3 (thus $J = 0$) and that $24 - e(F_\infty) = 20$, are either 5 fibers of type IV or 10 fibers of type II.

  By the holomorphic Lefschetz formula and equations (4.3) we find that either $a = (0, 1, 0, 0, 3, 0, 0, 0)$ or $a = (0, 0, 0, 0, 3, 3, 1, 0)$. If $a = (0, 1, 0, 0, 3, 0, 0, 0)$, then it follows from equations (4.2) that either $d = (2, 0, 2, 2)$ or $d = (1, 2, 1, 4)$. The first case has been excluded in the proof of Proposition 4.5.1. In the second case by (4.2) and Table 2.3, $\chi_3 = 9$ and the fixed locus of σ_3 contains at least two curves. We now exclude this case as well.

The automorphism σ_{15} fixes four points: three of them lie on the unique curve F_0 fixed by σ_5 and the other one is an isolated fixed point for σ_5. By the previous description, it follows that σ_3 must fix the center of the fiber F_∞ and permutes the other three fixed points of σ_5 on it (and thus the three components of the fiber F_∞). Moreover, being of types $A_{2,15}$ and $A_{5,15}$, the fixed points of σ_{15} are all contained in a curve C fixed by σ_3. Since C passes through the center of the fiber F_∞, then it is connected and by the Riemann-Hurwitz formula it is the unique fixed curve of σ_3 which is transversal to the fibers of π. On the other hand, σ_3 can not fix a curve R contained in a fiber of π, since the other singular fibers are either of type II, or of type IV, and in both cases R would intersect C, a contradiction. Thus σ_3 fixes at most one (connected) curve, so that the case $d = (1, 2, 1, 4)$ is not possible.

If $a = (0, 0, 0, 0, 3, 3, 1, 0)$, then it follows from equations (4.2) that either $d = (2, 0, 1, 4)$ or $d = (1, 2, 0, 6)$. If $d = (2, 0, 1, 4)$, then by (4.2) and Table 2.3, $\chi_3 = -3$ and the fixed locus of σ_3 consists either of the disjoint union of a genus three curve and one point or the disjoint union of a curve of genus four, a rational curve and one point. The first case is not possible by Remark 4.11.3.

If $d = (1, 2, 0, 6)$, then by (4.2) and Table 2.3, $\chi_3 = 12$ and the fixed locus of σ_3 consists either of the union of three disjoint rational curves and 6 points or the disjoint union of a curve of genus one, three rational curves and six points. We now exclude the second case. Observe that in this case σ_{15} fixes three points on F_0 and four isolated points in the fiber F_∞. Six of these points are contained in a curve fixed by σ_3, which will intersect each fiber of π at three points counting multiplicity. The same argument as before shows that σ_3 can not fix a curve contained in a fiber of π. Thus σ_3 fixes at most three (connected) curves.

To conclude, the only possible cases have $a = (0,0,0,0,3,3,1,0)$ and either $d = (2,0,1,4)$ with σ_3 fixing a genus four curve, a rational curve and one point, or $d = (1,2,0,6)$ with σ_3 fixing three smooth rational curves and six points.

- **Case C**: in this case σ_5 fixes exactly four points, more precisely $a_{1,5} = 3$ and $a_{2,5} = 1$. As before, by the holomorphic Lefschetz formula one obtains that either $a = (0,1,0,0,3,0,0,0)$ or $a = (0,0,0,0,3,3,1,0)$. In both cases $a_{4,15} + a_{5,15} > 0$, thus this case is not possible by Remark 4.11.2.

- **Case D**: in this case the fixed locus of σ_5 contains an elliptic curve, a smooth rational curve R and 7 isolated fixed points, with $a_{1,5} = 5, a_{2,5} = 2$. The holomorphic Lefschetz formula with the restrictions of (4.3) gives four solutions for the vector a:

$$(0,0,0,0,3,3,1,0),(0,0,1,2,2,0,0,0),(0,1,0,0,3,0,0,0),(3,2,2,3,0,0,0,1). \tag{4.10}$$

The only one compatible with equations in (4.2) is $a = (3,2,2,3,0,0,0,1)$. By Remark 4.11.2 a solution with $\alpha = 1$ means that only R is fixed by σ_{15}. By (4.2), this gives $\chi(\mathrm{Fix}(\sigma_3)) = 9$. According to Table 2.3, there are two possibilities for $\mathrm{Fix}(\sigma_3)$:

D1 $\mathrm{Fix}(\sigma_3)$ is the union of 3 smooth rational curves and 6 points;

D2 $\mathrm{Fix}(\sigma_3)$ is the union of an elliptic curve, 3 smooth rational curves and 6 points.

We now show that case D2 is not possible. Let $\mathrm{Fix}(\sigma_3) = E \cup R_1 \cup R_2 \cup R_3 \cup \{p_1, p_2, \ldots, p_6\}$ and consider the elliptic fibration $\pi : X \to \mathbb{P}^1$ defined by the linear system $|E|$. The automorphism $\bar{\sigma}_3$ induced by σ_3 on $\mathbb{P}^1$ is not the identity, since otherwise σ_3 should act on the general fiber of π either as a translation (which is impossible since σ_3 is non-symplectic) or with fixed points (impossible, since otherwise σ_3 should fix a curve which is transverse to all fibers, and thus intersecting E). Thus $\bar{\sigma}_3$ has order three and fixes two points in $\mathbb{P}^1$, one of them corresponding to the fiber E. The smooth rational curves and the isolated points fixed by σ_3 must be components of the other invariant fiber. This implies that such fiber is of type $I_6^* = \tilde{D}_{10}$.

Since the curve E is preserved by σ_5, thus the fibration π is preserved too. The fixed locus of σ_5 contains a curve of genus one E'. The curve E' can not be transverse to the fibers of π, since otherwise the general fiber of π would have an order five automorphism with a fixed point, which is impossible by [Har77, Corollary 4.7, IV]. Thus E' is one of the fibers of π. A similar reasoning to the one used for σ_3 implies that σ_5 induces an order 5 automorphism of $\mathbb{P}^1$, thus it preserves exactly two fibers of π. Observe that σ_5 must preserve both E, since it commutes with σ_3, and the fiber of type $I_6^* = \tilde{D}_{10}$, since an elliptic fibration of a K3 surface can not have five fibers of this type (the Euler number of the fiber is 12). This implies that $E = E'$, thus E would be a fixed curve of σ_{15}, a contradiction.

- **Case E**: as in the previous case, $a_{1,5} = 5, a_{2,5} = 2$ and the holomorphic Lefschetz formula with the restrictions of (4.3) has the four solutions of (4.10). Since in each case $a_{4,15} + a_{5,15} > 0$, then by Remark 4.11.2 the only curve fixed by σ_5 is not fixed by σ_{15} and $\alpha = 0$. For each one of the three possibles a's with $\alpha = 0$, the system (4.2) has no solutions. Thus there are no σ_{15} such that σ_5 has invariants as in case E.

- **Case F**: in this case $\mathrm{Fix}(\sigma_5)$ contains two rational curves R_1, R_2 and ten points with $a_{1,5} = 7, a_{2,5} = 3$. Lefschetz formula with the restrictions of (4.3) gives nine solutions, all of them with $\alpha = 0$ or 1. Thus at most one of the two curves R_i is contained in $\mathrm{Fix}(\sigma_{15})$.

  If $\mathrm{Fix}(\sigma_{15})$ contains a rational curve, then $\alpha = 1$ and combining the nine solutions of the Lefschetz formula with (4.2) one gets the possibilities F1-F7 of Table 4.6. If $\mathrm{Fix}(\sigma_{15})$ only contains points, then $\alpha = 0$ and by (4.2) we get possibilities F8 and

F9.

By Remark 4.11.3 we exclude cases F4 and F9.

Case F1 has to be excluded for the following reason: the total number of fixed points for σ_{15} is 9 and σ_{15} fixes a rational curve. Thus, $a_{2,15}+a_{3,15}+a_{5,15}+a_{6,15} = 5$ of the isolated fixed points for σ_{15} lie on curves fixed by σ_3. However, $\mathrm{Fix}(\sigma_3)$ contains just one rational curve, which is fixed by σ_{15}, giving a contradiction.

Case F2 has to be excluded for the following reason: σ_{15} acts as an automorphism of order 5 on the elliptic curve in $\mathrm{Fix}(\sigma_3)$ and it contains fixed points, which is not possible by [Har77, Corollary 4.7, IV]. Case F6 is analogous.

In case F5, the total number of fixed points for σ_{15} is 12: 5 of them are isolated for σ_3, thus 7 points should lie on the rational curve in $\mathrm{Fix}(\sigma_3)\backslash\mathrm{Fix}(\sigma_{15})$. This is not possible by the Riemann-Hurwitz formula.

- **Case G**: in this case $\mathrm{Fix}(\sigma_5)$ contains three rational curves and all solutions of the Lefschetz formula with the restrictions of (4.3) have $\alpha = 0$ or 1. Thus at most one of the three rational curves in $\mathrm{Fix}(\sigma_5)$ is contained in $\mathrm{Fix}(\sigma_{15})$. Checking (4.2) for all solutions in both cases $\alpha = 0, 1$ we find no solutions. Thus there are no possible σ_{15} such that $\mathrm{Fix}(\sigma_5)$ is as in case G.

In the following we will provide Examples for all cases, thus completing the proof. Examples of cases A and B1 can be found in Section 4.5. □

Example 4.11.4. (Case B2). The elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)^2$$

has six fibers of type IV, over $t = \infty$ and over the zeroes of $t^5 - 1$. It carries the order 15 automorphism

$$\sigma_{15} : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_5 t).$$

The fixed locus of σ_5 is contained in the union of the smooth fiber over $t = 0$ and in the fiber over $t = \infty$. The fixed locus of σ_3 contains the section at infinity, the two sections defined by $x = y \pm (t^5 - 1) = 0$ and the six centers of the fibers of type IV.

Example 4.11.5. (Case D1). This surface appears in [Bra19]. Let X be the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t^5 x + 1.$$

The fibration has one fiber of type $III^* = \tilde{E}_7$ over $t = \infty$ and 15 fibers of type I_1. It carries the order 15 automorphism

$$\sigma_{15} : (x, y, t) \mapsto (\zeta_{15}^{10} x, y, \zeta_{15} t).$$

The automorphism $\sigma_5 = \sigma_{15}^3$ fixes the smooth fiber E over $t = 0$, the smooth rational curve of multiplicity 4 of the fiber over $t = \infty$ and 7 isolated points in the same reducible fiber. Thus the invariants of σ_5 are $(g_5, k_5) = (1, 1)$, which corresponds to case D. The elliptic curve E is not fixed by $\sigma_3 = \sigma_{15}^5 : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_3 t)$. The automorphism σ_3 fixes three smooth rational curves and 3 isolated points in the fiber over $t = \infty$, and 3 points in the curve E.

Example 4.11.6. (Case F3) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_2(x_0^2 x_1^3 + x_2^5 + x_0^5).$$

The branch sextic B is the union of a line L and a quintic curve Q. The surface Y has four rational double points: one point of type D_7 at $(0, 1, 0, 0)$ and three points of type A_1 at $(-\zeta_3^i, 1, 0, 0)$, for $i = 0, 1, 2$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15} : (x_0, x_1, x_2, y) \mapsto (x_0, \zeta_3 x_1, \zeta_5 x_2, \zeta_5^3 y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus two curve C_2 defined by $x_1 = 0$ and the singular point $(0, 1, 0, 0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of C_2 and the union of two components and four isolated points in the exceptional divisor of type D_7. Thus we are in case F3.

Example 4.11.7. (Case F7) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_2(x_2^5 + x_1^5 + x_0^3 x_1 x_2).$$

The branch sextic B is the union of a line L and a quintic curve Q. The surface Y has a rational double point of type D_{10} at $(1, 0, 0, 0)$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15} : (x_0, x_1, x_2, y) \mapsto (\xi_5^2 x_0, \xi_{15}^7 x_1, \xi_3^2 x_2, y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus two curve C_2 defined by $x_0 = 0$ and the point $(1, 0, 0, 0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of C_2 and the union of three components and five isolated points in the exceptional divisor of type D_{10}. Thus we are in case F7.

Example 4.11.8. (Case F8) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_0^5 x_1 + (x_1^3 - x_2^3)^2.$$

The surface Y has three rational double points of type A_4 at $(0, 1, \zeta_3^i, 0)$, with $i = 0, 1, 2$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15} : (x_0, x_1, x_2, y) \mapsto (\zeta_5 x_0, x_1, \zeta_3 x_2, y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus two curve C_2 defined by $x_2 = 0$ and the smooth points $(0, 0, 1, \pm 1)$. Thus we are either in case F8 or in case A1. The automorphism σ_5 fixes the two smooth rational curves defined by $x_0 = y \pm (x_1^3 - x_2^3) = 0$ and the point $(1, 0, 0, 0)$. Thus its lifting $\tilde{\sigma}_5$ fixes two smooth rational curves, so we are in case F8.

	n	d	i	g_i	k_i	N_i	NS
11a	11	$(2,2)$	11	1	1	2	U
11b	11	$(2,2)$	11	-	0	2	$U(11)$
22	22	$(2,0,0,2)$	22	-	0	6	
			11	1	1	2	U
			2	10	2	0	
15a	15	$(2,1,0,2)$	15	-	0	5	
			5	2	1	1	$U(3)\oplus A_2\oplus A_2$
			3	2	1	2	
15b	15	$(2,0,1,4)$	15	-	0	7	
			5	1	1	4	$H_5\oplus A_4$
			3	4	2	1	
30a	30	$(2,0,1,0,0,0,1,1)$	30	-	0	1	
			15	-	0	5	
			5	2	1	1	$U(3)\oplus A_2\oplus A_2$
			3	2	1	2	
			2	10	1	0	
30b	30	$(2,0,0,1,0,0,1,3)$	30	-	0	3	
			15	-	0	7	
			5	1	1	4	$H_5\oplus A_4$
			3	4	2	1	
			2	9	1	0	
16a	16	$(2,0,0,0,6)$	16	0	1	6	
			8	0	1	6	$U\oplus D_4$
			4	0	1	6	
			2	7	3	0	
16b	16	$(2,0,0,2,4)$ or $(2,1,1,3,5)$	16	-	0	4	
			8	0	1	6	$U(2)\oplus D_4$
			4	0	1	6	
			2	6	2	0	
20	20	$(2,0,1,0,0,2)$	20	-	0	3	
			10	-	0	7	
			5	2	1	1	$U(2)\oplus D_4$
			4	0	1	6	
			2	6	2	0	
24	24	$(2,0,0,0,0,1,0,4)$	24	-	0	5	
			12	-	0	5	
			6	0	1	11	$U\oplus D_4$
			3	4	2	1	
			2	7	3	0	

Table 4.1 Non-symplectic automorphisms with $\varphi(n)=8,10$

d_{30}	d_{15}	d_{10}	d_6	d_5	d_3	d_2	d_1	χ_{30}	χ_{15}	χ_5	χ_3	χ_2
2	0	1	0	0	0	1	1	1	5	-1	0	-18
2	0	1	0	0	0	0	2	3	5	-1	0	-16
2	0	0	0	1	0	0	2	1	5	-1	0	-8

Table 4.2 Proof of Proposition 4.6.2, I

d_{30}	d_{15}	d_{10}	d_6	d_5	d_3	d_2	d_1	χ_{30}	χ_{15}	χ_5	χ_3	χ_2
2	0	0	1	0	0	2	2	1	7	4	-3	-16
2	0	0	1	0	0	1	3	3	7	4	-3	-14
2	0	0	0	0	1	1	3	1	7	4	-3	-10
2	0	0	1	0	0	0	4	5	7	4	-3	-12
2	0	0	0	0	1	0	4	3	7	4	-3	-8

Table 4.3 Proof of Proposition 4.6.2, II

d_{16}	d_8	d_4	d_2	d_1	χ_{16}	χ_8	χ_4	χ_2
2	1	0	1	1	2	4	0	-8
2	0	2	1	1	2	2	8	-8
2	0	1	3	1	0	5	8	-8
2	0	0	5	1	-2	8	8	-8
2	1	0	0	2	4	4	0	-8
2	0	2	0	2	4	2	8	-8
2	0	1	2	2	2	5	8	-8
2	0	0	4	2	0	8	8	-8
2	0	1	1	3	4	5	8	-8
2	0	0	3	3	2	8	8	-8
2	0	1	0	4	6	5	8	-8
2	0	0	2	4	4	8	8	-8
2	0	0	1	5	6	8	8	-8
2	0	0	0	6	8	8	8	-8

Table 4.4 Proof of Theorem 4.7.1

d_{20}	d_{10}	d_5	d_4	d_2	d_1	χ_{20}	χ_{10}	χ_5	χ_4	χ_2
2	0	0	0	0	6	8	12	4	8	-8
2	0	0	0	1	5	6	12	4	6	-8
2	0	0	0	2	4	4	12	4	4	-8
2	0	0	0	3	3	2	12	4	2	-8
2	0	0	0	4	2	0	12	4	0	-8
2	0	0	0	5	1	-2	12	4	-2	-8
2	0	0	1	0	4	6	8	4	6	-12
2	0	0	1	1	3	4	8	4	4	-12
2	0	0	1	2	2	2	8	4	2	-12
2	0	0	1	3	1	0	8	4	0	-12
2	0	0	2	0	2	4	4	4	4	-16
2	0	0	2	1	1	2	4	4	2	-16
2	0	1	0	0	2	3	7	-1	8	-8
2	0	1	0	1	1	1	7	-1	6	-8
2	1	0	0	0	2	5	7	-1	0	-8
2	1	0	0	1	1	3	7	-1	-2	-8

Table 4.5 Proof of Proposition 4.8.1

	$a_{1,5}$	$a_{2,5}$	g_5	k_5	$a_{1,3}$	g_3	k_3	$a_{1,15}$	$a_{2,15}$	$a_{3,15}$	$a_{4,15}$	$a_{5,15}$	$a_{6,15}$	$a_{7,15}$	α
F1	7	3	0	1	4	0	0	3	3	1	1	1	0	0	1
F2	7	3	0	1	4	1	1	3	3	1	1	1	0	0	1
F3	7	3	0	1	4	2	2	3	3	1	1	1	0	0	1
F4	7	3	0	1	4	3	3	3	3	1	1	1	0	0	1
F5	7	3	0	1	5	0	1	3	2	1	1	1	3	1	1
F6	7	3	0	1	5	1	2	3	2	1	1	1	3	1	1
F7	7	3	0	1	5	2	3	3	2	1	1	1	3	1	1
F8	7	3	0	1	2	2	0	0	0	1	2	2	0	0	0
F9	7	3	0	1	2	3	1	0	0	1	2	2	0	0	0

Table 4.6 Case F

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