Universidad de Concepción

Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas
Programa de Doctorado en Ciencias Aplicadas con mención en Ingeniería Matemática

MÉTODOS NUMÉRICOS PARA LA SIMULACIÓN Y LA ESTABILIZACIÓN DE ALGUNOS SISTEMAS VIBRATORIOS Y ONDULATORIOS.

NUMERICAL METHODS FOR THE SIMULATION AND STABILIZATION OF SOME VIBRATORY AND ONDULATORY SYSTEMS.

Rodrigo Adolfo Véjar Asem

Tesis para optar al grado de Doctor en Ciencias Aplicadas con mención en Ingeniería Matemática

Concepción, 16 de Diciembre de 2019

Métodos numéricos para la simulación y la estabilización de algunos sistemas vibratorios y ondulatorios.

Rodrigo Adolfo Véjar Asem

Directores de Tesis:	Prof. Mauricio Sepúlveda Cortés
	Prof. Marcelo Moreira Cavalcanti
Director de Programa:	Prof. Rodolfo Rodríguez Alonso

COMISIÓN EVALUADORA

Prof. Christophe Besse, Institut de Mathématiques de Toulouse, Francia.

Prof. Eduardo Cerpa, Universidad Técnica Federico Santa María, Chile.

Prof. Aissa Guesmia, Université de Lorraine, Francia.

Prof. Lionel Rosier, Université Côte d'Opale, Francia.

COMISIÓN EXAMINADORA

Firma: ______________________________

Prof. Raimund Bürger, Universidad de Concepción, Chile

Firma: ______________________________

Prof. Marcelo M. Cavalcanti, Universidade Estadual de Maringá, Brasil.

Firma: ______________________________

Prof. Eduardo Cerpa, Universidad Técnica Federico Santa María, Chile.

Firma: ______________________________

Prof. Wellington J. Corrêa, Universidade Tecnológica Federal do Paraná, Brasil.

Firma: ______________________________

Prof. Mauricio Sepúlveda, Universidad de Concepción, Chile.

La moda actual del hormiguero dice que las obreras deben llevar hojas podridas a la colonia, en nombre de aquellas que no tienen bosques cerca de donde viven.

Esta obra está dedicada a todas las hormigas que viven en el ostracismo después de opinar que está mal llevar hojas podridas a casa.

Agradecimientos

Agradezco infinitamente el apoyo recibido y la (enorme) paciencia que tuvo mi tutor de tesis, Prof. Mauricio Sepúlveda Cortés. Si no fuese por la confianza que ha depositado en mi persona, las constantes discusiones y sugerencias, y la apertura a responder dos o tres veces la misma pregunta; simplemente no estaría donde ahora estoy.

También estoy enormemente agradecido por la hospitalidad y constante apoyo de parte del Prof. Marcelo M. Cavalcanti (mi cotutor) y de Wellington J. Corrêa durante mi estadía en Maringá y en Campo Mourão, respectivamente. No sé si sea posible devolverles la mano en el futuro, ya sea académicamente o mediante botellas de vino; más sólo me consuela el hacer la retribución con otro estudiante en caso que llegue a tener la oportunidad.

Agradezco de igual manera el financiamiento recibido por Beca CONICYT-PCHA/Doctorado Nacional/2015-21150799.

Mi trabajo tampoco pudo haber llegado a buen puerto de no ser por el gran apoyo en infraestructura entregado por el Centro de Investigación en Ingeniería Matemática, CI^{2}MA. No sé si vuelva a tener una oficina tan agradable y tranquila en el futuro; espero haberla aprovechado al máximo. Agradezco, a propósito, al equipo de guardias de la Universidad de Concepción por su constante y permanente colaboración en preservar la paz del lugar, manteniéndola libre de ruidos y olores molestos. ¡Sepan desde acá que el anexo 4205 y su gente son lo máximo!

Quiero dar las gracias al Departamento de Ingeniería Matemática de la Universidad de Concepción, por haberme dado la oportunidad de mantener el contacto con la docencia a través de ayudantías y mediante mi primera experiencia como docente responsable. Hago una mención especial al Prof. Manuel Campos Pareja por haber confiando en mí durante varios semestres como ayudante de sus cursos. La experiencia recibida ha sido importantísima para mi persona.

Mi más sincero agradecimiento a los docentes asociados al programa de doctorado, y aquellos que tuvieron impacto en mi formación. Agradezco también a toda la juventud CI^{2}MA que me aguantó y chismeó conmigo en estos cinco años de doctorado. Tampoco olvidaré la ayuda y amistad brindadas por Lorena y Jorge, ambos jóvenes pertenecientes al personal de apoyo CI^{2}MA. ¡Los quiero mucho!

Finalmente, nada de esto hubiese sucedido de no ser por la oportunidad y apoyo brindados por mis seres queridos. A Karina y Jaime, mis padres, por haberme dado vida, valores ético-morales (que hoy escasean), y por haberme apoyado material y moralmente durante toda mi formación académica. A mi hermana Paulina por su paciencia, cariño, y apoyo logístico como receptora de encomiendas de dudosa procedencia. Y a Liliana por haberme endulzado la existencia y hacerme ver que no soy un ser **tan** deleznable después de todo. Yo no sería nada sin ustedes.

A todos, y por todo: ¡muchísimas gracias!

Métodos numéricos para la simulación y la estabilización de algunos sistemas vibratorios y ondulatorios.

Rodrigo Adolfo Véjar Asem

Departamento de Ingeniería Matemática &
Centro de Investigación en Ingeniería Matemática
Universidad de Concepción
Concepción, Chile
2019

RESUMEN

La presente tesis contiene contribuciones asociadas a tres contextos distintos. La primera propone un esquema de diferencias finitas que resuelve una ecuación nolineal de Schrödinger de alto orden en una dimensión, esquema que además está dotado con propiedades de conservación, y de estabilización de la norma L^2 en caso que el problema presente ciertos elementos disipativos. El segundo aborda un problema nolineal de Schrödinger en dos dimensiones, usando un esquema de volúmenes finitos que aproxima la solución cuando el dominio presenta disipación localizada. Los resultados numéricos replican un resultado de estabilización exponencial demostrado por Cavalcanti, Corrêa, Özsari, Sepúlveda y Véjar-Asem. El tercero resuelve numéricamente un problema de puente colgante usando un esquema de diferencias finitas, que también logra replicar un resultado de estabilización previamente demostrado en [DCMCC+].

Numerical methods for the simulation and stabilization of some vibratory and ondulatory systems.

Rodrigo Adolfo Véjar Asem

Departamento de Ingeniería Matemática &
Centro de Investigación en Ingeniería Matemática
Universidad de Concepción
Concepción, Chile
2019

ABSTRACT

The present thesis contains contributions associated to three different contexts. The first one proposes a finite difference scheme that solves a High Order Nonlinear Schrödinger equation (HNLS) in one dimension, scheme that also has conservation and stabilization properties, if a certain damping function is present. The second one deals with a Nonlinear Schrödinger equation (NLS) in two dimensions, where a finite volume scheme was used to approximate the solution when a localized damping function is present. The scheme replicates a stabilization result proved by Cavalcanti, Corrêa, Özsari, Sepúlveda y Véjar-Asem. In the third contribution, a hanging bridge problem is solved numerically using a finite difference scheme. The scheme also manages to replicate a stabilization result proved in [DCMCC+].

Table of Contents

List of Figures

Introduction

In this chapter, we will introduce the problems to be studied through the dissertation.

The Schrödinger Equation

The first deduction of the Schrödinger Equation

Erwin Schrödinger, in his series of papers *Quantisation as a Problem of Proper Values*, proposed the now celebrated *Schrödinger Equation.* Originally, the equation models a Hydrogen-like[1] Atom without external forces, magnetic fields, and relativistic effects. It can be constructed using the traditional approach of building the Hamilton function of the system; but rather than considering a scalar function as in Classical Mechanics, Schrödinger considered it as an operator acting over a given space of functions. When obtaining its eigenvalues, we can predict the correct energy values for an unperturbed Hydrogen atom, previously discovered using atomic spectroscopy. Another interesting feature is that the equation sees the system as a wave, different from what the Classical Mechanics deals with where the atom is viewed as a system of particles.

The equation comes from considering the Hamiltonian of the hydrogen atom

$$\hat{H}(x) = \hat{K}(x) + \hat{V}(x) \tag{1}$$

where $\hat{K}$ is its kinetic energy operator, and $\hat{V}$ is its potential energy operator. The Hamiltonian operator is such that its eigenvalues corresponds to the energy of the described system:

$$\hat{H}\psi = E\psi$$

where ψ is assumed to be the state function that describes the system, in the sense that quantities like the energy, the velocity, or the position of an electron can be computed from ψ. From the quantum mechanics, the kinetic energy operator is defined by

$$\hat{K} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2\mu}$$

where $\hat{p}_x := \frac{\hbar}{i}\frac{\partial}{\partial x}$, $\hbar$ is the Planck constant[2] and μ is the reduced mass[3] of the system. $\hat{p}_y$ and $\hat{p}_z$ are defined in a similar fashion. Hence,

$$\hat{K} = -\frac{\hbar^2}{2m}\nabla^2.$$

[1] A Hydrogen-like system is composed by two particles: the proton at the center, and the electron orbiting outside.

[2] In the International System of Units (SI), $\hbar = 1.054571800 \times 10^{-34}\ J \cdot s$

[3] For two masses m_1 and m_2, $\mu = \frac{m_1 m_2}{m_1+m_2}$. It allows a two-body problem to be considered as one whose mass is given by μ.

On the other hand, for the case of the Hydrogen atom composed by one proton and one electron, its potential energy will come from the Coulomb potential

$$\hat{V} = -\frac{e^2}{4\pi\epsilon_0 r}$$

where e is the electric charge of the electron[4], ϵ_0 is the permittivity of vacuum[5], and $r = \sqrt{x^2 + y^2 + z^2}$. Hence, the Hamiltonian will be given by

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

and due to its properties for a state function $\psi = \psi(x)$, we have

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi.$$

We've deduced, then, the so called *time independent Schrödinger equation* for a Hydrogen atom:

$$\frac{\hbar^2}{2m}\nabla^2\psi + \Big(\frac{e^2}{4\pi\epsilon_0 r} + E\Big)\psi = 0. \tag{2}$$

Another similar expression was postulated by Schrödinger in order to get a representation of mechanics using a wave equation. Modelling an atomic-level system through waves comes from the failure of Classical Mechanics to describe quantum-scale phenomena such as the Zeeman effect. The *time dependent Schrödinger equation* is given by[6]

$$\frac{\hbar^2}{2m}\nabla^2\psi - \frac{1}{E^2}\Big(\frac{e^2}{4\pi\epsilon_0 r} + E\Big)\frac{\partial^2\psi}{\partial t^2} = 0.$$

where the function ψ depend on time exclusively through a periodic factor given by

$$\psi \sim Re\Big(e^{i\frac{E}{\hbar}t}\Big) \tag{3}$$

However, Erwin Schrödinger stated that the main drawback of this equation is that it remains valid only for previously known values of E. It also fails when dealing with non-conservative problems; this is, for $\hat{V} = \hat{V}(x,t)$. For this reason, Schrödinger attempted to modify the equation through getting rid of the energy term E. From (3), we have

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{E^2}{\hbar^2}\psi$$

Applying two times (1) to some state function $\psi = \psi(x,t)$ and for some potential $\hat{V} = \hat{V}(x,t)$, and after using the recently obtained result, we get

$$\Big(-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\Big)^2\psi + \hbar^2\frac{\partial^2\psi}{\partial t^2} = 0.$$

[4]In SI units, $e = 1.602176634 \times 10^{-19}\ C$

[5]In SI units, $\epsilon_0 = 8.8541878128 \times 10^{-12} F \cdot m^{-1}$

[6]A complete and reasonable deduction of this expression can be found in Sakurai [SC95]

Thus, the problem of the energy was solved at the price to deal with a fourth-order PDE which is difficult to solve if $\hat{V}$ depends on time. To overcome this difficulty, and from (3), Schrödinger added its imaginary part to obtain

$$\frac{\partial \psi}{\partial t} = i\frac{E}{\hbar}\psi,$$

replacing this in (2) and re-ordening

$$i\frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2}\nabla^2\psi - \hat{V}\psi. \tag{4}$$

This is the first of many forms of the Schrödinger Equation. In addition to hydrogen-like systems, variations of equation (4) were used to describe waves propagating in plasmas, or solitons propagating in optical fibers. For this dissertation, we will put our focus on the second case.

The Nonlinear Schrödinger Equation

Many years after the publication of Erwin Schrödinger's papers, similar equations came to light in different areas of physics. One of the most remakful and still in use up to this day, is the propagation of an electromagnetic wave while produces its own dielectric guide. One of the first proposals comes from Chiao, Garmire and Townes [CGT64], where they proposed the NonLinear Schrödinger (NLS) Equation to describe the propagation of an electromagnetic beam travelling inside a dielectric such that its diameter has a size comparable to is wavelength. The dielectric responds with nonlinearly over the beam, where self-focusing and self-phase modulation effects appear.

Regarding the self-focusing effect, it manifests after a certain distance travelled by the light beam, and is such that it will tend to focus over the regions with higher intensities; as such, the intensity at the center of the beam will increase. This is because the light beam changes the refraction index of the dielectric as it travels through it. This effect is called the *Kerr effect*, where the change in the refraction index Δn can be computed using the following equation

$$\Delta n = \lambda K|E|^2$$

where λ is the wavelength of the beam, K is the Kerr constant which depends on the material, and E is the electric field carrying the light beam. In Kelley's work [Kel65], a deduction of the distance needed for the pulse to experiment self-focusing is obtained, and then the NLS equation is solved using a Finite Difference scheme proposed by Harmuth [Har57] in 1957 using cyllindrical coordinates.

A reasonable form of the NLS equation reads as follows from the work of Zakharov and Shabat [ZS72]

$$2ik\frac{\partial E}{\partial z} + \frac{\partial E}{\partial x^2} + k^2\frac{\lambda K}{n_0}|E|^2E = 0 \tag{5}$$

where $k = \frac{\omega}{c}\sqrt{\epsilon_0}$, ω is the frequency of the beam, c is the speed of light, and n_0 is the initial refraction index of the dielectric. Variable z is considered to have the role of the time coordinate. This equation, however, can be rewritten as

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + \kappa|E|^2 E = 0 \tag{6}$$

for some κ constant containing the frequency of the soliton and the Kerr effect. In the literature, it is common to find equations similar to (6) with the time and space coordinates exchanged. In Agrawal [Agr00], and starting from Maxwell equations, a clear deduction of the NLS equation can be found for a light pulse travelling in an optical fiber. It reads as

$$i\frac{\partial A}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \gamma|A|^2 A = 0 \tag{7}$$

where A is the envelope of the electromagnetic pulse; β_2 is a parameter that accounts a phase shift of the beam, which could in turn lead to dissipative effects; and γ accounts for the Kerr effect. The z coordinate is the distance travelled by the light pulse through the fiber. On the other hand; while (7) has a strong physical meaning, the form (6) is suitable for the study of well-possedness and numerical integration. Through this dissertation in particular, will stick to a more general form of (6) for $u = u(x,t)$ and a_1, a_2 some real constants:

$$i\frac{\partial u}{\partial t} + a_1\frac{\partial^2 u}{\partial x^2} + a_2|u|^2 u = 0 \tag{8}$$

This model was proved to be a successful model for the transmission of light pulses through dielectric guides [MSG80], which with time turned into the study of information transmission in optical fibers. We must emphasize that equation (7) no longer models the physical situation of the original Schrödinger Equation (4). However, this equation rettained its name due to its ressemblance to the original expression.

Equation (8) presents an infinite number of conservation laws [ZS72], from where two of them will be recurrently used through this dissertation: the mass or the L^2- norm

$$\mathcal{H}_1 = \int_\Omega |u(x,t)|^2 d\Omega \tag{9}$$

and the energy

$$\mathcal{H}_2 := a_1\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 d\Omega - \frac{a_2}{2}\int_\Omega |u|^4 d\Omega. \tag{10}$$

This is a key feature that is recurrently used when working with numerical simulations; as they are an indicator of the quality of the approximation obtained by the scheme. This is an element that will also be considered in this thesis.

In Chen and Liu [CL76], equation (6) is solved for $\kappa = 2$ and assuming $E(x,t) = A(x,t)e^{i\varphi(x,t)}$ for real functions A and φ given by

$$A(x,t) = 2\eta\,\mathrm{sech}\left(2\eta(x - 4\xi t - x_0)\right), \quad \varphi(x,t) = 2\xi x - 4(\xi^2 - \eta^2)t + \varphi_0$$

which is a travelling soliton. Parameter η represents the amplitude and pulse width, ξ represents the speed, and φ_0 is its initial phase [Has89]. Another way yo obtain solutions is through an inverse scattering problem, first implemented by Zakharov and Shabat [ZS72]. They also proved that the solution can be written as a combination of N solitons. Glassey proved [Gla77] that, under certain conditions, the solution may suffer blow-up in finite time; this is, there exist $T_0 \in \mathbb{R}^+ : \lim_{t\to T_0} ||u(t)||_{L^2(\Omega)} = +\infty$.

Dark solitons can also rise a solutions for (7) for $\beta_2 > 0$. They were first observed by Emplit et al. [EHR+87], and Krökel et al. [KHGG88] as gaps forming from a single dip travelling though a fiber. The initial gap is formed after manipulating the modulator while it is emitting a longer signal. In Krökel et al., they compared their results with numerical experiments using a splitting scheme, obtaining a reasonable agreement. Those dark solitons have the following form when $\beta_2 = 1$ and $\gamma = 1$:

$$A(z,t) = \eta\big(B\tanh(\zeta) - i\sqrt{1-B^2}\big)e^{i\eta^2 z}$$

for $\zeta = \eta B\big(t - t_0 - \eta B\sqrt{1-B^2}\big)$. The parameter η denotes the background amplitude of the soliton, while t_0 indicates the position of the gap; B controls the depth of the gap.

Equation (8) can be extended to 2D, where it can be written as follows

$$i\frac{\partial u}{\partial t} + a_1\Delta u + a_2|u|^2 u = 0. \tag{11}$$

It can model the amplitude of progressive waves in superposed fluids under the presence of magnetic fields (see, for instance, [KM82]). As in the one-dimensional case, those waves can also behave as travelling solitons [KCS03]. This form also has applications in small-amplitude gravity waves on the surface of deep inviscid water, Langmuir waves in hot plasmas, slowly varying packets of quasi-monochromatic waves in weakly nonlinear dispersive media, Bose-Einstein condensates, Davydov's alpha-helix solitons, and plane-diffracted wave beams in the focusing regions of the ionosphere (see for instance [SS99], [Mal05], [PS03], [Bal85], and [Gur78]).

The High-order Nonlinear Schrödinger Equation

Hasegawa and Kodama, in [HK81], hinted out that equation (8) is no longer valid when the pulse has a wide smaller than a picosecond. In that same publication (and also [Kod85], [KH87]), another NLS equation was deduced which takes into account high order dispersion, soliton splitting, self-steeping, and retarded Raman effect for ultra-short light pulses; this is, pulses shorter than 100 [fs]. The now called High-order NonLinear Schrödinger (HNLS) equation obtained in Hasegawa and Kodama is the following

$$i\frac{\partial q}{\partial z} + \frac{1}{2}\frac{\partial^2 q}{\partial t^2} + |q|^2 q + i\varepsilon\Big(\beta_1\frac{\partial^3 q}{\partial t^3} + \beta_2\frac{\partial}{\partial t}(|q|^2 q) + \beta_3 q\frac{\partial}{\partial t}|q|^2\Big) = 0 \tag{12}$$

A more general version can be written for $u = u(x,t)$

$$i\frac{\partial u}{\partial t} + a_1\frac{\partial^2 u}{\partial x^2} + a_2|u|^2 u + i\Big(a_3\frac{\partial^3 u}{\partial x^3} + a_4\frac{\partial}{\partial x}(|u|^2 u) + a_5 u\frac{\partial}{\partial x}|u|^2\Big) = 0 \tag{13}$$

for some real or complex constants $a_1, \ldots, a_5$. One of the most remarkable properties of this equation is to be able to model the propagation of solitons trough optical fibers without being modified in its shape or intensity, which opens a new way to send information at high speeds and with minimal losses [Pot89]. Most of the literature consider the HNLS equation as a one dimensional evolution problem, due to the symmetries of the fibers.

In (13), the third order dispersion term a_3 is associated with the group dispersion velocity of the soliton, and can take an approximate value of -0.002 if the wavelength of the pulse is near 1.3 [μm] [HK81]. If certain conditions are met, this term can decompose the bound-state of the soliton, leading to its splitting into individual solitons [KH87]. The imaginary part of the a_5 term can produce a frequency shift in the soliton, proportional to the distance travelled [KH87] inside the fiber. This, in turn, will lead to a change in the propagation velocity, which is what describes the Raman effect observed by Mitschke and Mollenauer [MM86] for 120 [fs] pulses. In Chapter 1, Example 1.5.5, we managed to numerically replicate this effect. Meanwhile, in Anderson and Lisak [AL83] is stated that the a_4 term can produce self-steeping effects, which could end in the formation of shocks in the trailing part.

Travelling solitons can be obtained as solutions only for some values of a_3, a_4 and a_5. Anderson and Lisak [AL83] managed to obtain sech-shaped soliton solutions for (13) for $a_1 = a_3 = a_5 = 0$. For $a_4 + a_5 = 0$ and $3a_2a_3 = a_1a_4$, Hirota [Hir73a] proposed an N soliton solution for (13). Sasa and Satsuma [SS91] also obtained explicit solutions when $a_3 : a_4 : a_4 + a_5 = 1 : 6 : 3$ through the Inverse Scattering Transform (IST) method. Potasek and Tabor [PT91a] found bright and dark soliton solutions for $a_2 = 1$, $a_4 = \rho$ and $a_5 = \delta - 2\rho$, for ρ and δ real parameters depending on the carrier frequency and the geometry of the fiber. Kumar and Chand [KC13] proposed a solitary wave solution as ansatz, using $2a_5 + 3a_4 = 0$ as a constraint.

If in (13) we consider u as a real function instead of complex, and make $a_1 = a_2 = a_5 = 0$, $a_3 = 1$, and $a_4 = 6$; we obtain a modified Korteweg de-Vried (KdV) equation:

$$u_t + u_{xxx} + 6u^2u_x = 0$$

this expression models surface waves on conducting nonviscous incompressible liquid under the presence of a transverse electric field [PFE74]. The KdV equation has also great importance in the study of surface water waves [KDV95]. In this sense, the study of the HNLS equation (13) gets an increased value of solving many problems at once.

In case the fiber presents some losses, equation (13) is modified to

$$i\frac{\partial u}{\partial t} + a_1\frac{\partial^2 u}{\partial x^2} + a_2|u|^2u + i\left(a_3\frac{\partial^3 u}{\partial x^3} + a_4\frac{\partial}{\partial x}(|u|^2u) + a_5u\frac{\partial}{\partial x}|u|^2 + a(x)u\right) = 0 \qquad (14)$$

where, generally speaking, the function $a(x)$ describes the loss of energy from the soliton, and could be acting over the whole fiber or some part of it. This is what the literature calls *the damping function*. Considering $\Omega \subseteq \mathbb{R}$ as the space domain and for some real parameter a_0: if $a(x) \geq a_0 > 0$, $\forall x \in \Omega$, then we say that the problem has full damping; otherwise, we say that the damping is localized. For an ideal fiber with small loss, $a_0 \approx 0.03$ [HK81]. Anderson and Lisak [AL83] obtained solutions for a full damping case with $a_1 = a_3 = a_5 = 0$; where the probability of creation of a shock due to self-steeping is still present, but delayed thanks to the damping term. The damping function can be considered as a *defocusing* damping when, for $a = a(x,t)$, $a \to 0$ when $|x| \to \infty$ and $t \to \infty$.

Cavalcanti et al. [CCSN10] proved existence and uniqueness of solutions for a *damped cubic NLS equation*:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + a_2|u|^2u + ia(x)u = 0, \qquad \mathbb{R} \times (0,\infty)$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}.$$

for $a(x)$ a function with the coercivity properties described previously. They also proved the exponential decay rate of the L^2-norm. An improvement was obtained in 2017, now for a damping function $a = a(x,t)$ such that $a \to 0$ when $|x| \to \infty$ and $t \to \infty$ [CCCT17].

No explicit solution for (14) with all his coefficients has been obtained up to this day.

Numerical Methods

Since the first appearance of computers, many attempts to solve equations like (4), (5) or (12) have been made, where the Finite Difference Method appears to be the most popular and the most successfull. One of the first efforts comes from Harmuth [Har57], where he proposed a finite difference scheme to solve equation (4). Assuming $V = V(x)$ in (4), and for a one-dimensional problem, the scheme proposed for some given Δx and Δt reads as follows

$$-i\hbar\frac{1}{2\Delta t}\big(\psi_{j,n+1} - \psi_{j,n-1}\big) = \frac{\hbar^2}{2\Delta x^2}\Big(\psi_{j+1,n} - 2\psi_{j,n} + \psi_{j-1,n}\big)\Big) - V_j\psi_{j,n}, \tag{15}$$

where $\psi_{j,n} = \psi(j\Delta x, n\Delta t)$. The scheme is stable if the following conditiond holds:

$$\Big(2\frac{\hbar\Delta t}{\Delta x^2}\big(1 + \frac{\Delta x^2}{2\hbar^2}V\big)\Big)^2 \leq 1$$

This scheme was used and adapted by Kelley [Kel65] to solve (7) in order to compute the increase in intensity of the a light beam with travelled axial distance due to self-focusing effects. However, the scheme cannot achieve the preservation of the L^2- norm when $V \equiv 0$; nor the energy of the numerical solution. This issue was solved by a Crank-Nicolson scheme proposed in Delfour, Fortin and Payre [DFP81], which solves (8) while preserving the L^2-norm and the energy. The numerical solution is computed by the following expression

$$i\frac{u_j^{n+1} - u_j^n}{\Delta t} + a_1\frac{u_{j-1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j+1}^{n+\frac{1}{2}}}{\Delta x^2} + a_2\frac{|u_j^{n+1}|^2 + |u_j^n|^2}{2}|u_j^{n+\frac{1}{2}}|^2 = 0 \tag{16}$$

where $u^{n+\frac{1}{2}} = \frac{1}{2}(u^{n+1} + u^n)$. Because the scheme is nonlinear, a fixed-point problem must be solved in each iteration. Nevertheless, its conservation properties makes it one of the most succesfull schemes proposed. Akrivis [Akr93] proved existence and uniqueness of numerical solutions for scheme (16). He also proved the local error estimate to be of order 2 in time and space variables for Δt small enough.

Conservation properties are a key feature when proposing a numerical scheme for NLS equations. Sanz-Serna and Verwer [SSV86] tested five numerical schemes for equation (7) under three initial conditions: a soliton, a solliton collision, and bounded-state solitons. A Crank-Nicolson scheme preserving the discrete L^2- norm, proposed in Verwer and Sanz-Serna [VSS84], was the one that performed better in comparison to an explicit scheme, a pseudo-linear conservative scheme, and two splitting schemes. In particular: the explicit scheme exhibits blow-up solutions for a single-soliton case, while the splitting schemes fail to replicate the collision of solitons due to the dominance of numerical dissipation over the self-focusing effect. Similar successfull results are obtained a relaxation scheme proposed by Besse [Bes04], which also preserves the numerical L^2- norm and the energy.

In Gao, Li, Liu and Wei [GZD13], a Finite Volume scheme was proposed to solve (7). The remaining derivatives were approximated using a Compact Finite Difference method [Lel92], which can handle arbitrary error estimates for the numerical solutions. No conservation properties were proved for the scheme; nevertheless, they managed to control numerical diffusion in their examples. Galerkin methods were also proposed to solve (7): the scheme proposed in Xu and Shu [XS05] can control the error and the numerical dissipation of the L^2 norm, but it cannot preserve it; while in Lu, Huang and Liu [LHL15] its proposal can preserve the discrete L^2- norm, but not the energy. Both schemes can be extended to the two dimensional case. Another attempts to solve (7) considered the Finite Element Method approach ([Zou01]), Spectral methods ([FFJS82]), Splitting methods ([Gra07], [BJM02]), multigrid and adaptive algorithms [CW90], among others.

Regarding numerical approximations of the HNLS equation (13), the literature is much less abundant than the NLS case. Smadi and Bahloul ([SB11], [SB15]) attempted to solve the problem with an extra fourth-order dispersion term using Compact Finite Differences scheme with split step. No conservation property or convergence of the numerical solution is proved. Furthermore: we will show in Chapter 1 that the scheme has some difficulties when dealing with some situations different from the travelling soliton.

From this lack of work in the field, it is clear that the formulation of a numerical scheme which solves (13) is a challenge in itself. The challenge is further increased if we want a scheme that achieves the numerical L^2- norm and energy preservation. From our knowledge, there is no numerical scheme that replicates, for the HNLS equation, the success achieved by Delfour, Fortin and Payre in their numerical proposal for the NLS equation. In addition, there are no detailed results for what happens when a damping function is present. Chapter 1 of this dissertation tries to fill one of these voids, where we will propose a Finite Difference scheme for equation (13) which not only preserves the discrete L^2- norm, but also can control the Energy. These results were published in the following article:

- M. M. Cavalcanti, W. J. Corrêa, M. Sepúlveda, R. Véjar Asem: *Finite Difference Scheme for a High Order Nonlinear Schrödinger Equation.* To appear in Calcolo, 2019.

Next, in Chapter 2, we will modify the scheme to replicate another novel result: exponential decay for the L^2 norm when a localized damping is present. This decay not only is proved, but it is also obtained from numerical experiments. Some of those results were published in the following article

- M. M. Cavalcanti, W. J. Corrêa, M. Sepúlveda, R. Véjar Asem: *Finite Difference Scheme for a High Order Nonlinear Schrödinger Equation with Localized Damping.* Studia Universitatis Babes-Bolyai, Mathematica, Vol. 64 (2019), No. 2.

Chapter 3 replicates an extension of these stabilization results to 2D through a Finite Volume scheme for equation (11) using a polygonal mesh. These results were published in the following pre-print:

- M. M. Cavalcanti, W. J. Corrêa, T. Özsari, M. Sepúlveda, R. Véjar Asem: *Exponential Stability for the Nonlinear Schrödinger Equation with Locally Distributed Damping.* Preprint, Departamento de Ingeniería Matemática, Universidad de Concepción, 2019.
`ftp://ftp.ci2ma.udec.cl/pub/ci2ma/pre-publicaciones/2019/pp19-14.pdf`

The Hanging Bridge Problem

Chapter 4 presents some new stabilization results regarding a hanging bridge problem, which will be presented in the following pages.

The Governing Equation

The last chapter of this thesis deals with a different topic: the modelling of a hanging bridge, which is given by a plate subject to a reasonable set of boundary conditions. The study of hanging bridges gain a significant impulse after the failure of the Tacoma Narrows Bridge in 1940. One of the reasons this problem is challenging is because the bridge can oscillate greatly with winds of moderate strength, while it can show no oscillation at all after strong winds [LM90]. The study of the collapse of the Tacoma Narrows Bridge shows also the formation of torsional oscillations after a long period of vertical oscillation, and shortly before its final collapse. McKenna and Walter [MW90] proved the existance of travelling wave solutions for this situation, motivated from a report written after the storms that affected the Golden Gate bridge in 1938 and 1941, entitled *Observations of motions of Golden Gate wind storms of February 9, 1938 and February 11, 1941.* Thus, any attempt to model this kind of situation must consider the possibility of transversal and torsional oscillations, as well as generation of oscillations using weak external forces. Many big bridges also use dampers to prevent further oscillations for the tresspassing vehicules.

Fig. 1: The Clifton Suspension Bridge in Bristol, UK.
`https://en.wikipedia.org/wiki/Clifton_Suspension_Bridge`

This problem can be approximated by long rectangular plate anchored at both ends, while the sides remain free to move but under the action of dampers in order to improve stability. To this end, vertical hangers are attached to the sides of the bridge, which in turn are connected via a horizontal cable anchored at both ends of the bridge. One of the numerous examples can be seen in the case of the Clifton Bridge shown in Figure 1.

We will consider a static rectangular plate at its initial state, whose domain can be given as $\Omega := [x_0, x_f]\times[y_0, y_f]$ for some real parameters x_0, x_f, y_0, y_f. It is assumed that the thickness of the plate is not altered during its deformation; and thus, we can neglect it in our calculations. After the proposals of Kirchhoff [Kir76], Lazer and McKenna [LM90], McKenna and Walter [MW87], and Al-Gwaiz et al [AGBG14], we will consider the model proposed in Gazzola [Gaz13], which starts from a Kirchoff-Love plate bending model, and adds the corresponding effects like external forces, vertical dampers, and internal friction and torsional forces. That model reads as follows:

$$\begin{cases} u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + \phi(u)u_{xx} + a(x,y)g(u_t(x,y,t)) = f(x,y,t), & \text{in } \ \Omega \times (0,+\infty), \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), & \text{in } \ \Omega, \end{cases} \tag{17}$$

where the function $u(x, y, t)$ denotes the amplitude of the oscillation of the plate at the point (x, y) and at the instant t; the function $\phi(u)$ carries a nonlinear effect to the model over the x coordinate and was first proposed by Kirchhoff in 1876 [Kir76] (see algo: [FGdS16]). It is given by

$$\phi(u) = -P + S \int_{\Omega} u_x^2 dx$$

where the parameter S depends on the elasticity of the material composing the plate, and $S \int_{\Omega} u_x^2 dx$ measures the geometric nonlinearity of the plate due to its stretching; the parameter P is the prestressing constant: when $P > 0$ the plate is compressed, while for $P < 0$ the plate is stretched [AGBG14].

The constant σ is called the Poisson Ratio, and it is related to the Poisson Effect where the material expands/contracts perpendicularly to the direction of compression. $f(x, y, t)$ represents the external force such as the wind or a seismic wave.

The function $a(x, y)g(u_t)$ carries the damping effect. It is responsible for the dissipation of energy from the structure, and it is of critical importance when the structure itself is oscillating. The bridge can vibrate thanks to vehicle/pedestrian transit, external wind, or seismic waves; among other factors. Furthermore: those oscillations can be larger if the bridge enters in resonance; this is, the external force transfers energy to the bridge in such a way that reinforces the oscillations along the bridge. Given that those factors are of common appreance, and because the resonance effect can lead to the collapse of the structure, it is important to consider special arrangements or devices along the bridge such that they can dissipate the energy contained in the oscillations; with that, we can prevent the collapse of the bridge if the oscillations are too big.

Because we are modelling a bridge, the plate is assumed to have two opposing edges much longer than the others; this is $|x_f - x_0| \gg |y_f - y_0|$. For simplicity, we assume that the bridge has a long equal to π, and has a wide equal to $2l$. Frequently, we have $l \approx \frac{\pi}{200}$. Thus, we will use $x_0 = 0$, $x_f = \pi$, and $y_f = -y_0 = l$.

The boundary conditions of this problem can be explained from two facts. One of those is that we are assuming the vibrating plate is such that two of its extremes are *simply supported*; this is, those two edges (here: x_0 and x_f) will remain sticked to ground level, and no bending moment will act over them. The other two edges located at y_0 and y_f will be considered as free; in those regions the edges will not be contracted, nor expanded towards the inner part of the bridge, and will not be twisted or bent over the respective edge. Kirchhoff made a modification to this condition, by adding the assumption that the conditions imposed on the twisting moment and the shear force applied over the free edges are not independent from each other. This leads to the two boundary conditions imposed for $y_0 = -l$ and $y_f = l$. A more detailed deduction of these boundary conditions can be seen in Ventsel and Krauthammer ([VK01]).

For the conservative case (i.e. $\phi(u) \equiv 0$, $a(x, y) \equiv 0$, and $f(x, y, t) \equiv 0$), the energy of the system is defined by

$$E(t) := \frac{1}{2}||u_t(t)||_{L^2}^2 + \frac{1}{2}||u(t)||_{H_*^2}^2$$

where it remains preserved $\forall t \geq 0$. The space H^2_* is defined as follows

$$H^2_*(\Omega) = \{w \in H^2(\Omega) : w = 0 \text{ on } \{0, \pi\} \times (-l, l)\}$$

which in turn is accompanied by the following inner product

$$(u, v)_{H^2_*} = \int_\Omega F(u, v) dxdy$$

where

$$F(u, v) = u_{xx}v_{xx} + u_{yy}v_{yy} + \sigma(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\sigma)u_{xy}v_{xy}.$$

The following Lemma was proved by Messaoudi and Mukiawa in [MM15]

Lemma 0.0.1. *For $v \in H^2_*(\Omega)$, and $\forall u \in H^4(\Omega) \cap H^2_*(\Omega)$ such that*

$$\begin{cases} u_{xx}(0, y) = u_{xx}(\pi, y) = 0 \\ u_{yy}(x, \pm l) + \sigma(u_{xx}(x, \pm l) = 0 \\ u_{yyy}(x, \pm l) + (2-\sigma)u_{xxy}(x, \pm l) = 0 \end{cases}$$

then the following identity holds

$$(\Delta^2 u, v)_{L^2(\Omega)} = (u, v)_{H^2_*(\Omega)} = \int_\Omega F(u, v) dxdy$$

Ferrero and Gazzola [FG15] studied the case when an external force, a restoring force h due to the action of hangers, and a global damping term $\delta \in \mathbb{R}$, are present:

$$\begin{cases} u_{tt}(x, y, t) + \Delta^2 u(x, y, t) + h(x, y, u) + \delta u_t(x, y, t) = f(x, y, t), \text{ in } \Omega \times (0, +\infty), \\ u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, \quad (y, t) \in (-l, l) \times (0, +\infty), \\ u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = 0, \quad (x, t) \in (0, \pi) \times (0, +\infty), \\ u_{yyy}(x, \pm l, t) + (2-\sigma)u_{xxy}(x, \pm l, t) = 0, \quad (x, t) \in (0, \pi) \times (0, +\infty), \\ u(x, y, 0) = u_0(x, y), \ u_t(x, y, 0) = u_1(x, y), \quad \text{in } \Omega, \end{cases} \tag{18}$$

where the following energy functional is defined

$$\mathbb{E}_T(u) = \int_\Omega \Big(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + H(x, y, u) - fu\Big) dxdy$$

where $H(x, y, s) := \int_s^s h(x, y, \tau)d\tau$, $s \in \mathbb{R}$. The following result holds

Theorem 0.0.2. *For $\sigma \in (0, \frac{1}{2})$, and for $f \in \mathcal{H}(\Omega) :=$ the dual space of $H^2_*(\Omega)$, then there exists a unique $u \in H^2_*(\Omega)$ such that*

$$\int_\Omega \Big[\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\Big] dxdy = \langle f, v\rangle, \quad \forall v \in H^2_*(\Omega).$$

Moreover: u is the minimum point of the convex functions $\mathbb{E}_T$. If $f \in L^2(\Omega)$, then $u \in H^4(\Omega)$; and if $u \in C^4(\overline{\Omega})$, then u is a classical solution of the static problem

$$\begin{cases} \Delta^2 u(x, y, t) = f(x, y, t), \ \text{in } \Omega, \\ u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, \quad (y, t) \in (-l, l) \times (0, +\infty), \\ u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = 0, \quad (x, t) \in (0, \pi) \times (0, +\infty), \\ u_{yyy}(x, \pm l, t) + (2-\sigma)u_{xxy}(x, \pm l, t) = 0, \quad (x, t) \in (0, \pi) \times (0, +\infty), \end{cases} \tag{19}$$

This is, both the conservative case and equation (18) admit a variational formulation; a key element when attempting to implement a Finite Element scheme for this problem. Solutions for the static case (19) has been found in [FG15]; for the case with full damping, solutions were computed in [AGBG14].

In the last years, numerous authors have studied well-posedness and time evolution of the energy on different situations. For instance: Bochicchio et al. [BGV10] considered a similar model using full damping over the domain. They managed to prove well-posedness and the existence of a global attractor. Messaoudi and Mukiawa [MM15] proved an exponential stability result for Problem (17) using full damping. Gazzola et al. [FGdS16] proved existence, uniqueness and asymptotic behavior for the solutions for all initial data in suitable functional spaces for the same model (17) using a full and constant damping.

It is worth of mention that the function $a(x, y)$, responsible for the location of the damping, should not be acting over the full domain, and its behavior could not be linear. This is why, in (17), we will consider the damping function $a : a(x, y) > 0$ for $(x, y) \in \omega$ such that $\omega \cap \Gamma \neq \emptyset$. The function $g(u_t)$, which could be linear or not linear, describes the behavior of the damping, and it can be due to special structures like dampers or internal friction effects due to different material composition over ω. We will consider a nonlinear function because it has been empirically proved that the damping depends nonlinearly on the oscillation amplitude and the velocity ([GZD13], [HS96]).

In Cavalcanti et al. [DCMCC$^+$], a new stability result was obtained for the model (17) using a minimal damping $a(x, y)$. In this regard, in Chapter 4 we will propose a Finite Difference scheme that solves this hanging bridge problem; where at the same time, numerically replicates the exponential decay of the energy that is theoretically proved. These results can be revised in the following preprint:

- A. D. Domingos Cavalcanti, M. Moreira Cavalcanti, W. J. Corrêa, Zayd Hajjej, M. Sepúlveda Cortés, R. Véjar Asem: *Uniform decay rates for a suspension bridge with locally distributed nonlinear damping.* Preprint, Departamento de Ingeniería Matemática, Universidad de Concepción, 2018.
  `ftp://ftp.ci2ma.udec.cl/pub/ci2ma/pre-publicaciones/2018/pp18-47.pdf`

Regarding the numerical modellation of Problem (17), the literature that deals with the problem in particular is rather scarce. For the PDE itself: one of the most traditional boundary value problems involving the biharmonic operator is given by

$$\begin{cases} \Delta^2 u(x, y) = f(x, y), \ (x, y) \in \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = g(x, y), \quad \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = h(x, y) \end{cases}$$

for some functions g and h defined over the boundary $\partial\Omega$. Most of the Finite Element software can solve the problem with ease[7] by using an auxiliar variable $z(x, y) = \Delta u(x, y)$, and solving

[7]See, for instante: `https://www.um.es/freefem/ff++/pmwiki.php?n=Main.ComputingTheBilaplacian`.

two coupled Poisson problems instead; idea which originally came from Ciarlet and Raviart [CR74] where they solved the problem using a mixed method. Other authors (see: [BOP80], [Mon87], [BG11]) kept studying the problem from a Finite Element perspective. Discontinuous Galerkin methods were also proposed to solve the biharmonic problem with Dirichlet boundary conditions (see: [GH09], [CDG09], [Bar18]). For the suspension bridge case, Preidikman and Mook [PM97] implemented a predictor-corrector method which solves the Euler-Lagrange equations of motion in one dimension; while Arena and Lacarbonara [AL12] solved for the shear and stress forces, which leads to a coupled problem of order 1 in space and order 2 in time.

Up to our knowledge, there are no other results regarding a numerical approximation of Problem (17) with any existing numerical method.

Chapter 1

Finite Difference Scheme for a conservative HNLS Equation.

1.1 Introduction

This chapter is devoted to the numerical analysis of the following HNLS equation

$$iu_t + a_1 u_{xx} + ia_2 u_{xxx} + a_3|u|^2 u + ia_4|u|^2 u_x + ia_5 u|u|_x^2 = 0 \tag{1.1}$$

where we will consider $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$, and $u = u(x,t)$, $x, t \in \mathbb{R}$ a complex valued function.

$$iu_t + a_1 u_{xx} + a_3|u|^2 u = 0 \tag{1.2}$$

Carvajal proved in [Car06] for a_3, $a_5 \neq 0$ the global well-posedness of the Cauchy Problem given by the equation (1.1), and an initial condition $u(x,0) = u_0(x)$ in $H^s(\mathbb{R}), s > \frac{1}{4}$ when $3a_2a_3 = a_1a_4$. Meanwhile, Takaoka proved in [Tak00], for $a_3 = 1$, the local well-posedness for the Cauchy Problem (1.1) in $H^s(\mathbb{T})$, $s > \frac{1}{2}$, where $\mathbb{T}$ is a unidimensional torus. Similar conclusions were obtained also by Takaoka in [Tak99] for $a_3 = 0$, where the well-posedness is over $H^{\frac{1}{2}}(\mathbb{R})$. Regularity properties were studied by Alves et al. [ASV09] when $\delta = \epsilon = 0$.

Several works regarding the equation (1.1), concerning the well-posedness and analytic behaviors of the solution are studied in the whole real line, or in the torus (for periodic boundary conditions) when it comes to bounded domains. In this chapter, we will focus our interest in the following initial value problem for equation (1.1), with null boundary conditions in a bounded domain $\Omega := [x_0, x_f] \subset \mathbb{R}$:

$$\begin{cases} iu_t + a_1 u_{xx} + a_2|u|^2 u + ia_3 u_{xxx} + ia_4|u|^2 u_x + ia_5 u|u|_x^2 = 0 \ \text{ in } \Omega \times (0,T) \subset \mathbb{R}^2 \\ u(x_0,t) = u(x_f,t) = 0, \quad u_x(x_f,t) = 0, \ t \geq 0 \\ u(x,0) = u_0 \text{ for } x \in \Omega \end{cases} \tag{1.3}$$

Null-boundary conditions like (1.3) are generally used in dispersive equations with third-order derivatives such as KdV ([BSZ03]), and not commonly in Schrodinger or HNLS. However, there are some works, such as for the Sasa-Satusma equation in that if boundary conditions of this type are considered in bounded domains [XZF18].

Exact solutions for (1.1) can be found using the Inverse Scattering Transform (IST) [AS81], proposed originally in Zakharov et al. [ZS72]. Its integration depends on the values of a_2, a_4 and a_5. In particular: for $a_1 = \frac{1}{2}$, $a_3 = 1$, and rewriting equation (1.1) as

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + i\varepsilon(\beta_1 u_{xxx} + \beta_2|u|^2 u_x + \beta_3|u|_x^2 u) = 0 \tag{1.4}$$

for some real constants $\beta_1, \beta_2, \beta_3, \varepsilon$; then exact solutions can be obtained via IST for the following cases:

- For the derivative NLS equation of type I: $\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1$ [AL83].
- For the derivative NLS equation of type II: $\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0$ [CLL79].
- For the Hirota equation: $\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 0$ [Hir73a].
- For the Sasa-Satsuma equation: $\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$ [SS91].

Exact solutions are all of solitonic form. N-soliton solutions can also be obtained [Hir73a]. Potasek [PT91b] shows some particular solutions that has been proven experimentally. But even when continuous solutions can be found for some specific initial conditions and some values for the real constants in (1.1), numerical solutions can prescinde from those requirements when computed. We can even use non-solitonic initial conditions in order to obtain a result. One way to compute numerical solutions is using the Finite Difference Method, whose computational implementation can be done in an fast and efficient way.

Other ways to obtain numerical solutions for (1.1) has been studied by different authors in the recents years. One of the first scheme were proposed by Delfour, Fortin and Payre [DFP81], which solves the NLS equation (1.2) proposing a rule to discretize powers of the nonlinearity multiplying the a_2 term. Their method has a strong property: it preserves the discrete versions of both the L^2 norm and the energy of the numerical solution, where their continuous versions are given by the following relations:

$$||u||^2_{L^2(\Omega)}(t) = \int_\Omega |u(x,t)|^2 dx$$
$$E(t) := \frac{a_1}{2} \int_\Omega |\nabla u(x,t)|^2 dx - \frac{a_3}{4} \int_\Omega |u(x,t)|^4 dx$$

for $u = u(x,t) \in \Omega \subset \mathbb{R} \times \mathbb{R}^+ \longmapsto \mathbb{C}$ the exact solution of (1.1). The convergence of the numerical method is proved in Matsuo and Furihata [FM11]. Pazoto et al [PSV10] proposed a finite difference scheme which solves the critical generalilzed Kortewetg-de Vries equation (GKdV-4) in a bounded domain. The higher-power term $u^4 u_x$ was rewritten as a linear combination of other derivatives in order to obtain specific conservation properties. Smadi and Bahloul [SB11] [SB15] combined a Compact Padé Finite Difference scheme [Lel92] with a fourth order Runge-Kutta (RK4) scheme. They splitted the problem in two parts: a linear section which is solved using the finite difference scheme; and the nonlinear, which is solved using the RK4 scheme. The method was implemented with an interesting success, but no analysis of the error, convergence, or preserved quantities was made.

The purpose of this chapter then, is to search for numerical solutions of the IVP (1.3) using a Finite Difference scheme which preserves the numerical L^2 norm and controls the energy.

1.2 Well posedness of weak and strong solutions

Multiplying (1.3) by $\overline{v}$ with $v \in H_0^2(\Omega) = \{w \in H^2(\Omega) \mid w(x_0) = w(x_f) = w_x(x_0) = w_x(x_f) = 0\}$, and integrating by part, we have

$$\begin{aligned} i < u_t; v > - a_1 \int_\Omega u_x \overline{v}_x \, dx + a_2 i \int_\Omega u_x \overline{v}_{xx} \, dx + a_3 \int_\Omega |u|^2 u \overline{v} \, dx \\ + i a_4 \int_\Omega |u|^2 u_x \overline{v} \, dx - i a_5 \int_\Omega |u|^2 \left(u_x \overline{v} + u \overline{v}_x \right) \, dx = 0. \end{aligned} \tag{1.5}$$

Definition 1.2.1. *$u \in L^2(0,T; H_0^1(\Omega))$, with $u_t \in L^2(0,T; H^{-2}(\Omega))$ is called weak solution of (1.3) if it verifies (1.5), for all $v \in H_0^2(\Omega)$.*

We have the following result of existence and uniqueness for the problem (1.3)

Theorem 1.2.2. *Let $u_0 \in H_0^1(\Omega)$. Then, there exists $T > 0$ and a weak solution of (1.3) in $L^2(0,T; H_0^1(\Omega))$. Moreover, if $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, then there exists a unique strong solution of (1.3), such that*

$u \in L^\infty(0,T; H^3(\Omega))$

$u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$

Proof. The existence of a weak solution is result of the convergence of the numerical solution and passing to the limits in the proof of Theorem 2.2.2 (see later in Section 4). In order to prove the existence and uniqueness of the strong solution, we consider the gauge transformation [Car06]:

$$v(x,t) = \exp(i\lambda x - i(a_1\lambda + 2a_2\lambda^3)t)\, u(x - (2a_1\lambda + 3a_2\lambda)t, t)$$

Then u solves (1.3) if and only if v satisfies the system

$$\begin{cases} iv_t + (a_1 + 3\lambda a_2)v_{xx} + ia_2 v_{xxx} + (a_3 + \lambda a_4)|v|^2 v \\ \qquad\qquad + ia_4|v|^2 v_x + ia_5 v|v|_x^2 = 0, & (x,t) \in Q_t, \\ v(x_0 + (2a_1\lambda + 3a_2\lambda)t, t) = v(x_f + (2a_1\lambda + 3a_2\lambda)t, t) = 0, & t \geq 0, \\ v_x(x_f + (2a_1\lambda + 3a_2\lambda)t, t) = 0, & t \geq 0, \\ v(x,0) = \exp(i\lambda x)u_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

with $Q_t = \{(x,t) \in \mathbb{R}^2 \mid x_0 < x - (2a_1\lambda + 3a_2\lambda)t < x_f\}$. Thus, if we take $\lambda = -\dfrac{a_1}{3a_2}$ and $a_3 = \dfrac{a_1 a_4}{3a_2}$, the function $v(x,t)$ satisfies a complex modified Korteweg-de Vries equation with moving boundaries. The result of existence and uniqueness of strong solution for KdV equation in domains with moving boundaries proved in [DL07] can be easily generalized for the complex modified Korteweg-de Vries equation (1.6) with $\lambda = -\dfrac{a_1}{3a_2}$ and $a_3 = \dfrac{a_1 a_4}{3a_2}$. Moreover, if $a_3 \neq \dfrac{a_1 a_4}{3a_2}$, the a priori estimates and the Faedo-Galerkin method argument of the proof in [DL07, Theorem 1] is also valid, with which Theorem 1.2.2 is proved. □

1.3 Numerical Scheme

1.3.1 Notation.

For the sake of the following analysis, and for a given $M \in \mathbb{N}$, we will introduce the vector space

$$X_M := \left\{u = [u_0\, u_1\, \dots\, u_M]^T \in \mathbb{C}^{M+1} : u_0 = u_{M-1} = u_M = 0\right\}$$

Let us introduce the classical finite differences operators for complex-valued arrays:

$$\left[\boldsymbol{D}^+ u\right]_j := \frac{u_{j+1} - u_j}{\Delta x}, \qquad \left[\boldsymbol{D}^- u\right]_j := \frac{u_j - u_{j-1}}{\Delta x},$$
$$\boldsymbol{D}u := \frac{1}{2}\Big(\boldsymbol{D}^+ u + \boldsymbol{D}^- u\Big),$$
$$\boldsymbol{D}^2 u := \boldsymbol{D}^+\boldsymbol{D}^- u, \qquad \boldsymbol{D}^3 u := \boldsymbol{D}\boldsymbol{D}^+\boldsymbol{D}^- u.$$

For $u, v \in X_M$, $L := x_f - x_0$, and $\Delta x := \frac{L}{M+1}$; we will make use of the following inner product and their respective norm

$$(u, v)_2 := \sum_{j=0}^{M} u_j \overline{v}_j \Delta x, \qquad ||u||_2^2 := (u, u)_2. \tag{1.7}$$

For $p \in [1, \infty]$, we will use other similar norms for u:

$$||u||_p := \left(\sum_{j=0}^{M} |u_j|^p \Delta x\right)^{\frac{1}{p}} \quad (p < \infty), \qquad\qquad ||u||_\infty := \max_{j\in[0,M]} |u_j|$$

For $u, v \in X_M$, we will introduce the following inner product and their respective norm:

$$(u, v)_x := \sum_{j=0}^{M} j\Delta x^2 u_j \overline{v}_j, \qquad ||u||_x^2 := (u, u)_x \tag{1.8}$$

1.3.2 Fundaments of the Numerical Scheme.

In order to construct the numerical method, we will write a similar form of equation (1.1). The modification of equation (1.1) will proceed by adding and substracting the same nonlinear term as follows:

$$iu_t + a_1 u_{xx} + a_3|u|^2 u + ia_2 u_{xxx} + ia_4\Big(|u|^2 u_x + (|u|^2 u)_x\Big) + ia_5 u|u|_x^2 - ia_4(|u|^2 u)_x = 0 \tag{1.9}$$

For the time being, we will not discretize this expression directly. We will focus instead on the last term in (1.9). For a sufficiently differentiable function $u(x, t) : \mathbb{R}^2 \longrightarrow \mathbb{C}$, we can write $(|u|^2 u)_x$ as a convex combination of itself, $(|u|_x^2 u + |u|^2 u_x)$ and $(u^2 \overline{u_x} + 2|u|^2 u_x)$. Hence,

$$\frac{\partial}{\partial x}\left(|u|^2 u\right) = \alpha_0 \frac{\partial}{\partial x}\left(|u|^2 u\right) + \beta_0\left(u^2 \frac{\partial \overline{u}}{\partial x} + 2|u|^2 \frac{\partial u}{\partial x}\right) + (1 - \alpha_0 - \beta_0)\left(u\frac{\partial}{\partial x}(|u|^2) + |u|^2 \frac{\partial u}{\partial x}\right) \tag{1.10}$$

where $\alpha_0, \beta_0 \in [0,1]$. Using (1.10), equation (1.9) can be written as

$$\begin{aligned} &iu_t + a_1 u_{xx} + a_3|u|^2u + ia_2u_{xxx} + ia_4\Big(|u|^2u_x + (|u|^2u)_x\Big) + ia_5u|u|^2_x \\ -ia_4\Big[&\alpha_0\left(|u|^2u\right)_x + \beta_0\left(u^2\overline{u}_x + 2|u|^2u_x\right) + (1-\alpha_0-\beta_0)\left(u|u|^2_x + |u|^2u_x\right)\Big] = 0 \end{aligned} \tag{1.11}$$

and using $\alpha_0 = \frac{3}{4} - \frac{\gamma_0}{2}$ and $\beta_0 = \frac{1}{4} - \frac{\gamma_0}{2}$ for $\gamma_0 \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned} &iu_t + a_1 u_{xx} + a_3|u|^2u + ia_2u_{xxx} + ia_4\Big(|u|^2u_x + (|u|^2u)_x\Big) + ia_5u|u|^2_x \\ &-ia_4\Big[(\tfrac{3}{4} - \tfrac{\gamma_0}{2})\left(|u|^2u\right)_x + (\tfrac{1}{4} - \tfrac{\gamma_0}{2})u^2\overline{u}_x + \tfrac{1}{2}|u|^2u_x + \gamma_0|u|^2_xu\Big] = 0 \end{aligned} \tag{1.12}$$

Which will be the expression to discretize using the finite differences, whose coefficients were choosen in order to help the preservation of the L^2 norm of the numerical solution. At this point, it is straightforward to write $u_j^n \approx u(x_j, t_n)$; this is, the approximation of the exact solution $u(x,t)$ at the time $t_n = n\Delta t$ and at the coordinate $x_j = j\Delta x$. We also write $u_j^{n+\frac{1}{2}} := \frac{1}{2}\left(u_j^{n+1} + u_j^n\right)$. Using the notation already presented, we will define the approximations that will lead us to the numerical scheme. The time derivative will be discretized using the forward finite difference quotient in time:

$$\frac{\partial u}{\partial t}(x_j, t_n) \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} = D_t(u_j^n)$$

The terms multiplied by a_1 and a_2 are discretized as follows:

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) \approx D^2(u_j^{n+\frac{1}{2}}), \quad \text{and} \quad \frac{\partial^3 u}{\partial x^3}(x_j, t_n) \approx D^3(u_j^{n+\frac{1}{2}})$$

The discretization of the term multiplied by a_3 will be given by:

$$|u(x_j, t_n)|^2 u(x_j, t_n) \approx |u_j^{n+\frac{1}{2}}|^2 \left(u_j^{n+\frac{1}{2}}\right)$$

For the terms multiplied by a_4 and $-a_4$, we write:

$$\begin{aligned} &|u(x_j,t_n)|^2\left(u(x_j,t_n)\right)_x + \Big(|u(x_j,t_n)|^2u(x_j,t_n)\Big)_x - \Big(|u(x_j,t_n)|^2u(x_j,t_n)\Big)_x \\ &\approx (\alpha_0 - \beta_0)|u^{n+\frac{1}{2}}|^2 D\Big(u^{n+\frac{1}{2}}\Big) + (1-\alpha_0)D\Big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}\Big) \\ &- \beta_0(u^{n+\frac{1}{2}})^2 D\Big(\overline{u^{n+\frac{1}{2}}}\Big) - (1-\alpha_0-\beta_0)D\Big(|u^{n+\frac{1}{2}}|\Big)^2 u^{n+\frac{1}{2}} \end{aligned}$$

using $\alpha_0 = \frac{3}{4} - \frac{\gamma_0}{2}$, $\beta_0 = \frac{1}{4} - \frac{\gamma_0}{2}$, and $\gamma_0 \in \left[0, \frac{1}{2}\right]$, we define the operator $F_{a_4} : \mathbb{C}^{\mathbb{Z}} \longrightarrow \mathbb{C}^{\mathbb{Z}}$

$$\begin{aligned} [F_{a_4}(u^{(p)})]_j := &\frac{1}{2}\left|\frac{u_j^p + u_j^n}{2}\right|^2 D\left(\frac{u_j^p + u_j^n}{2}\right) + \left(\frac{1}{4} + \frac{\gamma_0}{2}\right) D\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2 \frac{u_j^p + u_j^n}{2}\right) \\ &+ \left(\frac{\gamma_0}{2} - \frac{1}{4}\right)\left(\frac{u_j^p + u_j^n}{2}\right)^2 D\left(\overline{\frac{u_j^p + u_j^n}{2}}\right) - \gamma_0\frac{u_j^p + u_j^n}{2} D\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2\right) \end{aligned} \tag{1.13}$$

The ϵ term will be discretized directly:

$$u(x_j, t_n)\Big|u(x_j, t_n)\Big|_x^2 \approx u_j^{n+\frac{1}{2}} D\left(|u_j^{n+\frac{1}{2}}|^2\right)$$

where we define its representing function $F_{a_5} : \mathbb{C}^{\mathbb{Z}} \longrightarrow \mathbb{C}^{\mathbb{Z}}$

$$[F_{a_5}(u^{(p)})]_j := \left(\frac{u_j^p + u_j^n}{2}\right) D\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2\right)$$

Hence, $\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, and for a given $u^0 \in X_M$ the numerical scheme will be given component-wise by

$$\begin{cases} iD_t u_j^n + a_1 D^2(u_j^{n+\frac{1}{2}}) + ia_2 D^3(u_j^{n+\frac{1}{2}}) + a_3 |u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} \\ \qquad\qquad + ia_4[F_{a_4}(u^{(n+1)})]_j + ia_5[F_{a_5}(u^{(n+1)})]_j = 0 \\ u^n \in X_M, \qquad n \in \mathbb{N} \end{cases} \tag{1.14}$$

where the initial condition u^0 is such that $u_j^0 = u_0(x_j)$, for $u_0 \in L^2(\Omega)$ an initial condition of Problem (1.3).

1.4 Properties of the scheme

1.4.1 Behavior of the numerical L^2-norm

This numerical scheme was designed to control the numerical dissipation of the L^2-norm. In order to demonstrate that property, we will start by proving the next lemma

Lemma 1.4.1. *$\forall \varphi \in X_M$, we have*

$$Im(\boldsymbol{D}^2\varphi, \varphi)_2 = 0 \tag{1.15}$$

$$Re(\boldsymbol{D}^3\varphi, \varphi)_2 = 0 \tag{1.16}$$

$$Re(|\varphi|^2\boldsymbol{D}\varphi + \boldsymbol{D}\left(|\varphi|^2\varphi\right), \varphi)_2 = 0 \tag{1.17}$$

$$(\varphi^2\boldsymbol{D}\overline{\varphi} + \boldsymbol{D}(|\varphi|^2\varphi), \varphi)_2 = 0 \tag{1.18}$$

$$(\varphi\boldsymbol{D}|\varphi|^2, \varphi)_2 = 0 \tag{1.19}$$

Proof. To get (1.15), note that

$$[\boldsymbol{D}^2(\varphi)]_j = D^2(\varphi_j) = \frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{\Delta x^2} = \frac{1}{\Delta x}\left(D_+\varphi_j - D_-\varphi_j\right)$$

Which can be extended to the other elements of $\boldsymbol{D}^2(\varphi)$. Hence, we have.

$$(\boldsymbol{D}^2(\varphi), \varphi)_2 = \frac{1}{\Delta x}(\boldsymbol{D}_+\varphi - \boldsymbol{D}_-\varphi, \varphi)_2 = \frac{1}{\Delta x}(-\overline{(\boldsymbol{D}_-\varphi, \varphi)_2} + \overline{(\boldsymbol{D}_+\varphi, \varphi)_2}) = \overline{(\boldsymbol{D}^2(\varphi), \varphi)_2}$$

from here, we get (1.15). A similar approach is used to prove (1.16). To obtain (1.17), we have

$$(\varphi^2 \boldsymbol{D}\varphi, \varphi)_2 = (|\varphi|^2\varphi, \boldsymbol{D}\overline{\varphi})_2 = -\Big(\boldsymbol{D}(|\varphi|^2\varphi), \overline{\varphi}\Big)_2$$

From here, both (1.17) and (1.18) can be concluded. A similar procedure can be applied to prove (1.19). □

Remark 1.4.2. *Lemma 1.4.1 will give us reasons to write the expression (1.9) and define the numerical scheme (1.14). From the convex combination (1.10), we want to re-write it in function of the conserved quantities of the Lemma 1.4.1. In other words, we want to write the following equality*

$$\frac{\partial}{\partial x}(|\varphi|^2\varphi) = \hat{\alpha}\left(|\varphi|^2\varphi_x + (|\varphi|^2\varphi)_x\right) + \hat{\beta}(\varphi^2\overline{\varphi}_x + (|\varphi|^2\varphi)_x) + \gamma_0(|\varphi|_x^2\varphi) \tag{1.20}$$

for some real constants $\hat{\alpha}$, $\hat{\beta}$ and γ_0 to be found. Comparing the previous expression with the convex combination (1.10), and solving the resulting linear system of equations, gives $\hat{\alpha} = \frac{1}{2}$ and $\hat{\beta} = \frac{1}{4} - \frac{\gamma_0}{2}$; or $\alpha_0 = \frac{3}{4} - \frac{\gamma_0}{2}$ and $\beta_0 = \frac{1}{4} - \frac{\gamma_0}{2}$, for $\gamma_0 \in \left[0, \frac{1}{2}\right]$. On the other hand, to simplify the calculations, we will preferably choose $\gamma_0 = 1/2$. In effect, with this value one of the terms in (1.13) is simplified, and the convergence proof of the numerical scheme also.

Theorem 1.4.3. *Let $u^0 \in X_M$. Then, $\forall n \in \mathbb{N}$, and for $u^n \in X_M$, we have*

$$||u^{n+1}||_2^2 = ||u^n||_2^2 \tag{1.21}$$

Proof. We will multiply the numerical scheme (1.14) by $\left(\overline{u}_j^{n+1} + \overline{u}_j^n\right)\Delta x$, sum for j, and extract the imaginary part. On the time derivative term, we have

$$i\sum_{j=0}^{M-1} \frac{u_j^{n+1} - u_j^n}{\Delta t}(\overline{u}_j^{n+1} + \overline{u}_j^n)\Delta x = \frac{i}{\Delta t}\sum_{j=0}^{M-1}\left(|u_j^{n+1}|^2 - |u_j^n|^2 + 2iIm(u_j^{n+1}\overline{u}_j^n)\right)\Delta x \tag{1.22}$$

and thus,

$$Im\Big(i\boldsymbol{D}_t u^n, u^{n+\frac{1}{2}}\Big)_2 = \frac{1}{\Delta t}\Big(||u^{n+1}||_2^2 - ||u^n||_2^2\Big) \tag{1.23}$$

The Theorem can be then concluded using the results obtained in Lemma 1.4.1 for $\varphi = u^{n+\frac{1}{2}}$. □

1.4.2 Convergence

Before presenting the next results, we will introduce some extension operators, presented already in [TTZ03], [PSV10]; and originally, [Lio69]. For $v \in X_M$ with $v = (v_j)_{j=0}^M$, for the space variable we define:

$$p_\Delta v_\Delta(x) = \begin{cases} \text{the continuous function, linear in each interval } [j\Delta x, (j+1)\Delta x] \\ \text{such that } p_\Delta v_\Delta(j\Delta x) = v_j, \quad j = 0, \dots, M \end{cases}$$

$$q_\Delta v_\Delta(x) = \begin{cases} \text{the step function, defined in each interval } \Big((j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x\Big) \cap (\Omega) \\ \text{such that } q_\Delta v_\Delta(j\Delta x) = v_j, \quad j = 0, \dots, M \end{cases}$$

and for the time variable, we have

$$P_\Delta v_\Delta(x,t) = \begin{cases} \text{the continuous function, linear in each interval } [n\Delta t, (n+1)\Delta t] \\ \text{such that } P_\Delta v_\Delta(x,t_n) = p_\Delta v_\Delta^n(x), \quad n \in \mathbb{N}, x \in (\Omega) \end{cases}$$

$$P_\Delta^{\frac{1}{2}} v_\Delta(x,t) = \begin{cases} \text{the continuous function, linear in each interval } [n\Delta t, (n+1)\Delta t] \\ \text{such that } P_\Delta^{\frac{1}{2}} v_\Delta(x,t_n) = \frac{1}{2}\Big(p_\Delta v_\Delta^n(x) + p_\Delta v_\Delta^{n+1}(x)\Big), \quad n \in \mathbb{N}, x \in (\Omega) \end{cases}$$

$$Q_\Delta u_\Delta(x,t) = \begin{cases} \text{the step function, linear in each interval } [n\Delta t, (n+1)\Delta t] \\ \text{such that } Q_\Delta v_\Delta(x,t_n) = q_\Delta v_\Delta^n(x), \qquad t_n \le t \le t_{n+1}, n \in \mathbb{N}, x \in (\Omega) \end{cases}$$

$$Q_\Delta^{\frac{1}{2}} u_\Delta(x,t) = \begin{cases} \text{the step function, linear in each interval } [n\Delta t, (n+1)\Delta t] \\ \text{such that } Q_\Delta^{\frac{1}{2}} v_\Delta(x,t_n) = \frac{1}{2}\Big(q_\Delta v_\Delta^n(x) + q_\Delta v_\Delta^{n+1}(x)\Big), \\ t_n \le t \le t_{n+1}, n \in \mathbb{N}, x \in (\Omega) \end{cases}$$

With this, it is easy to see that

$$\begin{aligned} ||Q_\Delta^{n+\frac{1}{2}} u_\Delta||^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_0^L |Q_\Delta^{n+\frac{1}{2}} u_\Delta(x,t)|^2 dxdt \\ &= \sum_{n=0}^{N-1} \sum_{j=0}^{M-1} |u_j^{n+\frac{1}{2}}|^2 \Delta x \Delta t = \sum_{n=0}^{N-1} ||u^{n+\frac{1}{2}}||_2^2 \Delta t \\ ||p_\Delta u_\Delta||^2_{H_0^1(\Omega)} &= \int_0^L |(p_\Delta u_\Delta)_x|^2 dx = \sum_{j=0}^{M-1} \left| \frac{u_{j+1} - u_j}{\Delta x} \right|^2 \Delta x \end{aligned}$$

In order to prove the main results of this section, we will present some lemmata:

Lemma 1.4.4. *For $u \in X_M$ such that $||u||_\infty < \infty$, we have*

$$||q_\Delta u_\Delta||^2_{L^\infty(\Omega)} \le 2||q_\Delta u_\Delta||_{L^2(\Omega)} ||p_\Delta u_\Delta||_{H_0^1(\Omega)} \tag{1.24}$$

$$||q_\Delta u_\Delta||^4_{L^4(\Omega)} \le 2||q_\Delta u_\Delta||^3_{L^2(0,T)} ||p_\Delta u_\Delta||_{H_0^1(\Omega)} \tag{1.25}$$

$$||q_\Delta u_\Delta||^6_{L^6(\Omega))} \le 4||q_\Delta u_\Delta||^4_{L^2(\Omega)} ||p_\Delta u_\Delta||^2_{H_0^1(\Omega)} \tag{1.26}$$

Proof. To prove (1.24), we will need the algebraic identity $(a^2 - b^2) + (a-b)^2 = 2a(a-b)$ for any constants $a, b \in \mathbb{C}$. For $u_i \in u = [u_0\, u_1\, \dots\, u_{M-1}\, u_M]^T \in \mathbb{C}^{M+1}$, we have:

$$\begin{aligned} u_i^2 &= \frac{1}{2}\left[\sum_{j=1}^{i} (u_j^2 - u_{j-1}^2) + \sum_{j=0}^{i-1} (u_{j+1}^2 - u_j^2) \right] \\ &= \frac{1}{2}\left[\sum_{j=1}^{i} 2u_j(u_j - u_{j-1}) - (u_j - u_{j-1})^2 \right] - \frac{1}{2}\left[\sum_{j=0}^{i-1} 2u_j(u_j - u_{j+1}) - (u_{j+1} - u_j)^2 \right] \\ &= \sum_{j=1}^{i} \left[\Delta x\, u_j D^- u_j - \frac{\Delta x^2}{2}(D^- u_j)^2 \right] + \sum_{j=0}^{i-1} \left[\Delta x\, u_j D^+ u_j + \frac{\Delta x^2}{2}(D^+ u_j)^2 \right] \\ &= \sum_{j=1}^{i} \Delta x\, u_j D^- u_j + \sum_{j=0}^{i-1} \Delta x\, u_j D^+ u_j + \sum_{j=0}^{i-1} \frac{\Delta x^2}{2}(D^+ u_j)^2 - \sum_{j=0}^{i-1} \frac{\Delta x^2}{2}(D^+ u_j)^2. \end{aligned}$$

Taking the modulus at both sides, using Hölder Inequality, and recalling that $u_0 = 0$,

$$\begin{aligned}
|u_i|^2 &\leq \sum_{j=1}^{i} \Delta x\,|u_j D^+ u_j| + \sum_{j=1}^{i-1} \Delta x\,|u_j D^+ u_j| \\
&\leq \sqrt{\sum_{j=1}^{i} \Delta x |u_j|^2}\sqrt{\sum_{j=1}^{i} \Delta x |D^- u_j|^2} + \sqrt{\sum_{j=1}^{i-1} \Delta x\,|u_j|^2}\sqrt{\sum_{j=1}^{i-1} \Delta x\,|D^+ u_j|^2} \\
&\leq 2\sqrt{\sum_{j=1}^{i} |u_j|^2 \Delta x}\sqrt{\sum_{j=1}^{i} |D^+ u_j|^2 \Delta x} \\
&\leq 2||u||_2||\boldsymbol{D}^+ u||_2.
\end{aligned}$$

Inequality (1.24) is then proved since this is valid to any $i = 0,\, 1,\, \dots,\, M$. To get (1.25), and combining Hölder Inequality with (1.24),

$$\begin{aligned}
||q_\Delta u_\Delta||^4_{L^4(\Omega)} &= \sum_{j=1}^{M-1} |u_j|^4 \Delta x \leq ||u||^2_\infty \sum_{j=1}^{M-1} |u_j|^2 \Delta x \\
&\leq 2||u||_2||\boldsymbol{D}^+ u||_2||u||^2_2 = 2||q_\Delta u_\Delta||^3_{L^2(0,T)}||p_\Delta u_\Delta||_{H^1_0(\Omega)}
\end{aligned}$$

In order to conclude (1.26), we will again use Hölder inequality with (1.24) and (1.25):

$$\begin{aligned}
||q_\Delta u_\Delta||^6_{L^6(\Omega))} &= \sum_{j=1}^{M-1} |u_j|^6 \Delta x \leq ||u||^2_\infty \sum_{j=1}^{M-1} |u_j|^4 \Delta x \\
&\leq 2||u||_2||\boldsymbol{D}^+ u||_2\, 2||u||^3_2||\boldsymbol{D}^+ u||_2 = 4||u||^4_2||\boldsymbol{D}^+ u||^2_2
\end{aligned}$$

which can then lead us to conclude (1.26), and hence, the lemma is proved. □

Another lemma is presented below:

Lemma 1.4.5. *For $z \in \mathbb{C}^M$, $w \in X_M$, we have*

$$\Big(\boldsymbol{D}^+ z, w\Big)_x = -\Big(z, \boldsymbol{D}^- w\Big)_x + \Delta x\Big(z, \boldsymbol{D}^- w\Big)_2 - \Big(z, w\Big)_2 \tag{1.27}$$

$$\Big(\boldsymbol{D}^- z, w\Big)_x = -\Big(z, \boldsymbol{D}^+ w\Big)_x - \Delta x\Big(z, \boldsymbol{D}^+ w\Big)_2 - \Big(z, w\Big)_2 \tag{1.28}$$

$$\Big(\boldsymbol{D} z, w\Big)_x = -\Big(z, \boldsymbol{D} w)_x + \frac{\Delta x}{2}(z, \boldsymbol{D}^- w\Big)_2 - \frac{\Delta x}{2}\Big(z, \boldsymbol{D}^+ w\Big) - \Big(z, w\Big)_2 \tag{1.29}$$

$$Re\Big(\boldsymbol{D}^+ w, w\Big)_x = -\frac{1}{2}||w||^2_2 - \frac{\Delta x}{2}||\boldsymbol{D}^+ w||^2_x \tag{1.30}$$

$$Re\Big(\boldsymbol{D}^3 w, w\Big)_x = \frac{3}{2}||\boldsymbol{D}^+ w||^2_2 \tag{1.31}$$

Proof. Starting with (1.27), and using the definition (1.8), we have

$$\left(\boldsymbol{D}^{+}z,w\right)_x=\sum_{j=2}^{M}\frac{1}{\Delta x}\Delta x^2(j-1)z_j\overline{w}_{j-1}-\sum_{j=1}^{M-1}\frac{1}{\Delta x}\Delta x^2 j z_j\overline{w}_j$$
$$=\frac{1}{\Delta x}\Delta x^2(M-1)z_M\overline{w}_{M-1}+\sum_{j=1}^{M-1}\frac{1}{\Delta x}\Delta x^2(j-1)z_j\overline{w}_{j-1}$$
$$-\frac{1}{\Delta x}\Delta x^2(0)z_1\overline{w}_0-\sum_{j=1}^{M-1}\frac{1}{\Delta x}\Delta x^2 j z_j\overline{w}_j$$

Since $w\in X_M$, we can conclude (1.27). The same argument can be used for (1.28) after observing that

$$\left(\boldsymbol{D}^{-}z,w\right)_x=-(z,\boldsymbol{D}^{+}w)_x-\Delta x(z,\boldsymbol{D}^{+}w)_2-(z,w)_2+M\Delta x z_{M-1}\overline{w}_M \tag{1.32}$$

On the other hand, (1.30) can be proved combining (1.27) with (1.28). For (1.30), we have

$$Re\left(\boldsymbol{D}^{+}z,z\right)_x=\sum_{j=1}^{M-1}\frac{1}{2\Delta x}j\Delta x^2|z_{j+1}|^2-\sum_{j=1}^{M-1}\frac{1}{2\Delta x}j\Delta x^2|z_j|^2-\sum_{j=1}^{M-1}\frac{1}{2\Delta x}j\Delta x^2|z_{j+1}-z_j|^2$$
$$=-\frac{1}{2}||z||_2^2-\frac{\Delta x}{2}||\boldsymbol{D}^{+}z||_x^2.$$

In order to obtain (1.31), we need to combine (1.27), (1.28) and (1.30). First: because $w\in X_M$, we have

$$\left(\boldsymbol{D}^{3}w,w\right)_x=-\frac{1}{2}(\boldsymbol{D}^{-}\boldsymbol{D}^{+}w,\boldsymbol{D}^{+}w)_x-\frac{1}{2}(\boldsymbol{D}^{-}\boldsymbol{D}^{+}w,\boldsymbol{D}^{-}w)_x \tag{1.33}$$
$$+\frac{\Delta x}{2}(\boldsymbol{D}^{+}\boldsymbol{D}^{-}w,\boldsymbol{D}^{-}w)_2-\frac{\Delta x}{2}(\boldsymbol{D}^{+}\boldsymbol{D}^{-}w,\boldsymbol{D}^{+}w)_2+||\boldsymbol{D}^{+}w||_2^2 \tag{1.34}$$

At this point, we can use (1.27), (1.28), and sum by parts if we assume that $w_{-1}=0$. This leads to $D^{+}w_{-1}=D^{+}w_0=0$. We also have $D^{+}w_M=D^{-}w_M=0$. Hence, so far we have

$$\left(\boldsymbol{D}^{3}w,w\right)_x=\frac{1}{2}(\boldsymbol{D}^{-}w,\boldsymbol{D}^{2}w)_x-\frac{\Delta x}{2}(\boldsymbol{D}^{-}w,\boldsymbol{D}^{2}w)_2+\frac{1}{2}(\boldsymbol{D}^{-}w,\boldsymbol{D}^{+}w)_2$$
$$+\frac{1}{2}(\boldsymbol{D}^{+}w,\boldsymbol{D}^{2}w)_x+\frac{\Delta x}{2}(\boldsymbol{D}^{+}w,\boldsymbol{D}^{2}w)_2+\frac{1}{2}(\boldsymbol{D}^{+}w,\boldsymbol{D}^{-}w)_2$$
$$+\frac{\Delta x}{2}(\boldsymbol{D}^{2}w,\boldsymbol{D}^{-}w)_2-\frac{\Delta x}{2}(\boldsymbol{D}^{2}w,\boldsymbol{D}^{+}w)+||\boldsymbol{D}^{+}w||_2^2$$

using again (1.27) and (1.28) and summation by parts,

$$\left(\boldsymbol{D}^{3}w,w\right)_x=-(w,\boldsymbol{D}^{3}w)_x+||\boldsymbol{D}^{+}w||_2^2+(\boldsymbol{D}^{-}w,\boldsymbol{D}^{+}w)_2+(\boldsymbol{D}^{+}w,\boldsymbol{D}^{-}w)_2$$

the identity (1.31) then follows. □

Lemma 1.4.6. *Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence in X_M induced by the numerical scheme (1.14) with $3a_3 \geq |a_4 + 2a_5|$, $\gamma_0 = 1/2$, and $u^0 \in X_M$. Then, there exist some constant $K = K(T, L) > 0$, independent of Δx and Δt, such that*

$$||P_\Delta^{\frac{1}{2}} u_\Delta||^2_{L^2((0,T);H_0^1(\Omega))} \leq K||u^0||_2^2 \tag{1.35}$$

$$||Q_\Delta^{\frac{1}{2}}(|u|^2 u_x)_\Delta||^2_{L^2(0,T;L^2(\Omega))} \leq K||q_\Delta u_\Delta^0||^6_{L^2(\Omega)} \tag{1.36}$$

$$||q_\Delta(|u|_x^2)_\Delta||^2_{L^2(\Omega)} \leq 32(1+\Delta x^2)||q_\Delta u_\Delta||^2_{L^\infty(\Omega)}||p_\Delta u_\Delta||^2_{H_0^1(\Omega)} \tag{1.37}$$

Proof. In order to prove (2.15), we need to multiply (1.14) component-wise by $j\Delta x u_j^{n+\frac{1}{2}}$, sum over $j = 0, 1, \ldots, M-1$, and extract the imaginary part. Initially we have

$$\begin{aligned} i\Big(\boldsymbol{D}_t u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + a_1\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_2\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x \\ +ia_3\Big(\boldsymbol{D}\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_4\Big(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x \\ +ia_5\Big(F_{a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = 0 \end{aligned} \tag{1.38}$$

We will study each term in (1.38). First, and using the definition (1.8), it is easy to see that

$$Im\left(i\Big(\boldsymbol{D}_t u^n, u^{n+\frac{1}{2}}\Big)_x\right) = \frac{1}{2\Delta t}(||u^{n+1}||_x^2 - ||u^n||_x^2) \tag{1.39}$$

Using (1.27), we can write

$$\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_x^2 + \Delta x||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 - \Delta x\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2.$$

Hence,

$$Im\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -\Delta x Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2. \tag{1.40}$$

We can also write

$$Im\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = 0. \tag{1.41}$$

Using now the identity (1.31) for $w = u^{n+\frac{1}{2}}$, we have

$$Im\left(i\Big(\boldsymbol{D}^3 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\right) = \frac{3}{2}||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \tag{1.42}$$

For the nonlinear terms F_{a_4} and F_{a_5} with $\gamma_0 = \frac{1}{2}$; this is,

$$\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x := \left(\frac{1}{2}|u^{n+\frac{1}{2}}|^2 \boldsymbol{D}\left(u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x + \left(\frac{1}{4}\boldsymbol{D}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x - \left(\frac{1}{4}\left(u^{n+\frac{1}{2}}\right)^2 \boldsymbol{D}\overline{\left(u^{n+\frac{1}{2}}\right)}, u^{n+\frac{1}{2}}\right)_x \tag{1.43}$$

$$\left(F_{a_5}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x := \left(u^{n+\frac{1}{2}} \boldsymbol{D}\left(|u^{n+\frac{1}{2}}|^2\right), u^{n+\frac{1}{2}}\right)_x \tag{1.44}$$

Using (1.29), we have

$$\left(\boldsymbol{D}(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}), u^{n+\frac{1}{2}}\right)_x = -\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x + \frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2 - \frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\right)_2 - \left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2 \tag{1.45}$$

and, at the same time,

$$\left((u^{n+\frac{1}{2}})^2 \boldsymbol{D}\overline{u}^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_x = \left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x \tag{1.46}$$

Combining (1.45) and (1.46) in (2.27), we get

$$\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2 \boldsymbol{D}\left(u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x - \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x + \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2 - \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2$$

and extracting the real part, we get

$$Re\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \frac{\Delta x}{8} Re\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2 - \frac{\Delta x}{8} Re\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4 \tag{1.47}$$

and recalling that $\boldsymbol{D}^2 u = \frac{\boldsymbol{D}^+ u - \boldsymbol{D}^- u}{\Delta x}$

$$Re\Big(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = -\frac{\Delta x^2}{8} Re\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\Big)_x - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4 \tag{1.48}$$

Finally, for the last nonlinear term, we have

$$\begin{aligned}
\Big(F_{a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x &= \Big(u^{n+\frac{1}{2}} \boldsymbol{D}(|u^{n+\frac{1}{2}}|^2), u^{n+\frac{1}{2}}\Big)_x \\
&= \Big(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, |u^{n+\frac{1}{2}}|^2\Big)_x \\
&= -\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\Big)_x + \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big)_2 \\
&\quad - \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 - ||u^{n+\frac{1}{2}}||_4^4
\end{aligned}$$

and thus,

$$Re\Big(F_{a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = \frac{\Delta x}{4}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{\Delta x}{4}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4$$

which, in turn, can be rewritten as

$$Re\Big(F_{a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = -\frac{\Delta x^2}{8}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4. \tag{1.49}$$

Combining together (1.39), (1.40), (1.41), (1.42), (1.48) and (1.49); multiplying by Δt, and summing over n, we obtain

$$\begin{aligned}
\frac{||u^0|_x^2}{2} &= \frac{||u^{N+1}||_x^2}{2} - a_1 \Delta x \sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t + \frac{3a_3}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \Delta t \\
&\quad - a_4 \frac{\Delta x^2}{8} \sum_{n=0}^{N} Re(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}})_2 \Delta t \\
&\quad - (a_5)\left(\frac{\Delta x^2}{8}\sum_{n=0}^{N} (|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2)_2 \Delta t\right) \\
&\quad - \frac{a_4 + 2a_5}{4}\sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t + \sum_{n=0}^{N}\Big(a u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x \Delta t
\end{aligned}$$

Let us recall the fact that $a_1, a_3 > 0$. This can let us drop some terms in the above equality to get

$$\frac{||u^0||_x^2}{2} \geq -a_1\Delta x \sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t + \frac{3a_3}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \Delta t \tag{1.50}$$
$$- a_4\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\big)_2 \Delta t$$
$$- (a_5)\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N} \big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2 |u^{n+\frac{1}{2}}|^2\big)_2 \Delta t\Bigg)$$
$$- \frac{a_4 + 2a_5}{4}\sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t$$

Using (1.24) and (1.25) along with Young, Cauchy-Schwarz and Hölder inequalities, and for $T = N\Delta t$, we can demonstrate with ease the following inequalities:

$$\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t \leq \frac{1}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \Delta t + \frac{T}{2}||u^0||_2^2 \tag{1.51}$$

$$\sum_{n=0}^{N} Re\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\Big)_2 \Delta t \leq \frac{1}{\Delta x}\sum_{n=0}^{N}\Big(||u^{n+\frac{1}{2}}||_2^4 + ||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2\Big)\Delta t. \tag{1.52}$$

Indeed: for (2.35), after using Cauchy-Schwarz and Young Inequalities, and due to the preservation of the discrete L^2 norm we have

$$\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t \leq \sum_{n=0}^{N} ||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2 ||u^{n+\frac{1}{2}}||_2 \Delta t$$
$$\leq \frac{1}{2}\sum_{n=0}^{N}\Big(||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 + ||u^{n+\frac{1}{2}}||_2^2\Big)\Delta t$$
$$\leq \frac{1}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 \Delta t + \frac{T}{2}||u^0||_2^2$$

this leads us to (2.35). We will now consider the following identity for $a, b \in \mathbb{C}$:

$$Re\Big(a(\overline{a} - \overline{b})\Big) = \frac{1}{2}\Big(|a|^2 - |b|^2\Big) + \frac{1}{2}|a - b|^2 \tag{1.53}$$

To prove (2.36), and thanks to (1.53), we have componentwise

$$Re\Big(|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} D^2 \overline{u^{n+\frac{1}{2}}}_j\Big) = \frac{|u_j^{n+\frac{1}{2}}|^2}{\Delta x^2}\Big(Re\big(u_j(\overline{u}_{j+1}^{n+\frac{1}{2}} - \overline{u}_j^{n+\frac{1}{2}})\big) - Re\big(u_j(\overline{u}_j^{n+\frac{1}{2}} - \overline{u}_{j-1}^{n+\frac{1}{2}})\big)\Big)$$
$$= \frac{1}{2\Delta x}\Big(|u_j^{n+\frac{1}{2}}|^2 D^+ |u_j^{n+\frac{1}{2}}|^2 - \Delta x |u_j^{n+\frac{1}{2}}|^2 |D^+ u_j^{n+\frac{1}{2}}|^2$$
$$- |u_j^{n+\frac{1}{2}}|^2 D^- |u_j^{n+\frac{1}{2}}|^2 - \Delta x |u_j^{n+\frac{1}{2}}|^2 |D^- u_j^{n+\frac{1}{2}}|^2\Big).$$

Hence, after summing by parts and due to Cauchy-Schwarz, Young, and Inverse Triangle inequalities,

$$\begin{aligned}
Re\Big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\Big)_2 &= \frac{1}{2\Delta x}\Bigg[\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 - \Delta x\Big(|u^{n+\frac{1}{2}}|^2, |\boldsymbol{D}^+u^{n+\frac{1}{2}}|^2\Big)_2 \\
&\quad - \Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big)_2 - \Delta x\Big(|u^{n+\frac{1}{2}}|^2, |\boldsymbol{D}^-u^{n+\frac{1}{2}}|^2\Big)_2\Bigg] \\
&\leq \frac{1}{\Delta x}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 \\
&\leq \frac{1}{\Delta x}(||u^{n+\frac{1}{2}}||_2^4 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2)
\end{aligned}$$

and after multiplying by Δt and summing for n, we get (2.36). Using (1.25), replacing (2.35) and (2.36) in (1.50), and summing by parts in the fourth term at the right hand side of (1.50),

$$\begin{aligned}
&\frac{||u^0||_x^2}{2} + a_1\frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t + a_1T\frac{\Delta x}{2}||u^0||_2^2 \\
&+ |a_4|\frac{\Delta x}{8}\sum_{n=0}^{N}\Big(||u^{n+\frac{1}{2}}||_2^4 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Big)\Delta t \\
&- a_5\frac{\Delta x^2}{8}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t \\
&+ \frac{1}{2}|a_4 + 2a_5|\left(\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_2^6 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t\right) \\
&\geq \frac{3a_3}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t
\end{aligned} \tag{1.54}$$

Reordening,

$$\begin{aligned}
&||u^0||_x^2 + \Big(a_1\Delta x + |a_4|\frac{\Delta x}{4}||u^0||_2 + |a_4 + 2a_5|||u^0||_2^4\Big)T||u^0||_2^2 \\
&+ \Big(a_1\Delta x + |a_4|\frac{\Delta x}{4} - a_5\frac{\Delta x^2}{4} + |a_4 + 2a_5|\Big)\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t \\
&\geq 3a_3\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t
\end{aligned}$$

From here, because $3a_3 \geq |a_4 + 2a_5|$, $||u^0||_x^2 \leq L||u^0||_2^2$, and considering $\Delta x \ll 1$ we can infere the existence of the needed constant $K = K(T, L)$ such that (2.15) holds. To prove (2.16), let us first note that:

$$\Big|\Big||u^{n+\frac{1}{2}}|^2\boldsymbol{D}u^{n+\frac{1}{2}}\Big|\Big|_2^2 = \sum_{j=0}^{M-1}|u_j^{n+\frac{1}{2}}|^4|Du_j^{n+\frac{1}{2}}|^2\Delta x \leq ||u^{n+\frac{1}{2}}||_\infty^4\sum_{j=0}^{M-1}|Du_j^{n+\frac{1}{2}}|^2\Delta x.$$

Hence, (2.16) can be obtained after summing over n and using (1.24) and (2.15). To prove (2.17), we will use the identity $(a^2 - b^2) + (a-b)^2 = 2a(a-b)$. For a $u_j \in u$, $i = 0, 1, \dots, M$, we have:

$$
\begin{aligned}
D|u_j|^2 &= \frac{|u_{j+1}|^2 - |u_{j-1}|^2}{2\Delta x} = \frac{1}{2\Delta x}\Big(|u_{j+1}|^2 - |u_j|^2 + |u_j|^2 - |u_{j-1}|^2\Big) \\
&= \frac{1}{2\Delta x}\Big[2|u_j|(|u_j| - |u_{j-1}|) - (|u_j| - |u_{j-1}|)^2 \\
&\quad + 2|u_j|(|u_{j+1}| - |u_j|) + (|u_{j+1}| - |u_j|)^2\Big] \\
&= |u_j|D^-|u_j| - \frac{\Delta x^2}{2}(D^-|u_j|)^2 + |u_j|D^+|u_j| + \frac{\Delta x^2}{2}(D^+|u_j|)^2.
\end{aligned}
$$

Taking the square at both sides, using inverse triangle inequality, and $D^2|u_j| \leq 4\frac{||u||_\infty}{\Delta x^2}$,

$$
\begin{aligned}
(D|u_j|^2)^2 &= \left(|u_j|D^-|u_j| - \frac{\Delta x^2}{2}(D^-|u_j|)^2 + |u_j|D^+|u_j| + \frac{\Delta x^2}{2}(D^+|u_j|)^2\right)^2 \\
&= \left(2|u_j|D|u_j| + \Delta x^3 D|u_j|D^2|u_j|\right)^2 \\
&\leq 4\left(4|u_j|^2|Du_j|^2 + \Delta x^6|Du_j|^2(D^2|u_j|)^2\right) \\
&\leq 16||u||_\infty^2|Du_j|^2 + 16\Delta x^2||u||_\infty^2|Du_j|^2.
\end{aligned}
$$

Summing over j will lead us to

$$
\sum_{j=0}^{M-1}(D|u_j|^2)^2\Delta x \leq 32||u||_\infty^2||\boldsymbol{D}^+u||_2^2(1+\Delta x^2)
$$

and hence, (2.17) is proved, and thus concluding the demonstration of the Lemma.

□

With these results, we can prove the convergence of the numerical scheme.

Theorem 1.4.7. *Let $u_\Delta = \{u_m^n\}_{m\in\mathbb{N}}$ a sequence in X_M of solutions induced by the numerical scheme (1.14), with $\gamma_0 = 1/2$ at a time $t_n = n\Delta t$, computed from a sequence of initial conditions $\{u_m^0\}_{m\in\mathbb{N}} \subset X_M$ using a timestep Δt and a spacestep Δx. If $3a_3 \geq |a_4 + 2a_5|$, then there is a subsequence, still denoted by $\{u_m^n\}_{m\in\mathbb{N}}$, such that*

$$
Q_\Delta u_\Delta \to u \ \ \textit{strongly in} \ \ L^2(0,T;L^2(\Omega)) \tag{1.55}
$$

when $\Delta t, \Delta x \to 0$, and for u the weak solution of (1.1)

Proof. From (1.24), we infer the existence of a u such that

$$Q_\Delta u_\Delta \to u \;\text{ weakly in }\; L^2(0,T;L^2(\Omega)) \tag{1.56}$$

From (1.24) and (2.15), we can also say that there exists a $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$\{Q_\Delta u_\Delta\} \;\text{ is bounded in }\; L^2(0,T;H_0^1(\Omega)) \tag{1.57}$$

and thus

$$Q_\Delta u_\Delta \rightharpoonup u \text{ weak in }\; L^2(0,T;H_0^1(\Omega)) \tag{1.58}$$

From (1.26) and (2.15), we have

$$\{Q_\Delta^{\frac{1}{2}}(|u|^2u)_\Delta\} \;\text{ is bounded in }\; L^2(0,T;L^2(\Omega)) \tag{1.59}$$

And from (2.16) and (2.17),

$$\{Q_\Delta F_{a_4}(u)_\Delta\} \;\text{ is bounded in }\; L^2(0,T;L^2(\Omega)) \tag{1.60}$$
$$\{Q_\Delta F_{a_5}(u)_\Delta\} \;\text{ is bounded in }\; L^2(0,T;L^2(\Omega)) \tag{1.61}$$

Let us now consider a $\varphi \in H_0^2(\Omega)$, with $\varphi_j^n = \varphi(x_j,t_n)$, $0 \le n \le N$, $0 \le j \le M$, . Multiplying (1.14) by $\Delta t \Delta x \overline{\varphi}_j$, sum over j and then sum over n. We then get

$$\begin{aligned}\sum_{n=0}^{N}\Big(\boldsymbol{D}_t u_m^n,\varphi\Big)_2\Delta t &= ia_1\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t - a_3\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t\\ &+ a_2\sum_{n=0}^{N}\Big(|u_m^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}},\varphi\Big)_2\Delta t - a_4\sum_{n=0}^{N}\Big(F_{a_4}(u_m^{n+1}),\varphi\Big)_2\Delta t\\ &- a_5\sum_{n=0}^{N}\Big(F_{a_5}(u_m^{n+1}),\varphi\Big)_2\Delta t\end{aligned} \tag{1.62}$$

Our aim is to prove that the left hand side of (2.42) is bounded. From (1.24) and (2.15), and summing by parts, we get

$$\begin{aligned}&\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t + \sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t\\ &\quad= \sum_{n=0}^{N} -\Big(\boldsymbol{D}^+ u_m^{n+\frac{1}{2}},\boldsymbol{D}^+\varphi\Big)_2\Delta t + \sum_{n=0}^{N}\Big(\boldsymbol{D}^+ u_m^{n+\frac{1}{2}},\boldsymbol{D}^+\boldsymbol{D}^-\varphi\Big)_2\Delta t\\ &\quad\le \sum_{n=0}^{N}||\boldsymbol{D}^+ u_m^{n+\frac{1}{2}}||_2||\boldsymbol{D}^+\varphi||_2\Delta t + \sum_{n=0}^{N}||\boldsymbol{D}^+ u_m^{n+\frac{1}{2}}||_2||\boldsymbol{D}^+\boldsymbol{D}^-\varphi||_2\Delta t\\ &\quad\le C_\varphi\Big(\sum_{n=0}^{N}||\boldsymbol{D}^+ u_m^{n+\frac{1}{2}}||_2\Delta t + \sum_{n=0}^{N}||\boldsymbol{D}^+ u_m^{n+\frac{1}{2}}||_2\Delta t\Big) \le 2C_\varphi K||u_m^0||_2\end{aligned}$$

since we are considering any $\varphi \in H_0^2(\Omega)$, and combining (1.57), (2.40), (2.41) and (1.61) after using Cauchy-Schwarz Inequality in (2.42) , we get

$$\Big\{\frac{\partial}{\partial t}P_\Delta u_\Delta\Big\} \;\text{ is bounded in }\; L^2(0,T;H^{-2}(\Omega)) \tag{1.63}$$

and as in the continuous case, because

$$H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-2}(\Omega),$$

and employing Aubin-Lions Theorem, there exists a subsequence of $\{u_m^n\}_{m\in\mathbb{N}}$, still denoted by the same form, such that,

$$Q_\Delta u_\Delta \longrightarrow u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega))\,. \tag{1.64}$$

Now we will prove that u is the weak solution of (1.1). Thanks to (2.43), we have

$$|u_m^{n+\frac{1}{2}}|u_m^{n+\frac{1}{2}} \longrightarrow |u|^2 u, \qquad \text{a.e. in } (\Omega)\times(0,T) \tag{1.65}$$

using (1.65), and recalling again Lions lemma [Lio69], we will get

$$Q_\Delta^{\frac{1}{2}}(|u|^2u)_\Delta \rightharpoonup |u|^2 u \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{1.66}$$

furthermore, combining (2.43) and (2.39),

$$Q_\Delta(F_{a_4}(u))_\Delta \rightharpoonup \frac{1}{2}|u|^2u_x + \frac{1}{4}(|u|^2u)_x - \frac{1}{4}u^2\overline{u}_x \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{1.67}$$

$$Q_\Delta(F_{a_5}(u))_\Delta \rightharpoonup u|u|_x^2 \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{1.68}$$

Multiplying componentwise the numerical scheme (1.14) by $\Delta x\Delta t\phi_k^n$, sum by parts, and passing to the limit, is easy to see that $u = u(t_n)$ is, indeed, the weak solution of problem (1.1), and hence the Theorem is proved. □

1.4.3 Behavior of the Energy

In this presentation, we will consider the energy $E^{(n)}$ of the numerical solution u^n, at a timestep n, as follows

$$E^{(n)} := \frac{a_1}{2}||D^+u^n||_2^2 - \frac{a_3}{4}||u^n||_4^4. \tag{1.69}$$

The following property holds:

Lemma 1.4.8. *$\forall\varphi \in X_M$, and for $\varphi_+, \in \mathbb{C}^{M+1}$: $(\varphi_+)_j = \varphi_{j+1}$, $j = 0,\dots,M$, and $\varphi_- \in \mathbb{C}^{M+1}$ $(\varphi_-)_j = \varphi_{j-1}$, $j = 1,\dots,M$, $(\varphi_-)_0 = 0$, , we have*

$$Re\left(|\varphi|^2\boldsymbol{D}\varphi, \boldsymbol{D}^2\varphi\right)_2 = \frac{1}{2}\left(|\varphi|^2, \boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\right)_2 \tag{1.70}$$

$$Re\left(\boldsymbol{D}(|\varphi|^2)\varphi, \boldsymbol{D}^2\varphi\right)_2 = \left(\frac{|\varphi_+|^2 + 2|\varphi|^2 + |\varphi_-|^2}{4}, \boldsymbol{D}^+\boldsymbol{D}^-|\varphi|^2\right)_2 \tag{1.71}$$

$$Re\left(\boldsymbol{D}(|\varphi|^2\varphi), \boldsymbol{D}^2\varphi\right)_2 = \left(\frac{|\varphi_+|^2 + |\varphi|^2 + |\varphi_-|^2}{2}, \boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\right)_2 + \frac{\Delta x^2}{2}\left(\boldsymbol{D}|\varphi|^2, |\boldsymbol{D}^2\varphi|^2\right)_2 \tag{1.72}$$

$$Re\left(\boldsymbol{D}^3\varphi, \boldsymbol{D}^2\varphi\right)_2 = 0 \tag{1.73}$$

Proof. starting with (1.70), we have

$$\begin{aligned}|\varphi_j|^2 D\varphi_j\overline{D^2\varphi_j} &= |\varphi_j|^2 D\varphi_j\overline{D^+D^-\varphi_j}\\ &= \frac{1}{2}|\varphi_j|^2\left(D^+\varphi_j\overline{D^-D^+\varphi_j}\right) + \frac{1}{2}|\varphi_j|^2\left(D^-\overline{D^+D^-\varphi_j}\right)\end{aligned} \tag{1.74}$$

using (1.53) over the real part of (1.74), we get

$$
\begin{aligned}
Re\left(|\varphi_j|^2 D\varphi_j \overline{D^2\varphi_j}\right) &= \frac{1}{4}|\varphi_j|^2 D^+\left(|D^-\varphi_j|^2\right) - \frac{\Delta x^2}{4}|D^+D^-\varphi_j|^2|\varphi_j|^2 \\
&\quad + \frac{1}{4}|\varphi_j|^2 D^-\left(D^-(|D^+\varphi_j|^2)\right) + \frac{\Delta x^2}{4}|D^-D^+\varphi_j|^2|\varphi_j|^2 \\
&= \frac{1}{4}|\varphi_j|^2\Big(D^+\left(|D^-\varphi_j|^2\right) + D^-\left(|D^+\varphi_j|^2\right)\Big).
\end{aligned}
$$

Summing over j, we get

$$
\begin{aligned}
Re\Big(|\varphi|^2\boldsymbol{D}\varphi, \boldsymbol{D}^2\varphi\Big)_2 &= \frac{1}{4}\Big(|\varphi|^2, \boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\Big)_2 - \frac{1}{4}\Big(\boldsymbol{D}^+|\varphi|^2, |\boldsymbol{D}^+\varphi|^2\Big)_2 \\
&= \frac{1}{2}\Big(|\varphi|^2, \boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\Big)_2
\end{aligned}
$$

hence, (1.70) is proved. To obtain (1.71), we will first require the following property for $a, b \in \mathbb{C}^{\mathbb{Z}}$:

$$
D(a_jb_j) = a_{j+1}\frac{D^+b_j}{2} + a_{j-1}\frac{D^-b_j}{2} + b_jDa_j \tag{1.75}
$$

$$
D^-(a_jb_j) = b_{j-1}D^-a_j + a_jD^-b_j \tag{1.76}
$$

Using this over all the components of φ in (1.71), we get

$$
\begin{aligned}
\Big(\boldsymbol{D}\varphi, \boldsymbol{D}^2\varphi\Big)_2 &= -\Big(\boldsymbol{D}^-\left(\boldsymbol{D}(|\varphi|^2)\varphi\right), \boldsymbol{D}^-\varphi\Big)_2 \\
&= -\Big(\boldsymbol{D}(|\varphi|^2)\boldsymbol{D}^-\varphi, \boldsymbol{D}^-\varphi\Big)_2 - \Big(\varphi_-\boldsymbol{D}^-\boldsymbol{D}(|\varphi|^2)\boldsymbol{D}^-\varphi\Big)_2 \\
&= -\Big(\boldsymbol{D}(|\varphi|^2), |\boldsymbol{D}^-\varphi|^2\Big)_2 - \Big(\varphi_-\boldsymbol{D}^-\boldsymbol{D}(|\varphi|^2), \boldsymbol{D}^-\varphi\Big)_2
\end{aligned}
$$

extracting the real part,

$$
\begin{aligned}
Re\Big(\boldsymbol{D}\varphi, \boldsymbol{D}^2\varphi\Big)_2 &= \Big(|\varphi|^2, \boldsymbol{D}(|\varphi|^2)\Big)_2 - \Big(\boldsymbol{D}^-\boldsymbol{D}(|\varphi|^2), (\frac{1}{2}\boldsymbol{D}^-|\varphi|^2 - \frac{\Delta x}{2}|\boldsymbol{D}^-\varphi|^2)\Big)_2 \\
&= \Big(|\varphi|^2, \boldsymbol{D}(|\varphi|^2)\Big)_2 - \frac{\Delta x}{2}\Big(\boldsymbol{D}^-(|\varphi|^2), \boldsymbol{D}(|\boldsymbol{D}^-\varphi|^2)\Big)_2 \\
&= \Big(|\varphi|^2, \boldsymbol{D}\Big[|\boldsymbol{D}^-\varphi|^2 + |\boldsymbol{D}^+\varphi|^2\Big]\Big)_2 \\
&= -\frac{1}{4}\Big(\boldsymbol{D}^-|\varphi_+|^2 + 2\boldsymbol{D}^-|\varphi|^2 + \boldsymbol{D}^-|\varphi_-|^2, |\boldsymbol{D}^-\varphi|^2\Big)_2 \\
&= \Big(\frac{|\varphi_+|^2 + 2|\varphi|^2 + |\varphi_-|^2}{4}, \boldsymbol{D}^+\left(|\boldsymbol{D}^-\varphi|^2\right)\Big)_2.
\end{aligned}
$$

Hence, (1.71) is proved. To prove (1.72), and starting by using (1.75), we have

$$
\Big(\boldsymbol{D}(|\varphi|^2\varphi), \boldsymbol{D}^2\varphi\Big)_2 = \Big(\boldsymbol{D}(|\varphi|^2)\varphi, \boldsymbol{D}^2\varphi\Big)_2 + \frac{1}{2}\Big(\varphi_+|^2\boldsymbol{D}^+\varphi + |\varphi_-|^2\boldsymbol{D}^-\varphi, \boldsymbol{D}^2\varphi\Big)_2 \tag{1.77}
$$

extracting the real part, and by (1.53), we have

$$
Re\Big(|\varphi_+|^2\boldsymbol{D}^+\varphi, \boldsymbol{D}^2\varphi\Big)_2 = \frac{1}{2}\Big(|\varphi_+|^2, \boldsymbol{D}^+|\boldsymbol{D}^+\varphi|^2\Big)_2 + \frac{\Delta x}{2}\Big(|\varphi_+|^2|\boldsymbol{D}^2\varphi|^2\Big)_2 \tag{1.78}
$$

$$
Re\Big(|\varphi_-|^2\boldsymbol{D}^-\varphi, \boldsymbol{D}^2\varphi\Big)_2 = \frac{1}{2}\Big(|\varphi_-|^2, \boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\Big)_2 - \frac{\Delta x}{2}\Big(|\varphi_-|^2, |\boldsymbol{D}^2\varphi|^2\Big)_2 \tag{1.79}
$$

replacing (1.78) and (1.79) over the real part of (1.77), we get

$$Re\Big(\boldsymbol{D}(|\varphi|^2\varphi),\boldsymbol{D}^2\varphi\Big)_2 = Re\Big(\boldsymbol{D}(|\varphi|^2)\varphi,\boldsymbol{D}^2\varphi\Big)_2 + \frac{1}{4}\Big((|\varphi_+|^2+|\varphi_-|^2),\boldsymbol{D}^+|\boldsymbol{D}^-\varphi|^2\Big)_2 + \frac{\Delta x^2}{2}\Big(\boldsymbol{D}|\varphi|^2,|\boldsymbol{D}^2\varphi|^2\Big)_2$$

and recalling (1.71), the conclusion follows. Finally for (1.73),

$$\Big(\boldsymbol{D}^3\varphi,\boldsymbol{D}^2\varphi\Big)_2 = \Big(\boldsymbol{D}\boldsymbol{D}^2\varphi,\boldsymbol{D}^2\varphi\Big)_2 = -\Big(\boldsymbol{D}^2\varphi,\boldsymbol{D}\boldsymbol{D}^2\varphi\Big)_2 = -\overline{\Big(\boldsymbol{D}^3\varphi,\boldsymbol{D}^2\varphi\Big)_2}$$

and hence, the proof of the Lemma is complete. □

Theorem 1.4.9. *Let $u^n \in X_M$ the numerical solution of (1.1) using scheme (1.14) for $\gamma_0 = \frac{1}{2}$ in (1.13), such that $\frac{\partial}{\partial t}(P_\Delta u_\Delta)$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. If in (1.1) $3a_2a_3 = a_1(a_4+2a_5)$, then there exists $C \in \mathbb{R}$, independent of the values of Δx and Δt, such that*

$$||\boldsymbol{D}^+u^n||_2^2 < C, \quad \forall n \in \mathbb{N} \tag{1.80}$$

if and only if

$$E^{(n+1)} = E^{(n)} + \mathcal{O}(\Delta t), \quad \forall n \in \mathbb{N} \tag{1.81}$$

where $\mathcal{O}(\Delta t)$ means that it admits the existence of a constant $K \in \mathbb{R}$ such that $|\mathcal{O}(\Delta t)| \leq \Delta t K$, where K does not depend on Δt and Δx.

Proof. If (1.81) is true, then (1.80) follows after considering (1.25). We will now focus on the case when (1.80) hols true. Multiplying (1.14) with $\Delta x \boldsymbol{D}_t\overline{u}_j^n$, sum over j, and extract the real part, will lead us to

$$\begin{aligned}0 =& \frac{a_1}{2a_4\Delta t}(||\boldsymbol{D}^+u^{n+1}||_2^2 - ||\boldsymbol{D}^+u^n||_2^2) - a_3\sum_{j\in\mathbb{Z}} Re\Big(|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} D_t\overline{u}_j^n\Big)\Delta x \\ &+ a_2\sum_{j\in\mathbb{Z}} Im\Big(D^3u_j^{n+\frac{1}{2}} D_t\overline{u_j}^n\Big)\Delta x + a_4\sum_{j\in\mathbb{Z}} Im\Big(F_{a_4}(u^{n+1})_j D_t\overline{u}_j^n\Big)\Delta x \\ &+ a_5\sum_{j\in\mathbb{Z}} Im\Big(F_{a_5}(u^{n+1})_j D_t\overline{u}_j^n\Big)\Delta x\end{aligned} \tag{1.82}$$

and replacing $D_tu_j^n$ from the numerical scheme on the last three products,

$$\begin{aligned}0 =& \frac{a_1}{2\Delta t}\Big(||\boldsymbol{D}^+u^{n+1}||_2^2 - ||\boldsymbol{D}^+u^n||_2^2\Big) - a_3\sum_{j\in\mathbb{Z}} Re\Big(|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} D_t\overline{u}_j^n\Big)\Delta x \\ &+ a_1a_2\sum_{j\in\mathbb{Z}} Re\Big(D^3u_j^{n+\frac{1}{2}}\overline{D^2u_j^{n+\frac{1}{2}}}\Big)\Delta x + a_1a_4\sum_{j\in\mathbb{Z}} Re\Big(F_{a_4}(u^{n+1})_j\overline{D^2u_j^{n+1\frac{1}{2}}}\Big)\Delta x \\ &+ a_1a_5\sum_{j\in\mathbb{Z}} Re\Big(F_{a_5}(u^{n+1})_j\overline{D^2u_j^{n+\frac{1}{2}}}\Big)\Delta x + a_2a_3\sum_{j\in\mathbb{Z}} Re\Big(D^3u_j^{n+\frac{1}{2}}\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}}\Big)\Delta x \\ &+ a_3a_4\sum_{j\in\mathbb{Z}} Re\Big(F_{a_4}(u^{n+1})_j\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}}\Big)\Delta x + a_3a_5\sum_{j\in\mathbb{Z}} Re\Big(F_{a_5}(u^{n+1})_j\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}}\Big)\Delta x\end{aligned} \tag{1.83}$$

We will study each term in (1.83) by components if its convenient. For the second term in (1.83), let us note that

$$|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} - \frac{|u_j^{n+1}|^2 + |u_j^n|^2}{2} u_j^{n+\frac{1}{2}} = -\frac{\Delta t^2}{8} u_j^{n+\frac{1}{2}} |D_t u_j^n|^2.$$

Hence,

$$Re\Big(|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}} \overline{D_t u_j^n}\Big) = \frac{1}{4\Delta t}\Big(|u_j^{n+1}|^4 - |u_j^n|^4\Big) - \frac{\Delta t^2}{8} Re\Big(u_j^{n+\frac{1}{2}} |D_t u_j^n|^2 \overline{D_t u_j^n}\Big) \tag{1.84}$$

Meanwhile, and thanks to (1.73) in Lemma 1.4.8, we can get rid of the third term in (1.83). For the fourth term in (1.83), we will need to use $\gamma_0 = \frac{1}{2}$ in (1.14) in order to recall (1.70), (1.71) and (1.72) in Lemma 1.4.8; then:

$$\begin{aligned} Re\Big(F_{a_4}(u^{n+1}), \overline{\boldsymbol{D}^2 u^{n+\frac{1}{2}}}\Big)_2 = {} & \left(\frac{|u_+^{n+\frac{1}{2}}|^2 + 2|u^{n+\frac{1}{2}}|^2 + |u_-^{n+\frac{1}{2}}|^2}{8}, \boldsymbol{D}^+ |\boldsymbol{D}^- u^{n+\frac{1}{2}}|^2\right)_2 \\ & + \frac{\Delta x^2}{4}\left(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, \left|\boldsymbol{D}^2 u^{n+\frac{1}{2}}\right|^2\right)_2 . \end{aligned} \tag{1.85}$$

where $u_+ \in \mathbb{C}^{\mathbb{Z}} : (u_+)_j = u_{j+1}$, and $u_- \in \mathbb{C}^{\mathbb{Z}} : (u_-)_j = u_{j-1}$. For the fourth term, and by (1.71) in Lemma 1.4.8:

$$\begin{aligned} Re\Big(F_{a_5}(u^{n+1}) \overline{\boldsymbol{D}^2 u^{n+1\frac{1}{2}}}\Big)_2 &= \Big(u^{n+\frac{1}{2}} \boldsymbol{D}|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\Big)_2 \\ &= \left(\frac{|u_+^{n+\frac{1}{2}}|^2 + 2|u^{n+\frac{1}{2}}|^2 + |u_-^{n+\frac{1}{2}}|^2}{8}, \boldsymbol{D}^2 |u^{n+\frac{1}{2}}|^2\right)_2 \end{aligned} \tag{1.86}$$

The sixth term can be worked thanks to (1.72) in Lemma 1.4.8:

$$\begin{aligned} Re\Big(\boldsymbol{D}^3 u^{n+\frac{1}{2}}, |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big)_2 = {} & -\Big(\frac{|u_+^{n+\frac{1}{2}}|^2 + |u^{n+\frac{1}{2}}|^2 + |u_-^{n+\frac{1}{2}}|^2}{2}, \boldsymbol{D}^+ |\boldsymbol{D}^- u^{n+\frac{1}{2}}|^2\Big)_2 \\ & + \frac{\Delta x^2}{2}\Big(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, \left|\boldsymbol{D}^2 u^{n+\frac{1}{2}}\right|^2\Big)_2 \end{aligned} \tag{1.87}$$

The last two terms in (1.83) require more effort. For the second to last one with $\gamma_0 = \frac{1}{2}$, and because $Re(\boldsymbol{D}u, u) = 0$, $\forall u \in \mathbb{C}^{\mathbb{Z}}$, we have

$$\begin{aligned} Re\Big(F_{a_4}(u^{n+1}), |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big)_2 &= \frac{1}{2} Re\Big(|u^{n+\frac{1}{2}}|^2 \boldsymbol{D} u^{n+\frac{1}{2}}, |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big)_2 \\ &\quad + \frac{1}{2} Re\Big(\boldsymbol{D}\big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\big), |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big)_2 \\ &\quad - \frac{1}{2} Re\Big(\boldsymbol{D}\big(|u^{n+\frac{1}{2}}|^2\big) u^{n+\frac{1}{2}}, |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big)_2 \\ &= \frac{1}{2} Re\left(\Big(|u^{n+\frac{1}{2}}|^4 \boldsymbol{D} u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 - \Big(|u^{n+\frac{1}{2}}|^4, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\Big)_2\right) \end{aligned} \tag{1.88}$$

let us consider the following identity for $a, b \in \mathbb{R}$

$$b^2(b-a) = \frac{1}{3}(b^3 - a^3) - \frac{1}{3}(b-a)^3 + b(b-a)^2.$$

Due to that, and after reordening the sum of the innter product $(\cdot)_2$, we have

$$\Big(|u^{n+\frac{1}{2}}|^4, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\Big)_2 = \frac{\Delta x^2}{6}\Big((\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2)^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 \tag{1.89}$$

On the other hand, after recalling identity (1.53),

$$Re\Big(|u^{n+\frac{1}{2}}|^4\boldsymbol{D}u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 = \frac{3}{2}\Big(|u^{n+\frac{1}{2}}|^4, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{\Delta x^2}{6}\Big((\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2)^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 \tag{1.90}$$

then, replacing (1.89) and (1.90) in (1.88),

$$Re\Big(F_{a_4}(u^{n+1}), |u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}\Big)_2 = -\frac{\Delta x^2}{24}\Big((\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2)^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 \tag{1.91}$$

Using the same technique, we can write

$$Re\Big(F_{a_5}(u^{n+1}), |u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}\Big)_2 = \frac{\Delta x^2}{6}\Big((\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2)^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2. \tag{1.92}$$

Replacing (1.84), (1.86), (1.85), (1.87), (1.91) and (1.92) in (1.83); multiplying by Δt, using the condition $3a_2a_3 = a_1(a_4 + 2a_5)$, and recalling definition (1.69), we get

$$\begin{aligned} E^{(n+1)} = E^{(n)} &- a_3\frac{\Delta t^3}{8}Re\Big(u^{n+\frac{1}{2}}|\boldsymbol{D}_tu^n|^2, \boldsymbol{D}_tu^n\Big)_2 \\ &+ \Delta x^2a_2a_3\frac{\Delta t}{4}\Bigg(\frac{1}{2}\Big(\boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|\boldsymbol{D}^-u^{n+\frac{1}{2}}|^2\Big)_2 \\ &+\Big(\frac{3a_5}{a_5+2a_4} - 2\Big)\Big(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, |\boldsymbol{D}^2u^{n+\frac{1}{2}}|^2\Big)_2\Bigg) \\ &+ a_3(4a_5 - a_4)\Delta t\frac{\Delta x^2}{24}\Big((\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2)^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2. \end{aligned} \tag{1.93}$$

Let us now consider the following inequality for $a, b \in \mathbb{R}$ due to Young Inequality

$$\frac{a^2 - b^2}{\Delta x} \le a^2 + b^2 + \frac{1}{2}\Big(\frac{a-b}{\Delta x}\Big)^2$$

using the previous inequality for $a = |u_{j+1}|$ and $b = |u_j|$, along with reverse triangle inequality,

$$D^+|u_j^{n+\frac{1}{2}}|^2 = \frac{|u_{j+1}^{n+\frac{1}{2}}|^2 - |u_j^{n+\frac{1}{2}}|^2}{\Delta x} \le |u_{j+1}^{n+\frac{1}{2}}|^2 + |u_j^{n+\frac{1}{2}}|^2 + \frac{1}{2}|D^+u_j^{n+\frac{1}{2}}|^2$$

$$D^-|u_j^{n+\frac{1}{2}}|^2 = \frac{|u_j^{n+\frac{1}{2}}|^2 - |u_{j-1}^{n+\frac{1}{2}}|^2}{\Delta x} \le |u_j^{n+\frac{1}{2}}|^2 + |u_{j-1}^{n+\frac{1}{2}}|^2 + \frac{1}{2}|D^-u_j^{n+\frac{1}{2}}|^2$$

thus, we have

$$D^+|D^-u_j^{n+\frac{1}{2}}|^2 \leq |D^+u_j^{n+\frac{1}{2}}|^2 + |D^-u_j^{n+\frac{1}{2}}|^2 + \frac{1}{2}|D^2u_j^{n+\frac{1}{2}}|^2 \tag{1.94}$$

$$D^2|u_j^{n+\frac{1}{2}}|^2 \leq \frac{1}{2\Delta x}\Big(|2u_{j+1}^{n+\frac{1}{2}}|^2 + 4|u_j^{n+\frac{1}{2}}|^2 + 2|u_{j-1}^{n+\frac{1}{2}}|^2 + |D^+u_j^{n+\frac{1}{2}}|^2 + |D^-u_j^{n+\frac{1}{2}}|^2\Big) \tag{1.95}$$

$$|D^2u_j^{n+\frac{1}{2}}|^2 = \left|\frac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{\Delta x^2}\right|^2 \leq \frac{1}{2\Delta x^2}\Big(|D^+u_j^{n+\frac{1}{2}}|^2 + |D^-u_j^{n+\frac{1}{2}}|^2\Big). \tag{1.96}$$

Assuming (1.80) is true, then the third, fourth and fifth term in (1.93) will be bounded by that C constant thanks to (1.94), (1.95) and (1.96). The second term at the right hand side of (1.93) is also bounded because $\frac{\partial}{\partial t}P_\Delta u_\Delta$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Thus, all terms in (1.92) will follow the behavior of Δt; and therefore, (1.93) admits the existence of a constant $K \in \mathbb{R}$ such that (1.81) holds. This concludes the proof. □

Remark 1.4.10. *Theorem 1.4.9 states that the energy of the numerical solution is not completely preserved, even when* $||\boldsymbol{D}^+u^n||_2^2$ *is bounded by a real constant. From (1.93) however, if* $||\boldsymbol{D}^2u^n||_2^2$ *is bounded, then this quantity can be controlled better. This can be seen as a regularity hypothesis necessary to improve the preservation; improvement which in turn is sufficient to conclude the regularity hypothesis.*

1.5 Numerical Experiments

In this section we will show some numerical experiments with results supporting the theorems demonstrated in the previous sections. In particular, we will test the scheme with some known examples whose exact solutions are previously known. Finally, we will test the code for an initial condition representing colliding solitons.

1.5.1 Computing Strategy

Some comments ought to be made if we consider that solutions of (1.1) are defined over the whole real line:

- A bounded domain $[x_0, x_f] \subset \mathbb{R}$ is considered because of computational limitations. As such, the space domain will be discretized using $2N$ equally spaced grids. Also, the numerical solution u^n will be considered as a complex-valued vector with $2M+1$ elements for each timestep. Theorems proved in the previous sections will still hold.
- The finite difference operators $\boldsymbol{D}$, $\boldsymbol{D}^2$ and $\boldsymbol{D}^3$ can be represented as matrices in $\mathbb{R}^{2M+1\times 2M+1}$ operating over complex-valued vectors. Furthermore: because these operators will act

only over the numerical solution u^n, and because of our choice on the boundary conditions, their matrix representations can be written as:

$$\boldsymbol{D} = \frac{1}{\Delta x}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}, \qquad \boldsymbol{D}^2 = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -2 & 1 \\ & & & 1 & -2 \end{bmatrix},$$

$$\boldsymbol{D}^3 = \frac{1}{\Delta x^3}\begin{bmatrix} 0 & -1 & \frac{1}{2} & & & & \\ 1 & 0 & -1 & \frac{1}{2} & & & \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ & & & -\frac{1}{2} & 1 & 0 & -1 \\ & & & & -\frac{1}{2} & 1 & 0 \end{bmatrix}.$$

- Therefore, the numerical scheme can be rewritten using matrix notation: for a given $u^0 \in \mathbb{C}^M$, the numerical scheme (1.14) allows us to compute $u^n \in \mathbb{C}^M$, the numerical approximation of the solution $u(\cdot, t_n) \in L^2(\Omega)$, as follows:

$$\left[-\frac{1}{\Delta t}\boldsymbol{I} + i\frac{a_1}{2}\boldsymbol{D}^2 - \frac{a_3}{2}\boldsymbol{D}^3\right] u^{n+1} + \left[\frac{1}{\Delta t}\boldsymbol{I} + i\frac{a_1}{2}\boldsymbol{D}^2 - \frac{a_3}{2}\boldsymbol{D}^3\right] u^n = a_4 F_{a_4}(u^{n+1}) + a_5 F_{a_5}(u^{n+1}) - ia_2 |u^{n+\frac{1}{2}}| u^{n+\frac{1}{2}} \quad (1.97)$$

To compute the numerical solution, we will use a Picard fixed-point iteration in order to solve equation (1.97) for each time-step, as applied in Delfour, Fortin and Payre [DFP81]. For a given $u^{p=1} = u^n \in \mathbb{C}^M$, we compute a sequence of complex vectors $\{u^p\}, p = 2, 3, 4, \ldots$, until a stopping criteria is verified. The sequence is given by

$$u^p = \left[-\frac{1}{\Delta t}\boldsymbol{I} + i\frac{a_1}{2}\boldsymbol{D}^2 - \frac{a_3}{2}\boldsymbol{D}^3\right]^{-1} \left[a_4 F_{a_4}(u^{p-1}) + a_5 F_{a_5}(u^{p-1}) - ia_2 \left|\frac{u^{p-1} + u^n}{2}\right|^2 \left(\frac{u^{p-1} + u^n}{2}\right) - \left(\frac{1}{\Delta t}\boldsymbol{I} + i\frac{a_1}{2}\boldsymbol{D}^2 - \frac{a_3}{2}\boldsymbol{D}^3\right) u^n\right] \quad (1.98)$$

In other words, we have to solve a linear system of equations many times per timestep until a stopping criterion is fulfilled, where the matrix to be inverted has a pentadiagonal structure. The fixed point scheme can stop by two reasons: first, if for some $p \in \mathbb{N}$ we have

$$||u^p - u^{p-1}||^2_{L^2(\Omega)} < \hat{\delta}$$

where $\hat{\delta}$ is given. In our computations: $\hat{\delta} = 10^{-6}$. In that case, we do $u^{n+1} = u^p$ and then proceed to the next timestep. The second reason used to stop the iteration is if (1.98) is not a contraction anymore; that is,

$$\frac{||u^p - u^{p-1}||_2}{||u^{p-1} - u^{p-2}||_2} \geq 1 \quad (1.99)$$

In that case, the scheme cannot guarantee the uniqueness of the numerical solution; and thus, the computation ends with no output. This conditions needs of al least three iterations of the fixed-point scheme in order to proceed with the verification. The scheme has linear convergence for Δt sufficiently small.

1.5.2 Single travelling soliton.

As initial condition, we will use the solution found in Potasek and Tabor [PT91b] when $t = 0$, this is,

$$u(x,t) = u_0 e^{it} sech(kx + lt)$$

where $k = 1$, $|u_0|^2 = 6$, and $l = -0.3$; while in (1.3), $a_1 = 3$, $a_2 = 1$, $a_3 = 0.3$, $a_4 = a_5 = 0.1$. In our computations, $t \in [0, 100]$, $x \in [-10, 50]$, $\Delta t = 0.0001$ and $\Delta x = \frac{60}{2^{12}} \approx 0.014$. The form of the travelling soliton can be found in Figure 1.6. The behavior of the preserved quantities can be seen in Figure 1.7; where $||u^0||_2 = 3.464101$ and the difference in time is equal to $5.7802 \cdot 10^{-10}$, and $E^0 = -12.00029$ with a difference in time equal to $6.2904 \cdot 10^{-4}$.

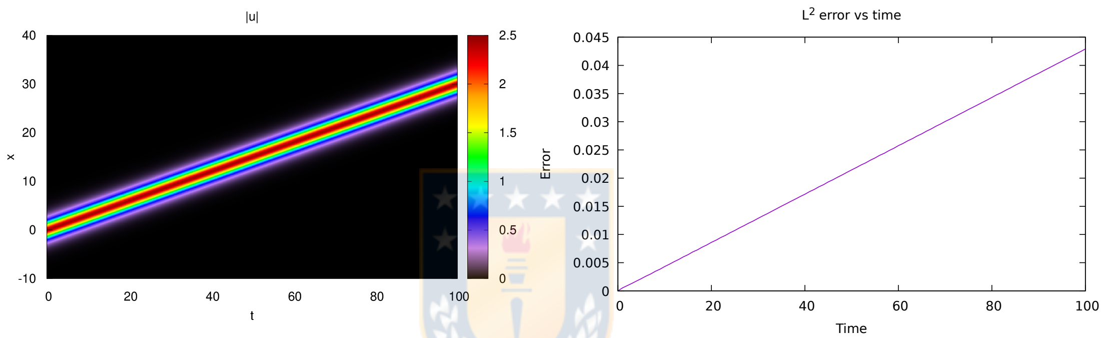

Fig. 1.1: First case results. Left: time evolution of the absolute value of the solution. Right: numerical error of the solution.

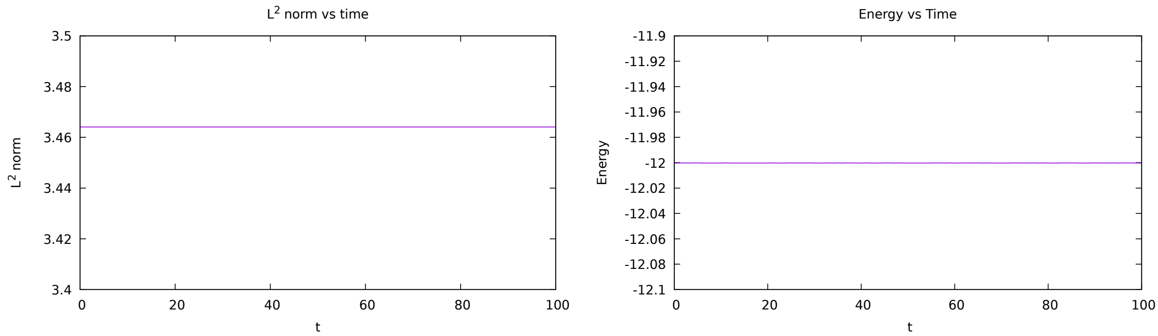

Fig. 1.2: Time evolution of the preserved quantities for the first case. Left: L^2 norm. Right: energy.

1.5.3 Collision of 2 Solitons for a HNLS equation.

In this second example, we will replicate a collision between two solitons. For this we consider the exact solution of Hirota described in [Hir73a]. Then we consider our numerical scheme to approximate the Hirota equation:

$$iu_t + u_{xx} + |u|^2 u + i\frac{1}{10}u_{xxx} + i\frac{3}{10}|u|^2 u_x = 0.$$

for $(x,t) \in [-50, 50] \times (0, 15]$. At that time, we consider as initial condition the solution obtained by Hirota for $t = 0$, that approximately corresponds to the sum of two hyperbolic secants of different amplitudes and centered at distant points. In this way, we calculate the numerical solution described in section 2 of this paper and we compare the result with the exact solution described in Hirota [Hir73a].

Because $3a_2a_3 = a_1a_4$, we conclude that there should not exist any energy decay at L^2 and H^1 levels. For our calculations, we have made $\Delta t = 0.0001$, and $\Delta x = \frac{100}{2^{15}} \approx 0.00305$.

Regarding the error, we can observe that the shape of the numerical solution moves away from the exact solution just at the moment of crossing between both solitons (see Figure 1.5 right). However, surprisingly we can notice that past the crossing, the numerical solution returns to reasonable levels of error ($t > 13$). The time evolution of the preserved quantities can be seen in Figure 1.4, where $\Delta E_{L^2} = 1.216 \times 10^{-8}$ and $\Delta E_{H^1} = 4.088 \times 10^{-5}$.

1.5.4 3 Soliton solution for a modified KdV problem.

We will use as initial condition the 3 soliton solution proposed by Hirota [Hir73b] for a modified Korteweg-de Vries (KdV) equation for $(x,t) \in [-80, 250] \times (0, 1500]$:

$$u_t + 24u^2 u_x + u_{xxx} = 0$$

given that $u(x,t) \in \mathbb{R}$ for thte KdV problem, we can use F_{a_4} or F_{a_5} to approximate the nonlinear term $u^2 u_x$. For the sake of this article, we will use the first one; hence, in (1.1), $a_1 = a_2 = a_5 = 0$, $a_3 = 1$, and $a_4 = 24$. The initial condition is given by the 3-soliton solution proposed by Hirota when $t = 0$:

$$\begin{aligned}
u(x,t) &= \frac{\partial \phi(x,t)}{\partial x} \\
\tan \phi(x,t) &= g(x,t)/f(x,t) \\
f(x,t) &= 1 + a_{12}\exp(\xi_1 + \xi_2) + a_{13}\exp(\xi_1 + \xi_3) + a_{23}\exp(\xi_2 + \xi_3) \\
g(x,t) &= \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_3) + a_{123}\exp(\xi_1 + \xi_2 + \xi_3) \\
a_{123} &= a_{13}a_{13}a_{23} \\
\xi_i &= P_i x + P_i^3 t - C_i, \quad i = 1,2,3 \\
a_{ij} &= -\frac{(P_i - P_j)^2}{(P_i + P_j)^2}, \quad i,j = 1,2,3, \quad i \neq j
\end{aligned}$$

in this calculation, we've used $P_1 = 0.4$, $P_2 = 0.3$, $P_3 = 0.2$, $C_1 = 12$, $C_2 = 0$, $C_3 = -8$, $\Delta x = \frac{330}{2^{12}} \approx 0.056$, $\Delta t = 0.1$. As seen in Figure 1.3, the numerical solution replicates the exact solution with small errors. Numerical L^2 norm is preserved, with a value equal to $||u^0||_2 = 0.67082039$ and a difference in time equal to $1.5419 \cdot 10^{-12}$.

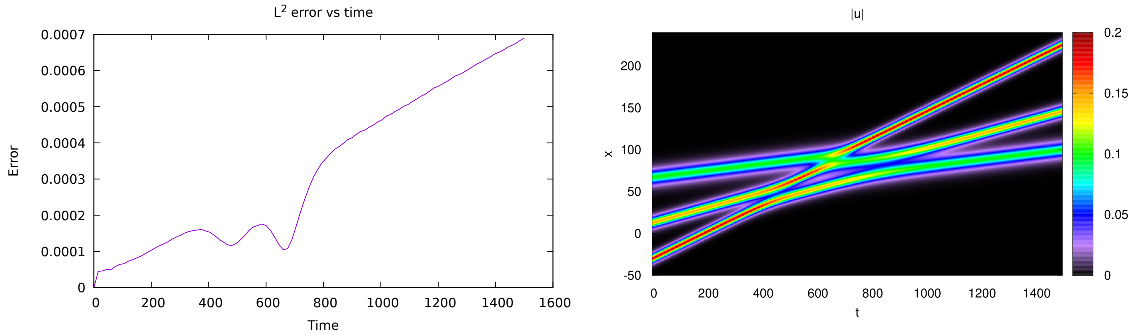

Fig. 1.3: Left: numerical error in function of time. Right: time evolution of the 3-soliton solution for the KdV equation.

1.5.5 HNLS equation with a imaginary parameter.

In the following example, we will solve equation (1.1) using $a_1 = a_2 = 1$, $a_3 = 0.01$, $a_4 = 0.1$ and $a_5 = -0.07 + 0.01i$. Here, $(x, t) \in [-10, 100] \times [0, 100]$ for $\Delta t = 0.001$ and $\Delta x = \frac{110}{2^{12}} \approx 0.0268$. Our intention with this last experiment is to test the code when some of the hypothesis of the theorems proved here are missing: when one of the parameters of the equation is not a real number. As initial condition, we will use again the solution found in Potasek and Tabor [PT91b] when $t = 0$

$$u(x, t) = u_0 e^{it} sech(kx + lt)$$

where $u_0 = \sqrt{2}$, $k = 1$ and $l = 0.0133 - 0.0067i$. Figure 1.6 illustrates the situation. Through the calculations, the unicity condition (1.99) was verified in each timestep, which gives us confidence in our results. As stated in the introduction, when $Im(a_5) \neq 0$, an additional nonlinear effect is added; the solitons travels following a nonlinear variation of the velocity due to a frequency shift. The energy at L^2 level keeps preserved up to a difference of the order of 10^{-3}, but not the one at H^1 level. This should be expected because of the variation of the velocity with time.

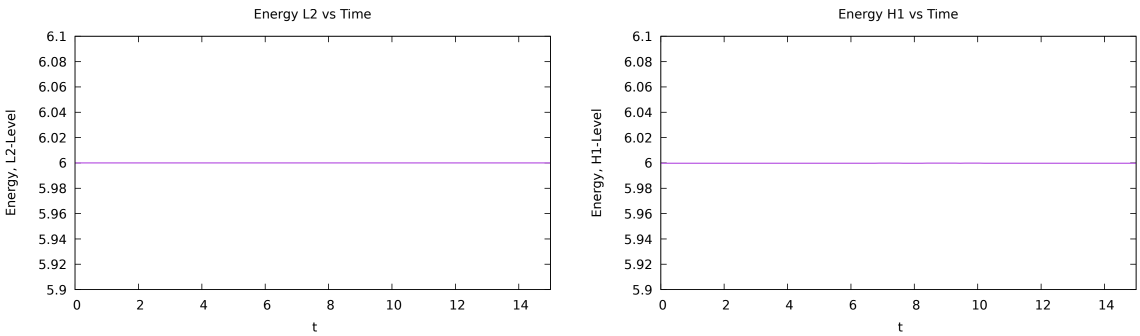

Fig. 1.4: Preserved quantities for the 2 soliton experiment (second case). Left: L^2 level energy. Right: H^1 level energy.

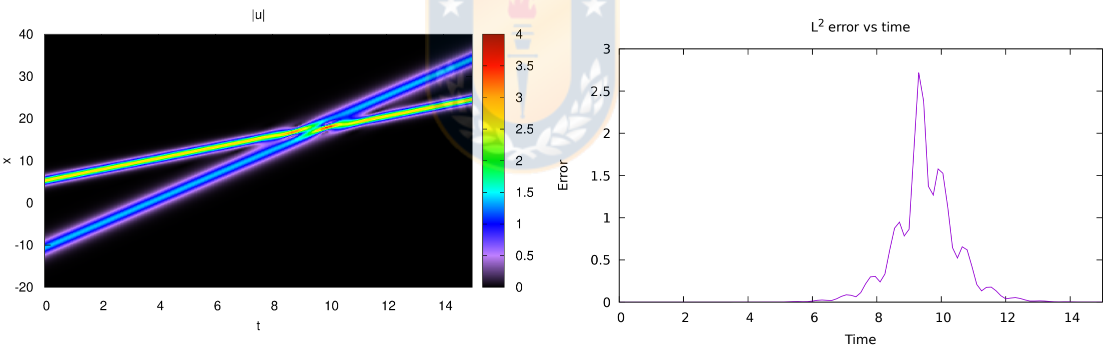

Fig. 1.5: Left: the 2 soliton solution over time. Right: numerical error.

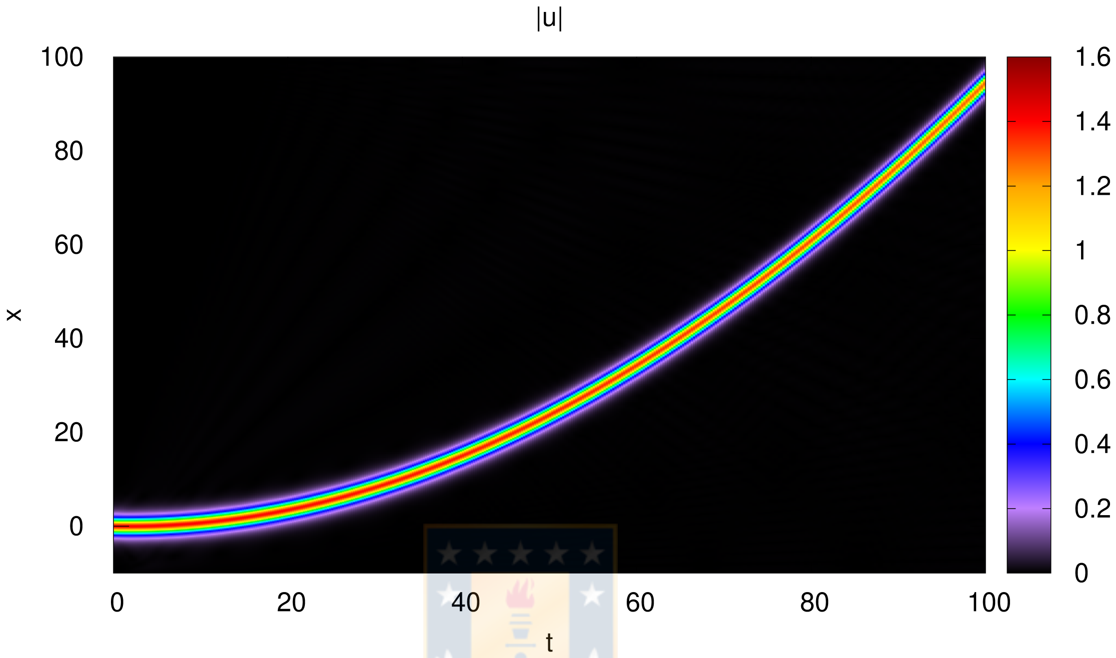

Fig. 1.6: Numerical solution when $Im(a_5) \neq 0$.

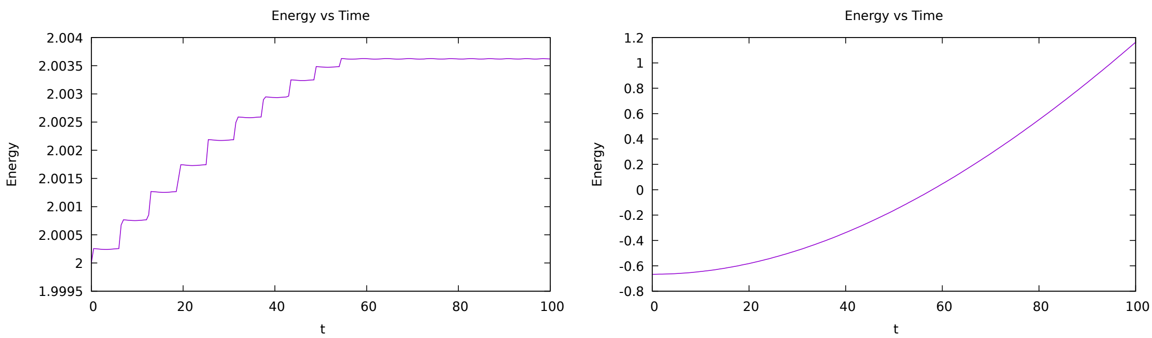

Fig. 1.7: Left: L^2 level energy. Right: H^1 level energy.

Chapter 2

Finite Difference scheme for a HNLS Equation with localized dissipation

2.1 Introduction

2.1.1 Description of the Problem.

In this chapter, we will study the following HNLS equation with damping:

$$\begin{cases} i\,u_t + a_1\,u_{xx} + a_2\,|u|^2\,u + i\left[a_3\,u_{xxx} + a_4\left(|u|^2u\right)_x + a_5\,u\left(|u|^2\right)_x + a(x)u\right] = 0 & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = 0, \qquad u_x(L,t) = 0 & \forall t \geqslant 0 \\ u(x,0) = u_0 & \text{in } (0,L) \end{cases} \tag{2.1}$$

so that the real constants $a_1, a_3 > 0$ and $a_i \neq 0,\ i = 2,4,5$. Let us assume that $a(x)$ is a non-negative real valued function belonging to $L^\infty(0,L)$; moreover, we will assume that

$$a(x) \geqslant a_0 > 0 \ \text{ a.e. in an open, non-empty subset } \omega \text{ of } (0,L), \tag{2.2}$$

where the damping is acting effectively. The main objective of the present chapter is to prove the exponential decay to zero of the L^2-norm of the numerical solutions in problem (2.1); that is, there exist positive constants C, γ, such that

$$||u^n||_2 \leq Ce^{-\gamma\,t_n}||u^0||_2,\ \forall\, t_n \geq T_0, \tag{2.3}$$

where, in this chapter, we will consider the initial data u_0 in bounded sets of $L^2(\Omega)$. This, in turn, is motivated from the result proved in [CCSVA], where the well-posedness of problem (2.1) is demonstrated, along with the exponential decay of the solution at L^2- level.

In what follows we would like to mention some important papers in connection with the subject of the present chapter. Let us start by considering the following initial boundary value problem of the HNLS equation with localized damping:

$$\begin{cases} i\,u_t + a_1\,u_{xx} + |u|^2\,u + i\,a_3\,u_{xxx} + i\,a(x)\,u = 0 & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = 0, \qquad u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,0) = u_0 & \text{in } (0,L), \end{cases} \tag{2.4}$$

where $a_1, a_3 \in \mathbb{R}, a_3 \neq 0$ and the damping $a \in C^\infty(0,L)$ satisfies (2.2). We observe that the problem (2.4) is a particular case of the problem (2.1) considering $a_2 = 1$ and $a_4 = a_5 = 0$. Bisognin et al., [BBV07] proved the exponential decay in L^2- level. Using compactness arguments, the smoothing effect of the KdV equation on the line and the unique continuation results, the authors deduced the exponential decay in time of the solutions of the linear equation and a local uniform stabilization result of the solutions of the nonlinear equation when the localized damping is active simultaneously only in a neighborhood of both extremes $x = 0, x = L$. In order to prove the result, the authors used multipliers together with compactness arguments and the Unique Continuation Principle valid for this problem given in Bisognin and Vera, [BV07].

Later, Alves, Sepúlveda and Vera, [ASV09] studied local and global existence and smoothing properties of the problem (2.4) with $a(x) \equiv 0$. In this situation, the authors verified gain in regularity for this equation. Specifically, they were proved conditions on this problem for which initial data u_0 possessing sufficient decay at infinity and minimal amount of regularity will lead to a unique solution $u(t) \in C^\infty(\mathbb{R})$ for $0 < t < T$, where T is the existence time of the solution.

Ceballos et al.,[CCPVV05], analyzed directly the exact boundary controllability problem for the higher order Schrödinger equation with $a \equiv 0$ by adapting a method which combines the Hilbert Uniqueness Method (HUM) and multiplier techniques.

Chen [CLL79] studied the internal stabilization of a simplified version for the HNLS equation: the nonlinear terms $|u|_x^2 u$ and $(|u|^2u)_x$ are replaced by a linear term u_x. In this case, a result similar to (2.3) was obtained by Chen using Carleman estimates, but nonetheless, there are no numerical experiments replicating the result.

A damping of the type $a(x)u$ was introduced in Menzala et al., [PMVZ02] to stabilize the KdV system inspired in the work of Rosier, [Ros97]. More precisely, considering the damping localized at a subset $\omega \subset (0, L)$ containing nonempty neighborhoods of the end-points of an interval, it was shown that solutions of both linear and nonlinear problems for the KdV equation decay, independently on $L > 0$. In Pazoto, [PSV10] it was proved that the same holds without cumbersome restrictions on $\omega \subset (0, L)$. Linares and Pazoto, [LP09] proved the exponential stabilization of the Korteweg-de Vries equation in the right half-line under the effect of the same localized damping term $a(x)\, u$. Araruna et al., [ACFD12] proved the exponential decay in L^2 - level for the modified Kawahara equation posed in a bounded bounded interval under the presence of a localized damping term $a(x)\, u$ satisfying where the function $a(\cdot)$ satisfies (2.2). Cavalcanti et al., [CCKR14] studied the well-posedness and the asymptotic behavior of solutions of a KdV- Burgers equation subject to a localized dissipation mechanism with indefinite sign:

$$u_t - u_{xx} + u_{xxx} + u\, u_x + \lambda(x)\, u = 0\, x \in \mathbb{R},\, t > 0,\, \lambda \in L^\infty(\mathbb{R})$$

such that a sufficient condition criteria for the exponential decay has been established. Rosier and Zhang [RZ06] obtained similar results for a generalized KdV equation

$$u_t + u_x + a(u)u_x + u_{xxx} + b(x)u = 0,\ x \in [0, L],\ t \geq 0$$

where $b(x)$ is a nonnegative function with support in $\omega \subset [0, L]$, and $a \equiv a(\mu)$ is a given smooth function satisfying the following growth condition

$$a(0) = 0, \quad |a^{(j)}(\mu)| \leq C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R}$$

for $j = 0, 1$ if $1 \leq p < 2$, and for $j = 0, 1, 2$ if $p \geq 2$. It is worth of mention that these results are valid for a damping located everywhere in the domain, and for initial data in $L^2([0, L])$.

The study of decay rate estimates for weakly full damped semilinear focusing and defocusing Schrödinger equations ($a_4 = a_5 = 0$)

$$iy_t + \Delta y \pm |y|^2 y + iay = 0 \text{ in } \Omega \times (0, \infty), \; a > 0, \tag{2.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with zero Dirichlet boundary condition, has been considered by Tsutsumi [Tsu90] where exponential stabilization of H^k-solutions ($k = 1, 2$) is established. For this purpose, smallness on the initial data is assumed. Later on, Özsarı, Kalantarov and Lasiecka [ÖKL11] generalized the previous result mentioned above (at least for the defocusing case) by considering inhomogeneous Dirichlet boundary conditions. Smallness on the initial data is also assumed for proving decay rates estimates in H^2-norm. In H^1-norm, no smallness is required. Indeed, the result for H^1-solutions obtained in [ÖKL11] is strong in the sense that it is independent of the dimension of the domain and the smallness of the initial data.

On the other hand, regarding the exponential stability for the semilinear defocusing Schrödinger equation, subject to a linear damping locally distributed and posed in unbounded domains, namely,

$$iy_t + \Delta y - |y|^2 y + ia(x)y = 0 \text{ in } \mathbb{R}^n \times (0, \infty), \; n = 1, 2, \tag{2.6}$$

(here $a(x) \geq a_0 > 0$ for $||x|| > R > 0$), we would like to mention the works of the authors Cavalcanti et al. [CCFN09], [CCSN10]. In order to achieve the desired goal, the authors make use of two main ingredients in the proof: (i) To establish an unique continuation property associated with regular and mild solutions of the non-damped problem $iy_t + \Delta y - |y|^2 y = 0$ restricted to a fixed ball of radius $r > R$; (ii) To employ a smoothing effect as established, for instance, in Constantin and Saut [CS88]. In the same spirit of Cavalcanti et al., [CCFN09] we can also mention the following works by Natali, [Nat15] regarding the one-dimensional case, and Natali [Nat16] in the two-dimensional case. Another stabiliation result can be found in Muñoz Rivera et al. [MRPS+19] for a problem with double power nonlinearity.

Cavalcanti et al., [CCCT17] proved the existence in H^1-norm as well as the stability for the damped defocusing Schrödinger equation using the following model:

$$\begin{cases} i\,\partial_t\, y + \Delta\, y - |y|^p\, y + i\,\lambda(x,t)\, y = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ y(0) = y_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{2.7}$$

where $n \geq 1, p > 0$. The damping coefficient $\lambda(x,t)$ may vanish at infinity and satisfies the following conditions:

$$\lambda \in C_b([0, \infty); W^{1,\infty}(\mathbb{R}^n)), \quad \lambda(x,t) > 0, \forall\, x \in \mathbb{R}^n, \forall t \geq 0. \tag{2.8}$$

To prove the existence, the authors employed the method devised by Özsari, et al. [ÖKL11]. In particular, when $n = 1$ or $n = 2$, the uniqueness is obtained. Decay estimates for the L^2-norm and $(H^1 \cap L^{p+2})-$norm are established with the help of direct multipliers method, coupled with refined energy estimates and a lower semi-continuity argument.

Concerning the numerical results, and regarding the finite difference method, we have to mention one of the first proposals from Delfour et al [DFP81] to solve the problem (2.5) for $a(x) \equiv 0$. Their main contribution was the way the term $|u|^2u$ was discretized in order to preserve the numerical energy. Meanwhile, Pazoto et al. [PSV10] proposed a finite differences scheme to solve the following KdV problem for $u = u(x,t)$:

$$\begin{aligned} u_t + u_{xxx} + u^4 u_x + u_x + a(x)u = 0, \qquad & (x,t) \in (0,L) \times (0,+\infty) \\ u(0,t) = u(L,t) = 0, \qquad u_x(L,t) = 0, \qquad & t \in (0,+\infty) \\ u(x,0) = u_0(x), \qquad & x \in (0,L), \end{aligned}$$

for $a \in L^\infty(0,L) : a(x) \geq a_0 > 0$, a. e. in Ω, and Ω a nonempty open subset of $(0,L)$. The power nonlinearity $u^4u_x =: F(u)$ was rewritten using algebraic identities widely used in finite differences analysis, and aiming to achieve two conservation properties:

$$(u, F(u)) = 0, \qquad (u, F(u))_x = -\frac{1}{6}|u|_6^6.$$

This was done in order to obtain H_0^1-estimates for the numerical solution of the problem. The exponential decay of the energy at L^2- level was also proved. A similar philosofy was applied in the previous chapter.

For the present chapter, we will modify the numerical scheme proposed previously in (1.14), so it considers the damping function $a(x)$. We will prove that, along with the continuous case, the numerical L^2-norm decays exponentially when $t \to \infty$. Convergence of the numerical solution will be also proved. Those results will be accompanied by numerical experiments.

2.2 Numerical Scheme

The numerical scheme used in this section is a slight modification of (1.14). Because of the difference between (1.3) and (2.1), we are in need to re-write it as

$$iu_t + a_1 u_{xx} + a_2|u|^2u + i\Big[a_3 u_{xxx} + a_4|u|^2u_x + (a_4+a_5)u\left(|u|^2\right)_x + a(x)u\Big] = 0 \qquad (2.9)$$

Let us make the comparison between this expression, and equation (1.3) from the previous chapter:

$$iu_t + a_1 u_{xx} + a_2|u|^2u + i\Big[a_3 u_{xxx} + a_4|u|^2u_x + a_5 u\left(|u|^2\right)_x\Big] = 0$$

The numerical scheme to be proposed will consider the case when $a(x) \equiv 0$ for $x \in \Omega$. As in the previous chapter, we need that the numerical L^2 norm remains preserved when $a(x) \equiv 0$. Keeping this in mind, the numerical method will be defined as follows: for a given $u^0 \in X_M$, and for $u^{n+\frac{1}{2}} := \frac{1}{2}(u^{n+1} + u^n)$, then $u^{n+1} \in X_M, n \in \mathbb{N}$, approximated solution of (2.1) at the time $t_{n+1} = (n+1)\Delta t$, $\Delta t < 1$, can be calculated as

$$\begin{aligned} i\boldsymbol{D}_t u^n + a_1 \boldsymbol{D}^2 u^{n+\frac{1}{2}} + a_2|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}} + ia_3 \boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}} + ia_4 F_{a_4}(u^{n+1}) & \\ +i(a_4+a_5)F_{a_4+a_5}(u^{n+1}) + ia(x)u^{n+\frac{1}{2}} = 0 & \qquad (2.10) \\ u^0 \in X_M \cap L^2_{\Delta x}(0,L) \text{ given.} & \end{aligned}$$

where the functions F_{a_4} and $F_{a_4+a_5}$ are given by

$$F_{a_4}(u^p) := \frac{1}{2}\left|\frac{u^p+u^n}{2}\right|^2 \boldsymbol{D}\left(\frac{u^p+u^n}{2}\right) + \frac{1}{4}\boldsymbol{D}\left(\left|\frac{u^p+u^n}{2}\right|^2 \frac{u^p+u^n}{2}\right) \tag{2.11}$$
$$- \frac{1}{4}\left(\frac{u^p+u^n}{2}\right)^2 \boldsymbol{D}\overline{\left(\frac{u^p+u^n}{2}\right)}$$
$$F_{a_4+a_5}(u^p) := \frac{u^p+u^n}{2}\boldsymbol{D}\left(\left|\frac{u^p+u^n}{2}\right|^2\right) \tag{2.12}$$
$$a \in \mathbb{R}^{M+1} : a_j = a(x_j), \quad x_j = x_0 + j\Delta x, \quad j = 0, 1, \dots, M.$$

Let us mention that this scheme uses $\gamma_0 = \frac{1}{2}$ in (1.14). The third derivative approximation was also changed in order to obtain estimates for D^2u^n. In order to solve this problem in each timestep, a Picard fixed-point iteration is used to treat the nonlinear part. In each iteration, a linear problem is solved until a suitable stopping criteria is fulfilled.

2.2.1 Convergence of the Numerical Solution

We will make use of the same extension operators presented in the previous chapter. The following lemma will show us some bounds of the numerical solution.

Lemma 2.2.1. *Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence in X_M induced by the numerical scheme (2.10) where $3a_3 \geq |3a_4 + 2a_5|$, and let $u^0 \in X_M$. Then, there exist some constant $K = K(T, L) > 0$ such that*

$$||Q_\Delta u_\Delta||^2_{L^\infty(0,T;L^2(0,L))} \leq ||u^0||_2^2, \quad \forall n \in \mathbb{N} \tag{2.13}$$
$$||Q_\Delta^{\frac{1}{2}} \boldsymbol{D}^2 u_\Delta||^2_{L^2(0,T;L^2(0,L))} \leq \frac{1}{2\Delta x}||u^0||_2^2 \tag{2.14}$$
$$||P_\Delta^{\frac{1}{2}} u_\Delta||^2_{L^2((0,T);H_0^1(0,L))} \leq K||u^0||_2^2 \tag{2.15}$$
$$||Q_\Delta^{\frac{1}{2}}(|u|^2 u_x)_\Delta||^2_{L^2(0,T;L^2(0,L))} \leq K||q_\Delta u_\Delta^0||^6_{L^2(0,L)} \tag{2.16}$$
$$||q_\Delta(|u|_x^2)_\Delta||^2_{L^2(0,L)} \leq 32(1+\Delta x^2)||q_\Delta u_\Delta||^2_{L^\infty(0,L)}||p_\Delta u_\Delta||^2_{H_0^1(0,L)} \tag{2.17}$$

Proof. We start by multiplying (2.10) component-wise by $\Delta x \overline{u}_j^{n+\frac{1}{2}}$, sum over j and extract the imaginary part. This will lead us to

$$\frac{1}{2\Delta t}\left(||u^{n+1}||_2^2 - ||u^n||_2^2\right) - a_3 Re\left(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2 + \left(a u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2 = 0 \tag{2.18}$$

We can re-write the second term in (2.18) as

$$Re\left(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2 = -\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2$$

and thus, (2.18) can be written as

$$\frac{||u^{n+1}||_2^2 - ||u^n||_2^2}{2\Delta t} + a_3\left(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\right) + \sum_{j=1}^{M-1} a_j |u_j^{n+\frac{1}{2}}|^2 \Delta x = 0. \tag{2.19}$$

Because the damping function is positive in all its domain, we can conclude (2.13). Multiplying (2.18) by $2\Delta t$, dropping some terms, and summing for $n = 0, 1, \ldots, N$ while considering (2.19), we get

$$a_3\Delta x \sum_{n=0}^{N} ||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 \Delta t \leq \sum_{n=0}^{N} ||u^n||_2^2 - ||u^{n+1}||_2^2 = ||u^0||_2^2 - ||u^{N+1}||_2^2$$

and thus, (2.14) can be concluded. In order to prove (2.15), we need to multiply (1.14) component-wise by $j\Delta x u_j^{n+\frac{1}{2}}$, sum over $j = 0, 1, \ldots, M-1$, and extract the imaginary part. Initially we have

$$\begin{aligned} i\Big(\boldsymbol{D}_t u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + a_1\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_2\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x \\ +ia_3\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_4\Big(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x \\ +i(a_4+a_5)\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x + i\Big(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = 0 \end{aligned} \tag{2.20}$$

We will study each term in (2.20). First, and using the definition (1.8), it is easy to see that

$$Im\left(i\Big(\boldsymbol{D}_t u^n, u^{n+\frac{1}{2}}\Big)_x\right) = \frac{1}{2\Delta t}(||u^{n+1}||_x^2 - ||u^n||_x^2) \tag{2.21}$$

Using (1.27) from the previous chapter, we can write

$$\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_x^2 + \Delta x||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 - \Delta x\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2.$$

Hence,

$$Im\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -\Delta x Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2. \tag{2.22}$$

The following identity is also straightforward

$$Im\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = 0. \tag{2.23}$$

For the third derivative term, we have that

$$\begin{aligned} Im\left(i\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\right) = -\tfrac{\Delta x}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \tfrac{3}{2}||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \\ +\tfrac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_x^2 - \tfrac{\Delta x^2}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 \end{aligned} \tag{2.24}$$

Indeed: for $u \in X_M$, we can write

$$\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u, u\Big)_x = -\Big(\boldsymbol{D}^+\boldsymbol{D}^- u, \boldsymbol{D}^- u\Big)_x + \Delta x\Big(\boldsymbol{D}^+\boldsymbol{D}^- u, \boldsymbol{D}^- u\Big)_2 + ||\boldsymbol{D}^+ u||_2^2 \tag{2.25}$$

Denoting $b := \boldsymbol{D}^- z$, and using (1.30), we can re-write the first term in the right hand side of (2.25) as

$$Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u, \boldsymbol{D}^- u\Big)_x = -\frac{1}{2}||\boldsymbol{D}^- u||_2^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u||_x^2$$

Hence,

$$Re\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u, u\Big)_x = \frac{3}{2}||\boldsymbol{D}^+ u||_2^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u||_x^2 + \Delta x Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u, \boldsymbol{D}^- u\Big)_2. \tag{2.26}$$

On the other hand

$$Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u, \boldsymbol{D}^- u\Big)_2 = -\frac{|D^- u_1|^2}{2} - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u||_2^2,$$

(1.42) can be then obtained combining this last result with (2.26). For the nonlinear terms F_{a_4} and $F_{a_4+a_5}$, we have

$$\begin{aligned}\Big(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x &:= \left(\frac{1}{2}|u^{n+\frac{1}{2}}|^2 \boldsymbol{D}\Big(u^{n+\frac{1}{2}}\Big), u^{n+\frac{1}{2}}\right)_x \\ &+ \left(\frac{1}{4}\boldsymbol{D}\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\Big), u^{n+\frac{1}{2}}\right)_x \\ &- \left(\frac{1}{4}\Big(u^{n+\frac{1}{2}}\Big)^2 \boldsymbol{D}\overline{\Big(u^{n+\frac{1}{2}}\Big)}, u^{n+\frac{1}{2}}\right)_x\end{aligned} \tag{2.27}$$

$$\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x := \left(u^{n+\frac{1}{2}}\boldsymbol{D}\Big(|u^{n+\frac{1}{2}}|^2\Big), u^{n+\frac{1}{2}}\right)_x \tag{2.28}$$

Using (1.29), we have

$$\begin{aligned}\Big(\boldsymbol{D}(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}), u^{n+\frac{1}{2}}\Big)_x = &-\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\Big)_x + \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\Big)_2 \\ &- \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\Big)_2 - \Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\end{aligned} \tag{2.29}$$

and, at the same time,

$$\Big((u^{n+\frac{1}{2}})^2 \boldsymbol{D}\overline{u}^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = \Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\Big)_x \tag{2.30}$$

Combining (2.29) and (2.30) in (2.27), we get

$$\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2 \boldsymbol{D}\left(u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x - \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D} u^{n+\frac{1}{2}}\right)_x$$
$$+ \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2$$
$$- \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2$$

and extracting the real part, we get

$$Re\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \frac{\Delta x}{8} Re\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\right)_2$$
$$- \frac{\Delta x}{8} Re\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^+ u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4 \tag{2.31}$$

and recalling that $\boldsymbol{D}^2 u = \frac{\boldsymbol{D}^+ u - \boldsymbol{D}^- u}{\Delta x}$

$$Re\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = -\frac{\Delta x^2}{8} Re\left(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\right)_x - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4 \tag{2.32}$$

Finally, for the last nonlinear term, we have

$$\left(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \left(u^{n+\frac{1}{2}} \boldsymbol{D}(|u^{n+\frac{1}{2}}|^2), u^{n+\frac{1}{2}}\right)_x$$
$$= \left(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, |u^{n+\frac{1}{2}}|^2\right)_x$$
$$= -\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\right)_x + \frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\right)_2$$
$$- \frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\right)_2 - ||u^{n+\frac{1}{2}}||_4^4$$

and thus,

$$Re\left(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = \frac{\Delta x}{4}\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\right)_2 - \frac{\Delta x}{4}\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\right)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4$$

which, in turn, can be rewritten as

$$Re\left(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = -\frac{\Delta x^2}{8}\left(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\right)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4. \tag{2.33}$$

Combining together (2.21), (2.22), (2.23), (2.24), (2.32) and (2.33); multiplying by Δt, and summing over n, we obtain

$$
\begin{aligned}
\frac{||u^0|_x^2}{2} &= \frac{||u^{N+1}||_x^2}{2} - a_1\Delta x\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^-u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t\\
&+ a_3\Bigg(- \frac{\Delta x}{2}\sum_{n=0}^{N}|D^-u_1^{n+\frac{1}{2}}|^2\Delta t + \frac{3}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t\\
&+ \frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_x^2\Delta t - \frac{\Delta x^2}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Delta t\Bigg)\\
&- a_4\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\big)_2\Delta t\\
&- (a_5)\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N}\big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\big)_2\Delta t\Bigg)\\
&- \frac{a_4+2a_5}{4}\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_4^4\Delta t + \sum_{n=0}^{N}\Big(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\Delta t
\end{aligned}
$$

Let us recall the fact that $a_1, a_3 > 0$. This can let us drop some terms in the above equality to get

$$
\begin{aligned}
\frac{||u^0||_x^2}{2} &\geq -a_1\Delta x\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^-u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t + \frac{3a_3}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t\\
&- a_3\frac{\Delta x^2}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Delta t - a_3\frac{\Delta x}{2}\sum_{n=0}^{N}|D^-u_j^{n+\frac{1}{2}}|^2\Delta t\\
&- a_4\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\big)_2\Delta t\\
&- (a_5)\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N}\big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\big)_2\Delta t\Bigg)\\
&- \frac{a_4+2a_5}{4}\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_4^4\Delta t
\end{aligned}
\tag{2.34}
$$

As in the previous chapter: using (1.24) and (1.25) along with Young, Cauchy-Schwarz and Hölder inequalities, and for $T = N\Delta t$, we have

$$
\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^-u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t \leq \frac{1}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t + \frac{T}{2}||u^0||_2^2 \tag{2.35}
$$

$$
\sum_{n=0}^{N} Re\Big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\Big)_2\Delta t \leq \frac{1}{\Delta x}\sum_{n=0}^{N}\Big(||u^{n+\frac{1}{2}}||_2^4 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Big)\Delta t. \tag{2.36}
$$

Using (1.25), replacing (2.35) and (2.36) in (2.34), and summing by parts in the sixth term at the right hand side of (2.34),

$$\frac{||u^0||_x^2}{2} + a_1\frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t + a_1 T\frac{\Delta x}{2}||u^0||_2^2$$
$$+ a_3\frac{\Delta x^2}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Delta t + a_3\frac{\Delta x}{2}\sum_{n=0}^{N}|D^-u_j^{n+\frac{1}{2}}|^2\Delta t$$
$$+ |a_4|\frac{\Delta x}{8}\sum_{n=0}^{N}\left(||u^{n+\frac{1}{2}}||_2^4 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\right)\Delta t$$
$$- a_5\frac{\Delta x^2}{8}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t$$
$$+ \frac{1}{2}|a_4 + 2a_5|\left(\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_2^6 + ||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t\right) \tag{2.37}$$
$$\geq \frac{3a_3}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t$$

Meanwhile, let us recall equality (2.19). Multiplying it by Δt, and summing from $n = 0$ to N, we get

$$-a_3\sum_{n=0}^{N}|\boldsymbol{D}^-u_1^{n+\frac{1}{2}}|^2\Delta t - a_3\frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Delta t \tag{2.38}$$
$$= \frac{||u^{N+1}||_2^2 - ||u^0||_2^2}{2} + \sum_{n=0}^{N}\left(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2\Delta t \geq -\frac{1}{2}||u^0||_2^2$$

therefore,

$$||u^0||_x^2 + \left(a_1\Delta x + (2a_3 + |a_4|)\frac{\Delta x}{4}||u^0||_2^2 + |a_4 + 2a_5|||u^0||_2^4\right)T||u^0||_2^2$$
$$+ \left(a_1\Delta x + |a_4|\frac{\Delta x}{4} - a_5\frac{\Delta x^2}{4} + |a_4 + 2a_5|\right)\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t$$
$$\geq 3a_3\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t$$

From here, because $3a_3 \geq |a_4 + 2a_5|$, $||u^0||_x^2 \leq L||u^0||_2^2$, and considering $\Delta x \ll 1$ we can infere the existence of the needed constant $K = K(T, L)$ such that (2.15) holds. To prove (2.16), let us first note that:

$$\left|\left||u^{n+\frac{1}{2}}|^2\boldsymbol{D}u^{n+\frac{1}{2}}\right|\right|_2^2 = \sum_{j=0}^{M-1}|u_j^{n+\frac{1}{2}}|^4|Du_j^{n+\frac{1}{2}}|^2\Delta x \leq ||u^{n+\frac{1}{2}}||_\infty^4\sum_{j=0}^{M-1}|Du_j^{n+\frac{1}{2}}|^2\Delta x.$$

Hence, using (2.13) and (2.15),

$$\sum_{n=0}^{N}\left|\left||u^{n+\frac{1}{2}}|^2\boldsymbol{D}u^{n+\frac{1}{2}}\right|\right|_2^2\Delta t \leq ||u^0||_2^4\sum_{n=0}^{N}||\boldsymbol{D}u^{n+\frac{1}{2}}||_2^2\Delta t \leq K||u^0||_2^6$$

proving then (2.16). To prove (2.17), we will again use the identity $(a^2-b^2)+(a-b)^2 = 2a(a-b)$. For a $u_j \in u$, $i = 0, 1, \ldots, M$,

$$D|u_j|^2 = |u_j|D^-|u_j| - \frac{\Delta x^2}{2}(D^-|u_j|)^2 + |u_j|D^+|u_j| + \frac{\Delta x^2}{2}(D^+|u_j|)^2.$$

Taking the square at both sides, using inverse triangle inequality, and $D^2|u_j| \le 4\frac{||u||_\infty}{\Delta x^2}$, and summing over j will lead us to

$$\sum_{j=0}^{M-1}(D|u_j|^2)^2\Delta x \le 32||u||_\infty^2||\boldsymbol{D}^+u||_2^2(1+\Delta x^2)$$

and thus concluding the proof of the Lemma. □

Now we are in conditions to state and prove the followig theorem:

Theorem 2.2.2. *Let $u_\Delta = \{u_m^n\}_{m\in\mathbb{N}}$ a sequence in X_M of solutions induced by the numerical scheme (2.10), at a time $t_n = n\Delta t$, computed from a sequence of initial conditions $\{u_m^0\}_{m\in\mathbb{N}} \subset X_M$ using a timestep Δt and a spacestep Δx. If $u^0 \in X_M$ and $6a_3 \ge |3a_4 + 2a_5|$, then there is a subsequence, still denoted by $\{u_m^n\}_{m\in\mathbb{N}}$, such that*

$$Q_\Delta u_\Delta \to u \text{ strongly in } L^2(0,T;L^2(0,L)), \qquad \text{when } \Delta t, \Delta x \to 0,$$

where u is the weak solution of (2.1)

Proof. We infer the existence of a u such that $Q_\Delta u_\Delta \to u$ weakly in $L^2(0,T;L^2(0,L))$. From (2.13) and (2.15), we can also say that there exists a $u \in L^2(0,T;H_0^1(0,L))$ such that $\{Q_\Delta u_\Delta\}$ is bounded in $L^2(0,T;H_0^1(0,L))$ and thus

$$Q_\Delta u_\Delta \overset{\star}{\rightharpoonup} u \text{ weak star in } L^2(0,T;H_0^1(0,L)) \tag{2.39}$$

From (1.26) and (2.15), we have

$$\{Q_\Delta^{\frac{1}{2}}(|u|^2u)_\Delta\} \text{ is bounded in } L^2(0,T;L^2(0,L)) \tag{2.40}$$

And from (2.16) and (2.17),

$$\begin{cases}\{Q_\Delta F_{a_4}(u)_\Delta\} \\ \{Q_\Delta F_{a_4+a_5}(u)_\Delta\}\end{cases} \text{ are bounded in } L^2(0,T;L^2(0,L)) \tag{2.41}$$

Let us now consider a $\varphi \in H_0^2(0,L)$, with $\varphi_j^n = \varphi(x_j,t_n)$, $0 \le n \le N$, $0 \le j \le M$, . Multiplying (2.10) by $\Delta t\Delta x\overline{\varphi}_j$, sum over j and then sum over n. We then get

$$\begin{aligned}\sum_{n=0}^{N}\Big(\boldsymbol{D}_t u_m^n,\varphi\Big)_2\Delta t &= ia_1\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^-u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t - a_3\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^-u_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t \\ &+ a_2\sum_{n=0}^{N}\Big(|u_m^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}},\varphi\Big)_2\Delta t - a_4\sum_{n=0}^{N}\Big(F_{a_4}(u_m^{n+1}),\varphi\Big)_2\Delta t \\ &- (a_4+a_5)\sum_{n=0}^{N}\Big(F_{a_4+a_5}(u_m^{n+1}),\varphi\Big)_2\Delta t - \sum_{n=0}^{N}\Big(au_m^{n+\frac{1}{2}},\varphi\Big)_2\Delta t\end{aligned} \tag{2.42}$$

Our aim is to prove that the left hand side of (2.42) is bounded. From (2.13) and (2.15), and summing by parts, we get

$$\begin{aligned}\sum_{n=0}^{N}\left(\boldsymbol{D}^{+}\boldsymbol{D}^{-}u_m^{n+\frac{1}{2}},\varphi\right)_2\Delta t+\sum_{n=0}^{N}\left(\boldsymbol{D}^{+}\boldsymbol{D}^{+}\boldsymbol{D}^{-}u_m^{n+\frac{1}{2}},\varphi\right)_2\Delta t \\ \leq \sum_{n=0}^{N}||\boldsymbol{D}^{+}u_m^{n+\frac{1}{2}}||_2||\boldsymbol{D}^{+}\varphi||_2\Delta t+\sum_{n=0}^{N}||\boldsymbol{D}^{+}u_m^{n+\frac{1}{2}}||_2||\boldsymbol{D}^{+}\boldsymbol{D}^{-}\varphi||_2\Delta t \\ \leq C_\varphi\left(\sum_{n=0}^{N}||\boldsymbol{D}^{+}u_m^{n+\frac{1}{2}}||_2\Delta t+\sum_{n=0}^{N}||\boldsymbol{D}^{+}u_m^{n+\frac{1}{2}}||_2\Delta t\right) \\ \leq 2C_\varphi K||u_m^0||_2\end{aligned}$$

since we are considering any $\varphi\in H_0^2(0,L)$, and combining (2.39), (2.40) and (2.41) after using Cauchy-Schwarz Inequality in (2.42) , we get

$$\left\{\frac{\partial}{\partial t}P_\Delta u_\Delta\right\}\ \text{is bounded in}\ \ L^2(0,T;H^{-2}(0,L)).$$

Using $H_0^1(0,L)\overset{c}{\hookrightarrow}L^2(0,L)\hookrightarrow H^{-2}(0,L)$, and employing Aubin-Lions Theorem, there exists a subsequence of $\{u_m^n\}_{m\in\mathbb{N}}$, still denoted by the same form, such that,

$$Q_\Delta u_\Delta\longrightarrow u\quad\text{strongly in}\quad L^2(0,T;L^2(0,L))\,. \tag{2.43}$$

Now we will prove that u is the weak solution of (2.1). Thanks to (2.43), we have

$$|u_m^{n+\frac{1}{2}}|u_m^{n+\frac{1}{2}}\longrightarrow|u|^2u,\qquad\text{a.e. in }(0,L)\times(0,T) \tag{2.44}$$

using (2.44), and recalling again Lion's lemma [Lio69], we will get

$$Q_\Delta^{\frac{1}{2}}(|u|^2u)_\Delta\rightharpoonup|u|^2u\text{ weakly in }L^2(0,T;L^2(0,L)).$$

furthermore, combining (2.43) and (2.39),

$$\begin{aligned}Q_\Delta(F_{a_4}(u))_\Delta&\rightharpoonup\frac{1}{2}|u|^2u_x+\frac{1}{4}(|u|^2u)_x-\frac{1}{4}u^2\overline{u}_x\ \text{ weakly in }\ L^2(0,T;L^2(0,L))\\ Q_\Delta(F_{a_4+a_5}(u))_\Delta&\rightharpoonup u|u|_x^2\ \text{ weakly in }\ L^2(0,T;L^2(0,L))\end{aligned}$$

Multiplying componentwise the numerical scheme (2.10) by $\Delta x\Delta t\phi_k^n$, sum by parts, and passing to the limit, is easy to see that $u=u(t_n)$ is, indeed, the weak solution of problem (2.1), and hence the Theorem is proved. □

2.2.2 Exponential Decay.

We will now prove the exponential decay of the numerical energy.

Theorem 2.2.3. *Consider a sequence $\{u^N\}_{N\in\mathbb{N}}\subset X_M$ induced by the numerical scheme (2.10), and consider the function $a(x)$ and the set ω as defined in (2.2). If $u^0\in X_M$, $3a_3\geq|3a_4+2a_5|$, and for $T_0=n\Delta t>0$, there exist a positive constant $C=C(T_0)$ and $\mu=\mu(T_0)$, both independent of Δx and Δt, such that the inequality*

$$||u^n||_2^2\leq C||u^0||_2^2e^{-\mu n\Delta t}$$

holds for all $n>0$.

From here, we will consider $T_0 = N\Delta t$ fixed with $N \in \mathbb{N}$. Before proceeding with this result, we will state and prove the next proposition:

Proposition 2.2.4. *Let $T_0 \geq 0$, $u = u(x,t)$ a solution of problem (1.1), and $a(x) \in H^1(0,L)$. If u is such that*

$$u \equiv 0 \text{ in } \omega \times (0,L)$$

then $u \equiv 0$ in $(0,L) \times (0,T_0)$.

Proof. Consider the following IVP for real functions $r = r(x,t)$ and $g = g(r)$:

$$\begin{cases} r_t + a_3\, r_{xxx} = g & (x,t) \in (0,L) \times (0,T_0) \\ r(0,t) = r(L,t) = r_x(L,t) = 0 & t \in (0,T_0) \\ r(x,0) = r_0(x) & x \in (0,L) \end{cases} \tag{2.45}$$

Regarding (2.45), the next property was proved in Chen [Che18], Proposition 3.1:

Proposition 2.2.5. *Consider $T_0 > 0$. There exists two constants $C > 0$ and $s_0 > 0$ such that for any $h \in L^2(0,T_0;\, L^2(0,L))$, any $r_0 \in L^2(0,L)$ and any $s \geq s_0$, the solution r of the problem (2.45) fulfills*

$$\begin{aligned} &\int_\varepsilon^{T_0} \int_0^L \left[s\,\psi_\varepsilon\, |r_{xx}|^2 + s^3\,\psi_\varepsilon\, |r_x|^2 + s^5\,\psi_\varepsilon^5\, |r|^2 \right] \mathrm{e}^{-2\,s\,\psi_\varepsilon}\, dx\, dt \\ &\leq C \left(\int_\varepsilon^{T_0} \int_\omega \left[s\,\psi_\varepsilon\, |r_{xx}|^2 + s^3\,\psi_\varepsilon\, |r_x|^2 + s^5\,\psi_\varepsilon^5\, |r|^2 \right] \mathrm{e}^{-2\,s\,\psi_\varepsilon}\, dx\, dt \right. \\ &\left. + \int_\varepsilon^{T_0} \int_0^L |g|^2\, \mathrm{e}^{-2\,s\,\psi_\varepsilon}\, dx\, dt \right), \; \varepsilon \in (0,T_0)\,. \end{aligned}$$

where the function ψ_ε is defined as follows: for $(l_1,l_2) \subset \omega$ such that $0 < l_1 < l_2 < L$., and for any $\varphi \in C^3([0,L])$ given by

$$\begin{cases} \varphi > 0 \text{ in } [0,L]; \\ |\varphi'| > 0,\; \varphi'' < 0,\; \varphi' \cdot \varphi'' < 0 \text{ in } [0,L]\backslash(l_1,l_2); \\ \varphi'(0) < 0 \text{ and } \varphi'(L) > 0; \\ \min\limits_{x \in [l_1,l_2]} \varphi(x) = \varphi(l_3) < \max\limits_{x \in [l_1,l_2]} \varphi(x) = \varphi(l_1) = \varphi(l_2), \\ \max\limits_{x \in [0,L]} \varphi(0) = \varphi(L) \text{ and } \varphi(0) < \dfrac{4}{3}\,\varphi(l_3) \text{ for some } l_3 \in (l_1,l_2)\,. \end{cases}$$

The existence of the function φ can be found in Capistrano-Filho et al. [CFPR15]. The function ψ_ε is then given by

$$\psi_\varepsilon(x,t) = \frac{\varphi(x,t)}{(t-\varepsilon) \cdot (T-\varepsilon)}\,.$$

Following the methods developed in [CFPR15], Proposition 2.2.5 holds for the following complex equation:

$$\begin{cases} u_t + a_3\, u_{xxx} = g & (x,t) \in (0,L) \times (0,T_0) \\ u(0,t) = y(L,t) = u_x(L,t) = 0 & t \in (0,T_0) \\ u(x,0) = y_0(x) & x \in (0,L) \end{cases} \tag{2.46}$$

where

$$g(u) = i\,a_1\,u_{xx} + i\,a_2\,|u|^2\,u - a_4\,\left(|u|^2\,u\right)_x - a_5\,u\,\left(|u|^2\right)_x - a(x)\,u\,.$$

Then,

$$\begin{aligned} &\int_\varepsilon^{T_0}\int_0^L \left[s\,\psi_\varepsilon\,|u_{xx}|^2 + s^3\,\psi_\varepsilon\,|u_x|^2 + s^5\,\psi_\varepsilon^5\,|u|^2\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \\ &\leq C\left(\int_\varepsilon^{T_0}\int_\omega \left[s\,\psi_\varepsilon\,|u_{xx}|^2 + s^3\,\psi_\varepsilon\,|u_x|^2 + s^5\,\psi_\varepsilon^5\,|u|^2\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \right. \\ &\left. + \int_\varepsilon^{T_0}\int_0^L |g|^2\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt\right),\ \varepsilon\in\,(0,T_0), \end{aligned} \tag{2.47}$$

We observe that

$$\begin{aligned} \int_\varepsilon^{T_0}\int_0^L |g|^2\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \leq C\int_\varepsilon^{T_0}\int_0^L \left[\,|u_{xx}|^2 + |u|^2\,\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt + \int_\varepsilon^{T_0}\int_0^L |u|^6\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \\ + \int_\varepsilon^{T_0}\int_0^L \left|\left(|u|^2\,u\right)_x\right|^2\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt + \int_\varepsilon^{T_0}\int_0^L \left|u\,\left(|u|^2\right)_x\right|^2\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \\ := \int_\varepsilon^{T_0}\int_0^L \left[\,|u_{xx}|^2 + |u|^2\,\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt + I\,. \end{aligned} \tag{2.48}$$

Using the arguments from Chen [Che18], it can be proved that for $T_0 > 0$, $s \in [0,3]$, and assuming that $a \in H^1(0,L)$, then for all $u_0 \in L^2(0,L)$., the solution of the problem (1.1) satisfies

$$u \in C(\varepsilon, T;\, H^3(0,L)) \cap L^2(\varepsilon, T;\, H^4(0,L)),\ \text{ for any } 0 < \varepsilon < T.$$

Due to this local strong smoothing effect, we can infer the existence of a constant $C > 0$ and $\tilde{C}$ such that

$$\begin{aligned} I &\leq C\,\|u\|^4_{L^\infty((0,L)\,\times\,(\varepsilon,T_0))}\int_\varepsilon^{T_0}\int_0^L \left[\,|u_x|^2 + |u|^2\,\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \\ &\leq \tilde{C}\left(\|u_0\|_{L^2(0,L)}\right)\int_\varepsilon^{T_0}\int_0^L \left[\,|u_x|^2 + |u|^2\,\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt\,. \end{aligned} \tag{2.49}$$

Combining (2.48) and (2.49), we obtain

$$\int_\varepsilon^{T_0}\int_0^L |g|^2\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \leq \tilde{C}\left(\|u_0\|_{L^2(0,L)}\right)\int_\varepsilon^{T_0}\int_0^L \left[\,|u_{xx}|^2 + |u_x|^2 + |u|^2\,\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt\,. \tag{2.50}$$

Taking s_0 large enough, the estimate (2.50) can be absorbed by the left-hand side of (2.47). Thus, it follows that

$$\begin{aligned} &\int_\varepsilon^{T_0}\int_0^L \left[s\,\psi_\varepsilon\,|u_{xx}|^2 + s^3\,\psi_\varepsilon\,|u_x|^2 + s^5\,\psi_\varepsilon^5\,|u|^2\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt \\ &\leq C\left(\int_\varepsilon^{T_0}\int_\omega \left[s\,\psi_\varepsilon\,|u_{xx}|^2 + s^3\,\psi_\varepsilon\,|u_x|^2 + s^5\,\psi_\varepsilon^5\,|u|^2\right]\,\mathrm{e}^{-2\,s\,\psi_\varepsilon}\,dx\,dt\right)\,. \end{aligned}$$

Since $u \equiv 0$ in $\omega\,\times\,(0,T_0)$, we infer that $u \equiv 0$ in $(0,L)\,\times\,(\varepsilon,T_0)$. By arbitrarily of $\varepsilon > 0$, we conclude that $u \equiv 0$ in $(0,L)\,\times\,(0,T_0)$ and Proposition 2.2.4 is proved. □

For the sake of the proof of this theorem and the lemma that is left to be presented, we will consider the sequence $\{u^N\}_{N\in\mathbb{N}}$ induced by the numerical scheme (2.10) such that $u^n = u(T_0 = N\Delta t)$. The proof of Theorem 2.2.3 follows after proving the following lemma:

Lemma 2.2.6. *Let $T_0 = N\Delta t$ fixed with $N \in \mathbb{N}$, and let $\{u^N\}_{N\in\mathbb{N}}$ a sequence in X_M induced by the numerical scheme (2.10) such that $||u^0||_2^2 < \infty$ and using $3a_3 \geq |3a_4 + 2a_5|$. Then, there exist a constant $C = C(T_0)$, independent of Δt and Δx, such that*

$$||u^0||_2^2 \leq C\left(a_3 \sum_{n=0}^{N}\left(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\right)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t\right) \tag{2.51}$$

Proof. Let $N \in \mathbb{N}$, and consider the numerical scheme (2.10). Multiplying it by $(N+1-n)\overline{u}^{n+\frac{1}{2}}\Delta t$ componentwise, extracting the imaginary part and summing over $n = 0, 1, \ldots, N$, we get

$$\begin{aligned}||u^0||_2^2 \leq \frac{1}{2T_0}\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t + a_3\sum_{n=0}^{N}\left(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\right)\Delta t \\ + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t.\end{aligned} \tag{2.52}$$

In order to prove (2.51) then, we must to prove the existence of a constant $C_1 = C_1(T_0)$ such that

$$\begin{aligned}\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t \leq C_1\Bigg(a_3\sum_{n=0}^{N}\left(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\right)\Delta t \\ + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t\Bigg).\end{aligned} \tag{2.53}$$

We will proceed by contradiction. Hence, we must assume as true the opposite of (2.53). Since $||Q_\Delta u_\Delta||_{L^\infty(0,T;L^2(0,L))} < \infty$, we can extract a subsequence $\{u^{N_m}\}_{m\in\mathbb{R}}$, still denoting it by $\{u^N\}_{N\in\mathbb{N}}$, such that

$$\lim_{\Delta x,\Delta t\to 0}\frac{\sum\limits_{n=0}^{N}||u^{n+1}||_2^2\Delta t}{a_3\sum\limits_{n=0}^{N}(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2)\Delta t + \sum\limits_{n=0}^{N}\sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t} = +\infty \tag{2.54}$$

Let $\lambda^N \geq 0, \forall N \in \mathbb{N}$ such that $(\lambda^N)^2 = \sum_{k=0}^{N+1}||u^k||_2^2\Delta t$, and let us define $v^n := \frac{u^n}{\lambda^n}$ for some $n \in \mathbb{N}$. This induces the following sequence of numerical problems: find $v^{n+1} \in X_M$ such that

$$\begin{aligned}0 &= i\boldsymbol{D}_t v^n + a_1\boldsymbol{D}^2 v^{n+\frac{1}{2}} + a_2(\lambda^n)^2|v^{n+\frac{1}{2}}|^2 v^{n+\frac{1}{2}} + ia_3\boldsymbol{D}^3 v^{n+\frac{1}{2}} \\ &\quad + (\lambda^n)^2 F_{a_4}(v^{n+1}) + (\lambda^n)^2 F_{a_4+a_5}(v^{n+1}) + iav^{n+\frac{1}{2}} \\ v^0 &= u^0,\ u^0 \in X_M\end{aligned}$$

where

$$\sum_{n=0}^{N} ||v^n||_2^2 \Delta t = 1 \tag{2.55}$$

Because $||u^0||_2 < \infty$, when $\Delta x, \Delta t \to 0$ in (2.54) we have to consider

$$a_3 \sum_{n=0}^{N} \Big(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x ||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Big)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j |u_j^{n+\frac{1}{2}}|^2 \Delta x \Delta t \to 0 \tag{2.56}$$

and due to (2.52), we conclude that $||v^0||_2$ is bounded. And by Theorem 2.2.2, we can extract a subsequence from $\{v^N\}_{N\in\mathbb{N}}$, still denoted by the same way, that $v^N \to v(t_N)$ strongly on $L^2(0,T_0;L^2(0,L))$, and by (2.55),

$$||v(t)||_{L^2(0,T_0,L^2(0,L))} = 1. \tag{2.57}$$

When passing to the limit in (2.56), we have

$$\begin{aligned} 0 &= \lim_{\Delta x,\Delta t\to 0} a_3 \sum_{n=0}^{N} \Big(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x ||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Big)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j |u_j^{n+\frac{1}{2}}|^2 \Delta x \Delta t \\ &= \int_0^{T_0} |v(0,t)|^2 dt + 2\int_0^{T_0}\int_0^L a(x)|v|^2 dx dt, \end{aligned}$$

and thus, $v(x,t) \equiv 0$ for $(x,t) \in (\omega \times (0,T_0))$. From here, we must distinguish two scenarios:

Case 1: Let us extract a subsequence from $\{\lambda^N\}_{N\in\mathbb{N}}$, denoted by the same way, such that $\lambda^N \to 0$ when $N \to \infty$. This induces the following linear problem:

$$\begin{aligned} iv_t + a_1 v_{xx} + ia_3 v_{xxx} + iav = 0, &\quad (x,t) \in (0,L)\times(0,T_0) \\ v(0,t) = v(L,t) = 0 & \\ v_x(L,t) = 0, &\quad t \in (0,T_0) \\ v(t=0) = u^0, &\quad u^0 \in L^2(0,L) \\ v(x,t) \equiv 0, &\quad (x,t) \in \omega\cdot(0,T_0) \end{aligned}$$

And by Holgrem's Theorem, we conclude that $v(x,t) \equiv 0$, for $(x,t) \in (0,L)\times(0,T_0)$, which contradicts (2.57).

Case 2: There is a subsequence from $\{\lambda^N\}_{N\in\mathbb{N}}$, still denoted by λ^N; and there is a $\lambda > 0$ such that $\lambda^N \to \lambda$. Thus, the sequence $\{v^N\}_{N\in\mathbb{N}}$ converges to the IVP (2.1), while $v(x,t) \equiv 0, (x,t) \in \omega\cdot(0,T_0]$. Due to Proposition 2.2.4, we conclude that $v \equiv 0$, $x \in (0,L)$, $t \in (0,T_0)$, which is again a contradiction.

This allows to conclude that the opposite of (2.53) is false, and hence, the lemma is proved. □

2.3 Numerical Examples

We will finally present some computational results using the numerical scheme proposed in this section.

2.3.1 Initial condition from Potasek and Tabor.

For a first numerical result, we will work with the following HNLS equation for $u = u(x,t), (x,t) \in (-100, 10) \times (0, 1000]$:

$$iu_t + 3u_{xx} + |u|^2 u + i\Big(0.03u_{xxx} + 0.05|u|^2 u_x - 0.025|u|_x^2 u + a(x)u\Big) = 0$$
$$u(x,0) = u^0(x) = A\, sech(kx) \qquad (2.1)$$

The initial condition is given by the analytical solution of the IVP (2.1) when $a(x) \equiv 0$, proposed by Potasek and Tabor [PT91b]; this is,

$$u(x,t) = A\, e^{int} sech(kx + lt)$$

where, for $\alpha = -2a_1$, $\rho = a_5$, $\delta = a_4 + \rho$, and $k = 1$,

$$\epsilon = -\frac{1}{6}\alpha(\delta + \rho), \quad l = -\epsilon k^3, \quad n = -\frac{1}{2}\alpha k^2, \quad |u^0|^2 = -\alpha k^2$$

On the other hand, $a(x) = 0.005$, $x < -5$, and in our calculations, $\Delta t = 0.0001$ and $\Delta x = \frac{110}{2^{12}} \approx 0.0268$.

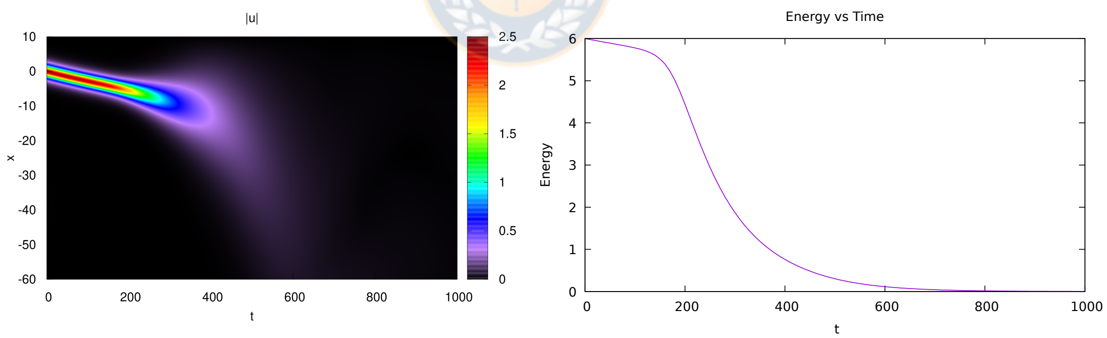

Fig. 2.1: First case results. Left: time evolution of the absolute value of the solution. Right: evolution of the energy.

As shown in Figure 3.1 right, the energy decays at an exponential rate to zero, which is what we expected from Theorems (**??**) and (2.2.3). While Figure 3.1 left shows how the soliton is getting *dissipated* after entering the damping zone, starting at $x = -5$.

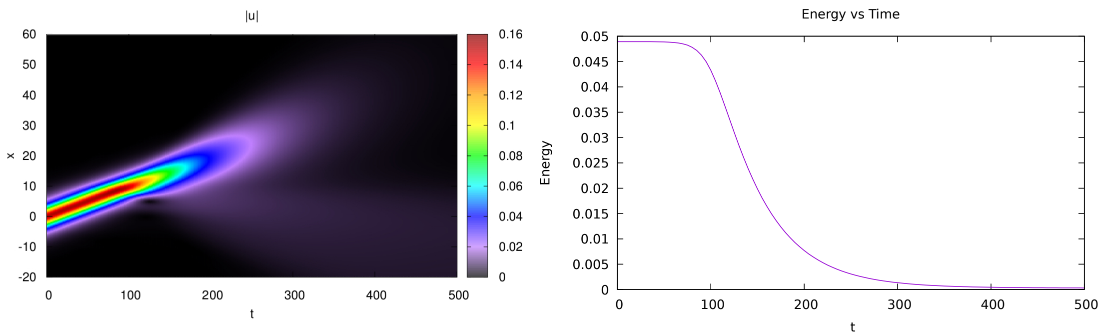

Fig. 2.2: Second case results. Left: time evolution of the absolute value of the solution. Right: evolution of the energy.

2.3.2 Initial condition from Kumar.

A last case will be presented, regarding the following equation for $u = u(x,t)$, $(x,t) \in (-60, 80) \times (0, 500]$

$$iu_t + 0.1u_{xx} + 2|u|^2u + i\Big(0.001u_{xxx} + 0.01|u|^2u_x + 0.1|u|_x^2u + a(x)u\Big) = 0$$
$$u(x,0) = u^0(x) = A\,e^{ix}\,sech(-Bx) \qquad (2.2)$$

where we used an initial condition based on a solution propsed by Kumar and Chand [KC13] when $a(x) \equiv 0$:

$$u(x,t) = A\,e^{i(-kt+\omega x)}\,sech(B(t-x))$$

where $v = 10$, $k = 0.001$, and

$$B = \pm\sqrt{\frac{k - a_1\omega^2 + a_3\omega^3}{3a_3\omega - a_1}} \qquad A = \pm\sqrt{\frac{2(k - a_1\omega^2 + a_3\omega^3)}{a_4\omega - a_2}} \qquad \omega = \frac{a_1v \pm \sqrt{a_1^2v^2 + 3a_3^2v^3B - 3a_3v}}{3a_3v}$$

Meanwhile, for the damping function we've considered $a(x) = 0.01$, $x > 10$; while for our computations we've used $\Delta t = 0.001$, $\Delta x = \frac{140}{2^{11}} \approx 0.068$.

As seen in Figure 3.2 right, the energy also decays following an exponential trend, while as seen in Figure 3.2 left, the soliton manages to enter the zone with the active damping, dissipating in the process. It is imperative to note that the parameters used on this example don't met the hypothesis requested on Theorem (2.2.2) and (2.2.3); furthermore, the parameters a_4 and a_5 don't met as well the requirement asked by Kumar and Chand in order to get a solution, this is, $3a_4 + 2a_5 = 0$. Nevertheless, the numerical solution converged to a result which is expected by Theorems (2.2.2) and (2.2.3). Hence, further reaserch on this topic is needed.

2.3.3 Effects of a strong damping.

We assume the following initial condition

$$u(x,0) = u_0 sech(kx)$$

where $k = 1$ and $u_0 = \sqrt{6}$. We consider additionally, that, $a_1 = 3$, $a_2 = 1$, $a_3 = 0.03$, $a_4 = 0.1$, $a_5 = -0.05$, and $a(x) \equiv 0$ (that is, without damping term). Then an exact solution of (2.1) is obtained, which corresponds to a soliton of a hyperbolic secant squared pulses often referred to as ”bright” pulses (see for more details Potasek and Tabor [PT91b]). Now, the effect that we want to show in this example is what happens with this solution when adding a strong damping term. For that, we introduce a damping function concentrated in a neighborhood of the boundary of the spatial interval, given by

$$a(x) = \begin{cases} 1000, & x \in (-15,-10) \cup (10,15) \\ 0, & \text{in other case.} \end{cases}$$

In our computations, $t \in [0, 1000]$, $x \in [-15, 15]$, $\Delta t = 0.00001$ and $\Delta x = \frac{30}{2^{13}} \approx 0.00366$. The form of the travelling soliton can be found in Figure 3.2. First, we observe that in the first times the wave propagates as the hyperbolic secant soliton predicted in Potasek and Tabor [PT91b], while does not touch the support of the damping function. However, once the soliton approaches the area of influence (approximately at t = 180), the damping function is so high that the soliton gets reflected instead of proceeding with his original path. In each reflection the soliton loses energy at L^2 level following the exponential rate predicted in the previous theorems, and illustrated in Figure 3.1 left. The energy at H^1 level also decays at an exponential rate in each reflection.

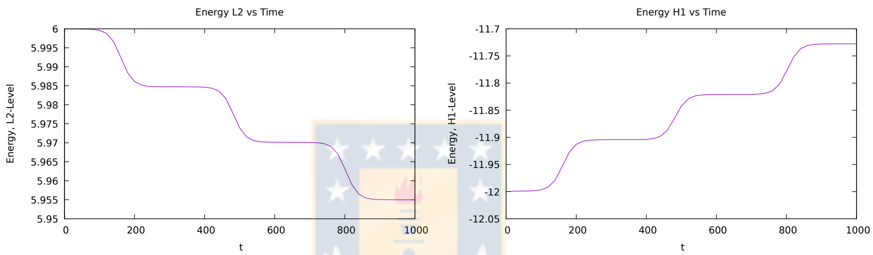

Fig. 2.3: First case results. Left: time evolution of the L^2 energy. Right: time evolution of the H^1 energy.

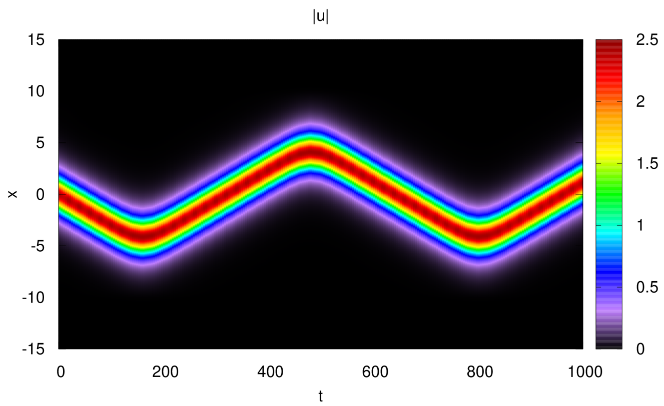

Fig. 2.4: Time evolution of the travelling soliton for the first case.

Chapter 3

Finite Volume scheme for a 2D NLS Equation with localized dissipation.

3.1 Introduction

This chapter is concerned with a numerical implementation of a stabilization result for a defocusing nonlinear Schrödinger equations (dNLS)

$$\begin{cases} i\,\partial_t y + \Delta y - |y|^p\, y + i\, a(x)\, y = 0 & \text{in } \Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is a general domain, and a is a nonnegative function that may vanish on some parts of the domain. We first study (dNLS) on a bounded domain Ω in $\mathbb{R}^N$ with boundary Γ of class C^2. In this case we assume $y = 0$ on Γ. Then, we extend the theory to unbounded domains in the particular cases $\Omega = \mathbb{R}^N$ and Ω being an exterior domain.

The NLS model without a damping term can describe an evolution without any mass and energy loss such as a laser beam propagated in the Kerr medium with no power losses. However, it is always true that some absorption by the medium is indispensable even in the visible spectrum [Fib15]. The effect of the absorption can be modelled by adding a linear (e.g., iay, $a > 0$) or nonlinear (e.g., $ia|y|^q y$, $a > 0$, $q > 0$) damping term into the model, depending on the physical situation. A localized damping, where the damping coefficient $a = a(x)$ depends on the spatial coordinate, can be used to obtain better physical information by distinguishing the spatial region where the absorption takes place or is detected, due to for example some impurity in the medium, from the rest of the domain. Throughout the following chapter (without any restatement), and regarding Problem (3.1), we will assume the following: The power index p can be taken as any positive number. The nonnegative real valued function $a(\cdot) \in W^{1,\infty}(\Omega)$ represents a localized dissipative effect.

If Ω is a bounded domain we will assume that a satisfies the geometric condition $a(x) \geq a_0 > 0$ (for some fixed $a_0 \in \mathbb{R}_+$) for a.e. x on a subregion $\omega \subset \Omega$ that contains $\overline{\Gamma(x^0)}$, where

$$\Gamma(x^0) = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}. \tag{3.2}$$

Here, $m(x) := x - x^0$ ($x_0 \in \mathbb{R}^N$ is some fixed point), and $\nu(x)$ represents the unit outward normal vector at the point $x \in \Gamma$.

On the other hand, if Ω is the whole space, we assume $a(x) \geq a_0 > 0$ in $\mathbb{R}^N \backslash B_{R'}$, where $B_{R'}$ represents a ball of radius $R' > 0$. We assume the same if Ω is an exterior domain: $\Omega := \mathbb{R}^N \setminus \mathcal{O}$, where $\mathcal{O} \subset\subset B_{R'}$ being $\mathcal{O}$ a compact star-shaped obstacle, namely, the following condition is verified: $m(x)\cdot\nu(x)\leq 0$ on Γ_0, where Γ_0 is the boundary of the obstacle $\mathcal{O}$ which is smooth and associated with Dirichlet boundary condition as in Lasiecka et al. [LTZ04]. In this case, the observer x_0 must be taken in the interior of the obstacle $\mathcal{O}$. Regarding to the localized dissipative effect, we consider $a(x)\geq a_0>0$ in $\Omega\backslash B_{R'}$.

Moreover, in all cases, we assume that the damping coefficient $a(\cdot)$ satisfies:

$$|\nabla a(x)|^2 \lesssim a(x),\ \forall\, x\in\Omega. \tag{3.3}$$

The above assumption on the function $a(\cdot)$ was used for the wave equation with Kelvin-Voight damping; see for instance Liu [LR06, Remark 3.1] and Burq and Christianson [BC15].

The assumption $p>0$ is in parallel with the general theory of defocusing nonlinear Schrödinger equations when the initial datum is considered at the H^1-level. On the contrary, it is well known that solutions of the focusing nonlinear Schrödinger equation (fNLS) may blow-up if $p\geq 4/N$ even in the presence of a weak damping acting on the whole domain for arbitrary initial data. The main result proved in [CCO+] can be extended to the case of the focusing problem via a Gagliardo-Nirenberg argument for the allowable range $p<4/N$. The critical case $p=4/N$ can also be treated with a smallness condition on the initial datum.

The main goal proposed in [CCO+] is to achieve stabilization with the (natural) weaker dissipative effect $ia(x)y$ instead of relying on a strong dissipation such as $ia(x)(-\Delta)^{1/2}a(x)y$. It will turn out that the assumption (3.3) enables us to avoid using such strong dissipation. The objective is to achieve stabilization in all dimensions $N\geq 1$ and for all power indices $p>0$. For this purpose, approximate solutions to a problem similar to (3.1) are constructed using the theory of monotone operators. It is proved that these approximate solutions decay exponentially fast in the L^2-sense by using the multiplier technique and a unique continuation property. Then, the global existence is proved, as well as the L^2-decay of solutions for the original model by passing to the limit and using a weak lower semicontinuity argument, respectively. Here it should be noted that the nonlinear structure $f(|y|^2)y$ ($f(s)=s^{p/2}$) is much more general than those treated to date in the context of stabilization with a locally supported damping. The manuscript [CCO+] complements the work of Aloui et al. [AKV13] on unbounded domains, because we prove the global exponential decay for dNLS, while [AKV13] obtained only a local exponential decay in the linear setting. In addition, a precise and efficient algorithm is implemented for studying the exponential decay established in the first part of the paper numerically. Simulations illustrate the efficacy of the proposed control design.

Before stating the maind result proved in [CCO+], we will mention the following notion of weak solutions for problem (3.1).

Definition 3.1.1. *Let $y_0\in L^2(\Omega)$ and set $\mathcal{X}=H_0^1(\Omega)\cap L^{p+2}(\Omega)$. Then, $y\in L^\infty(0,T;\mathcal{X})\cap C([0,T];L^2(\Omega))$ is said to be a weak solution of problem (3.1) if y satisfies $y(0,\cdot)=y_0(\cdot)$ in $L^2(\Omega)$, and*

$$\begin{aligned}&\int_0^T\left[-(y(t),\partial_t\varphi(t))_{L^2(\Omega)}+i\,(\nabla y(t),\nabla\varphi(t))_{L^2(\Omega)}\right]dt \\ &+i\int_0^T\left[\langle|\,y(t)\,|^p\,y(t),\varphi(t)\rangle_{L^{(p+2)'}(\Omega);\,L^{p+2}(\Omega)}-i(a(x)\,y(t),\varphi(t))_{L^2(\Omega)}\right]dt=0\end{aligned} \tag{3.4}$$

for all $\varphi\in C_0^\infty(0,T;\mathcal{X})$.

Meanwhile, the mass functional for the defocusing NLS is given by $E_0(y(t)) := \frac{1}{2}\,||y(t)||^2_{L^2(\Omega)}$.

The main contribution in this regard is to numerically replicate the following stabilization result:

Theorem 3.1.2 (Existence and stabilization)**.** *Let $y_0 \in \mathcal{X} = H^1_0(\Omega) \cap L^{p+2}(\Omega)$. Then, (3.1) admits a weak solution y in the sense of Definition 3.1.1. Moreover, there are $C, \gamma > 0$ (depending on $||y_0||_{H^1_0(\Omega)}$) such that the following exponential decay rate estimate*

$$E_0(y(t)) \leq Ce^{-\gamma t} E_0(y_0), t \geq T_0,$$

holds true for this weak solution provided $T_0 > 0$ is sufficiently large.

To achieve this, we will implement a Finite Volume scheme which solves Problem (3.1) in 2 dimensions. Given the variety of domains that can be considered for the mentioned problem, the Finite Volume method is a reasonable choice because it can be adapted to a great number of domains, while guaranteeing the preservation of quantities like the mass or the energy. Even when the numerical version of the mass functional $E(t)$ is not proved to decay exponentially, the numerical solutions show nevertheless the same behavior. This not only confirms the result proved in [CCO+], but also confirms the robustness of the numerical scheme.

3.2 Numerical Approximation

3.2.1 Presentation of the Scheme.

We consider that the domain $\Omega \subset \mathbb{R}^2$ in (3.1). We will approximate the domain using an admisible mesh (see [EGH00]) composed by a set $\mathcal{T}$ of convex polygons, denoted as the *control volumes* or *cells*, a set of faces $\mathcal{E}$ contained in hyperplanes of $\mathbb{R}^2$, and a set of points $\mathcal{P}$, representing the centroids of the control volumes. The size of the mesh will be given by $h := \max_{K\in\mathcal{T}}\{diam(K)\}$.

To generate the mesh, we have made use of the open-source code `PolyMesher` [TPPM12], which contructs Voronoï tessellations iteratively refined through a Lloyd's method in order to guarantee its regularity.

We will denote by $K \in \mathcal{T}$ a control volume or cell inside the mesh, which in turn has centroid $x_K \in \mathbb{R}^2$, a measure $m(K)$ (in our case: the area of K), a set of neighboring cells $\mathcal{N}(K)$, and a set $\mathcal{E}_K$ of faces $\sigma \in \mathcal{E}_K \subset \mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$, where $\mathcal{E}_{int}$ is the set of inner faces and $\mathcal{E}_{ext}$ is the set of boundary faces. We will also write $t_n = n\Delta t$ for a given timestep Δt. We will denote y^n_K as the numerical approximation of the solution of problem (3.1) over the cell K at the time t_n. We will also write $y_K^{n+\frac{1}{2}} := \frac{y_K^{n+1}+y_K^n}{2}$. $\forall K \in \mathcal{T}$, the proposed Finite Volume scheme for this problem will be defined as follows;

$$\begin{cases} im(K_i)\frac{y_K^{n+1}-y_K^n}{\Delta t} + \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{n+\frac{1}{2}} - \frac{m(K)}{2p}\frac{|y_K^{n+1}|^{2p}-|y_K^n|^{2p}}{|y_K^{n+1}|^2-|y_K^n|^2}(y_K^{n+1}+y_K^n) + im(K)a(x_K)y_K^{n+\frac{1}{2}} = 0 \\ F_{K,\sigma}^n = \tau_\sigma(y_L^n - y_K^n), \quad \sigma\in\mathcal{E}_{int},\ \sigma = K|L, \quad L\in\mathcal{T} \\ F_{K,\sigma}^n = -\tau_\sigma y_K^n, \quad \sigma\in\mathcal{E}_{ext} : \sigma\in\mathcal{E}_K \\ \tau_\sigma = m(\sigma)/|x_K - x_L|, \quad \sigma\in\mathcal{E}_{int}, \quad L\in\mathcal{T} : \sigma = K|L \\ \tau_\sigma = m(\sigma)/d(x_K,\sigma), \quad \sigma\in\mathcal{E}_{ext} : \sigma\in\mathcal{E}_K \end{cases} \tag{3.5}$$

The discretization of the nonlinear term comes from the work of Delfour, Fortin and Payre [DFP81], which was proposed in order to preserve the Energy at H^1 level if there is no damping term. The numerical solution over he whole domain $[0,T]\times\Omega$ will de denoted by $y_{\mathcal{T},\Delta t}$, such that $y_{\mathcal{T},\Delta t}(x_K,t_n) = y_K^n$. In some cases, we will write y^n instead of $y_{\mathcal{T},\Delta t}(t_n)$ for the sake of clarity.

Given the symmetric structure of the matrix involved in the induced linear system of equations, a GMRES method is used to solve it. The nonlinear problem is solved using a Picard Fixed Point iteration with a tolerance equal to 10^{-6} before moving to the next timestep.

3.2.2 Properties and convergence analysis.

In order to state the properties of the scheme (3.5), we will need some notation. We will denote the discrete L^2 norm as follows:

$$||y^n||^2_{L^2_{\mathcal{T}}(\Omega)} := \sum_{K\in\mathcal{T}} |y_K^n|^2 m(K).$$

In a similar fashion, we define the discrete L^{2p} norm as

$$||y^n||^{2p}_{L^{2p}_{\mathcal{T}}(\Omega)} := \sum_{K\in\mathcal{T}} |y_K^n|^{2p} m(K).$$

The discrete version of the H_0 norm will be defined as:

$$||y^n||^2_{H^1_{0,\mathcal{T}}(\Omega)} = \sum_{\sigma\in\mathcal{E}} \tau_\sigma |D_\sigma y^n|^2,$$

where τ_σ is defined as in (3.5), and for $K\in\mathcal{T}$ and $L\in\mathcal{N}(K)$,

$$D_\sigma y^n = \begin{cases} y_L^n - y_K^n, & \text{if } \sigma = K|L\in\mathcal{E}_{int} \\ -y_K^n, & \text{if } \sigma\in\mathcal{E}_{ext}. \end{cases}$$

The following property holds:

Theorem 3.2.1. *The numerical scheme* (3.5) *admits the existence of a unique solution* $y_{\mathcal{T},\Delta t}$.

Proof. For a given $n\in\{0,1,\dots,N\}$, and assuming that $y_K^n = 0, \forall K\in\mathcal{T}$, we take (3.5) and multiply it by $\overline{y}_K^{n+1}$, sum over $K\in\mathcal{T}$, and extract the imaginary part. This will lead us to conclude that $y_K^{n+1} = 0$, $\forall K\in\mathcal{T}$, and hence the existence of solutions is proved. Uniqueness follows after noticing that the linear system induced by the numerical scheme has finite dimension with respect to the vector of unknowns y_K^{n+1}, and hence has unique solution. □

Let us define the discrete version of the mass functional $E_0(y(t))$ as follows:

$$E_0^{(n)} := \frac{1}{2} \sum_{K \in \mathcal{T}} |y_K^n|^2 m(K), \quad n \in \mathbb{N}.$$

If we multiply the numerical scheme by $\overline{y}_K^{n+\frac{1}{2}}$, sum over $K \in \mathcal{T}$, and extract the imaginary part, we get the following result:

Theorem 3.2.2. *If $a(x) \equiv 0$, $\forall x \in \Omega$ in (3.5), then the following property is true $\forall n \in \mathbb{N}$:*

$$E_0^{(n)} = E_0^{(n+1)} \tag{3.6}$$

If $a(x) \geq a_0 > 0$, $x \in \omega \subset \Omega$, then

$$E_0^{(0)} \geq E_0^{(n)}, \quad \forall n \in \mathbb{N}.$$

A consequence of the previous procedure reads as follows:

Corollary 3.2.3. *Let $y_{\mathcal{T},\Delta t}$ be the solution of (3.5) such that $E_0^{(0)} < \infty$. Then, there exists a constant C_∞, depending on y^0 and T, such that*

$$||y_{\mathcal{T},\Delta t}||_\infty < C_\infty \tag{3.7}$$

where $||y^n||_\infty := \max_{K \in \mathcal{T}} |y_K^n|$.

We will also define the discrete version of the Energy functional at H^1 level:

$$E_1^{(n)} := \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \tau_\sigma |D_\sigma y^n|^2 + \sum_{K \in \mathcal{T}} \frac{1}{2p} |y_K^n|^{2p} m(K) \tag{3.8}$$

The following property holds:

Theorem 3.2.4. *Let $y_{\mathcal{T},\Delta t}$ be the numerical solution induced by the scheme (3.5) such that $||y^0_{\mathcal{T},\Delta t}||^2_{L^2_{\mathcal{T}(\Omega)}} < \infty$. If $a(x) \equiv 0, \forall x \in \Omega$ in (3.5); then the following property holds true $\forall n \in \mathbb{N}$:*

$$E_1^{(n+1)} = E_1^{(n)}. \tag{3.9}$$

If $a(x) \geq a_0 > 0$, $x \in \omega \subset \Omega$ and $a(x) \in W^{1,\infty}(\Omega)$, then there exists a constant $C > 0$, depending on T, $a(x)$, and y^0, such that

$$E_1^{(n)} \leq E_1^{(0)} + C. \tag{3.10}$$

Proof. We multiply (3.5) by $\frac{\overline{y}_K^{n+1} - \overline{y}_K^n}{\Delta t}$, sum over $K \in \mathcal{T}$, and extract the real part. We get

$$Re\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+\frac{1}{2}} \frac{\overline{y}_K^{n+1} - \overline{y}_K^n}{\Delta t} \right) - \sum_{K \in \mathcal{T}} \frac{m(K)}{2p\Delta t} \left(|y_K^{n+1}|^{2p} - |y_K^n|^{2p} \right) \tag{3.11}$$
$$+ Re\left(i \sum_{K \in \mathcal{T}} m(K) a(x_K) y_K^{n+\frac{1}{2}} \frac{\overline{y}_K^{n+1} - \overline{y}_K^n}{\Delta t} \right) = 0.$$

After using the identity $Re(a(\overline{b}-\overline{a})) = \frac{1}{2}\big(|b|^2 - |a|^2 - |b-a|^2\big)$ for $a, b \in \mathbb{C}$, and reordering the sum, the first term in (3.11) becomes

$$Re\Bigg(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{n+\frac{1}{2}}\frac{\overline{y}_K^{n+1}-\overline{y}_K^n}{\Delta t}\Bigg) = \sum_{\sigma\in\mathcal{E}}\frac{\tau_\sigma}{2}\Big(|y_L^n - y_K^n|^2 - |y_L^{n+1}-y_K^{n+1}|^2\Big)$$
$$= \sum_{\sigma\in\mathcal{E}}\frac{\tau_\sigma}{2}\Big(|D_\sigma y^n|^2 - |D_\sigma^{n+1}|^2\Big).$$

With this, (3.11) turns into the following:

$$\frac{1}{\Delta t}E_1^{(n+1)} = \frac{1}{\Delta t}E_1^{(n)} + Re\Big(i\sum_{K\in\mathcal{T}} m(K)a(x_K)y_K^{n+\frac{1}{2}}\frac{\overline{y}_K^{n+1}-\overline{y}_K^n}{\Delta t}\Big). \tag{3.12}$$

If $a(x) \equiv 0$, then we get (3.9). If not, then we will need to recall the following from the numerical scheme:

$$\frac{y_K^{n+1}-y_K^n}{\Delta t} = \frac{i}{m(K)}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{n+\frac{1}{2}} - \frac{i}{2p}\frac{|y_K^{n+1}|^{2p}-|y_K^n|^{2p}}{|y_K^{n+1}|^2-|y_K^n|^2}(y_K^{n+1}+y_K^n) - a(x_K)y_K^{n+\frac{1}{2}}. \tag{3.13}$$

Replacing (3.13) in (3.12) will lead us to study the following:

$$\begin{aligned} i\sum_{K\in\mathcal{T}} m(K)a(x_K)y_K^{n+\frac{1}{2}}\frac{\overline{y}_K^{n+1}-\overline{y}_K^n}{\Delta t} &= \sum_{K\in\mathcal{T}} a(x_K)y_K^{n+\frac{1}{2}}\sum_{\sigma\in\mathcal{E}_K}\overline{F}_{K,\sigma}^{n+\frac{1}{2}} \\ &- \sum_{K\in\mathcal{T}} a(x_K)\frac{m(K)}{p}\frac{|y_K^{n+1}|^{2p}-|y_K^n|^{2p}}{|y_K^{n+1}|^2-|y_K^n|^2}|y_K^{n+\frac{1}{2}}|^2 \\ &- i\sum_{K\in\mathcal{T}} m(K)(a(x_K))^2|y_K^{n+\frac{1}{2}}|^2. \end{aligned} \tag{3.14}$$

After extracting the real part in (3.14) the third term at the right hand side vanishes and the second term is a strictly negative number. For the first term, again using the identity $Re(a(\overline{b}-\overline{a})) = \frac{1}{2}\big(|b|^2-|a|^2-|b-a|^2\big)$ and reordering the sum, we get

$$\begin{aligned} Re\Bigg(\sum_{K\in\mathcal{T}} a(x_K)y_K^{n+\frac{1}{2}}\sum_{\sigma\in\mathcal{E}_K}\overline{F}_{K,\sigma}^{n+\frac{1}{2}}\Bigg) &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\frac{\tau_\sigma}{8}\Big(|y_K^{n+1}|^2+|y_K^n|^2\Big)\big(a(x_L)-a(x_K)\big) \\ &- \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\frac{\tau_\sigma}{8}a(x_K)\Big(|y_L^{n+1}-y_K^{n+1}|^2+|y_L^n-y_K^n|^2\Big) \\ &+ \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\frac{\tau_\sigma}{4}a(x_K)Re\Big(y_K^{n+1}(\overline{y}_L^n-\overline{y}_K^n)+y_K^n(\overline{y}_L^{n+1}-\overline{y}_K^{n+1})\Big). \end{aligned} \tag{3.15}$$

The second term in (3.15) is strictly negative. Hence, and given the regularity condition of the damping function $a(x) \in W^{1,\infty}(\Omega)$, we can infer the existence of a constant C_1, depending on $a(x)$, such that

$$Re\left(\sum_{K\in\mathcal{T}} a(x_K)y_K^{n+\frac{1}{2}} \sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma}^{n+\frac{1}{2}}\right) \leq C_1 \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \frac{\tau_\sigma}{8}\left(|y_K^{n+1}|^2 + |y_K^n|^2\right)$$
$$+ C_1 \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \frac{\tau_\sigma}{4}\left|y_K^{n+1}(\overline{y}_L^n - \overline{y}_K^n) + y_K^n(\overline{y}_L^{n+1} - \overline{y}_K^{n+1})\right|$$
$$\leq \frac{C_1}{8}\left(||y^{n+1}||^2_{L^2_\mathcal{T}(\Omega)} + ||y^n||^2_{L^2_\mathcal{T}(\Omega)}\right)$$
$$+ \frac{C_1}{4}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma\left(4|y_K^{n+1}|^2 + 4|y_K^n|^2\right)$$
$$\leq \frac{9}{4}C_1||y^0||^2_{L^2_\mathcal{T}(\Omega)}.$$

Hence, (3.12) will turn into

$$\frac{1}{\Delta t}E_1^{(n+1)} = \frac{1}{\Delta t}E_1^{(n)} + Re\left(i\sum_{K\in\mathcal{T}} m(K)a(x_K)y_K^{n+\frac{1}{2}}\frac{\overline{y}_K^{n+1} - \overline{y}_K^n}{\Delta t}\right)$$
$$\leq \frac{1}{\Delta t}E_1^{(n)} + \frac{9}{4}C_1||y^0||^2_{L^2_\mathcal{T}(\Omega)}.$$

Multiplying the previous result by Δt and repeating the upper bound n times will lead us to

$$E_1^{(n+1)} \leq E_1^{(0)} + \frac{9C}{4}n\Delta t||y^0||^2_{L^2_\mathcal{T}(\Omega)},$$

and because $||y^0||^2_{L^2_\mathcal{T}(\Omega)} < \infty$, we can infer the existence of a constant C, depending on T, y^0, and $a(x)$, such that

$$E_1^{(n+1)} \leq E_1^{(0)} + C.$$

Thus, the theorem is proved. □

On the other hand, if we go back to (3.10) and compare it with the definition (3.8), we get the following result:

Corollary 3.2.5. *Let y^n be the solution of (3.5) such that $||y^0||^2_{L^2_\mathcal{T}(\Omega)} < \infty$ and $E_1^{(0)} < \infty$. Then, there exist some constants C_1 and C_2, depending on y^0, $a(x)$, and T, such that*

$$||y^n||_{H^1_{0,\mathcal{T}}(\Omega)} < C_1, \quad \forall n \in \mathbb{N}. \tag{3.16}$$

and

$$||y^n||_{L^{2p}_\mathcal{T}(\Omega)} < C_2, \quad \forall n \in \mathbb{N}. \tag{3.17}$$

This upper bound will help us to prove the convergence of the numerical scheme.

Theorem 3.2.6. *For $m \in \mathbb{N}$, let $\{y_m\}_{m\in\mathbb{N}}$, $y_m = y_{\mathcal{T}_m,\Delta t_m}(x,t)$ be a sequence of solutions of (3.5) induced by their respective initial conditions $\{y_m^0\}_{m\in\mathbb{N}} \subset \mathcal{X}$, while using a sequence of admissible meshes $\mathcal{T}_m$ and timesteps Δt_m such that $h_m \to 0$ and $\Delta t_m \to 0$ when $m \to \infty$. Then, there exists a subsequence of the sequence of numerical solutions, still denoted by $\{y_m\}_{m\in\mathbb{N}}$, which converges to the weak solution $y(t)$ given by the Definition 3.1.1 when $m \to \infty$.*

Proof. We will start by proving that $\partial_t y_m$ is bounded in $\mathcal{X}'$; this is

$$
\begin{aligned}
||\partial_t y_m||_{\mathcal{X}'_m} &:= \sup_{||\varphi||_{\mathcal{X}_m}=1} \left\{ \left| \left(\partial y_m, \varphi\right)_{L^2_{\mathcal{T}_m}(\Omega)} \right| \right\} \\
&= \sup_{||\varphi||_{\mathcal{X}_m}=1} \left\{ \left| i \Big(\sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma \big(y_L^{n+\frac{1}{2}} - y_K^{n+\frac{1}{2}}\big)\overline{\varphi}_K \Big) \right.\right. \\
&\quad \left.\left. - \frac{i}{2p} \sum_{K\in\mathcal{T}_m} \left(\frac{|y_K^{n+1}|^{2p} - |y_K^n|^{2p}}{|y_K^{n+1}|^2 - |y_K^n|^2} (y_K^{n+1} + y_K^n)\overline{\varphi}_K m(K) \right) - \sum_{K\in\mathcal{K}} \left(a(x_K) y_K^{n+\frac{1}{2}} \overline{\varphi}_K m(K)\right) \right| \right\} \qquad (3.18) \\
&< \infty.
\end{aligned}
$$

The first term in the right hand side of (3.18) can be rewritten as follows

$$
\sum_{n=0}^{N} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma (y_L^{n+\frac{1}{2}} - y_K^{n+\frac{1}{2}})\overline{\varphi}_K \Delta t = \\
\sum_{n=0}^{N} \sum_{K|L\in\mathcal{E}_{int}} m(K|L)(y_L^{n+\frac{1}{2}} - y_K^{n+\frac{1}{2}}) \frac{\overline{\varphi}_K - \overline{\varphi}_L}{d_{K|L}} \Delta t.
$$

After (3.16) and the regularity of φ, we can write

$$
\sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma \big(y_L^{n+\frac{1}{2}} - y_K^{n+\frac{1}{2}}\big)\overline{\varphi}_K < \infty. \qquad (3.19)
$$

For the second term in (3.18), we will consider three cases.

- If $p \leq 1$, we have

$$
\begin{aligned}
\Big| \sum_{K\in\mathcal{T}_m} \frac{|y_K^{n+1}|^{2p} - |y_K^n|^{2p}}{|y_K^{n+1}|^2 - |y_K^n|^2} (y_K^{n+1} + y_K^n)\overline{\varphi}_K m(K) \Big| &\leq \sum_{K\in\mathcal{T}_m} |(y_K^{n+1} + y_K^n)\overline{\varphi}_K| m(K) \\
&\leq 2||\varphi||_{L^\infty_{\mathcal{T}_m}(\Omega)} ||y^0||^2_{L^2_{\mathcal{T}_m}(\Omega)}
\end{aligned}
$$

  which is bounded.

- If $1 < p < 2$, then

$$
\Big| \sum_{K\in\mathcal{T}_m} \frac{|y_K^{n+1}|^{2p} - |y_K^n|^{2p}}{|y_K^{n+1}|^2 - |y_K^n|^2} (y_K^{n+1} + y_K^n)\overline{\varphi}_K m(K) \Big| \leq 2||\varphi||_{L^\infty_{\mathcal{T}_m}} ||y^0||_{L^\infty_{\mathcal{T}_m}} \sum_{K\in\mathcal{T}_m} \Big(|y_k^{n+1}|^{2p-2} + |y_K^n|^{2p-2} \Big) m(K).
$$

  Using Young's inequality, we get

$$
\sum_{K\in\mathcal{T}_m} \Big(|y_k^{n+1}|^{2p-2} + |y_K^n|^{2p-2} \Big) m(K) \leq \sum_{K\in\mathcal{T}_m} \left(\left(\frac{2p-2}{2p} \right) (|y_k^{n+1}|^{2p} + |y_K^n|^{2p}) + \frac{2}{p} \right) m(K)
$$

  which is also bounded due to (3.17), (3.7), and by the fact that $|\Omega| < \infty$.

- If $p \geq 2$, then we have

$$\Big| \sum_{K\in\mathcal{T}_m} \frac{|y_K^{n+1}|^{2p} - |y_K^n|^{2p}}{|y_K^{n+1}|^2 - |y_K^n|^2} (y_K^{n+1} + y_K^n)\overline{\varphi}_K m(K)\Big| \leq 2||\varphi||_{L^\infty_{\mathcal{T}_m}} ||y^0||_{L^\infty_{\mathcal{T}_m}} \sum_{K\in\mathcal{T}_m} \frac{p}{2}\Big(|y_k^{n+1}|^{2p-2} + |y_K^n|^{2p-2}\Big)m(K)$$

which is bounded by the same reasons argued in the previous point.

Hence, we conclude that the second term in (3.18) is bounded for any $p > 0$; this is,

$$\Big| \sum_{K\in\mathcal{T}_m} \frac{|y_K^{n+1}|^{2p} - |y_K^n|^{2p}}{|y_K^{n+1}|^2 - |y_K^n|^2} (y_K^{n+1} + y_K^n)\overline{\varphi}_K m(K)\Big| < \infty. \tag{3.20}$$

Regarding the third term in (3.18): thanks to (3.6), and the regularity properties of $a(x)$, we observe that

$$\sum_{K\in\mathcal{K}} a(x_K) y_K^{n+\frac{1}{2}} \overline{\varphi}_K m(K) \leq \frac{C_2}{2}\Big(||y^{n+\frac{1}{2}}||^2_{L^2_{\mathcal{T}_m}(\Omega)} + ||\varphi||^2_{L^2_{\mathcal{T}_m}(\Omega)}\Big) < \infty \tag{3.21}$$

where C_2 is a constant depending on $a(x)$. Combining (3.19), (3.20) and (3.21), we conclude that

$$\{\partial_t y_m\} \quad \text{is bounded in} \quad L^\infty(0,T;\mathcal{X}'). \tag{3.22}$$

Therefore, due to the fact that

$$H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-2}(\Omega),$$

and thanks to the Aubin-Lions Theorem, we can extract a subsequence, still denoted by $\{y_m\}_{m\in\mathbb{N}}$, such that

$$y_m \to y \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.23}$$

We will now prove that this y is the weak solution given by Definition 3.1.1. Let $\varphi \in C_0^\infty(0,T;\mathcal{X})$ such that $\nabla\varphi\cdot\hat{\mathbf{n}} = 0$ in $\partial\Omega\times[0,T]$. Multiplying the numerical scheme (3.5) by $\frac{\Delta t}{2}\Big(\overline{\varphi}(x_K, n\Delta t) + \overline{\varphi}(x_K,(n+1)\Delta t)\Big) =: \frac{\Delta t}{2}\overline{\varphi}(x_K, t_{n+\frac{1}{2}})$, and summing over $K\in\mathcal{T}$ and over $n = 0, \dots, N$ with $T = N\Delta t$, we get:

$$\begin{aligned} i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(y_K^{n+1} - y_K^n)\overline{\varphi}(x_K, t_{n+\frac{1}{2}}) + \sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\mathcal{N}(K)} \tau_{K|L}(y_L^{n+\frac{1}{2}} - y_K^{n+\frac{1}{2}})\overline{\varphi}(x_K, t_{n+\frac{1}{2}})\Delta t \\ - \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} |y_k^{n+\frac{1}{2}}|^p y_K^{n+\frac{1}{2}}\overline{\varphi}(x_K, t_{n+\frac{1}{2}})\Delta t + i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} a(x_K) y_K^{n+\frac{1}{2}}\overline{\varphi}(x_K, t_{n+\frac{1}{2}})\Delta t = 0. \end{aligned} \tag{3.24}$$

We can re-write the first term in (3.24), after using summation by parts and recalling that $\varphi \in C_0^\infty(0,T;\mathcal{X})$:

$$i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(y_K^{n+1} - y_K^n)\overline{\varphi}(x_K, t_{n+\frac{1}{2}}) = -i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K) y_K^n \Big(\frac{\overline{\varphi}(x_K, t_{n+1}) - \overline{\varphi}(x_K, t_{n-1})}{2}\Big).$$

Hence, because $\{y_m\}_{m\in\mathcal{N}}$ is bounded in $L^\infty((0,T)\times, L^2(\Omega))$, then as $m\to\infty$,

$$-i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K) y_K^n \Big(\frac{\overline{\varphi}(x_K, t_{n+1}) - \overline{\varphi}(x_K, t_{n-1})}{2}\Big) \to -i\int_0^T\int_\Omega y(x,t)\overline{\varphi}_t(x,t)\,dxdt. \tag{3.25}$$

The second term in (3.24) can also be re-written as follows:

$$\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}(y_L^{n+\frac{1}{2}}-y_K^{n+\frac{1}{2}})\overline{\varphi}(x_K,t_{n+\frac{1}{2}})\Delta t =$$
$$\sum_{n=0}^{N}\sum_{K|L\in\mathcal{E}_{int}} m(K|L)(y_L^{n+\frac{1}{2}}-y_K^{n+\frac{1}{2}})\frac{\overline{\varphi}(x_K,t_{n+\frac{1}{2}})-\overline{\varphi}(x_L,t_{n+\frac{1}{2}})}{d_{K|L}}\Delta t. \tag{3.26}$$

On the other hand

$$\sum_{n=0}^{N}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Omega} y_{\mathcal{T},\Delta t}(x,t)\Delta\overline{\varphi}(x,n\Delta t)dxdt = \sum_{n=0}^{n}\sum_{K\in\mathcal{T}} y_K^{n+\frac{1}{2}}\int_K \Delta\overline{\varphi}(x,t_{n+\frac{1}{2}})dx\Delta t \tag{3.27}$$
$$= \sum_{n=0}^{N}\sum_{K|L\in\mathcal{E}_{int}}\left(y_K^{n+\frac{1}{2}}-y_L^{n+\frac{1}{2}}\right)\int_{K|L}\nabla\overline{\varphi}(x,t_{n+\frac{1}{2}})\cdot\mathbf{n}_{K,L}d\gamma. \tag{3.28}$$

By the same reasons argued in (3.25), we have that

$$\sum_{n=0}^{N}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Omega} y_{\mathcal{T},\Delta t}(x,t)\Delta\overline{\varphi}(x,n\Delta t)dxdt \to \int_0^T\int_{\Omega} y(x,t)\Delta\overline{\varphi}(x,t)dxdt \tag{3.29}$$

as $m\to\infty$. Now, subtracting the right hand side of (3.26) from (3.28),

$$\sum_{n=0}^{N}\sum_{K|L\in\mathcal{E}_{int}} m(K|L)\left(y_K^{n+\frac{1}{2}}-y_L^{n+\frac{1}{2}}\right)\left(\int_{K|L}\nabla\overline{\varphi}(x,t_{n+\frac{1}{2}})\cdot\mathbf{n}_{K,L}d\gamma-\frac{\overline{\varphi}(x_K,t_{n+\frac{1}{2}})-\overline{\varphi}(x_L,t_{n+\frac{1}{2}})}{d_{K|L}}\right)\Delta t. \tag{3.30}$$

Because of the regularity properties of φ, we have that (3.30) goes to 0 when $m\to\infty$. Hence, and thanks to (3.27) and (3.29),

$$\sum_{n=0}^{N}\sum_{K|L\in\mathcal{E}_{int}} m(K|L)(y_L^{n+\frac{1}{2}}-y_K^{n+\frac{1}{2}})\frac{\overline{\varphi}(x_K,t_{n+\frac{1}{2}})-\overline{\varphi}(x_L,t_{n+\frac{1}{2}})}{d_{K|L}}\Delta t \to \int_0^T\int_{\Omega} y(x,t)\Delta\overline{\varphi}(x,t)dxdt.$$

The third and fourth terms in (3.24) can be treated in a similar way because $y_m\in L^\infty(0,T;\mathcal{X})$; hence, and due to (3.23), we have

$$\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} |y_K^{n+\frac{1}{2}}|^p y_K^{n+\frac{1}{2}}\overline{\varphi}(x_K,t_{n+\frac{1}{2}})\Delta t \to \int_0^T\int_{\Omega}|y(x,t)|^p y(x,t)\overline{\varphi}(x,t)dxdt, \quad \text{as } m\to\infty.$$

Finally,

$$i\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} a(x_K)y_K^{n+\frac{1}{2}}\overline{\varphi}(x_K,t_{n+\frac{1}{2}})\Delta t \to i\int_0^T\int_{\Omega} a(x)y(x,t)\overline{\varphi}(x,t)dxdt, \quad \text{as } m\to\infty.$$

Thus, when passing to the limit in (3.24) and integrating by parts, we conclude that y is the weak solution of (3.1); concluding the proof. □

3.2.3 Example I

In the following example, we will use the given numerical scheme to solve equation (3.1) for $p=2$, $T=500$, Ω being disk with ratio $r=10$, $\omega\subset\Omega : x^2+y^2>8^2$, and a damping function

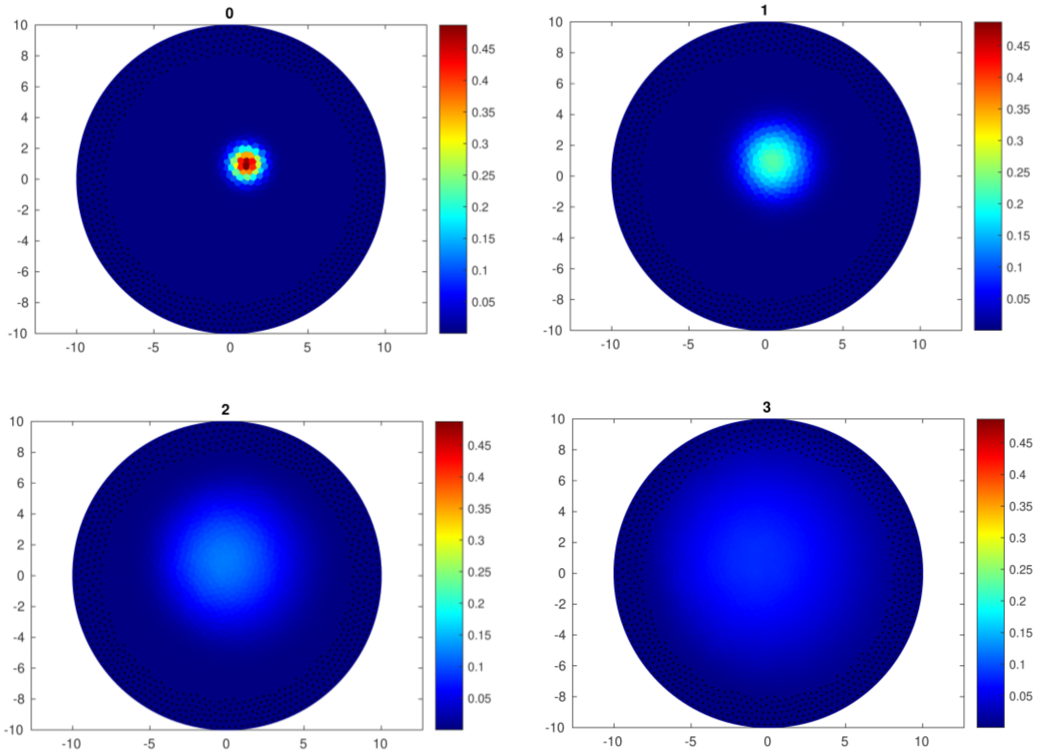

Fig. 3.1: Numerical solution at different timesteps. Cells with black dots indicate the zone where the damping function is in place.

defined as follows:

$$a(x,y) = \begin{cases} \left(\sqrt{(x^2+y^2)} - 8\right)^2, & 8^2 \leq x^2+y^2 \leq 10^2 \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the damping fulfills condition (3.3). The initial condition is given by

$$y_0 = \frac{1}{2}\exp\left(- \left((x-1)^2 + (y-1)^2 + \frac{i}{2}(x-1)\right)\right). \tag{3.31}$$

In our computations, we've used $\Delta t = \frac{1}{2^6} = 0.015625$ and $h = 0.64851$, where 2000 polygons were used to approximate the domain. Figure 3.1 illustrates the state of the numerical solution at different times, while Figure 3.2 left shows the evolution of the energy with time. In this case the decay is exponential, as expected from Theorem 3.1.2.

3.2.4 Example II

As a second experiment, we will repeat Example I but using $p = 2$, $T = 500$, and the damping function

$$a(x,y) = \begin{cases} \left(\exp(\sqrt{x^2+y^2} - 8) - 1\right)^2, & 8^2 \leq x^2+y^2 \leq 10^2 \\ 0, & \text{in other case.} \end{cases}$$

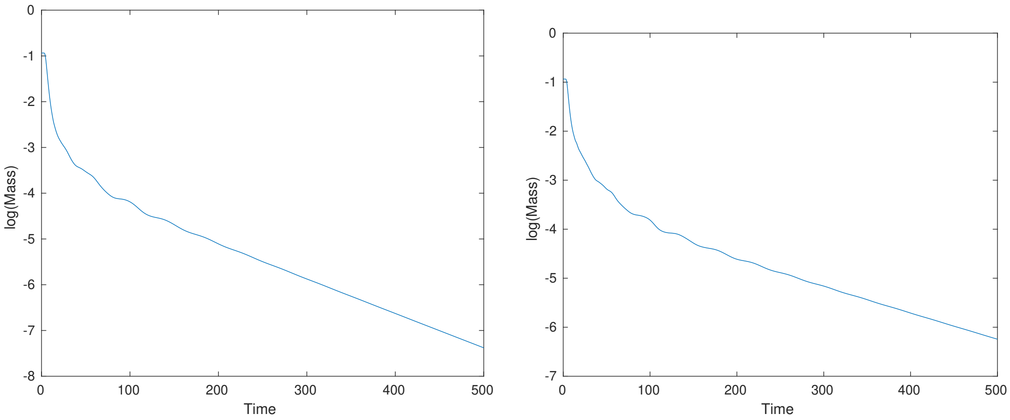

Fig. 3.2: Energy decays for both examples. Left: decay for Example I. Right: decay for Example II.

This function also fulfills condition (3.3). Figure 3.2 right shows the time evolution of the energy. The decay in this case is also exponential, replicating the theoretical result (3.1.2) proved in the previous sections.

3.2.5 Example III

We will now consider an exterior domain. The new domain Ω will be defined as:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 5 \leq \sqrt{x^2 + y^2} \leq 20\},$$

while the effective damping subset will be given by

$$\omega = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 17\}.$$

The initial condition to be used is

$$y(x, 0) = \exp\Big(- \big(x^2 + (y - 10)^2 + \frac{i}{2}x\big)\Big).$$

For these calculations, we've done $\Delta t = \frac{1}{2^6} = 0.0156$, and the domain was approximated using 5000 polygons with $h = 0.76172$. Figure 3.3 illustrates the initial condition and the time evolution of the mass functional. Its decay follows an exponential trend, as expected.

3.2.6 Example IV

As a final experiment, we will repeat the previous case but using the following domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 7 \leq \sqrt{x^2 + y^2} \leq 20\}.$$

The effective damping subset will be given by $\omega = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 17 \wedge \alpha \in (-\pi, 0)\}$, where α is the angle of the point (x, y) with respect to the positive x axis. This is equivalent to the geometric condition (3.2) for a point $x^0 = (0, y)$ such that $y \to +\infty$.

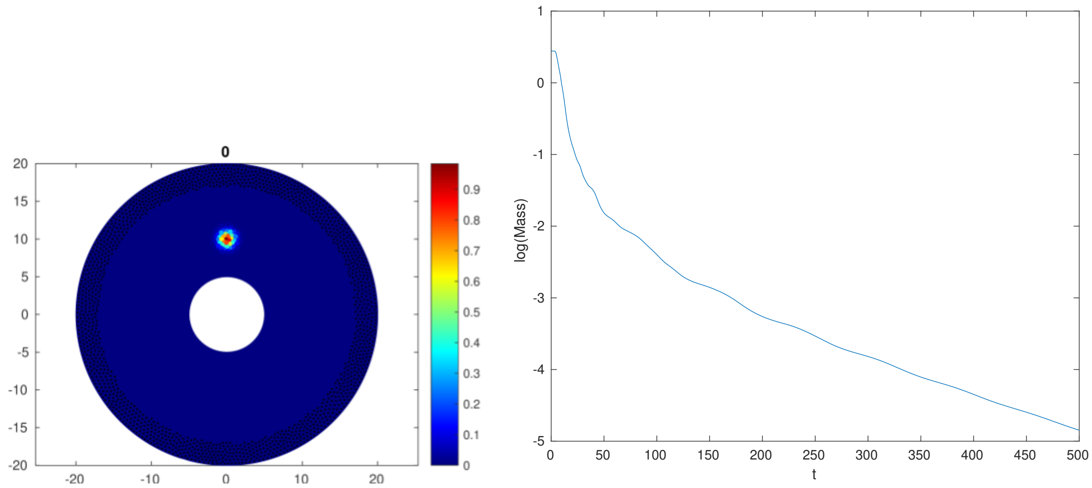

Fig. 3.3: Results for the experiment with an exterior domain. Left: the initial condition. Right: semi-log plot for the time-evolution of the mass function.

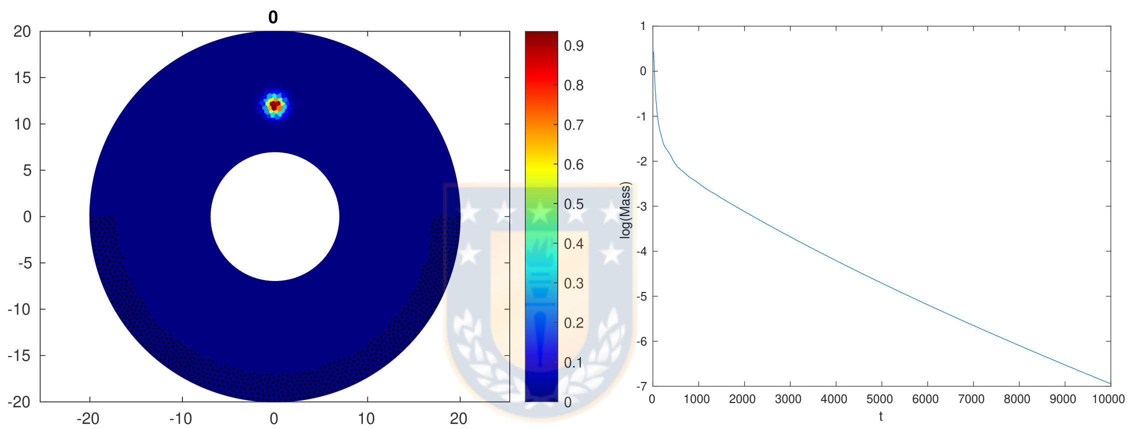

Fig. 3.4: Left: the initial condition. Black dots denote the cells where the damping function is acting effectively. Right: time evolution of the mass functional, at semi-log scale.

For our calculations, we've used $\Delta t = \frac{1}{2^5} = 0.0312$, $T = 10000$, and $h = 0.80958$ for a domain approximated using 5000 polygons. The left panel of Figure 3.4 shows the initial condition and the zone where the damping is acting effectively; while the right panel shows the decay of the Mass funcional in semi-log scale. We can clearly see the exponential decay rate, as expected.

Chapter 4

Finite Diffference scheme for a bridge with localized nonlinear damping.

The present chapter aims to present numerical results for a hanging bridge model. Let us consider a thin and narrow rectangular plate where the two short edges are hinged, whereas the two long edges are free. In absence of forces, the plate lies flat horizontally and is represented by the planar domain $\Omega = (0, \pi) \times (-l, l)$ where $l << \pi$, with boundary Γ. Then, the nonlocal evolution equation modeling the deformation of the plate reads as follows:

$$\begin{cases} u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + \phi(u)u_{xx} + a(x,y)g(u_t(x,y,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u(x,y,0) = u_0(x,y),\ u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where the nonlinear term ϕ, which carries a nonlocal effect into the model, is defined by

$$\phi(u) = -P + S \int_\Omega u_x^2 \, dx,$$

and the constant σ is the Poisson ratio: for metals its value lies around 0.3 while for concrete it is between 0.1 and 0.2. For this reason we shall assume that $0 < \sigma < \frac{1}{2}$, $a = a(x,y) \in L^\infty(\Omega)$ is assumed to be a nonnegative essentially bounded function such that

$$a \geq a_0 > 0 \quad \text{a.e. in } \omega,$$

for some non empty open subset ω around the boundary Γ of Ω and some positive constant $a_0 > 0$; and the function g verifies some conditions to be announced in the following pages.

$S > 0$ depends on the elasticity of the material composing the deck, the term $S \int_\Omega u_x^2 \, dx$ measures the geometric nonlinearity of the plate due to its stretching, and $P > 0$ is the prestressing constant: one has $P > 0$ if the plate is compressed and $P < 0$ if the plate is stretched. Here, we are considering the absence of a vertical loading accross the bridge.

The present chapter is inspired from the work published in [DCMCC+], whose main goal is to establish uniform decay rates estimates to problem (4.1) with a minimum amount of damping which represents less cost of material. This minimum refers a small 'collar' ω around the whole boundary Γ of Ω. In addition, the nonlinear feedback $a(x,y)g(u_t)$ can be superlinear, sublinear or linearly bounded at infinity according to the terminology given in [DCMCC+]. In particular: we aim to show numerical solutions for Problem 4.1 obtained from a Finite

Difference scheme, while also numerically replicate the stabilization result. The boundary conditions are incorporated into the matrix differential operators. Preliminary results show that the scheme is conservative when no damping funcion is used.

4.1 Well-posedness and stabilization.

The present section describes the well-possednes of the problem, as well as presenting the stabilization result. We define the Hilbert space $\mathcal{H} := H^2_*(\Omega) \times L^2(\Omega)$ endowed with the inner product $(U, V)_{\mathcal{H}} = (u, \tilde{u})_{H^2_*(\Omega)} + (v, \tilde{v})_{L^2(\Omega)}$, where $U = (u, v)^T$; $V = (\tilde{u}, \tilde{v})^T \in \mathcal{H}$. Inspired in [AB05], [AB10], [ABA11], [CCCT17] and [LT93], let h be a concave, strictly increasing function, with $h(0) = 0$, and such that $h(s\,g(s)) \geq s^2 + g^2(s)$, for $|s| < 1$. Problem (4.1) can be rewritten as

$$\begin{cases} U_t + \mathcal{A}U = G, \\ U(0) = U_0, \end{cases}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix};\ \mathcal{A}U := \begin{pmatrix} -v \\ \Delta^2 u + a(\cdot)g(v) \end{pmatrix};\ G(U) = \begin{pmatrix} 0 \\ -\phi(u)u_{xx} \end{pmatrix} \text{ and } U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

Using standard nonlinear semigroup theory, it can be proved that $\mathcal{A}$ is maximal monotone operator in $\mathcal{H}$ (see, for instance, [CSC14]). Thus, in order to prove that problem (4.1) is wellposed it is sufficient to prove that:

Lemma 4.1.1. *G is locally Lipschitz in $\mathcal{H}$.*

Proof: See Cavalcanti et al. [DCMCC+].

After proving that G is locally Lipschitz in $\mathcal{H}$, and according to standard semigroup properties, the following results follows

Theorem 4.1.2. *For $U_0 \in \mathcal{H}$ given, problem (4.1) possesses a unique solution $U \in C([0,\infty);\mathcal{H})$. In addition, if $U_0 \in D(A)$, then problem (4.1) has a unique regular solution $U \in C([0,\infty); D(A)) \cap C^1([0,\infty);\mathcal{H})$.*

On the other hand, the energy associated to problem (4.1) is now defined by

$$E_u(t) = \underbrace{\frac{1}{2}||u_t(t)||^2_{L^2(\Omega)}}_{\mathcal{K}_u(t)} + \underbrace{\frac{1}{2}||u(t)||^2_{H^2_*(\Omega)} - \frac{P}{2}||u_x(t)||^2_{L^2(\Omega)} + \frac{S}{4}||u_x(t)||^4_{L^2(\Omega)}}_{\mathcal{P}_u(t)}, \tag{4.2}$$

where $t \geq 0$. Here, $\mathcal{K}_u(t)$ and $\mathcal{P}_u(t)$ represent, respectively, the kinetic and the elastic potential energy of the model. Moreover, one has the identity of the energy

$$E_u(t_2) - E_u(t_1) = -\int_Q a(x,y)g(u_t(x,y,t))u_t(x,y,t)\,dx\,dy\,dt, \tag{4.3}$$

so that $0 \leq t_1 \leq t_2 < +\infty$, which shows that the energy is monotonic (non increasing). The main result proved in [DCMCC+] reads as follows:

Theorem 4.1.3. *For any $R > 0$ there exist constants C and $T_0 > 0$, depending on R, such that, if $E_u((0)) \leq R$, then*

$$E_u(T) \leq C \int_0^T \int_\Omega a(x,y) \left[|u_t(x,y,t)|^2 + |g(u_t(x,y,t))|^2 \right] dx\, dy\, dt, \ \forall T > T_0. \tag{4.4}$$

The previous theorem has the following consequence:

Theorem 4.1.4. *Denote by (u, u_t) a weak solution of the problem (4.1). Suppose the $a = a(x,y) \in L^\infty(\Omega)$ is assumed to be a nonnegative bounded function such that $a(x,y) \geq a_0 > 0$ a.e. in ω for some non empty open subset ω around the boundary $\partial\Omega$ of Ω and some positive constant $a_0 > 0$. Define h to be concave, strictly increasing function, vanishing at 0 and such that*

$$h(sg(s)) \geq s^2 + g(s)^2, \ \textit{for} \ |s| \leq 1, \tag{4.5}$$

(which can always be constructed since g is continuous increasing $g(0) = 0$).

In addition, if g is not linearly bounded at infinity, then let the Assumption 5.1.1 (see: [DCMCC+]) be satisfied with the corresponding integrability indices p_0. Next, define

$$\boldsymbol{C} = \|u_t\|_{L^\infty\left(_+; L^{p_0}(\Omega)\right)}^{\frac{|1-(g)|}{p_0-1-(g)}}, \qquad h(s) = s^{\frac{p_0-2\max\{(g),1\}}{p_0-1-(g)}}.$$

Then, there exist constants $T_0 \geq T > 0$ such that the energy $E(t)$ given by (4.2) satisfies

$$E_u(t) \leq S\left(\frac{t}{T} - 1\right), \quad \forall t > T_0,$$

where $\lim_{t\to\infty} S(t) = 0$.

This result is one of the key contributions in [DCMCC+]. The proof is beyond the scope of this manuscript, where it can be found in the previously cited article.

4.2 Numerical Results

4.2.1 Description of the numerical scheme.

Will replicate numerically the results obtained in Theorem 4.1.4. In particular, and given the boundary conditions we have to deal with, our proposal consist on the approximation of the solution of Problem (4.1) using the finite differences method. To achieve this, the x domain $[0, \pi]$ will be subdivided in $J+1$ equally spaced sub-intervals with length Δx each, while the y domain $[-l, l]$ will be subdivided in $K+1$ sub-intervals, each of length Δy.

The domain Ω will be then discretized using rectangles of area $\Delta x \Delta y$. We will also write $x_j := j\Delta x, \ j = 0, 1, \ldots, J+1$ and $y_k := -l + k\Delta y, \ k = 0, 1, \ldots, K+1$.

Integrating from $t = 0$ to some $t = T \in \mathbb{R}^+$ using N timesteps of length $\Delta t := \frac{T}{N}$, the solution at a timestep n will be approximated by a vector

$$U^n \in \mathbb{R}^{(J+2)(K+2)} : U^n = [U_0^n \; U_1^n \; \dots \; U_{(K+1)}^n]^T$$

where each U_k^n is such that

$$U_k^n \in \mathbb{R}^{(J+2)} : \; U_k^n = [U_{0,k}^n \; U_{1,k}^n \; \dots \; U_{J+1,k}^n] \tag{4.6}$$

this is, each U_k^n describes, for each node k on the y coordinate, the solution for all of the nodes on the x coordinate.

Discretization of the bilaplacian.

Recalling that $\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$, we will proceed to discretize directly each term using centered finite differences. Given a function $f(x)$ defined over $[0, \pi]$, we will write $f_i := f(x_i)$, $x_i \in (0, \pi)$, $i = 0, 1, \dots, J+1$. Ignoring the boundary for now, its fourth derivative at the j-th node can be approximated as follows

$$f_{xxxx}(x_i) \approx \frac{f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2}}{\Delta x^4}, \qquad i = 0, 1, \dots, J+1$$

this can be also represented as a matrix-vector product:

$$f_{xxxx} \approx \frac{1}{\Delta x^4} D^4 f := \frac{1}{\Delta x^4} \begin{bmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & 4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{J-1} \\ f_J \\ f_{J+1} \end{bmatrix}$$

where

$$D^4 := \begin{bmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & 4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 6 \end{bmatrix}$$

The second derivative receives also the same treatment: for a centered scheme, we have

$$f_{xx}(x_i) \approx \frac{f_{i-1} - 2f_i + f_{i+1}}{\Delta x^2} \tag{4.7}$$

and as a matrix-vector product, we have

$$f_{xx} \approx \frac{1}{\Delta x^2} D^2 f := \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_J \\ f_{J+1} \end{bmatrix} \tag{4.8}$$

where, as in the previous case,

$$D^2 := \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \tag{4.9}$$

This can be extended to further dimensions, while analog definitions can be given for f_{yy} and f_{yyyy}. With this in consideration, and given the structure of the numerical solution U, its bilaplacian can be approximated as a pentadiagonal block matrix:

$$\Delta^2 U = D_x^4 U + D_y^4 U + 2 D_x^2 D_y^2 U \tag{4.10}$$

where, for the identity matrix $I \in \mathbb{R}^{(J+2)(K+2)\times(J+2)(K+2)}$,

$$D_x^4 U = \frac{1}{\Delta x^4} I \otimes D^4, \quad D_y^4 U = \frac{1}{\Delta y^4} D^4 \otimes I$$
$$D_x^2 U = \frac{1}{\Delta x^2} I \otimes D^2, \quad D_y^2 U = \frac{1}{\Delta y^2} D^2 \otimes I$$

4.2.2 Treatment of the boundary

Given the boundary conditions of Problem (4.1), we must proceed to modify the discretized bilaplacian. On the x coordinate, we know that $u(0, y, t) = u(\pi, y, t) = 0$. Hence, we get $U_{0,k}^n = U_{J+1,k}^n = 0$, $\forall k \in [0, K+1]$, $\forall n \in [0, N]$. This doesn't alter the form of the matrix representing the second derivative if we consider the array U such that $U_{i,k}^n$, $i \in [1, J]$, but this also forces us to do the same for the fourth derivative matrix. From here, we will denote $[U^n]_{j,k}$ as the j-th element of the vector U_k^n defined in (4.6). Regarding that case, for $i = 1$ and $i = J$ we have:

$$[D_x^4 U^n]_{1,k} = \frac{U_{-1,k}^n - 4U_{0,k}^n + 6U_{1,k}^n - 4U_{2,k}^n + U_{3,k}^n}{\Delta x^4}$$
$$[D_x^4 U^n]_{J,k} = \frac{U_{J-2,k}^n - 4U_{J-1,k}^n + 6U_{J,k}^n - 4U_{J+1,k}^n + U_{J+2,k}^n}{\Delta x^4}.$$

In order to get the values of $U_{-1,k}^n$ and $U_{J+2,k}^n$, we have to take a look at the discretized second derivative on the boundary. Because $u_{xx}(0, y, t) = u_{xx}(\pi, y, t) = 0$, we can write

$$[D_x^2 U^n]_{0,k} = \frac{U_{-1,k}^n - 2U_{0,k}^n + U_{1,k}^n}{\Delta x^2} = 0, \quad [D_x^2 U^n]_{J+1,k} = \frac{U_{J,k}^n - 2U_{J+1,k}^n + U_{J+2,k}^n}{\Delta x^2} = 0$$

and thus, $U_{-1,k}^n = -U_{1,k}^n$ and $U_{J+2,k}^n = U_{J,k}^n$. Hence, the matrix representation will be given with the aid of a matrix $\hat{D}^4 \in \mathbb{R}^{J\times J}$ such that

$$D_x^4 = \frac{1}{\Delta x^4} I \otimes \begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & 4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{bmatrix} =: \frac{1}{\Delta x^4} I \otimes \hat{D}^4 \tag{4.11}$$

On the y coordinate, the second derivative can be modified with ease when considering he boundary condition $u_{yy}(x,\pm l,t)+\sigma u_{xx}(x,\pm l,t)=0$. With this, we have for $j\in[1,J]$ and for $n\in[0,N]$ that

$$[D_y^2U^n]_{j,0}=-\sigma[D_x^2U^n]_{j,0},\quad [D_y^2U^n]_{j,K+1}=-\sigma[D_x^2U^n]_{j,K+1}$$

and thus, the matrix representation of the second derivative over y will be

$$D_y^2=\frac{1}{\Delta y^2}\begin{bmatrix}-\frac{\sigma}{\Delta x^2}D^2 & & & & \\ I & -2I & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & -2I & I \\ & & & & -\frac{\sigma}{\Delta x^2}D^2\end{bmatrix} \tag{4.12}$$

For the fourth derivative, we will have the same problem as in the x coordinate case; this is,

$$[D_y^4U^n]_{j,0}=\frac{U^n_{j,-2}-4U^n_{j,-1}+6U^n_{j,0}-4U^n_{j,1}+U^n_{j,2}}{\Delta y^4}$$
$$[D_y^4U^n]_{j,K+1}=\frac{U^n_{j,K-1}-4U^n_{j,K}+6U^n_{j,K+1}-4U^n_{j,K+2}+U^n_{j,K+3}}{\Delta y^4}$$

To compute $U^n_{j,k}$ when $k=-2,-1,K+2,K+3$, we need to combine the fourth derivative discretization at the boundary with the one obtained from the second derivative discretization. For $k=0$, and approximating the boundary conditions using centered finite differences, we get

$$\frac{-U^n_{j,-2}+2U^n_{j,-1}-2U^n_{j,1}+U^n_{j,2}}{2\Delta y^3}+(2-\sigma)\frac{D_x^2U^n_{j,1}-D_x^2U^n_{j,-1}}{2\Delta y}=0$$
$$\frac{U^n_{j,-1}-2U^n_{j,0}+U^n_{j,1}}{\Delta y^2}+\sigma D_x^2U^n_{j,0}=0$$

this leads to

$$U^n_{j,-2}=2U^n_{j,-1}-\Delta y^2(2-\sigma)D_x^2U^n_{j,-1}-2U^n_{j,1}+\Delta y^2(2-\sigma)D_x^2U^n_{j,1}+U^n_{j,2}$$
$$U^n_{j,-1}=2U^n_{j,0}-\Delta y\sigma D_x^2U^n_{j,0}-U^n_{j,1}$$

hence,

$$D_y^4U^n_{j,0}=\frac{1}{\Delta y^4}\Big(2U^n_{j,0}+4\Delta y^4(\sigma-1)D_x^2U^n_{j,0}+\Delta y^4\sigma(2-\sigma)D_x^4U^n_{j,0}-4U^n_{j,1}+2\Delta y^2(2-\sigma)D_x^2U^n_{j,1}+2U^n_{j,2}\Big)$$
$$D_y^4U^n_{j,1}=\frac{1}{\Delta y^4}\Big(-2U^n_{j,0}-\Delta y^2\sigma D_x^2U^n_{j,0}+5U^n_{j,1}-4U^n_{j,2}+U^n_{j,3}\Big)$$

in a similar fashion, we get

$$D_y^4U^n_{j,K+1}=\frac{1}{\Delta y^4}\Big(2U^n_{j,K-1}-4U^n_{j,K}+2\Delta y^2(2-\sigma)D_x^2U^n_{j,K}+2U^n_{j,K+1}$$
$$+4\Delta y^2(\sigma-1)D_x^2U^n_{j,K+1}+\Delta y^4\sigma(2-\sigma)D_x^4U^n_{j,K+1}\Big)$$
$$D_y^4U^n_{j,K}=\frac{1}{\Delta y^4}\Big(U^n_{j,K-2}-4U^n_{j,K-1}+5U^n_{j,K}-2U^n_{j,K+1}-\Delta y^2\sigma D_x^2U^n_{j,K+1}\Big)$$

This gives the following matrix representation

$$D_y^4 = \frac{1}{\Delta y^4}\begin{bmatrix} 2I + 4\sigma_1 D^2 + \sigma_3 D^4 & -4I + 2\sigma_2 D^2 & 2I & & & \\ -2I - \Delta y^2 \sigma D^2 & 5I & -4I & I & & \\ I & -4I & 6I & -4I & I & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & I & -4I & 6I & -4I & I \\ & & I & -4I & 5I & -2I - \Delta y^2 \sigma D^2 \\ & & & 2I & -4I + 2\sigma_2 D^2 & 2I + 4\sigma_1 D^2 + \sigma_3 D^4 \end{bmatrix} \tag{4.13}$$

where $\sigma_1 = \Delta y^4(\sigma - 1)/\Delta x^2$, $\sigma_2 = 2\Delta y^2(2 - \sigma)/\Delta x^2$, and $\sigma_3 = \Delta y^4 \sigma(2 - \sigma)/\Delta x^4$. The bilaplacian matrix then will be a block pentadiagonal matrix of size $J(K + 2) \times J(K + 2)$, where it is defined by the sum (4.10) using the modified matrices given by (4.11) for D_x^4, (4.12) forD_y^2, and (4.13) for D_y^4.

4.2.3 Integration over time.

Given the definition of the function $\varphi(u)$ on Problem (4.1), the first order derivative will be approximated using a centered finite different scheme, and the integral will be computed using a Simpson rule for each value on the y coordinate. Meanwhile, the time derivative will be approximated using a finite difference scheme, analog the one used in (4.7). Finally, we will consider a Crank-Nicholson discretization for the bilaplacian; this is, we will approximate the bilaplacian over time using $\Delta^2\Big(\frac{U^{n+1}+U^n}{2}\Big)$.

This lead us to the numerical scheme which we will use on this work: for $U_{j,k}^n$ the numerical solution of Problem (4.1) on (x_j, y_k, t_n) with $h(x, y, t) = 0$, the solution at the timestep $n + 1$ will be given by

$$\Big[\Big(I + \frac{\Delta t^2}{2}(D_x^4 + D_y^4 + 2D_x^2 D_y^2)\Big)U^{n+1}\Big]_{j,k} \tag{4.14}$$

$$= 2U_{j,k}^n - U_{j,k}^{n-1} - \frac{\Delta t^2}{2}[(D_x^4 + D_y^4 + 2D_x^2 D_y^2)U^n]_{j,k} - \Delta t^2\Big(\varphi(U_{j,k}^n) + a(x_j, y_k)g\Big(\frac{U_{j,k}^n - U_{j,k}^{n-1}}{\Delta t}\Big)\Big)$$

if $a(x, y) = 0, \forall(x, y) \in \Omega$, and $P = S = 0$, then this scheme can control numerical diffusion of the energy if a sufficiently small value of Δt is used. If $g(s) = s$, then a Newmark scheme can be used to compute the numerical solution, which will preserve the energy for any value given for $\Delta t < 1$.

This scheme was implemented on a `MATLAB` script, where the linear equation system present in (4.14) was solved using its default solver. When solving the static problem $\Delta^2 u(x, y) = f(x, y)$, and using values of $\Delta x \approx 0.02$ and $\Delta y \approx 0.015$, the code can approximate the solution of the problem with errors of magnitude 10^{-6} for the numerical L^2 norm.

4.2.4 Numerical experiments for a static problem.

As a way to test the numerical scheme, we will attempt to solve the following static problem:

$$\begin{cases} \Delta^2 u(x,y) = f(x), & \text{in } \Omega, \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, & (y,t) \in (-l,l), \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, & x \in (0,\pi) \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, & x \in (0,\pi) \end{cases} \tag{4.15}$$

Solutions for this problem can be found in Ferrero and Gazzola ([FG15]), and are given by

$$u(x,y) = \sum_{m=1}^{+\infty} \left[\frac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my) \right] \sin(mx) \tag{4.16}$$

where A and B are defined as follows

$$A = \frac{\sigma}{1-\sigma} \frac{\beta_m}{m^4} \frac{(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell}$$

$$B = \sigma \frac{\beta_m}{m^4} \frac{\sinh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell}$$

and the β_m coefficients come from the following Fourier series of the source function

$$f(x) = \sum_{m=1}^{+\infty} \beta_m \sin(mx), \qquad \beta_m = \frac{2}{\pi} \int_0^\pi f(x)\sin(mx)dx$$

Here, we will use different functions $f(x)$ in order to compare the performance of the scheme.

Case 1: $f(x) = 4H(\frac{\pi}{2} - x) - 5H(x - \frac{\pi}{2})$

As a first attempt, we will use $f(x) = 4H(\frac{\pi}{2} - x) - 5H(x - \frac{\pi}{2})$. We have used $\ell = \frac{\pi}{250}$, $\Delta x = \frac{\pi}{500}$, $\Delta y = \frac{2\ell}{50}$, $\sigma = 0.1$. Figure 4.1 left shows the numerical solution, while Figure 4.1 right shows the difference obtained between the numerical and the exact solutions. Here, $||e||_2 = 7.9707E - 4$.

Case 2: $f(x) = sech(x - \pi)$

In this second experiment, we use $f(x) = sech(x - \pi)$. For our computations, $\ell = \frac{\pi}{200}$, $\Delta x = \frac{\pi}{600}$, $\Delta y = \frac{2\ell}{50} \approx 8.377 \cdot 10^{-4}$, $\sigma = 0.3$. Figure 4.2 left shows the numerical solution, while Figure 4.2 right shows the difference obtained between the numerical and the exact solutions. Here, $||e||_2 = 3.6613E - 4$.

Case 3: $f(x) = \sin(x - \frac{\pi}{2})$

In this second experiment, we use $f(x) = \sin(x - \frac{\pi}{2})$. We will use the same parameters as in Case 2. The error obtained was $||e||_2 = 1.1837E - 5$. Results can be seen in Figure 4.3.

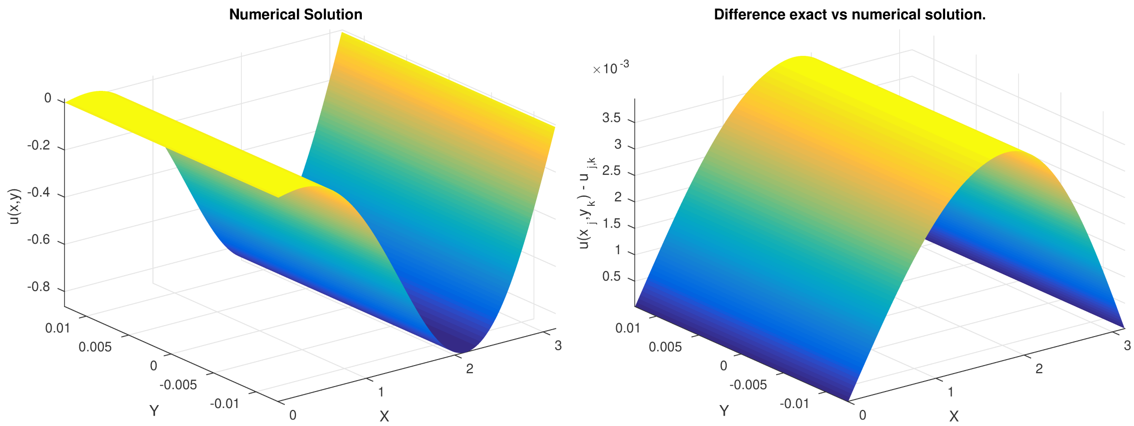

Fig. 4.1: Left: numerical solution obtained for problem (4.15). Right: difference between the exact and the numerical solution at each node.

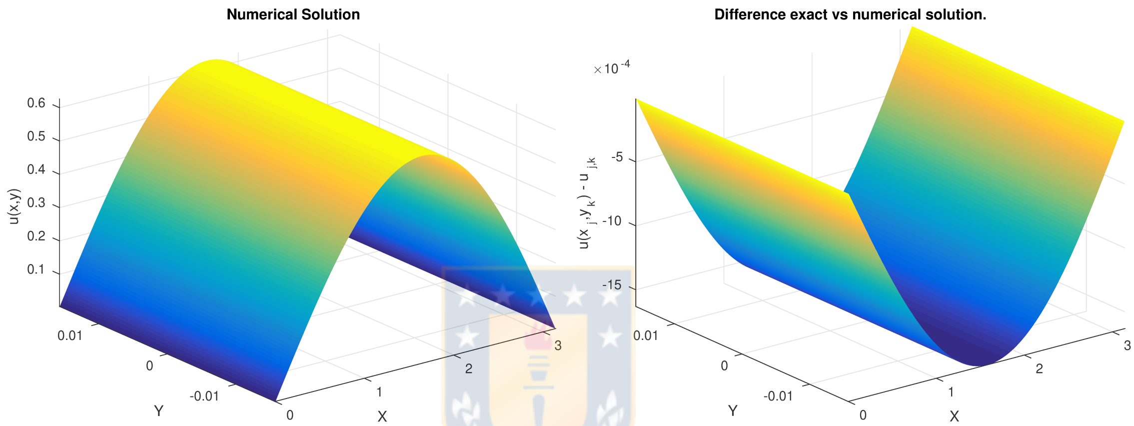

Fig. 4.2: Left: numerical solution obtained for problem (4.15). Right: difference between the exact and the numerical solution at each node.

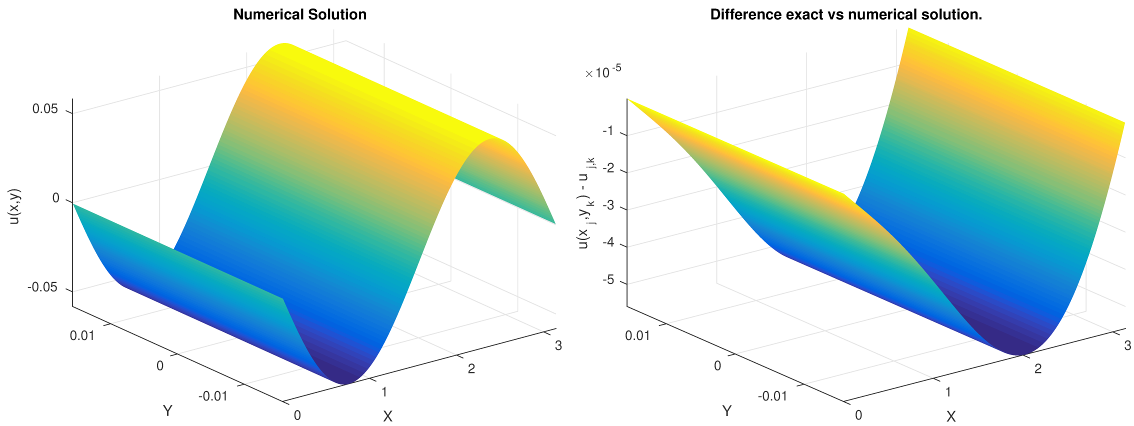

Fig. 4.3: Left: numerical solution obtained for problem (4.15). Right: difference between the exact and the numerical solution at each node.

4.2.5 Numerical experiments for a conservative problem.

We will now attempt to solve the following problem

$$
\begin{cases}
u_{tt}(x,y,t) + \Delta^2 u(x,y,t) = 0, & \text{in } \Omega \times (0,T], \\
u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, & (y,t) \in (-l,l), \\
u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, & x \in (0,\pi) \\
u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, & x \in (0,\pi) \\
u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y) & \text{in } \Omega.
\end{cases} \tag{4.17}
$$

where the initial condition will be given by (4.16) for $f(x) = \sin(2x)$. We will consider $t \in (0, T = 20]$, where $\ell = \frac{\pi}{150}$, $h = 0$, $\Omega = [0,\pi]\times[-\ell,\ell]$, $\sigma = 0.2$, and $u_1(x,y) = 0, \forall (x,y) \in \Omega$. For our simulation, $\Delta t = 10^{-5}$, $\Delta x = \frac{\pi}{150} \approx 0.021$, and $\Delta y = \frac{2\ell}{50} \approx 8.3775E-4$. Figure 4.4 shows the behavior in time of the numerical energy. The difference between its maximum and minimum values is[1] $8.83627E-3$.

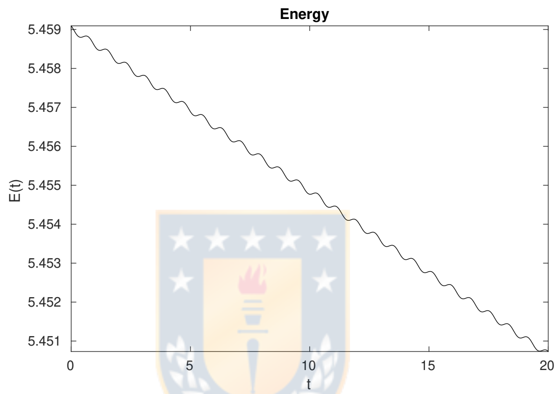

Fig. 4.4: Time behavior of the numerical energy when using $a(x,y) \equiv 0, \forall x \in \Omega$.

4.2.6 Numerical experiments with active damping.

For the following experiments, we will solve Problem (4.1) using $h = 0$, $\sigma = 0.2$, $S = 10^{-5}$, $P = 10^{-3}$, $l = \frac{\pi}{150}$, and $u_1 = 0$. Function u_0 will be given by the solution of the following static problem

$$
\begin{cases}
\Delta^2 u(x,y) = 50\sin(2x), \quad \text{in } \ \Omega \times (0,+\infty), \\
u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, & (y,t) \in (-l,l), \\
u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = 0, & x \in (0,\pi) \\
u_{yyy}(x,\pm l) + (2-\sigma)u_{xxy}(x,\pm l) = 0, & x \in (0,\pi)
\end{cases} \tag{4.18}
$$

[1] A much better choice could be a Newmark scheme, instead of classical Crank-Nicolson. If we repeat this same experiment using that method, then the difference in energy will get values near 10^{-7}. Sadly, the method is effective only when the function $g(s)$ is linear.

The solution is given in [FG15], Theorem 3.2. It can also be computed using this same numerical scheme. The function $a(x, y)$ is defined as follows:

$$a(x,y) = \begin{cases} 1, & (x,y) \in (0, 5\Delta x) \cup (\pi - 5\Delta x, \pi) \times (-l, -l + 5\Delta y) \cup (l - 5\Delta y, l) \\ 0, & \text{otherwise.} \end{cases}$$

where, on the numerical scheme, $\Delta x = \frac{\pi}{150} \approx 0.02$, $\Delta y = \frac{l}{50} \approx 0.015$, and $\Delta t = 0.01$. We will use three differents forms for the feedback function $g(s)$; or three different cases:

- **Case 1:** $g(s) = \sqrt{|s|}$
- **Case 2:** $g(s) = s$
- **Case 3:** $g(s) = \begin{cases} s^2, & \text{if } s \geq 0 \\ s^3, & \text{if } s < 0 \end{cases}$

Energy evolutions can be seen in Figure 4.5 in a semilogarithmic plot. The exponential decay, while oscillating, is clearly visible; as expected from Theorem 4.1.4. Figure 4.6 shows the numerical solutions at four different instants.

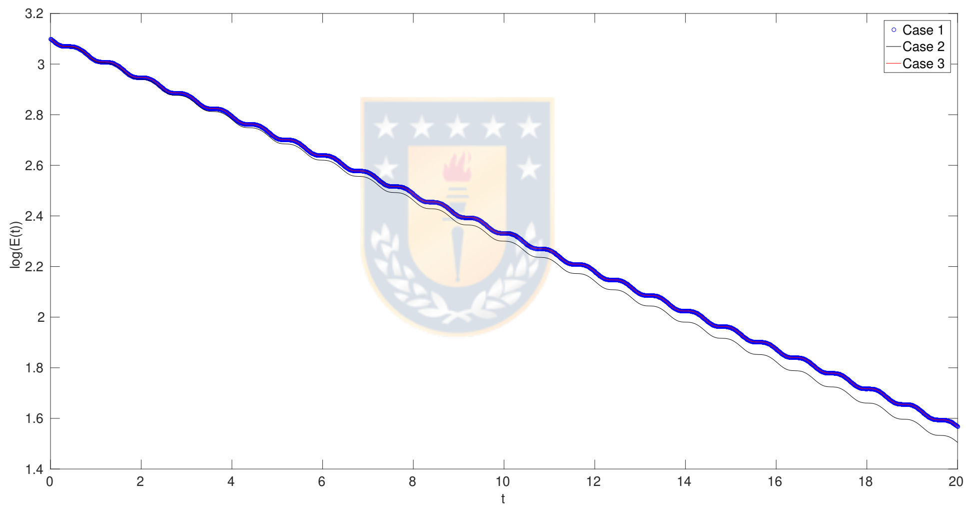

Fig. 4.5: Energy evolution for all three forms of $g(s)$.

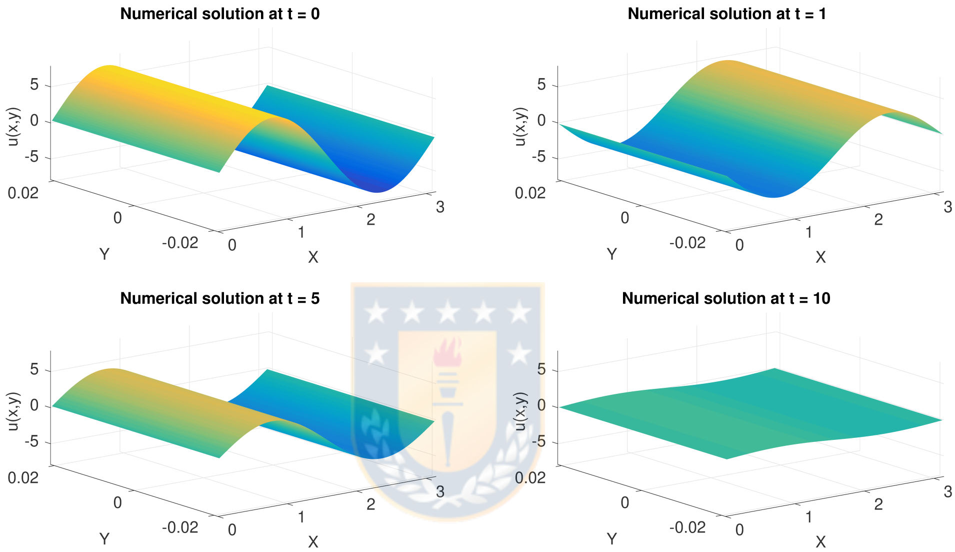

Fig. 4.6: Numerical solution when using $g(s) = s$ at four different instants.

Bibliography

[AB05] F. Alabau-Boussouira. Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. *Applied Mathematics & Optimization*, 51(1):61–105, 2005. 78

[AB10] F. Alabau-Boussouira. A unified approach via convexity for optimal energy decay rates of finite and infinite dimensional vibrating damped systems with applications to semi-discretized vibrating damped systems. *Journal of Differential Equations*, 248:1473–1517, 2010. 78

[ABA11] F. Alabau-Boussouira and K. Ammari. Sharp energy estimates for nonlinearly locally damped PDEs via observability for the associated undamped system. *Journal of Functional Analysis*, 260(8):2424–2450, 2011. 78

[ACFD12] F. D. Araruna, R. A. Capistrano-Filho, and G. G. Doronin. Energy decay for the modified Kawahara equation posed in a bounded domain. *Journal of Mathematical Analysis and Applications*, 2(385):743–756, 2012. 46

[AGBG14] M. Al-Gwaiz, V. Benci, and F. Gazzola. Bending and stretching energies in a rectangular plate modeling suspension bridges. *Nonlinear Analysis*, 106:18–34, 2014. 11, 12, 14

[Agr00] G. P. Agrawal. Nonlinear fiber optics. In *Nonlinear Science at the Dawn of the 21st Century*, pages 195–211. Springer, 2000. 5

[Akr93] G. D. Akrivis. Finite difference discretization of the cubic Schrödinger equation. *IMA Journal of Numerical Analysis*, 13(1):115–124, 01 1993. 8

[AKV13] L. Aloui, M. Khenissi, and G. Vodev. Smoothing effect for the regularized Schrödinger equation with non-controlled orbits. *Communications in Partial Differential Equations*, 38(2):265–275, 2013. 65

[AL83] D. Anderson and M. Lisak. Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides. *Physical Review A*, 27(3):1393–1398, 1983. 7, 8, 17

[AL12] A. Arena and W. Lacarbonara. Nonlinear parametric modeling of suspension bridges under aeroelastic forces: torsional divergence and flutter. *Nonlinear Dynamics*, 70(4):2487–2510, 2012. 15

[AS81] M. J. Ablowitz and H. Segur. *Solitons and the Inverse Scattering Transform*. SIAM, 1981. 16

[ASV09] M. Alves, M. Sepúlveda, and O. Vera. Smoothing properties for the higher-order nonlinear Schrödinger equation with constant coefficients. *Nonlinear Analysis*, 71:948–966, 2009. 16, 46

[Bal85] R. Balakrishnan. Soliton propagation in nonuniform media. *Physics Review A*, 32:1144–1149, 1985. 6

[Bar18] A. Barrios. A discontinuous Galerkin method for the biharmonic problem. *Undergraduate Thesis, Universidad de Concepción*, 2018. 15

[BBV07] E. Bisognin, V. Bisognin, and O. Villagran. Stabilization of solutions to higher-order nonlinear Schrodinger equation with localized damping. *Electronic Journal of Differential Equations*, 2007(06):1–18, 2007. 45

[BC15] N. Burq and H. Christianson. Imperfect geometric control and overdamping for the damped wave equation. *Communications in Mathematical Physics*, 336(1):101–130, 2015. 65

[Bes04] C. Besse. A relaxation scheme for the nonlinear Schrödinger equation. *SIAM Journal on Numerical Analysis*, 42(3):934–952, 2004. 9

[BG11] E. M. Behrens and J. Guzmán. A mixed method for the biharmonic problem based on a system of first-order equations. *SIAM Journal on Numerical Analysis*, 49(2):789–817, 2011. 15

[BGV10] I. Bochicchio, C. Giorgi, and E. Vuk. Long-term damped dynamics of the extensible suspension bridge. *International Journal of Differential Equations*, 383420, 2010. 14

[BJM02] W. Bao, S. Jin, and P. A. Markowich. On time-splitting spectral approximations for the schrödinger equation in the semiclassical regime. *Journal of Computational Physics*, 175(2):487–524, 2002. 9

[BOP80] I. Babuška, J. Osborn, and J. Pitkäranta. Analysis of mixed methods using mesh dependent norms. *Mathematics of Computation*, 35(152):1039–1062, 1980. 15

[BSZ03] J. L. Bona, S. M. Sun, and B. Y. Zhang. A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. *Communications in Partial Differential Equations*, 28(7-8):1391–1436, 2003. 16

[BV07] V. Bisognin and O. Villagrán. On the unique continuation property for the higher order nonlinear Schrödinger equation with constant coefficients. *Turkish Journal of Mathematics*, 31(3):265–302, 2007. 46

[Car06] X. Carvajal. Sharp Global Well-Posedness for a Higher Order Schrödinger Equation. *The Journal of Fourier Analysis and Applications*, 12(1):53–70, 2006. 16, 18

[CCCT17] M. M. Cavalcanti, W. J. Corrêa, V. N. Domingos Cavalcanti, and L. Tebou. Well-posedness and energy decay estimates in the Cauchy problem for the damped defocusing Schrödinger equation. *Journal of Differential Equations*, 262(3):2521 – 2539, 2017. 8, 47, 78

[CCFN09] MM Cavalcanti, VN Domingos Cavalcanti, R Fukuoka, and F Natali. Exponential stability for the 2-D defocusing Schrödinger equation with locally distributed damping. *Differential and Integral Equations*, 22(7/8):617–636, 2009. 47

[CCKR14] MM Cavalcanti, VN Domingos Cavalcanti, V Komornik, and JH Rodrigues. Global well-posedness and exponential decay rates for a KdV–Bürgers equation with indefinite damping. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 31, pages 1079–1100. Elsevier, 2014. 46

[CCO$^+$] M. Cavalcanti, W. Corrêa, T. Özsari, M. Sepúlveda, and R. Véjar-Asem. Exponential stability for the nonlinear Schrödinger equation with locally distributed damping. *Submitted.* 65, 66

[CCPVV05] V Ceballos, J Carlos, F Pavez, and O Vera Villagrán. Exact boundary controllability for higher order nonlinear Schrödinger equations with constant coefficients. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 2005, 2005. 46

[CCSN10] MM Cavalcanti, VN Domingos Cavalcanti, JA Soriano, and F Natali. Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: exponential and polynomial stabilization. *Journal of Differential Equations*, 248(12):2955–2971, 2010. 8, 47

[CCSVA] M. Cavalcanti, W. Corrêa, M. Sepúlveda, and R. Véjar-Asem. Well-posedness, exponential decay estimate, and numerical results for the high order nonlinear Schrödinger equation with localized dissipation. *Submitted.* 45

[CDG09] B. Cockburn, B. Dong, and J. Guzmán. A hybridizable and superconvergent discontinuous Galerkin method for biharmonic problems. *Journal of Scientific Computing*, 40(1-3):141–187, 2009. 15

[CFPR15] R. A Capistrano-Filho, A. F. Pazoto, and L. Rosier. Internal controllability of the Korteweg–de Vries equation on a bounded domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 21(4):1076–1107, 2015. 57

[CGT64] R. Y. Chiao, E. Garmire, and C. H. Townes. Self-trapping of optical beams. *Physical Review Letters*, 13(15):479–482, 1964. 4

[Che18] M. Chen. Stabilization of the higher order nonlinear Schrödinger equation with constant coefficients. *Proceedings-Mathematical Sciences*, 128(3):39, 2018. 57, 58

[CL76] H. H. Chen and C. S. Liu. Solitons in nonuniform media. *Physical Review Letters*, 37(11):693–697, 1976. 5

[CLL79] H. H. Chen, Y. C. Lee, and C. S. Liu. Integrability of nonlinear hamiltonian systems by inverse scattering method. *Physica Scripta*, 20:490–492, 1979. 17, 46

[CR74] P. G Ciarlet and P. A. Raviart. A mixed finite element method for the biharmonic equation. In *Mathematical aspects of finite elements in partial differential equations*, pages 125–145. Elsevier, 1974. 15

[CS88] P. Constantin and J. Saut. Local smoothing properties of dispersive equations. *Journal of the American Mathematical Society*, 1(2):413–439, 1988. 47

[CSC14] MM Cavalcanti, FRD Silva, and VD Cavalcanti. Uniform decay rates for the wave equation with nonlinear damping locally distributed in unbounded domains with finite measure. *SIAM Journal on Control and Optimization (Print)*, 52:545–580, 2014. 78

[CW90] Q. Chang and G. Wang. Multigrid and adaptive algorithm for solving the nonlinear Schrödinger equation. *Journal of Computational Physics*, 88(2):362–380, 1990. 9

[DCMCC+] A. D. Domingos Cavalcanti, M. M. Cavalcanti, W. J. Corrêa, Z. Hajjej, M. Sepúlveda Cortés, and R. Véjar Asem. Uniform decay rates for a suspension bridge with locally distributed nonlinear damping. *ArXiv: https://arxiv.org/abs/1902.09963, 2019.* vi, vii, 14, 77, 78, 79

[DFP81] M. Delfour, M. Fortin, and G. Payre. Finite-Difference Solutions of a Non-linear Schrödinger Equation. *Journal of Computational Physics*, 44:277–288, 1981. 8, 17, 39, 48, 67

[DL07] G. G. Doronin and N. A. Larkin. KdV equation in domain with moving boundaries. *Mathematical Analysis and Applications*, 328:503–527, 2007. 18

[EGH00] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. *Handbook of numerical analysis*, 7:713–1018, 2000. 66

[EHR+87] P. Emplit, J. Hamaide, F. Reynaud, C. Froehly, and A Barthelemy. Picosecond steps and dark pulses through nonlinear single mode fibers. *Optics communications*, 62(6):374–379, 1987. 6

[FFJS82] MD Feit, JA Fleck Jr, and A Steiger. Solution of the Schrödinger equation by a spectral method. *Journal of Computational Physics*, 47(3):412–433, 1982. 9

[FG15] A. Ferrero and F. Gazzola. A partially hinged rectangular plate as a model for suspension bridges. *Discrete & Continuous Dynamical Systems A*, 35:5879–5908, 2015. 13, 14, 84, 87

[FGdS16] V. Ferreira, F. Gazzola, and E. Moreira dos Santos. Instability of modes in a partially hinged rectangular plate. *Journal of Differential Equations*, 261(11):6302–6340, 2016. 11, 14

[Fib15] G. Fibich. *The nonlinear Schrödinger equation*, volume 192 of *Applied Mathematical Sciences*. Springer, Cham, 2015. 64

[FM11] D. Furihata and T. Matsuo. *Discrete variational derivative method : a structure-preserving numerical method for partial differential equations*. Chapman and Hall, 2011. 17

[Gaz13] F. Gazzola. Nonlinearity in oscillating bridges. *Electronic Journal of Differential Equations*, 2013(211):1–47, 2013. 11

[GH09] E. H Georgoulis and P. Houston. Discontinuous Galerkin methods for the biharmonic problem. *IMA journal of numerical analysis*, 29(3):573–594, 2009. 15

[Gla77] R. T. Glassey. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *Journal of Mathematical Physics*, 18(9):1794–1797, 1977. 6

[Gra07] V. Gradinaru. Strang splitting for the time-dependent Schrödinger equation on sparse grids. *SIAM Journal on Numerical Analysis*, 46(1):103–123, 2007. 9

[Gur78] A. Gurevich. *Nonlinear Phenomena in the Ionosphere*, volume 10 of *Physics and Chemistry in Space*. Springer-Verlag Berlin Heidelberg, 1978. 6

[GZD13] G. Z. Gao, L. D. Zhu, and Q. S. Ding. Identification of nonlinear damping and stiffness of spring-suspended sectional model. *Proceedings of The Eighth Asia-Pacific Conference on Wind Engineering*, 2013. 9, 14

[Har57] H. F. Harmuth. On the solution of the Schrödinger and the Klein-Gordon equations by digital computers. *Journal of Mathematics and Physics*, 36(1-4):269–278, 1957. 4, 8

[Has89] A. Hasegawa. Optical solitons in fibers. In *Optical Solitons in Fibers*, pages 1–74. Springer, 1989. 6

[Hir73a] R. Hirota. Exact envelope-soliton solutions of a nonlinear wave equation. *Journal of Mathematical Physics*, 14:805–809, 1973. 7, 17, 41

[Hir73b] R. Hirota. Exact solution of the Modified Korteveg-de Vries Equation for Multiple Collisions of Solitons. *Journal of the Physical Society of Japan*, 33(5):1456–1458, 1973. 41

[HK81] A. Hasegawa and Y. Kodama. Signal transmission by optical solitons in monomode fiber. *Proceedings of the IEEE*, 69(9):1145–1150, 1981. 6, 7

[HS96] A. M. Horr and L. C. Schmidt. Modelling of nonlinear damping characteristics of a viscoelastic structural damper. *Engineering Structures*, 18(2):154–161, 1996. 14

[KC13] H. Kumar and F. Chand. Dark and Bright Solitary Wave Solutions of the Higher Order Nonlinear Schrödinger Equation with Self-Steepening and Self-Frequency Shift Effects. *Journal of Nonlinear Optical Physics & materials*, 22(1):1350001, 2013. 7, 62

[KCS03] AH Khater, DK Callebaut, and AR Seadawy. Nonlinear dispersive instabilities in Kelvin–Helmholtz magnetohydrodynamic flows. *Physica Scripta*, 67(4):340–349, 2003. 6

[KDV95] D. J. Korteweg and G. De Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philosophical Magazine*, 539:422–443, 1895. 7

[Kel65] PL Kelley. Self-focusing of optical beams. *Physical Review Letters*, 15(26):1005–1008, 1965. 4, 8

[KH87] Y. Kodama and A. Hasegawa. Nonlinear pulse propagation in a monomode dielectric guide. *IEEE Journal of Quantum Electronics*, QE-23:510–524, 1987. 6, 7

[KHGG88] D Krökel, NJ Halas, G Giuliani, and D Grischkowsky. Dark-pulse propagation in optical fibers. *Physical review letters*, 60(1):29–32, 1988. 6

[Kir76] G. R. Kirchhoff. *Vorlesungen über mathematische physik: Mechanik*, volume 1. Teubner, 1876. 11

[KM82] R Kant and SK Malik. Nonlinear waves in superposed fluids. *Astrophysics and Space Science*, 86(2):345–360, 1982. 6

[Kod85] Y. Kodama. Optical solitons in a monomode fiber. *Journal of Statistical Physics*, 39(5-6):597–614, 1985. 6

[Lel92] S. K. Lele. Compact finite difference schemes with spectral-like resolution. *Journal of Computational Physics*, 103:16–42, 1992. 9, 17

[LHL15] W. Lu, Y. Huang, and H. Liu. Mass preserving discontinuous Galerkin methods for Schrödinger equations. *Journal of Computational Physics*, 282:210–226, 2015. 9

[Lio69] J. L. Lions. *Quelques Métodes de Résolution des Problèmes Aux Limites Non Linéaires*. Dunod, Paris, 1969. 22, 33, 56

[LM90] A. C. Lazer and P. J. McKenna. Large-amplitude periodic oscillations in suspension bridges: some new connections with non-linear analysis. *SIAM Review*, 32(4):537–578, 1990. 10, 11

[LP09] F. Linares and A. F. Pazoto. Asymptotic behavior of the Korteweg–de Vries equation posed in a quarter plane. *Journal of Differential Equations*, 246(4):1342–1353, 2009. 46

[LR06] K. Liu and B. Rao. Exponential stability for the wave equations with local Kelvin-Voigt damping. *Journal of Applied Mathematics and Physics.*, 57(3):419–432, 2006. 65

[LT93] I. Lasiecka and D. Tataru. Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping. *Differential and integral Equations*, 6:507–533, 1993. 78

[LTZ04] I. Lasiecka, R. Triggiani, and X. Zhang. Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. I. $H^1(\Omega)$-estimates. *Journal of Inverse and Ill-Posed Problems*, 12(1):43–123, 2004. 65

[Mal05] B. A. Malomed. *Nonlinear Schrödinger equations*. Encyclopedia of Nonlinear Science. New York: Routledge, 2005. 6

[MM86] F. M. Mitschke and L. F. Mollenauer. Discovery of the soliton self-frequency shift. *Optics letters*, 11(10):659–661, 1986. 7

[MM15] SA Messaoudi and SE Mukiawa. A suspension bridge problem: existence and stability. In *International Conference on Mathematics and Statistics*, pages 151–165. Springer, 2015. 13, 14

[Mon87] P. Monk. A mixed finite element method for the biharmonic equation. *SIAM Journal on Numerical Analysis*, 24(4):737–749, 1987. 15

[MRPS+19] J. Muñoz Rivera, V. Poblete, M. Sepúlveda, H. Vargas, and O. Vera. Remark on the stabilization for a Schrödinger equation with double power nonlinearity. *Applied Mathematics Letters*, 2019. 47

[MSG80] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon. Experimental observation of picosecond pulse narrowing and solitons in optical fibers. *Physical Review Letters*, 45(13):1095–1098, 1980. 5

[MW87] P. J. McKenna and W. Walter. Non-linear oscillations in a suspension bridge. *Archive for Rational Mechanics and Analysis*, 98(2):167–177, 1987. 11

[MW90] P. J. McKenna and W. Walter. Travelling waves in a suspension bridge. *SIAM Journal on Applied Mathematics*, 50(3):703–715, 1990. 10

[Nat15] F. Natali. Exponential stabilization for the nonlinear Schrödinger equation with localized damping. *Journal of Dynamical and Control Systems*, 21(3):461–474, 2015. 47

[Nat16] F. Natali. A note on the exponential decay for the nonlinear Schrödinger equation. *Osaka Journal of Mathematics*, 53(3):717–729, 2016. 47

[ÖKL11] T. Özsarı, V. K. Kalantarov, and I. Lasiecka. Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control. *Journal of Differential Equations*, 251(7):1841–1863, 2011. 47

[PFE74] T. L. Perel'man, a: Kh. Fridman, and M. M. El'yashevich. A modified Korteweg-de Vries equation in electrohydrodynamics. *Soviet Physics JETP*, 39(4):643–646, 1974. 7

[PM97] S Preidikman and D. T. Mook. Numerical simulation of flutter of suspension bridges. *Applied Mechanics Reviews*, 50(11S):S174–S179, 1997. 15

[PMVZ02] G. Perla Menzala, C. F. Vasconcellos, and E. Zuazua. Stabilization of the Korteweg-de Vries equation with localized damping. *Quarterly of Applied Mathematics*, 60(1):111–129, 2002. 46

[Pot89] M. J. Potasek. Novel femtosecond solitons in optical fibers, photonic switching, and computing. *Journal of Applied Physics*, 65(3):941–953, 1989. 7

[PS03] L. Pítajevskíj and S. Stringari. *Bose-Einstein Condensation*. Clarendon Press, Oxford, 2003. 6

[PSV10] A. F. Pazoto, M. Sepúlveda, and O. Vera. Uniform stabilization of numerical schemes for the critical generalized Korteweg-de Vries equation with damping. *Numerische Mathematik*, 116:317–356, 2010. 17, 22, 46, 48

[PT91a] M. J. Potasek and M. Tabor. Exact solutions for an extended nonlinear Schrödinger equation. *Physics Letters A*, 154(9):449–452, 1991. 7

[PT91b] M. J. Potasek and M. Tabor. Exact solutions for an extended nonlinear Schrödinger equation. *Physics Letters A*, 154(9):449–452, 1991. 17, 40, 42, 61, 63

[Ros97] L. Rosier. Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 2:33–55, 1997. 46

[RZ06] L. Rosier and B. Y. Zhang. Global Stabilization of the Generalized Korteweg–de Vries Equation Posed on a Finite Domain. *SIAM Journal on Control and Optimization*, 45(3):927–956, 2006. 46

[SB11] M. Smadi and D. Bahloul. A compact split step Padé scheme for higher-order nonlinear Schrödinger equation (HNLS) with power law nonlinearity and fourth order dispersion. *Computer Physics Communications*, 182:366–371, 2011. 9, 17

[SB15] M. Smadi and D. Bahloul. Dynamic of HNLS Solitons using Compact Split Step Padé Scheme. *Journal of Physics: Conference Series*, 574, 2015. 9, 17

[SC95] J. J. Sakurai and E. D. Commins. Modern quantum mechanics, revised edition, 1995. 3

[SS91] N. Sasa and J. Satsuma. New-Type of Soliton Solutions for a Higher-Order Nonlinear Schrödinger Equation. *Journal of The Physical Society of Japan*, 60(2):409–417, 1991. 7, 17

[SS99] C. Sulem and P. L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. 6

[SSV86] J. M. Sanz-Serna and J. G. Verwer. Conerservative and nonconservative schemes for the solution of the nonlinear Schrödinger equation. *IMA Journal of Numerical Analysis*, 6(1):25–42, 1986. 9

[Tak99] H. Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Advances in Differential Equations*, 4(4):561–580, 1999. 16

[Tak00] H. Takaoka. Well-posedness for the Higher Order Nonlinear Schrödinger Equation. *Advances in Mathematical Sciences and Applications*, 10(1):149–171, 2000. 16

[TPPM12] C. Talischi, G. H. Paulino, A. Pereira, and I. FM. Menezes. PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab. *Structural and Multidisciplinary Optimization*, 45(3):309–328, 2012. 66

[Tsu90] M. Tsutsumi. On global solutions to the initial-boundary value problem for the damped nonlinear Schrödinger equations. *Journal of Mathematical Analysis and Applications*, 145(2):328–341, 1990. 47

[TTZ03] R. Tcheugoué, H. Tébou, and E. Zuazua. Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity. *Numerische Mathematik*, 95:563–598, 2003. 22

[VK01] E. Ventsel and T. Krauthammer. *Thin Plates and Shells: Theory: Analysis, and Applications*. CRC Press, 2001. 12

[VSS84] J. G. Verwer and J. M. Sanz-Serna. Convergence of method of lines approximations to partial differential equations. *Computing*, 33(3-4):297–313, 1984. 9

[XS05] Y. Xu and C. W. Shu. Local discontinuous Galerkin methods for nonlinear Schrödinger equations. *Journal of Computational Physics*, 205(1):72–97, 2005. 9

[XZF18] J. Xu, Q. Zhu, and E Fan. The initial-boundary value problem for the Sasa-Satsuma equation on a finite interval via the Fokas method. *Journal of Mathematical Physics*, 59(7):073508–1 – 073508–19, 2018. 16

[Zou01] G. E. Zouraris. On the convergence of a linear two-step finite element method for the nonlinear Schrödinger equation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 35(3):389–405, 2001. 9

[ZS72] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34(1):62–69, 1972. 4, 5, 6, 16