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ANALYTIC SOLUTIONS IN THE GAUGED SKYRME MODEL AND CONSERVATION LAWS IN GAUGE THEORIES WITH APPLICATIONS TO BLACK HOLE MECHANICS

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BY : DIEGO ROBERTO HIDALGO TECAY

Thesis Advisor : Dr. Fabrizio Canfora

Agosto - 2021

Concepción, Chile

Thesis Advisor : Dr. Fabrizio Canfora

Laboratorio de Física Teórica,
Centro de Estudios Científicos (CECs),
Valdivia, Chile.

Assessment Committee : Dr. Fernando Izaurieta

Departamento de Física,
Universidad de Concepción (UdeC),
Concepción, Chile.

: Dr. Julio Oliva

Departamento de Física,
Universidad de Concepción (UdeC),
Concepción, Chile.

Assessment Exterior Committee : Dr. Joaquim Gomis
Departament de Física Quàntica i Astrofísica
and Institut de Ciències del Cosmos (ICCUB),
Universitat de Barcelona, Martí i Franquès 1
E-08028 Barcelona, Spain.

Dr. Jorge Zanelli

Laboratorio de Física Teórica,
Centro de Estudios Científicos (CECs),
Valdivia, Chile.

Dedicated to my beloved family and Cristina...

"El autor no responde de las molestias que puedan ocasionar sus escritos:
Aunque le pese
El lector tendrá que darse siempre por satisfecho..."

Nicanor Parra

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Abstract

The present thesis consists of two parts. Part I is devoted to the construction of the first analytic examples of topologically non-trivial solutions of the $U(1)$ gauged Skyrme model within a finite box in $(3+1)$-dimensional flat space-time. There are two types of gauged solitons. The first type corresponds to gauged Skyrmions living within a finite volume. The second corresponds to gauged time-crystals (smooth solutions of the $U(1)$ gauged Skyrme model whose periodic time-dependence is protected by a winding number). The notion of electromagnetic duality can be extended for these two types of configurations in the sense that some of the electric and magnetic field components can be interchanged. These analytic solutions show very explicitly the Callan-Witten mechanism (according to which magnetic monopoles may "swallow" part of the topological charge of the Skyrmion) since the electromagnetic field contribute directly to the topological charge of the gauged Skyrmions. As it happens in superconductors, the magnetic field is suppressed in the core of the gauged Skyrmions. On the other hand, the electric field is strongly suppressed in the centre of gauged time crystals.

Part II is concerned with studying a new derivation of surface charges in gauge theories. Part of the focus is on reviewing the method to compute quasi-local surface charges for gauge theories to clarify conceptual issues and their range of applicability. The surface charges found are quasi-local, explicitly coordinate independent, and gauge invariant. Many surface charge formulas for gravity theories are expressed in metric, tetrads-connection, and even Chern-Simons connection. For most of them, the language of differential forms is exploited and contrasted with the more popular metric components language. The study focuses on General Relativity theory coupled with matter fields as Maxwell, Skyrme, and spinors. To derive the surface charges, we specify the phase space by identifying the symplectic structure. We use the formulation of the covariant phase space method. Here the symplectic structure has two parts: the standard Lee-Wald term plus a contribution from the boundary term read from the action. The latter is fixed by requiring the on-shell and linearized equations of motion condition, and exact symmetry condition. These conditions guarantees the conservation of the symplectic structure in phase space, and leads to the new concept of "symplectic symmetry". Given the "conservation law" satisfying the symplectic structure, we construct the corresponding charges, the "symplectic symmetry generators". The explicit expression of the charges corresponds to a function over the phase space.

We find the remarkable property that, in contrast with usual Noether procedures to compute charges, the boundary terms and even topological terms do not affect the surface charges. On the other hand, by studying two concrete examples, we also examine how torsion affects the surface charges. Both of them conclude that the torsion field does not affect the general formula for the surface charges. Furthermore, three examples with ready-to-download *Mathematica* notebook codes show the method in full action. The charges and their associated first law of thermodynamics are derived for: the BTZ black hole, the charged rotating $(3+1)$-black hole, and the Lorentzian rotating Taub-NUT space-time.

Resumen

La presente tesis consiste de dos partes. La Parte I está dedicada a la construcción de los primeros ejemplos analíticos de soluciones topológicamente no-triviales del modelo $U(1)$-gauged-Skyrme dentro de una caja finita en un espacio plano $(3+1)$-dimensional. Existen dos tipos de gauged-solitones: El primer tipo corresponde a gauged-Skyrmions viviendo dentro de un volumen finito. El segundo tipo corresponde a gauged-cristales temporales, soluciones suaves del $U(1)$-gauged-Skyrme cuya dependencia temporal periódica es sustentada por un "winding number" no-trivial. Existe una noción de una dualidad electromagnética para estos dos tipos de configuraciones en tanto algunas de las componentes del campo eléctrico y el magnético pueden ser intercambiadas. Estas soluciones analíticas muestran claramente manifestaciones del mecanismo de Callan-Witten (el cual señala que los monopolos magnéticos pueden "tragar" parte de la carga topológica del Skyrmion) ya que el campo electromagnético contribuye directamente a la carga topológica de los gauged-Skyrmions. Tal como sucede en superconductores, el campo magnético es suprimido en el centro de los gauged-Skyrmions. Mientras que el campo eléctrico es fuertemente suprimido en el centro de los gauged-cristales temporales.

La parte II está dedicada al estudio de una nueva derivación de cargas superficiales en teorías de gauge. Parte del enfoque consiste en revisar el método para

calcular cargas superficiales cuasi-locales para teorías de gauge con el fin de aclarar los problemas conceptuales y su rango de aplicabilidad. Las cargas superficiales son cuasi-locales, explícitamente independiente de las coordenadas, e invariantes de gauge. Muchas de las fórmulas de cargas superficiales para teorías de gravedad son expresadas en variables métricas, formas diferenciales e incluso conexiones de tipo Chern-Simons. Para la gran mayoría de las teorías de gauge se utiliza el lenguaje de formas diferenciales para resaltar y contrastar con el lenguaje más popular de componentes métricas. El estudio se focaliza en la teoría de Relatividad General acoplada a campos de materia como: campo electromagnético de Maxwell, campo de Skyrme y espinores. Para derivar las cargas superficiales usamos la formulación del método de espacio de fase covariante. Encontramos que la estructura simpléctica posee dos partes: el estándard término de Lee-Wald más una contribución del término de borde derivado de la acción. Esto último término es fijado requiriendo las condiciones on-shell y ecuaciones de movimiento linealizadas, y la condición de simetría exacta. Estas condiciones garantizan la conservación de la estructura simpléctica en el espacio de fase, resultando al nuevo concepto de "simetría simpléctica". Dada la ley de conservación satisfecha por la estructura simpléctica, construimos las correspondientes cargas conservadas: los "generadores de simetría simplécticos". La expresión explícita de las cargas corresponden a funciones definidas sobre el espacio de fase.

Encontramos la importante propiedad de que, en contraste a los usuales procedimientos de Noether para calcular cargas, los términos de borde e incluso los términos topológicos añadidos a la acción no afectan a las cargas superficiales. Por otra parte, analizando dos ejemplos concretos, estudiamos cómo el campo de torsión afecta las cargas supericiales. Para ambos casos concluímos que el campo de torsión no afecta a las cargas superficiales.

Además, tres ejemplos con códigos del programa *Mathematica* muestran el método en acción. Las cargas y su primera ley de termodinámica son derivadas para: el agujero negro BTZ, el $(3+1)$-agujero negro cargado y rotante, y el espacio-tiempo Taub-NUT Lorentziano rotante.

Overview of the Thesis

The first part of this thesis was motivated by understanding the phase diagram of low energy Quantum Chromodynamics (QCD) by coupling a $U(1)$ gauge field with the Skyrme model. This is sustained by the well-known reason that the Skyrme model describes the low energy limit of QCD, one of the most successful theories in physics. This first part possesses an introduction that explains the role of topology in physics and the birth of Skyrmions. Emphasis is put on the concepts of topology and the construction of an effective field theory from this mathematical approach.

In the second chapter of the first part, we study the Skyrme model and explain its main topological properties, *e.g.* the discrete number: the topological charge. We emphasize how important it is to have solutions, particularly analytic ones, to understand specific underlying properties in this model. The next chapter is concerned with constructing analytical and topologically non-trivial solutions of the $U(1)$ gauged Skyrme model in $(3+1)$-dimensional flat space-times at finite volume. Some properties of these solutions would reveal an essential characterization of the phase diagram of QCD at this regime.

The second part of this thesis addresses an entirely different topic: the treatment of conservation laws in gauge theories through the surface charge method. Firstly, a general overview of symmetries in physics and their relations to conservation laws is presented in the introduction. Next, we offer the derivation of the surface charge method as an alternative to deal with the computations of charges in gauge theories. To exhibit the transparency of this method to compute physical charges, we present three different examples of a family of solutions in General Relativity with and without matter. The thermodynamics properties of these space-times and especially the derivation of the first law of black holes mechanics are studied. Furthermore, three examples with ready-to-download *Mathematica* notebook codes, show the method in full action. For each of them, we compute step-by-step the charges associated to the exact symmetries of the solution. New results as the presence of boundary and topological terms and torsion are highlighted through the notes.

Chapter 1

Preamble

The two topics addressed in this thesis become are related through the study of the thermodynamics properties of the most intriguing and strange objects in the Universe: *black holes*. The baryonic (topological) charge ables of characterizing baryonic matter, and the charges derived from the isometries of space-time will be the two ingredients of the Einstein-Skyrme system that we try to connect. We do not intend to show a concrete structure of this connection but only propose how this could turn out.

The first law of black hole thermodynamics is the energy balancing equation among small variations of the different quantities characterizing a black hole. Each term appearing in the first law is interpreted as standard physical quantities like energy, angular momentum, electric charge, electric potential or angular velocity. These quantities are normally defined in a non-gravitational context then physical interpretation is easier for asymptotic observers (*e.g.* space-time energy). Besides that, the true thermodynamical character of the first law emerges once quantum fields are considered on top of the black hole classical background. The emergence of the Hawking temperature almost complete the thermodynamic picture for black holes [1]. The missing piece is a statistical description of the black hole entropy, which is by now an open question almost a half-century old.

From a semi-classical point of view, the first law of thermodynamics of four-dimensional asymptotically flat black holes might be obtained through the geometric derivation of the Komar integrals [2], but as we will see throughout this thesis, it has problems. In many cases the charges obtained with this method must be compared with others, *e.g.* with the Arnowitt-Deser-Misner (ADM) expression.

A generalization of the Noether theorem for gauge theories, *the surface charges*, could better pave this road. In this approach, for each exact symmetry, $\delta_\epsilon \Phi = 0$, there is a conservation law, $dk_\epsilon \approx 0$, and therefore a conserved charge, $\delta Q_\epsilon \equiv \oint k_\epsilon$. In the gravity context, the integration is over any surface that encloses the black hole singularity. The symmetry parameter, $\epsilon = \epsilon_1 + \epsilon_2 + \cdots$, may be partitioned in several independent exact symmetries and each of them may produce a different surface charge δQ_{ϵ_1}, $\delta Q_{\epsilon_2}, \ldots$. Then, in this setup, the first law takes the form

$$\alpha_1 \delta Q_{\epsilon_1} + \alpha_2 \delta Q_{\epsilon_2} + \cdots = \delta Q_{\epsilon_0} \, , \tag{1.1}$$

where the charges can be arranged, with the help of the phase space functions α_i, such that at the right hand side we obtain a surface charge δQ_{ϵ_0}. A particular version of the previous equation for an electrically charged and rotating black hole family is the usual balancing equation

$$\delta M = T_h \delta S + \Omega_h \delta J + \Phi_h \delta Q \, , \tag{1.2}$$

where the energy/mass, $M(S, J, Q)$, is a function of the charges: entropy S, angular momentum J, and the electric charge Q. While $T_h \equiv \frac{\partial M}{\partial S}$, $\Omega_h \equiv \frac{\partial M}{\partial J}$, and $\Phi_h \equiv \frac{\partial M}{\partial Q}$ are identified as the temperature, the angular velocity, and the electric potential, respectively. At the heart of the surface charges method there is the gauge symmetry: Exact symmetries are a particular choice of gauge parameters that, for a family of solutions, do not affect the variables.

Undoubtedly, there are motivations for connecting the geometry of space-time with the thermodynamic properties of black holes. In 1977, Ruffini and Wheeler noted that the formation of a black hole by a gravitational collapse is uniquely specified by the conserved quantities previously mentioned: mass, angular momentum, and electric charge. This is known as the famous conjecture "Black holes have no hair".

During a long time there has been an interest in classical black hole hair [3–7]. In this context, one of the most interesting results are the solutions of the Einstein-Skyrme equations [8]. The so-called "skyrmion black holes" has validity as long as the classical parameter of those solutions are chosen such the black hole event horizon is smaller than a characteristic length scale associated to the Skyrmion [9]. Considering the fact the Skyrme theory corresponds to the low energy limit of QCD, the black holes with Skyrmion hair would be interpreted as black holes with a baryon hair.

Then, a natural question arise: *How the standard first law of black hole mechanics should be generalized to include topological charges?*

In the following, we design a possible answer to this question.

These families of black hole solutions with different topological charges may be very different in their symmetry properties. For instance, black hole solutions with different baryonic charges do not have, in general, a continuous symmetry. Instead, they may have different discrete symmetries. This fact makes a black hole solution of one baryonic number very different from another. However, there are families of black hole solutions with high baryonic numbers. One expects that small changes of the baryonic charge do not drastically change the physics of the problem (for example, the physics as seen by an observer far away from the source). These are the approximation regime we are interested in. In these cases, one may expect that small but discrete changes of the topological charge to be described similarly as the continuous charges even though there is no continuous symmetry to define it as a surface charge.

We first note that the surface charges in (1.2) are infinitesimal variation of the finite charges over the phase space. This is even assumed true for the electric charge which is known to be discrete. However, topological charges split the phase space into disconnected regions. They are insensitive to infinitesimal variations of the gauge variables. Then, a possible transformation that allows us to jump from two solutions with different topological charges may be a discrete finite transformation on the phase space. Let us define a "variation" that only change discretely the topological charge $\tilde{\delta}N = 1$ and let us define a generalized variation on the phase space by $\Delta \equiv \delta + \tilde{\delta}$ such that $\delta N = 0$ but $\Delta N = 1$. This variation allows us to explore the tangent space of a given solution but also to jump between solutions with different topological charges.

Thus, *a* possible generalization of the usual first law of black hole mechanics is reached by adding an additional term related with this new variation that connect in principle topologically disconnected sectors of the phase space, namely

$$\Delta M \; = \; T_h \Delta S + \mu_{N} \Delta N \, , \tag{1.3}$$

where μ_N is the baryonic *chemical potential*. For simplicity we consider the electrically neutral and static families of black hole solutions: $Q = 0$ and $J = 0$.

To fully complete the previous picture, one should have a description of the dynamical physical processes−in the same spirit of the known physical process of the first law of black holes mechanics that relates a change in entropy of a perturbed Killing horizon to the matter flow into the horizon−that change the baryonic number of a black hole solution. For instance, to consider a baryonic soliton free falling into a black hole. However, this interesting question is out of the scope of the present preamble. Here, the best we can say is that if such a process exists, then a first law as written in (1.3) holds.

Conventions and notations

We employ units such that the speed of light c, the reduced Planck's constant $\hbar$, and the magnetic and electric permeability, μ_0 and ϵ_0, are equal to one. But we keep manifest the Newton's constant G.

For a D-dimensional space-time $\mathcal{M}$, we use the signature convention for the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$. The dimension of space-time is generally D. The Levi-Civita totally anti-symmetric symbol $\varepsilon_{\mu_1 \cdots \mu_D}$ such that $\varepsilon_{01\cdots(D-1)} = +1$, we also have

$$\varepsilon_{\mu_1 \cdots \mu_D} = \tilde{\varepsilon}^{\mu_1 \cdots \mu_D} , \tag{1.4}$$

$$g^{\mu_1 \nu_1} \cdots g^{\mu_D \nu_D} \varepsilon_{\mu_1 \cdots \mu_D} = \frac{1}{g} \tilde{\varepsilon}^{\nu_1 \cdots \nu_D} = \varepsilon^{\nu_1 \cdots \nu_D} , \tag{1.5}$$

$$g_{\mu_1 \nu_1} \cdots g_{\mu_D \nu_D} \tilde{\varepsilon}^{\mu_1 \cdots \mu_D} = g \, \varepsilon_{\nu_1 \cdots \nu_D} , \tag{1.6}$$

with the space-time metric determinant $g \equiv \det(g_{\mu\nu})$. We also introduced $\tilde{\varepsilon}^{\mu_1 \cdots \mu_D}$ such that $\tilde{\varepsilon}^{\,01\cdots(D-1)} = +1$, this twiddle symbol is exactly the Levi-Civita symbol but with indices written upstairs. In contrast, $\varepsilon^{\mu_1 \cdots \mu_D}$ is a space-time function, not the Levi-Civita symbol, its indices are raised with the space-time metric. Thus, similarly to (1.5), to raise and lower indices with the internal flat metric, yields

$$\eta^{a_1 b_1} \cdots \eta^{a_D b_D} \varepsilon_{a_1 \cdots a_D} = \det(\eta^{-1}) \tilde{\varepsilon}^{b_1 \cdots b_D} = -\tilde{\varepsilon}^{b_1 \cdots b_D} = \varepsilon^{b_1 \cdots b_D} . \tag{1.7}$$

The introduction of the object $\tilde{\varepsilon}$ is highly recommended as a way to keep consistent the Einstein notation of index contraction and thus to avoid some usual confusions on the computations. This object will be extremely useful for the second part of the thesis.

Greek letters $\alpha, \beta, \gamma, ...$ generally are space-time indices, $\mu = 0, 1, 2, ..., D-1$. The index 0 represents a time-like coordinate. The space-time coordinates are denoted by x^μ. Latin indexes will designate space-like coordinates $x^i, i = 1, ..., D$. Capital latin letters will be used to denote internal indexes of a certain group. The differentials dx^μ anti-commute, and the volume element is denoted by $d^D x$

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \,, \qquad d^D x = \sqrt{-g} dx^0 \wedge \cdots \wedge dx^{D-1} \,. \tag{1.8}$$

For example a $(D-2)$-space-time form $\boldsymbol{k}$ reads as

$$\boldsymbol{k} = \frac{1}{2(D-2)!} k^{\mu\nu} \varepsilon_{\mu\nu\alpha_3 \ldots \alpha_D} dx^{\alpha_3} \wedge \cdots \wedge dx^{\alpha_D} \,. \tag{1.9}$$

The Einstein summation convention over repeated upper or lower indices generally applies, namely $p_\mu x^\mu = p_0 x^0 + p_1 x^1 + \cdots$.
Complete symmetrization is denoted by ordinary brackets $(\cdots)$, complete anti-symmetrization by square brackets $[\cdots]$ including the normalization factor

$$M_{(\mu_1 \cdots \mu_k)} = \frac{1}{k!} \sum_{\sigma \in S^k} M_{\mu_{\sigma(1)} \cdots \mu_{\sigma(k)}} \,, \qquad M_{[\mu_1 \cdots \mu_k]} = \frac{1}{k!} \sum_{\sigma \in S^k} (-1)^\sigma M_{\mu_{\sigma(1)} \cdots \mu_{\sigma(k)}} \,, \tag{1.10}$$

where the sums run over all elements σ of the permutation group S^k of k objects and $(-1)^\sigma$ is 1 for an even and -1 for an odd permutation. For example, for $k = 2$ we have

$$M_{(\mu\nu)} = \frac{1}{2}(M_{\mu\nu} + M_{\nu\mu}) \,, \qquad M_{[\mu\nu]} = \frac{1}{2}(M_{\mu\nu} - M_{\nu\mu}) \,. \tag{1.11}$$

For the second part of this thesis, we note that waved equalities $\approx$ represent any equation that holds if and only if the equations of motion of the theory.

Part I

Chapter 2

Introduction to Part I

The principles of many physical theories have been constructed from geometry. The geometrical character of a physical system can imply a deep understanding and considerable generality of the theory, and naturally, any physical law will inherit these properties. Analogously, the topology of a system is also another property equally crucial for understanding a theory.

Topology is an area of mathematics which analyze the equivalence of different objects under continuous deformations. This analysis allows studying different confined structures in physics, such as models in condensed matter physics, for example the Bose-Einstein condensates [10,11], antiferromagnetic superconductors [12]; models in quantum mechanics as the Dirac's analysis of magnetic monopoles [13], the Aharonov-Bohm effect [14], applications in topologically quantized edge and surface conducting states manifesting in quantum Hall effect and its variations [15], and more importantly for this thesis: baryonic matter described by the Skyrme model.

Probably, one of the first study into using conserved topological properties for describing matter was made by Kelvin's vortex atom model [16]. In this model, Kelvin thought to the atoms as knots in a perfect homogeneous fluid that cannot be smoothly transformed from one to another. Each knot could represent an atom of different chemical elements so that the atomic spectra would arise from the dynamics of the perfect fluid producing vibrations of the knots. Although

Kelvin's idea was never carried out, it was maybe the first step to involve topology in physics. This inspired to T. Skyrme to develop his model later.

Figure 2.1: Tony. H. R. Skyrme in 1946.

Image taken from *Wikipedia*.

In the 1950s, the British high energy physicist Tony. H. R. Skyrme proposed a radical new field theory [17–19] based on the "degree" of the various topological soliton solutions, known now as Skyrmions, which can be identified with the baryon number of the solution. This model is a highly non-linear field theory of pions that possesses topologically non-trivial solutions to its field equations: Skyrmions. These topological configurations are static and spatially localized field configurations labelled by an integer-valued topological charge. Skyrme's idea was to identify these Skyrmions as nuclei, with the topological charge being the baryon number. In his original paper, Skyrme studied the simplest configuration with a topological charge equal to one. This configuration was a spherically symmetric solution of the equations of motion with dynamics identical to those of a point particle at low energy.

The importance of the Skyrme model was not widely recognized at the time as research moved towards the culmination of the currently accepted theory of strong interactions: QCD theory. A few decades after Skyrme's paper, Witten showed the remarkable result that as the number of quark colours becomes large, the baryons act like solitons based on their mass scale [20] (see also [21]). Witten's development revived the interest in the Skyrme model. Since then, it has been a booming field theory in describing the structures of nuclei and outside the realm of particle physics broadly.

Despite the successful features of the Skyrme model in describing particle physics, the (highly) non-linear character of its field equations produces difficult tasks to obtain exact solutions with a topological charge greater than one. Usually, the most common way to tackle this issue is by adopting a particular *ansatz* to make the equations of motion more tractable. However, this guarantees neither the reduction of the number of equations nor a consistent system of equations. The first and best-known ansatz was given by Skyrme himself in order to reduce the field equations to only a single scalar differential equation.

Until very recently, no analytic solution with non-trivial topological properties was known. In particular, the lack of explicit solutions with a topological charge on flat space-times made very difficult the analysis of the corresponding phase diagram of the low energy limit of QCD at finite density and low temperatures.

On the other hand, due to the importance of the Skyrme model as a low energy limit of QCD, it is a mandatory task to analyze the effects of the coupling of a $U(1)$ gauge field with the Skyrme theory. The so-called gauged Skyrme model−which describes the low energy limit of QCD minimally coupled with Maxwell theory at the leading order in the 't Hooft expansion−can describe the decay of nuclei due to the coupling with weak interactions [22]. Such a model also describes baryons' electric and magnetic properties and allows to study the decay of nuclei in the vicinity of a monopole. Similarly, there have also been studies trying to understand the coupling of the $SU(2)_L$ gauged model with weak interactions [23, 24]. The gauged Skyrme model is expected to have exciting applications in nuclear and particle physics and astrophysics when the coupling of Baryons with strong electromagnetic fields cannot be neglected (see a recent work on how large amount of Baryon charge present within a finite volume, so-called nuclear pasta phase, can be described through the Skyrme model [25]).

Recently, in [26–28], (see also [29–31] and references therein) a more general hedgehog ansatz allowing to depart from spherical symmetry has been introduced for the Skyrme model. Such

an approach gave rise to the first $(3+1)-$dimensional analytic and topologically non-trivial solutions of the Skyrme-Einstein system in [29] as well as the Skyrme model without spherical symmetry living within a finite box in flat space-times in [32].
In this thesis, by using the generalized hedgehog approach, we construct the first analytic, and topologically non-trivial solutions of the $U(1)$ gauged Skyrme model. The two different topological configurations found here are: Firstly, gauged Skyrmions living within a finite volume (which appear as the natural generalization of the usual Skyrmions living within a limited volume). Secondly, the $U(1)$ gauged Skyrme model also admits smooth solutions whose periodic time-dependence is protected by a topological conservation law. These solitons manifest fascinating similarities with superconductors as well as with dual superconductors.

The first part of this thesis is organized as follows. In Chapter 3 we shall review the $SU(2)$ Skyrme model, discussing its Lagrangian, topological charge, energy stability, and the simplest analytic solution to its field equations. This will provide background material for the original research presented in Chapter 4. In the latter, we start with Section 4.2 about a review of the properties of the $(3+1)$-dimensional Skyrme model at finite volume both with and without isospin chemical potential. Then, in Section 4.2.1 the gauged solitons are constructed and their main physical properties are discussed. In Section 4.2.2, it is discussed time crystal gauged Skyrmions which exhibit similar properties to those that appear in low energy limit of QCD. Next, Section 4.3 is devoted to discuss how electromagnetic duality can be extended to include these gauged solitons. In Section 4.4 a physically interesting approximation is discussed in which the Skyrme field is considered as fixed and the electromagnetic field is slowly turned on. We end with Chapter 5 giving some conclusions and outlook about the results found.

Chapter 3

$SU(2)$ Skyrme model

The Skyrme model corresponds to the low energy limit of QCD. In this limit, the quark and gluon degrees of freedom are *frozen out*, and therefore the only relevant degrees of freedom are those of the mesons: neutrons, protons and pions. In this chapter, we describe the $(3+1)-SU(2)$ Skyrme model. In particular, we will focus on its topological properties and analytic solutions for its field equations.

3.1 $SU(2)$ Skyrme model

As it is well-known, QCD has asymptotic freedom. At low energies, the running coupling constant is large; at high energies, it becomes small. Therefore, QCD cannot be studied with the usual perturbative methods. A way to overcome this problem is the construction of an *effective field theory*. Usually, the construction of the effective field theory is based on the (known) symmetries of the interactions. We follow this road.

As far as the mesonic degrees of freedom are concerned, we will focus on only the lightest mesons: the pions. In order to construct a principle action for these fields, we will denote the pion fields by $\pi^a = (\pi^-, \pi^+, \pi^0)$ plus an auxiliary scalar field σ grouped in terms of the $SU(2)$ matrix

$$U(t,\vec{x}) \,=\, \sigma(t,\vec{x})\mathbb{I}_{2\times 2} + \sum_{a=1}^{3} \pi^a(t,\vec{x})\tau_a\,, \tag{3.1}$$

where $\mathbb{I}_{2\times 2}$ is the identity matrix of 2×2, $\tau_a \equiv i\,\sigma_a$, with $\sigma_a = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices and the fields satisfy $\sigma^2 + \sum_{a=1}^{3}\pi_a\pi^a \,=\, 1$ (see Appendix C for conventions). The expected low energy action should inherit Lorentz symmetry, and in the approximation of zero quark masses, it should also have an *approximated* chiral symmetry.[1] Since chiral symmetry is only approximated, we therefore expect, by Goldstone's theorem, to Goldstone to be lightest mesons of the Standard Model (if the broken symmetry were exact, the Goldstone particles should be massless).

For $N = 2$, there are three (lightest) Goldstone bosons: $\pi^{\pm}$, π^0. The *pions* whose masses are given by $\pi^{\pm} \approx 140$ MeV, $\pi^0 \approx 135$ MeV. For $N = 3$, we have eight (lightest) Goldstone bosons: the three pions, four Kaons $K^{\pm}, K^0, \bar{K}^0$, and the particle η. For this thesis, we put attention on the former case.

A chirally symmetric action in terms of U means $U' \longrightarrow C\,U\,D$, with $C, D \in SU(2)$. All of these conditions combined with the constraint of having at most second order time derivatives in the equations of motion, it leads us to write the following effective field action

$$I_{NL\sigma}[U] \,=\, \frac{F_\pi^2}{16}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left(L_\mu L^\mu\right)\,, \tag{3.2}$$

with F_π corresponding to the decay constant of the pions, $\sqrt{-g}$ is the square root of minus the determinant of the space-time metric, Tr stands for the trace, and the tensor L_μ an algebra valued field $L_\mu = L_\mu^i\,\tau_i$ corresponding to the left-invariant Maurer-Cartan (MC) one-forms given by

[1]The chiral symmetry is partially realized in nature. The symmetry group in nature H is much smaller than the chiral symmetry group G. It is because the quark masses are too light to cause such a large breaking of the symmetry. However, they break some of the chiral symmetry. Then, we shall say that the Lagrangian is approximately invariant under G, but with the vacuum only invariant under H symmetry.

$$L_\mu = U^{-1}\nabla_\mu U\,, \tag{3.3}$$

with ∇_μ the covariant derivative constructed from the Christoffel symbols. This model is called the *non-linear sigma model* (NLσ). Indeed, this action has $SU(2)\times SU(2)$ symmetry, that is, when $U \longrightarrow U' = CUD^{-1}$, we get

$$I_{NL\sigma}[U'] = \frac{F_\pi^2}{16}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left[\nabla^\mu(CU^{-1}D)\,\nabla_\mu(DUC^{-1})\right] = I_{NL\sigma}[U]\,. \tag{3.4}$$

In the last equality, remember that the $SU(2)$ elements C and D are constants, and the trace has cyclic symmetry.

For a long time, this model was regarded as the simplest effective model for baryons. However, the simple fact that in three spatial dimensions, the static energy corresponding to the pion field $U = U(\vec{x})$ decreases with the increase of the space scaling implied that the pion field were not stable [33]. To overcome this problem, T. H. Skyrme introduced an additional term to (3.2), the so-called Skyrme term, leading to the action [17–19]

$$I[U] = \int d^4x\sqrt{-g}\,\mathrm{Tr}\left(\frac{F_\pi^2}{16}\,L_\mu L^\mu + \frac{1}{32e^2}\,[L_\mu, L_\nu][L^\mu, L^\nu]\right)\,. \tag{3.5}$$

When the fundamental field $U(t,\vec{x})$ is static, in three spatial dimensions it is thought as the map

$$U : \mathbb{R}^3 \longmapsto S^3\,. \tag{3.6}$$

Due to that S^3 is the group manifold of $SU(2)$, we can consider $U(t,\vec{x})$ to be an static $SU(2)$-valued scalar. The parameters of the model, experimentally determined, have the values $F_\pi = 186$ [MeV] and $e = 5.45$, with the latter introduced for the stabilization of the Skyrmions [34,35]. Notice that from the action principle's point of view we could have added any term that were of degree four or higher in spatial derivatives. However, it is easy to check that the Skyrme term is the only term of degree four that is Lorentz invariant and that remain the equations of motion of second order in the time derivatives.

Introducing the new parameters $K = F_\pi^2/4$ and $\lambda = 4/(e^2 F_\pi^2)$, the action (3.5) can be rewritten as

$$I_{Skyrme}[U] = \frac{K}{2}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left(\frac{1}{2}L^\mu L_\mu + \frac{\lambda}{16}[L_\mu, L_\nu][L^\mu, L^\nu]\right). \tag{3.7}$$

Throughout this thesis, we will refer to (3.7) as the Skyrme action. As we mentioned above, the action (3.7) possesses two terms: the first term corresponds to the standard kinetic term of the non-linear sigma model describing the low-energy dynamics of the pions, whereas the second one, the so-called Skyrme term, is the simplest covariant term by maintaining the second degree of the field equations and providing the existence of stable topological solitons with finite energy. Furthermore, the wide range of applications of this theory in other areas (such as astrophysics, Bose-Einstein condensates, nematic liquids, multi-ferromagnetic materials, chiral magnets and condensed matter physics in general [36–45]) is well recognized by now.

It is possible to add a mass term in the action (3.7) given by [46, 47]

$$I_{SKmass} = \frac{F_\pi^2 m_\pi^2}{8}\int d^4x\,\mathrm{Tr}\left(U - \mathbb{I}_{2\times 2}\right). \tag{3.8}$$

This introduces a pion mass m_π which can be fixed experimentally. As we mentioned above, when m_π is zero, the action (3.7) is invariant under the full chiral symmetry, namely

$$U \longrightarrow C\,U\,D^T\,, \qquad C, D \in SU(2)\,. \tag{3.9}$$

A non-zero pion mass m_π restricts to take $C = D$, breaking the $SU(2)\times SU(2)$ symmetry down to an $SU(2)$ symmetry. This is called the isospin symmetry and is given by

$$U \longrightarrow C\,U\,C^T\,, \qquad C \in SU(2)\,. \tag{3.10}$$

The last transformation is also known as isorotation because in the internal space it rotates the triplet of pion fields via $\vec{\pi} \to M\vec{\pi}$, where $M_{ij} = \frac{1}{2}\mathrm{Tr}\left(\tau_i\,C\,\tau_j\,C^T\right)$ are the components of an $SO(3)$ matrix. Once quantized, it gives rise the isospin, the responsible quantity that distinguish protons and neutrons. However, for the purposes of this thesis, we will consider $m_\pi = 0$.

The three-non-linear coupled field equations that follows of varying with respect to U the action (3.7) are[2]

$$\nabla_\mu \left(L^\mu + \frac{\lambda}{4}\,[L_\nu, [L^\mu, L^\nu]] \right) \;=\; 0\,. \tag{3.11}$$

Despite the great interest of this model (due to the many applications in different branches of physics), it is not easy to analytically solve the Skyrme field equations (3.11) due to their highly non-linear form. Even for the simplest physical $SU(2)$ case, the system (3.11) for U constitutes a system of three coupled non-linear differential equations which cannot be solved *in principle* in a closed-form. In Skyrme's original works [17–19], he found a way to make this problem more handle by adopting a particular ansatz for the Skyrme field $U(t,\vec{x})$. It led him to find the first topologically non-trivial solution to (3.11): the so-called *Skyrmions*. As we will see later, only suitable ansatz for the Skyrme field U allows reducing the number of equations of motion keeping alive the topological charge. Therefore, the search for solutions to this system becomes a fascinating subject of study. Together with the coupling of a Maxwell field, the latter constitutes the main axis of this first part of this thesis.

[2]Remember that the Skyrme fields U are expanded in terms of the traceless Pauli matrices. Therefore, the traceless condition reduces the number of field equations from four to three.

3.1.1 Energy stabilization and topological charge

The stress-energy-momentum tensor is obtained from the Lagrangian density $\mathcal{L}$ associated to the action (3.7) as

$$
\begin{aligned}
T_{\mu\nu}^{SK} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}\,, \\
&= -\frac{K}{4}\mathrm{Tr}\left(2L_\mu L_\nu - g_{\mu\nu}L_\alpha L^\alpha + \frac{\lambda}{2}\left(g^{\alpha\beta}[L_\mu, L_\alpha][L_\nu, L_\beta] - \frac{1}{4}g_{\mu\nu}[L_\alpha, L_\beta][L^\alpha, L^\beta]\right)\right)\,.
\end{aligned}
\tag{3.12}
$$

Restricting to static $SU(2)$ Skyrme fields, say $U(\vec{x})$, the static energy functional can be obtained as the integral of T_{00}

$$
E = -\frac{K}{2}\int d^3x\sqrt{-g}\,\mathrm{Tr}\left(\frac{1}{2}L_iL^i + \frac{\lambda}{16}[L_i, L_j][L^i, L^j]\right)\,, \qquad i,j = 1,2,3\,. \tag{3.13}
$$

Here a few comments deserve mention: from left to right, the first term arises from the non-linear sigma model term in the action (3.7) and is quadratic in spatial derivatives. In contrast, the second term originated from the Skyrme term is quartic in spatial derivatives. We note that by making a spatial rescaling in the spatial coordinates via $\vec{x} \to \eta\vec{x}$, the static Skyrme energy E becomes

$$
E(\eta) \;=\; \frac{1}{\eta}\,E_\sigma + \eta\,E_{SK\,term}\,. \tag{3.14}
$$

Because the non-linear sigma term of the energy, E_σ, scales in the opposite way to the Skyrme one $E_{SK\,term}$ when the parameter scale η increases, then there is a minimal value of $E(\eta)$ for a finite $\eta \neq 0$. In other words, the soliton−Skyrmion− will not expand or contract indefinitely in order to lower its energy. We say that the Skyrme term comes to *stabilize* the configuration. Naively, instead of the Skyrme term as the second term in the action (3.7) that stabilize the non-linear sigma term, we could have considered any term that is of degree four or higher in spatial derivatives. However, the Skyrme term is the only one of degree four that is: (i) Lorentz invariant and (ii) that the resulting field equations remain second order in the time derivatives.

Demanding the existence of an unique vacuum in the theory, one imposes the boundary condition

$$U(x^\mu) \longrightarrow \mathbb{I}_{2\times 2} \qquad \text{as} \qquad |x^\mu| \to \infty\,. \tag{3.15}$$

It implies the compactification of the physical space, $\mathbb{R}^3 \cup \{\infty\} \cong S^3$, resulting that the target space of $SU(2)$ has group manifold S^3, and therefore all finite energy configurations are maps from S^3 to S^3. In this case the homotopy group is $\pi_3(S^3) = \mathbb{Z}$. Such maps, S^3 to itself, are characterized by a *degree* that measures how many times the sphere is wrapped around itself. Explicitly, it is given by

$$B = \frac{1}{24\pi^2}\int_\Sigma \epsilon^{ijk}\,\mathrm{Tr}\left[\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right]\,, \tag{3.16}$$

which is a topological invariant identified by Skyrme as the baryon number of the configuration. Because the latter B is also known as the *topological charge* or *baryonic charge*. For topological invariant, we mean that by doing a continuous deformation of the field U, say δU, the topological charge keeps unchanged. It is not necessary to demand that δU holds the equations of motion. The normalization factor $1/24\pi^2$ is just for later convenience. The hyper-surface Σ where one integrates is usually considered to be space-like, but as we will see in the next chapter, there exists specific soliton configurations where it can be time-like as well. The topological charge B depends on the (time/space-invariant) boundary conditions of the Skyrme field U. For example, for the boundary conditions (3.15), the topological charge is not zero, and the associated field configuration is stable because it cannot be deformed through continuum transformations to the trivial vacuum. As a final comment, we stress that this conserved charge does not arise from an invariance of the Skyrme action under any symmetry transformation, but rather it comes from the non-trivial topology of the Skyrme field equations. Then, it is not a Noether charge.

Completing the square in the integrand of the expression the static energy (3.13) and by using the Cauchy-Schwarz inequality $a^2 + b^2 \geq 2ab$, one finds

$$E \geq 12\pi^2 K\sqrt{\lambda}|B| \,, \tag{3.17}$$

namely, the static energy is bounded from below. This is known as the Bogomol'ny bound and it is saturated when $E = 12\pi^2 K\sqrt{\lambda}|B|$, *i.e* when L_i is a self-dual tensor

$$L_i = \frac{4}{\sqrt{\lambda}}\epsilon_{ijk}L^j L^k \,. \tag{3.18}$$

This means that for any baryon number there is a lower bound for the energy of Skyrmion solutions. This Bogomol'ny bound was also found by Skyrme himself in his paper [18]. It is worth noticing that this bound cannot be saturated for any non-trivial finite energy configuration in flat space-time. This latter was one of the main difficulties to explore solutions to the Skyrme field equations in contrast to, for instance, monopoles and vortices.

In the following, we will analyze some properties of the best known ansatz for the Skyrme field that allow to simplify the system (3.11).

3.1.2 Spherical hedgehog ansatz

Despite Skyrmions are not known analytically for any baryon number B, the case $B = 1$ allows to the Skyrmion minimizes the energy functional (3.13) with a spherical symmetry. Let us consider the flat space-time metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, and the Skyrme field U with the following $SU(2)$ standard parametrization [17–19]

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbb{I}_{2\times 2} \pm Y^i(x^\mu)\tau_i\,, \qquad (Y^0)^2 + Y^iY_i = 1\,, \tag{3.19a}$$

$$Y^0 = \cos\alpha\,, \qquad Y^i = \hat{n}^i \sin\alpha\,, \tag{3.19b}$$

$$\hat{n}^1 = \sin\theta\cos\phi\,, \qquad \hat{n}^2 = \sin\theta\sin\phi\,, \qquad \hat{n}^3 = \cos\theta\,, \tag{3.19c}$$

where θ, ϕ are the spherical coordinates, the signs $\pm$ in the exponent of the U field denote the field itself and its inverse, and $\alpha(r)$ is the radial profile subject to the constraints $\alpha(0) = \pi$ and $\alpha(\infty) = 0$ due to the boundary condition (3.15). Notice that the element U is not spherically symmetric (because it depends on all the coordinates), but rather the energy-momentum tensor $T_{\mu\nu}$, namely $\mathscr{L}_{\vec{L}} T_{\mu\nu} = 0$, where $\mathscr{L}_{\vec{L}}$ denotes the dragging of $T_{\mu\nu}$ along the $SO(3)$ generators.[3] This ansatz allows reduce the three non-linear coupled partial differential equations to a single scalar equation for the function $\alpha(r)$ keeping alive the topological charge equal to one. Usually, this is also known as the *hedgehog solution* because the pion fields point radially outwards from the origin at all points in space.

Substituting the hedgehog ansatz (3.19) into (3.11), it reduces to a single equation

$$(r^2 + 2\sin^2\alpha)\alpha'' + \lambda\sin(2\alpha)(\alpha')^2 + 2r\alpha' - \sin(2\alpha) - \frac{1}{r^2}\sin^2\alpha\sin(2\alpha) = 0\,, \tag{3.20}$$

where prime stands for derivative with respect to r. This second order non-linear ordinary differential equation cannot be solved in a closed form, but numerical solutions have been found [18, 48].[4] The energy of this configuration is given by

$$E = K\pi \int r^2 dr \left[\frac{2\sin^2\alpha}{r^2} + (\alpha')^2 + \frac{\lambda\sin^2\alpha}{r^2}\left(\frac{\sin^2\alpha}{r^2} + 2(\alpha')^2\right)\right]\,. \tag{3.21}$$

Particularly, the asymptotic behaviour of α can be achieved by linearizing the Euler-Lagrange equations (3.11), and leading to

[3] Spherical symmetry on U implies to have a radial-dependent field $U = U(r)$. But, this field configuration has a vanishing topological charge B.

[4] See also a numerical computations for $B = 2$ in [49–51].

$$\alpha(r) \sim \frac{c_0}{r}\, e^{-c_1 r}\,, \qquad r \to \infty \quad \text{with} \quad c_0, c_1 \in \mathbb{R}\,. \tag{3.22}$$

This means that the pion fields are of the form

$$\vec{\pi}(\vec{x}) \sim \frac{c_0\, \hat{n}}{r} e^{-c_1 r}\,, \qquad r \to \infty\,. \tag{3.23}$$

Therefore, at large distances the $B = 1$ Skyrmion approximates to a triplet of Yukawa dipoles. Finally, notice that for $c_1 = 0$ the pion fields take the form

$$\vec{\pi}(\vec{x}) \sim \frac{c_0\, \hat{n}}{r}\,, \qquad r \to \infty\,, \tag{3.24}$$

namely, we have now an approximate triplet of Coulomb triplets.

3.1.3 Generalized hedgehog ansatz

Inspired in the amazing characteristic of the spherically symmetric hedgehog ansatz (3.19) that reduces a system of coupled non-linear partial differential equations to a single scalar equation, Canfora and Salgado-Rebolledo in [52] (in the context of Yang-Mills theories) generalized this study for non-spherical geometrical conditions of the spherical hedgehog ansatz that also allow to reduce the number of non-linear coupled partial differential equations (PDEs) to a single scalar PDE. Remarkably, this idea works to Skyrme theory works as well. Let us consider the most general $SU(2)$ parametrization for the Skyrme field

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbb{I}_{2\times 2} \pm Y^i(x^\mu)\tau_i\,, \qquad (Y^0)^2 + Y^i Y_i = 1\,, \tag{3.25a}$$

$$Y^0 = \cos\alpha\,, \qquad Y^i = \hat{n}^i \sin\alpha\,, \tag{3.25b}$$

$$\hat{n}^1 = \sin\Theta\cos\Phi\,, \qquad \hat{n}^2 = \sin\Theta\sin\Phi\,, \qquad \hat{n}^3 = \cos\Theta\,, \tag{3.25c}$$

where the profile function $\alpha = \alpha(x^\mu)$, and the angular functions $\Theta = \Theta(x^\mu)$, and $\Phi = \Phi(x^\mu)$ depend now on all the four space-time coordinates. Replacing back into the action (3.7) these three functions can be interpreted as a theory of three interacting scalar fields.

The search of a suitable ansatz for the functions Θ and Φ that reduces the coupled system of PDEs to a single non-linear equation for the profile α defines the *generalized hedgehog ansatz* [52]. The Skyrme field equations (3.11) can be written as a 2×2-matrix representation, *i.e* $E \equiv E^i \tau_i$, $i = 1, 2, 3$, where the linearly independent three coupled non-linear field equations correspond to the entries of the matrix E. So, substituting (3.25) into (3.11) leads to the system

$$\begin{gathered}
\left(-\Box\alpha + \sin(\alpha)\cos(\alpha)\left(\nabla_\mu\Theta\nabla^\mu\Theta + \sin^2\Theta\nabla_\mu\Phi\nabla^\mu\Phi\right)\right) \\
+\lambda\begin{pmatrix}
\sin(\alpha)\cos(\alpha)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Theta\nabla^\nu\Theta) - (\nabla_\mu\alpha\nabla^\mu\Theta)^2\right) \\
+\sin(\alpha)\cos(\alpha)\sin^2(\Theta)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\alpha\nabla^\mu\Phi)^2\right) \\
+2\sin^3(\alpha)\cos(\alpha)\sin^2(\Theta)\left((\nabla_\mu\Theta\nabla^\mu\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\Theta\nabla^\mu\Phi)^2\right) \\
-\nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\Theta\nabla^\nu\Theta)\nabla^\mu\alpha\right) + \nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\Theta)\nabla^\mu\Theta\right) \\
-\nabla_\mu\left(\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi)\nabla^\mu\alpha\right) + \nabla_\mu\left(\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\Phi)\nabla^\mu\Phi\right)
\end{pmatrix} = 0\ ,
\end{gathered} \tag{3.26}$$

The variation of the Skyrme action with respect to Θ leads to the equation of motion

$$\begin{gathered}
\left(-\sin^2(\alpha)\Box\Theta - 2\sin(\alpha)\cos(\alpha)\nabla_\mu\alpha\nabla^\mu\Theta + \sin^2(\alpha)\sin(\Theta)\cos(\Theta)\nabla_\mu\Phi\nabla^\mu\Phi\right) \\
+\lambda\begin{pmatrix}
\sin^2(\alpha)\sin(\Theta)\cos(\Theta)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\alpha\nabla^\mu\Phi)^2\right) \\
+\sin^4(\alpha)\sin(\Theta)\cos(\Theta)\left((\nabla_\mu\Theta\nabla^\mu\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\Theta\nabla^\mu\Phi)^2\right) \\
-\nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\alpha)\nabla^\mu\Theta\right) + \nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\Theta)\nabla^\mu\alpha\right) \\
-\nabla_\mu\left(\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi)\nabla^\mu\Theta\right) + \nabla_\mu\left(\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Phi)\nabla^\mu\Phi\right)
\end{pmatrix} = 0\ ,
\end{gathered} \tag{3.27}$$

while the variation of the Skyrme action with respect to Φ yields to

$$\begin{aligned}&\left(-\sin^2(\alpha)\sin^2(\Theta)\Box\Phi-2\sin(\alpha)\cos(\alpha)\sin^2(\Theta)\nabla_\mu\alpha\nabla^\mu\Phi-2\sin^2(\alpha)\sin(\Theta)\cos(\Theta)\nabla_\mu\Theta\nabla^\mu\Phi\right)\\&+\lambda\left(\begin{array}{c}-\nabla_\mu\left[\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\alpha)\nabla^\mu\Phi\right]+\nabla_\mu\left[\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\Phi)\nabla^\mu\alpha\right]\\-\nabla_\mu\left[\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Theta)\nabla^\mu\Phi\right]+\nabla_\mu\left[\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Phi)\nabla^\mu\Theta\right]\end{array}\right)=0\,.\end{aligned} \tag{3.28}$$

On the other hand, the topological charge becomes

$$B=\frac{1}{24\pi^2}\int_\Sigma\sqrt{-g}\,d^3x\,\rho_B\,,\qquad\rho_B=12\sin^2(\alpha)\sin(\Theta)\,d\alpha\wedge d\Theta\wedge d\Phi\,. \tag{3.29}$$

Here Σ is considered to be a space-like hyper-surface. We observe that a necessary condition to have a non-vanishing topological charge is that the three functions must be independent, *i.e* $d\alpha\wedge d\Theta\wedge d\Phi\neq 0$. For example, a sector with non-vanishing winding number is the original Skyrme ansatz

$$\alpha(x^\mu)=\alpha_{SK}(r)\,,\qquad\Theta=\theta\,,\qquad\Phi=\phi\,, \tag{3.30}$$

in the flat metric in spherical coordinates $ds^2=-dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$. As we will see, this approach have allowed the construction of analytical solutions that describe multi-solitons at finite density for both, in the non-linear sigma model [53] as well as in the Skyrme model coupled to Maxwell theory [54]. Even recently, configurations of analytic multi-Skyrmions with crystalline order were constructed in [55] and similar features for the non-linear sigma model in [56].

Chapter 4

The Gauged Skyrme Model

Due to the importance of the Skyrme model as a low energy limit of QCD, it is natural to extend this study to analyze the effects of the coupling of a electromagnetic gauge field with the Skyrme theory. This theory is known as the Maxwell-Skyrme theory or $U(1)$ *Gauged Skyrme Model*. In the following, we will describe the main properties of this model and the first analytic examples of gauged soliton solutions in flat space-time.

4.1 The $U(1)$ Gauged Skyrme Model

Let us consider the $U(1)$ gauged Skyrme model in four space-time dimensions with global $SU(2)$ isospin internal symmetry. The action of the system is

$$S[U, A_\mu] = S_{Skyrme} + S^{U(1)} , \tag{4.1}$$

$$S_{Skyrme} = \frac{K}{2}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left(\frac{1}{2}L_\mu L^\mu + \frac{\lambda}{16}[L_\mu, L_\nu][L^\mu, L^\nu]\right) , \tag{4.2}$$

$$S^{U(1)} = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} , \tag{4.3}$$

where the tensor L_μ is now defined by

$$L_\mu = U^{-1} D_\mu U \,, \qquad\qquad D_\mu(\cdot) = \nabla_\mu(\cdot) + A_\mu \left[\tau_3, \, \cdot\,\right]. \tag{4.4}$$

Since the neutral pion π^0 is associated to the generator τ_3, the interaction with the electromagnetic force will occur only with the pions $\pi^{\pm}$. This is the reason because the commutator $[\tau_3, \, \cdot]$ in the covariant derivative D_μ is only along the generator τ_3.

The electromagnetic field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ the electromagnetic potential. The stress-energy-momentum tensor is

$$T_{\mu\nu} = T^{SK}_{\mu\nu} + T^{A}_{\mu\nu} \,, \tag{4.5}$$

with $T^{SK}_{\mu\nu}$ defined in (3.12) (taking into account the new definition of the tensor (4.4)), and the electromagnetic stress-energy momentum given by

$$T^{A}_{\mu\nu} \;=\; F_{\mu\alpha} F_{\nu}^{\;\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \,. \tag{4.6}$$

The field equations that follows from varying the action with respect to U and A_μ (4.1) are, respectively, given by

$$D_\mu \left(L^\mu + \frac{\lambda}{4} \left[L_\nu, \left[L^\mu, L^\nu\right]\right] \right) \;=\; 0 \,, \tag{4.7a}$$

$$\nabla_\mu F^{\mu\nu} \;=\; J^\nu \,, \tag{4.7b}$$

where J^ν is the variation of the action (4.1) with respect to A_ν, getting

$$J^\mu = \frac{K}{2} \mathrm{Tr} \left(\widehat{O} L^\mu + \frac{\lambda}{4} \widehat{O} \left[L_\nu, \left[L^\mu, L^\nu\right]\right] \right) \,, \tag{4.8}$$

where

$$\widehat{O} = U^{-1} \tau_3 U - \tau_3 \,.$$

It is worth to note that when the gauge potential reduces to a constant along the time-like direction, the system (4.7) describes the Skyrme model at a finite isospin chemical potential. Hence, the term *gauged Skyrmions* (or, more generically, *gauged topological configurations* of the $U(1)$ gauged Skyrme model in $(3+1)$-space-time dimensions) refers to smooth regular solutions of the coupled system in (4.7) possessing a non-vanishing winding number.

4.1.1 Gauged topological charge

By construction, the topological charges are conserved whether or not the equations of motion are satisfied. When the coupling to a $U(1)$ gauge field is considered, the expression of the topological charge in (3.29) cannot be correct since it is not gauge invariant under $U(1)$ gauge transformations. Naively, one could replace in (3.29) all the derivatives with covariant derivatives leading to a gauge invariant expression, but that topological current would not be conserved. The correct solution was constructed in [22] (see also the pedagogical analysis in [57]), and the expression for the gauge invariant and conserved topological charge reads

$$B = \frac{1}{24\pi^2}\int_{\Sigma}\epsilon^{ijk}\mathrm{Tr}\left(\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right) - \partial_i\left(3A_j\tau_3\left(U^{-1}\partial_k U + \partial_k U U^{-1}\right)\right)\right). \tag{4.9}$$

The topological charge gets one extra contribution which, at the end, is responsible for the Callan-Witten effect [22]. Notice that the extra contribution from electromagnetism is a total derivative that makes it still an integer.

As we described, the usual situation considered in the literature corresponds to a space-like hyper-surface Σ in which case B is the Baryon charge. However, in [32] it has been proposed to also consider cases in which Σ is time-like or light-like. If $B \neq 0$ (whether Σ is space-like, time-like or light-like) then one cannot deform continuously the corresponding ansatz into the trivial vacuum $U = \mathbb{I}_{2\times 2}$. Consequently, when Σ is time-like and $B \neq 0$ one gets a Skyrmionic configuration whose time-dependence is topologically protected as it cannot decay in static solutions. These kind of solitons have been named topologically protected *time crystals* in [32]. The computations below show that such an effect (according to which, roughly speaking, a magnetic monopole may "swallow" part of the topological charge) is more general and, in principle, strong magnetic fields may be able to support it even without magnetic monopoles.

4.2 Brief review of Skyrmions at finite volume

As a warm-up section, we analyze an extension of the generalized hedgehog ansatz, which also works in situations in which the Skyrme model (without a Maxwell field) is analyzed within a finite volume V in a flat metric. One of the reasons to study Skyrmions within a limited volume comes from the fact that the Skyrme model corresponds to the low-energy limit of QCD. Therefore, this type of study could reveal essential thermodynamics properties of the phase diagram of QCD at this energy regime (for example, such as the pressure $P = -\partial E/\partial V$). The first study of a Skyrmion in a limited region using a phenomenological approach was given by Klebanov in the eighties [58].

On the other hand, by considering this point of view, a non-vanishing isospin chemical potential was introduced in [59–61]. In all these cases, numerical tools were needed to implement on the analysis since the chemical potential breaks explicitly the spherical symmetry. Then, the spherical hedgehog ansatz does not work.

Later on, by considering the generalized ansatz, the first analytic examples of Skyrmions as well as Skyrmions-anti-Skyrmions bound states were found within a finite box in $(3+1)$-dimensional flat space-time [32]. In the following, wee will briefly describe this last approach.

Let us start considering the flat space-time metric

$$ds^2 = -dt^2 + l^2 \left(dr^2 + d\gamma^2 + d\phi^2 \right) , \tag{4.10}$$

where l denotes the size of the box side where the Skyrmion lives, and the coordinates r, γ and Φ have the range

$$0 \leq r \leq 2\pi \,, \qquad 0 \leq \gamma \leq 4\pi \,, \qquad 0 \leq \phi \leq 2\pi \,. \tag{4.11}$$

The Skyrme ansatz is based on the following parametrization of the functions

$$\Theta = \frac{\gamma + \phi}{2} \,, \qquad \tan \Phi = \frac{\tan H}{\cos A} \,, \qquad \tan \alpha = \frac{\sqrt{1 + \tan^2 \Phi}}{\tan A} \,, \tag{4.12}$$

where,

$$A = A(\gamma, \phi) = \frac{\gamma - \phi}{2} \,, \qquad H = H\left(t, r\right) \,. \tag{4.13}$$

The Skyrme field equations (3.11) reduce to a single ordinary differential sine-Gordon equation for the function $H(t, r)$

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{1}{l^2}\frac{\partial^2}{\partial r^2} \right) H - \frac{\lambda}{8l^2(\lambda + 2l^2)} \sin(4H) \;=\; 0 \,. \tag{4.14}$$

It can be verified directly that the topological density ρ_B is non-zero, and is given by

$$B \;=\; \frac{1}{24\pi^2} \int_{t_0 = cte} \rho_B \,, \qquad \rho_B = 3 \sin(2H) \, dH \wedge d\gamma \wedge d\phi \,, \tag{4.15}$$

where the integral is computed on a hyper-surface at fix time t_0. For certain boundary conditions on H, we get three different topological sectors

$$B = \begin{cases} -1 \quad \text{if} \qquad H(t_0, 0) = -\frac{\pi}{2} \quad \text{and} \quad H(t_0, 2\pi) = 0 \\ \;\;0 \quad \text{if} \qquad H(t_0, 0) - H(t_0, 2\pi) = 0 \\ +1 \quad \text{if} \qquad H(t_0, 0) = 0 \quad \text{and} \quad H(t_0, 2\pi) = \frac{\pi}{2} \end{cases} . \tag{4.16}$$

In particular, the sector $B = 0$ is relevant for the construction of Skrymion-anti-Skyrmion bound states. Hence, the original $(3 + 1)$-dimensional Skyrme field equations, energy density and effective action in a topologically non-trivial sector (as $\rho_B \neq 0$) can be reduced to the corresponding quantities of the $(1+1)$-dimensional sine-Gordon model. Following [32], this allows constructing Skyrmions as well as Skyrmions-anti-Skyrmions bound states.[1] The effective coupling sine-Gordon density Lagrangian reads

$$\mathcal{L}(\Phi) = -\frac{1}{2}\nabla^{\mu}\Phi\nabla_{\mu}\Phi + \frac{\alpha}{\beta^2}\left(\cos\left(\beta\Phi\right) - 1\right) \ , \tag{4.17}$$

$$\alpha = \frac{\lambda}{2l^2\left(\lambda + 2l^2\right)} \, , \qquad \beta = \frac{4l}{\sqrt{\lambda + 2l^2}} \, . \tag{4.18}$$

Therefore, the Skyrme model within the finite volume defined above always satisfies the Coleman bound $\beta^2 < 8\pi$.

It is worth emphazing that Skyrme and Perring [62] used sine-Gordon model in $(1+1)$-dimensions as a "toy model" for the $(3+1)$-dimensional Skyrme model. The analogies between (a simplified version of) the Skyrme model and the sine-Gordon model have also been emphasized in [63] and references therein. The very surprising feature of the results in [32] is that there is a non-trivial topological sector of the full $(3+1)$-dimensional Skyrme model in which it is exactly equivalent to the sine-Gordon model in $(1+1)$-dimensions.[2]

4.2.1 Gauged Skyrmion

In this section, we extend the Skyrmion configurations constructed in the previous section to the cases in which the minimal coupling with the $U(1)$ gauge field cannot be neglected. We will also analyze the most interesting physical properties of these gauged configurations.

For this analysis we start by considering the following parametrization of the $SU(2)$-valued

[1] Indeed, a quite remarkable prediction of the Skyrme model at finite volume discussed in [32] is that the model possesses around $8\pi/\beta^2 - 1$ Skyrmion-anti-Skyrmion bound states with β of Eq. (4.18). When the size of the box is large compared with $1fm$ one gets that the number of these bound states is between 5 and 6 (in good agreement with the number of Baryon-anti-Baryon resonances appearing in particles physics).

[2] The semi-classical quantization in the present sector of the Skyrme model can be analyzed following [34,64,65]: since *principle of symmetric criticality* applies (see also [32]).

scalar U

$$U(x^{\mu}) = e^{\tau_3 \alpha} e^{\tau_2 \beta} e^{\tau_3 \rho} \,, \tag{4.19}$$

where α, β and ρ are the Euler angles which in a single covering of space take the values $\alpha \in [0, 2\pi]$, $\beta \in \left[0, \frac{\pi}{2}\right]$ and $\rho \in [0, \pi]$. Like in the case without an electromagnetic field we consider the flat space-time metric (4.10), where the ordering of the coordinates that we assume is

$$x^{\mu} = (t, r, \gamma, \phi) \,,$$

and where again we fix the dimension of the spatial box by requiring

$$0 \leq r \leq 2\pi \,, \qquad 0 \leq \gamma \leq 4\pi \,, \qquad 0 \leq \phi \leq 2\pi \,. \tag{4.20}$$

As before, l represents the size of the box while r, γ and ϕ are dimensionless angular coordinates, and t represents the time coordinate. It is possible to choose an ansatz for the Skyrme configuration in the following manner

$$\alpha(\gamma) = p\frac{\gamma}{2} \,, \qquad \beta(r) = H(r) \,, \qquad \rho(\phi) = q\frac{\phi}{2} \,, \qquad p,\ q \in \mathbb{N} \,, \tag{4.21}$$

where p and q must be integer in order to cover $SU(2)$ an integer number of times. In this context we assume an electromagnetic potential of the form

$$A = A_{\mu} dx^{\mu} \,, \qquad A_{\mu} = (b_1(r),\, 0,\, b_2(r),\, b_3(r)) \ . \tag{4.22}$$

Under the previous setting, the ensuing Maxwell equations (4.7b) become

$$b_I''(r) \ = \ \frac{K}{2} \left(M_{IJ} b_J(r) + N_I \right) , \quad I, J = 1, 2, 3 \,, \tag{4.23}$$

where the non-vanishing components of the 3×3 matrix M_{IJ} are

$$M_{11} = 4\sin^2(H)\left(2\lambda H'^2 + \frac{\lambda\left(p^2+q^2\right)}{2}\cos^2(H) + 2l^2\right), \tag{4.24a}$$

$$M_{23} = M_{32} = -\frac{pq}{2}\lambda\sin^2(2H), \tag{4.24b}$$

$$M_{22} = M_{11} + \frac{p}{q}M_{23}, \tag{4.24c}$$

$$M_{33} = M_{11} + \frac{q}{p}M_{23}, \tag{4.24d}$$

and

$$N = \left(0,\ \frac{p}{4}M_{11} - \frac{q^2-p^2}{4q}M_{23},\ -\frac{q}{4}M_{11} - \frac{q^2-p^2}{4p}M_{23}\right). \tag{4.25}$$

Remarkably, the hedgehog property is not destroyed by the coupling to the above $U(1)$ gauge field since the Skyrme equations continue lead to a single equation for the profile $H(r)$,

$$\begin{aligned}&4\left(X_1\sin^2(H) + \frac{\lambda(p^2+q^2)}{2} + 2l^2\right)H'' + 2X_1\sin(2H)(H')^2 + 4\sin^2(H)X_1'H'\\ &+\left(2\lambda\left(pb_2+qb_3\right)\left(pb_2+qb_3+\frac{p^2-q^2}{2}\right) - \frac{\lambda p^2q^2}{2} - \frac{p^2+q^2}{2}X_1\right)\sin(4H) - \frac{2l^2X_1}{\lambda}\sin(2H) = 0,\end{aligned} \tag{4.26}$$

where

$$X_1(r) = 4\lambda\left(-2l^2b_1^2 + b_2(2b_2+p) + b_3(2b_3-q)\right). \tag{4.27}$$

Therefore, we have reduced the whole system to four coupled non-linear ordinary differential equations. But we also want to solve these remaining equations. Trying to solve the Eq. (4.26) analytically is a difficult task. However, there is hope, by considering $X_1(r)$ to be constant. In that case, we see that (4.26) turns out to an equation of the form $a(H)H''(r) + b(H)(H')^2 + c(H) = 0$, with a, b and c arbitrary functions, which can be solved through quadrature formulas.

Therefore, a slight pause on Skyrme equation (4.26) and Maxwell equations (4.23) tells us that it is possible to go further in the simplification by imposing the following relations

$$X_1 = -\frac{\lambda(p^2+q^2)}{2} \,, \tag{4.28}$$

and

$$b_2(r) = -\frac{q}{p} b_3(r) - \frac{p^2-q^2}{4p} \,. \tag{4.29}$$

Combining these two relations with (4.27), we can also express $b_1(r)$ in terms of $b_3(r)$ as

$$b_1(r) \;=\; \pm \frac{(4b_3(r)-q)\sqrt{p^2+q^2}}{4lp} \,. \tag{4.30}$$

Surprisingly, the Maxwell field equations (4.23) reduce to the following single scalar equation for $b_3(r)$

$$b_3'' + \frac{K}{4}(q-4b_3)\sin^2(H)\left(4l^2+4\lambda(H')^2+\lambda\left(p^2+q^2\right)\cos^2(H)\right) \;=\; 0 \,, \tag{4.31}$$

whereas the corresponding equation for the profile H reads

$$\left(\frac{8l^2}{p^2+q^2} + 2\lambda\cos^2(H)\right) H'' + \sin(2H)\left(l^2 - \lambda(H')^2\right) \;= 0 \,. \tag{4.32}$$

Interestingly enough, the above equation for the profile H interacting with a $U(1)$ gauge field is equivalent to the Skyrme field equation with a chemical potential possessing a value $\bar{\mu}_0^2 = \frac{p^2q^2}{4l^2(p^2+q^2)}$. An interesting non-trivial topological sector of Eq. (4.32) is the simplest solution $H(r) = h_0 + h_1 r$, with h_0, h_1 constants. Here, the coupled field equations of the gauged Skyrme model (which, in principle, are seven coupled non-linear PDEs) reduce to the Heun equation, which for some particular choice of the parameters, can be further reduced to the Whittaker-Hill equation [53].

The remarkable result of reducing the full coupled Skyrme-Maxwell system (4.7) in a topologically non-trivial sector (as it will be shown below) is that, in the finite box defined in Eq. (4.10), it can be reduced consistently to a solvable system of two ordinary differential equations. Then, gauged Skyrmions can be constructed explicitly (see Appendix D for the details of the derivation of this result).

We can summarize the previous procedure as follows: to use the static ansatz in Eq. (4.21) for the Skyrme configuration and the ansatz in Eqs. (4.22), (4.28) and (4.29) for the $U(1)$ gauge field. Thus, one can determine the Skyrme profile $H(r)$ from Eq. (4.32) and then Eq. (4.31) for the gauge potential $b_3(r)$ becomes a simple linear non-homogeneous equation in which there is an effective potential which depends on $H(r)$. The other components of the gauge potential are determined solving the simple algebraic conditions (4.28) and (4.29). The above system allows to clearly disclose many features of the $U(1)$ gauged Skyrme model which are close to superconductivity (for plots of these analytic solutions, see Figure 4.5).

As a final remark, it is worth mentioning that the Skyrme profile equation (4.32) coupled with the $U(1)$ gauge field looks like the Skyrme field equations with isospin chemical potential. On other hand, we know that a non-vanishing isospin chemical potential suppresses the Skyrmion until it reaches the critical value when the Skyrmion completely disappears [32]. Therefore, we conclude that the coupling with the Maxwell field suppresses in a certain space-time region (but without destroying) the Skyrmion.

As a consistency check, if one considers $b_i \to 0 \Rightarrow X_1 \to 0$, then (4.26) reduces to

$$H''(r) - \frac{\lambda p^2 q^2 \sin(4H(r))}{4\left(4l^2 + \lambda\left(p^2+q^2\right)\right)} = 0\,, \tag{4.33}$$

with a first integral given by

$$(H')^2 + \frac{\lambda\, p^2 q^2 \cos(4H(r))}{8\left(4l^2 + \lambda(p^2+q^2)\right)} = I_0 = cte\,, \tag{4.34}$$

and whose general solution is

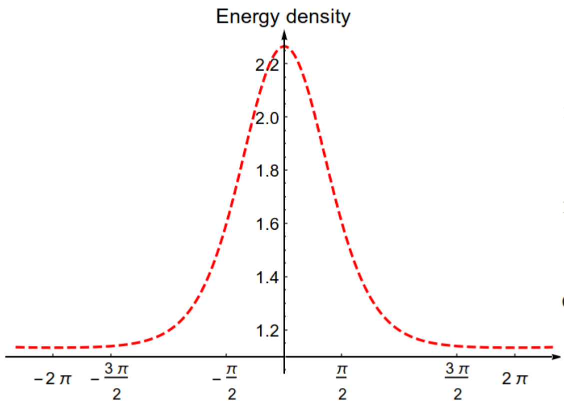

Figure 4.1: Energy density T_{00} of the Skyrmion.

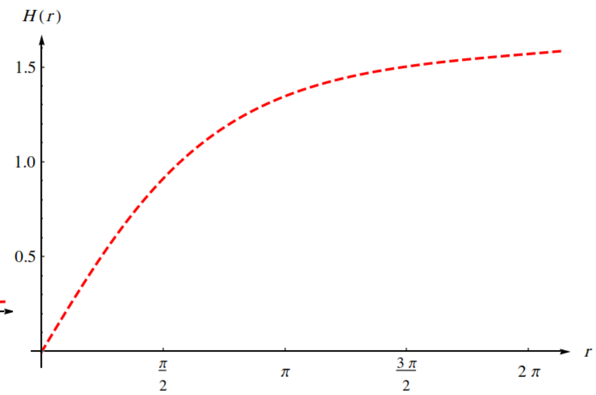

Figure 4.2: Skyrme profile.

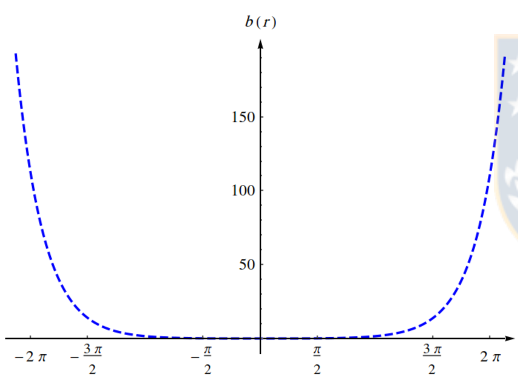

Figure 4.3: Gauge potential A_μ.

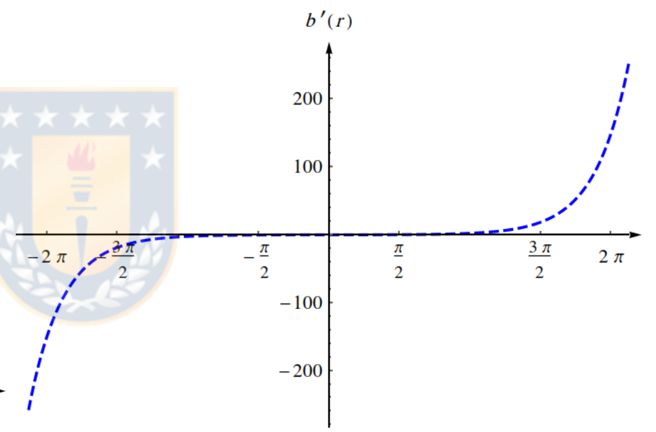

Figure 4.4: Magnetic field $B \equiv b'(r)$.

Figure 4.5: The solutions for the Eqs. (4.31) and (4.32) correspond to the values: $\lambda = 0.04$, $l = 0.47$, $K = 1.0$, $p = 1.0$, and $q = 1.0$. Solving for $b \equiv -b_2 = b_3$. The above plots clearly show the suppression of the magnetic field (which is non-vanishing only in the γ and ϕ directions) in the core of the Skyrmion.

$$H(r) = \pm\frac{1}{2}\text{am}\left(2c_0 \pm ir\sqrt{\frac{\lambda_{eff}}{2}},\, 2\right) , \qquad \lambda_{eff} \equiv \frac{\lambda p^2 q^2}{8\left(4l^2 + \lambda(p^2+q^2)\right)} , \tag{4.35}$$

where am denotes the Jacobi amplitude, and c_0 an integration constant.

Topological charge

For this configuration, we also compute the temporal component of the baryon density as it is modified by the Maxwell field following the steps of [22]. The full density reads

$$
\begin{aligned}
B_0 &= \frac{\epsilon_{0ijk}}{24\pi^2}\left(\mathrm{Tr}\left(L_i L_j L_k\right) - 3\partial_i\left(A_j \mathrm{Tr}\left(\tau_3\left(U^{-1}\partial_k U + \partial_k U U^{-1}\right)\right)\right)\right) ,\\
&= -\frac{pq}{8\pi^2}\sin(2H)H' + \frac{pq}{4\pi^2}\partial_r\left(\cos^2(H)\left(b_2 - b_3\right)\right) .
\end{aligned} \tag{4.36}
$$

So, the Baryon number for the gauged Skyrmion is

$$
B = \int_{t=t_0} B_0\, dr\wedge d\gamma \wedge d\phi \;=\; -p\,q\int_0^{2\pi}\sin(2H)dH + \left[2\cos^2(H)\left(q\,b_2 - p\,b_3\right)\right]\Big|_0^{2\pi} , \tag{4.37}
$$

which depending on the boundary conditions for the profile function $H(r)$, we get

$$
B = \begin{cases} -pq - 2\left(q\,b_2(0) - p\,b_3(0)\right) , & \text{if} \quad H(2\pi) = \frac{\pi}{2} \quad \text{and} \quad H(0) = 0 \\ pq + 2\left(q\,b_2(2\pi) - p\,b_3(2\pi)\right) , & \text{if} \quad H(2\pi) = 0 \quad \text{and} \quad H(0) = \frac{\pi}{2} \end{cases} . \tag{4.38}
$$

Clearly, B depends now on the size of the system through p and q, as well as on the boundary values set for b_2 and b_3 which are related to the magnetic components of $F_{\mu\nu}$. The solutions we have found with $p = q = 1$ for b_3 (and the corresponding values of b_2) of Eq. (4.31) have $b_2(0) = b_3(0)$ so that the topological charge reduces to the usual integer value. However, it is clear that there are much more general possibilities and one could try to construct configurations in which the topological charged is "shared" by the Skyrmion and the electromagnetic field.

It is worth notice that by combining these tools with the techniques introduced in [30], one can construct multi-layered configurations of the gauged Skyrme model such that each layer corresponds to the present gauged-Skyrmion configuration with Baryon charge the product pq, while the number of layers is related to the number of peaks of the (energy density associated to the) profile $H(r)$. This observation suggests that the present formalism could be used to describe analytically the regular patterns known to appear in the Skyrme model when fine-density effects are taken into account.

4.2.2 Gauged Time Crystals

In [66,67], Wilczek and Shapere made the following deep observation. One can construct simple models in which it is possible to break spontaneously time translation symmetry.

Although it is well-known that no-go theorems [68,69] ruled out the original proposals, new research fields started trying to realize in a concrete system the ideas presented in [66,67,70] (see [71] for a review on time crystals). Many examples have been found since then in condensed matter physics [72–77]. The first example in nuclear and particles physics has been found in [32] in the Skyrme model at finite volume.

Namely, the $(3+1)$-dimensional Skyrme model supports exact time-periodic configurations which cannot be deformed continuously to the trivial vacuum as they possess a non-trivial winding number. Consequently, these time crystals are only allowed to decay into other time-periodic configurations: hence, the name *topologically protected time crystals.*

Following [32], a very efficient choice to describe the finite box is the line element (4.10), where the coordinate γ now plays the role of time. The Skyrme configuration reads

$$\alpha = \frac{\phi}{2}\,, \qquad \beta = H(r)\,, \qquad \rho = \frac{\omega\gamma}{2}\,, \tag{4.39}$$

where ω again is a frequency so that ρ is a dimensionless quantity. Once more we assume an electromagnetic potential of the form (4.22), but now we have to consider that the coordinate ordering is

$$x^{\mu} = (\gamma,\, r,\, z,\, \phi)\,. \tag{4.40}$$

The Maxwell field equations have the same form as (4.23), but the entries of the matrix M now read

$$M_{11} = 2\sin^2(H(r))\left(4\lambda(H')^2 + \lambda\cos^2(H) + 4l^2\right)\ , \tag{4.41a}$$

$$M_{13} = -\frac{\lambda\omega}{2}\sin^2(2H)\,, \tag{4.41b}$$

$$M_{22} = M_{11} + \frac{l^2\omega^2+1}{\omega}M_{13}\,, \tag{4.41c}$$

$$M_{33} = M_{11} + l^2\omega M_{13}\,, \tag{4.41d}$$

$$M_{31} = -l^2 M_{13}\,, \tag{4.41e}$$

while

$$N = \left(\frac{1}{4}(M_{13} - \omega M_{11}),\ 0,\ \frac{1}{4}\left(\frac{(2l^2\omega^2+1)}{\omega}M_{13} + M_{11}\right)\right)\ . \tag{4.42}$$

Interestingly enough, also in this case the hedgehog property is not lost. Namely, the full Skyrme field equations for the time-crystal ansatz defined above coupled to the $U(1)$ gauge field in Eq. (4.22) (taking into account that the coordinates are as in Eq. (4.40)) reduce to a single ordinary differential equation for the profile function $H(r)$ (for more details see Appendix D)

$$\begin{gathered}4\left(l^2\left(4-\lambda\omega^2\right)+X_2\sin^2(H)+\lambda\right)H''+2X_2\sin(2H)(H')^2+4\sin^2(H)X_2'H'\\+\left[\frac{1}{4}\left(l^2\omega^2-1\right)X_2+\lambda\left(2l^2\omega b_1-2b_3-1\right)\left(2l^2\omega b_1-2b_3-l^2\omega^2\right)\right]\sin(4H)-\frac{2l^2X_2}{\lambda}\sin(2H)=0\,,\end{gathered} \tag{4.43}$$

where

$$X_2(r)\ =\ 8\lambda\left(l^2b_1(\omega-2b_1)+2b_2^2+b_3(1+2b_3)\right)\ . \tag{4.44}$$

The closeness with the situation in which one has (instead of the dynamical Maxwell field) a non-vanishing chemical potential is useful in this case as well. Indeed, by requiring

$$X_2 = \lambda\left(l^2\omega^2 - 1\right) , \tag{4.45}$$

and

$$b_3(r) = l^2\omega b_1(r) - \frac{l^2\omega^2}{4} - \frac{1}{4} , \tag{4.46}$$

not only the equation for the time-crystal profile becomes solvable (as it is reduced to a quadrature) but also the full Maxwell equations reduce consistently to a scalar ordinary differential equation for $b_1(r)$.

All in all, using the ansatz in Eqs. (4.22), (4.45) and (4.46) (in the line element in (4.10)) with coordinates (4.40)) the full coupled Skyrme Maxwell system made by Eqs. (4.7) and (4.7b) in a topologically non-trivial sector can be reduced consistently to the following solvable system of two coupled ODEs for $H(r)$ and $b_1(r)$ (a detailed derivation of this result can be encountered in the Appendix D)

$$b_1'' - \frac{K}{8}(\omega - 4b_1)\sin^2(H)\left(l^2(\lambda\omega^2 - 8) - \lambda + \lambda\left(l^2\omega^2 - 1\right)\cos(2H) - 8\lambda(H')^2\right) = 0\,, \tag{4.47}$$

$$H'' + \frac{\left(l^2\omega^2 - 1\right)\sin(2H)\left(l^2 - \lambda(H')^2\right)}{2\left(\lambda\left(l^2\omega^2 - 1\right)\cos^2(H) - 4l^2\right)} = 0\,. \tag{4.48}$$

Hence, also in this case the recipe is to determine the profile $H(r)$ (as the corresponding equation (4.48) is solvable) and then to replace the result into the equation for (4.47) which becomes a simple linear non-homogeneous equation in which there is an effective potential which depends on $H(r)$. It is worth noticing that there are two simple topologically non-trivial sectors: the first case is when $l^2 - \lambda(H')^2 = 0$, and the sector $l^2\omega^2 - 1 = 0$. In this thesis, we do not explore those sectors.

The remaining components of the gauge potential are determined solving the simple algebraic conditions in Eqs. (4.45) and (4.46). The above system allows to clearly disclose many features of the gauged time-crystals (and, more in general, of the $U(1)$ gauged Skyrme model) which are close to a "dual superconductivity".

The first integral of (4.48), which allows to reduce it to quadratures, is given by

$$\left(4l^2 + \lambda\left(1 - l^2\omega^2\right)\cos^2(H)\right)(H')^2 - \frac{1}{2}l^2\left(1 - l^2\omega^2\right)\cos(2H) = I_0\,, \tag{4.49}$$

where I_0 is determined by the boundary conditions. As we did in the previous section for the gauged Skyrmion, we also calculate here for the time crystal, the non-vanishing topological charge density which for this case becomes

$$\begin{aligned} B_2 &= \frac{\epsilon_{2ijk}}{24\pi^2}\left(\mathrm{Tr}\left(L_i L_j L_k\right) - 3\partial_i\left(A_j \mathrm{Tr}\left(\tau_3\left(U^{-1}\partial_k U + \partial_k U U^{-1}\right)\right)\right)\right)\,, \\ &= \frac{1}{4\pi^2}\left[\frac{\omega}{2}\sin(2H)H' + \partial_r\left(\cos^2(H)\left(b_1 - \omega b_3\right)\right)\right]\,, \end{aligned} \tag{4.50}$$

where the Latin indices of the previous relation assume the values $0, 1, 3$ and the resulting integral is

$$W = \int B_2\, dr \wedge d\left(\omega\gamma\right) \wedge d\phi = 1 + 2\Big[\cos^2(H(r))\left(\frac{b_1(r)}{\omega} - b_3(r)\right)\Big]\Big|_0^{2\pi} = 1 - 2\left(\frac{b_1(0)}{\omega} - b_3(0)\right)\,, \tag{4.51}$$

if we consider $r \in [0, 2\pi]$, $\omega\gamma \in [0, 4\pi]$, $\phi \in [0, 2\pi]$ and $H(2\pi) = \pi/2$, $H(0) = 0$.

However, a "normal" topological charge is also present here due to the correction from the electromagnetic potential. By taking B_0 as defined in (4.36) as an integral over spatial slices we obtain

$$B = \int B_0\, dr \wedge dz \wedge d\phi = -2\Big[\cos^2(H(r))b_2(r)\Big]\Big|_0^{2\pi} = 2b_2(0)\,,$$

with the same boundary values used as in (4.51) with the difference now that we have z in place of γ for which we consider $z \in [0, 2\pi]$. The charge B is non-zero as long as $b_2(0) \neq 0$.

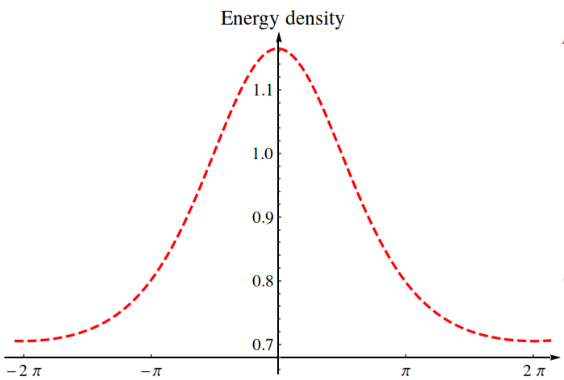

Figure 4.6: Energy density T_{00} of the Time Crystal.

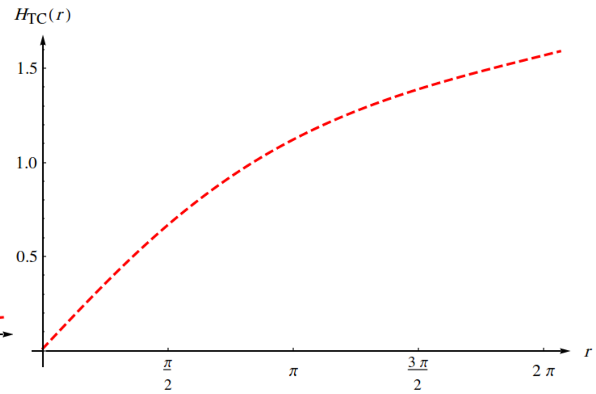

Figure 4.7: Time Crystal profile.

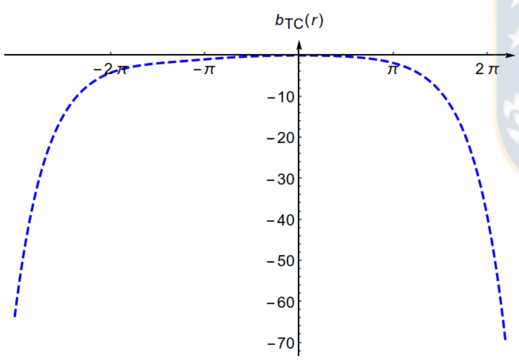

Figure 4.8: Gauge potential A_μ.

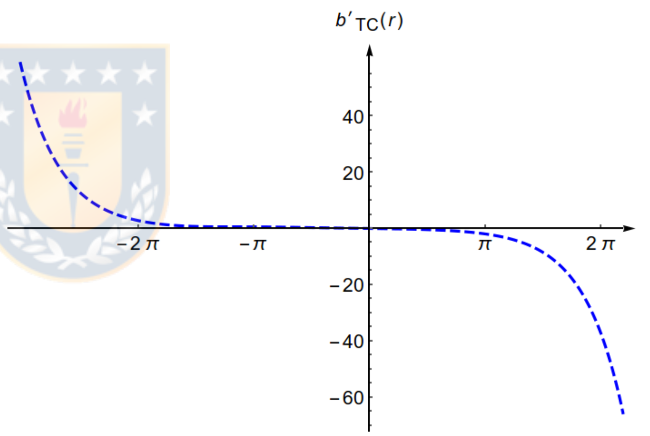

Figure 4.9: Electric field $E \equiv b'(r)$.

Figure 4.10: The solutions for the Eqs. (4.47) and (4.48) correspond to the values: $\lambda = 0.04$, $l = 0.47$, $\omega = 0.95$, $K = 1.00$, $p = 1.00$, and $q = 1.00$. Solving for $b_{TC} \equiv b_1$. Unlike the gauged Skyrmions, here the electric field suffers a suppression in the core of the time crystal.

4.3 Extended duality

In this section, we show that an extended electromagnetic duality[3] exists between the gauged Skyrmion and the gauged time-crystal constructed above. This means that to disclose such a duality, one needs to interchange electric and magnetic components suitably and transform specific parameters of the gauged solitons. In other words, the question we want to answer in this section is: how do the usual duality transformations of the electromagnetic field have to be generalized (so as to act on the Skyrmions and time-crystals considered here) in such a way that the field equations Eqs. (4.31) and (4.32), corresponding to the gauged Skyrmion, are mapped into the field equations Eqs. (4.47) and (4.48) of the gauged time-crystal?

Let us take the simplest non-trivial cases of gauged configurations we examined above. For the Skyrmion we have the profile equation (4.32), which for $p = q = 1$ reduces to

$$2\left(2l^2 + \lambda\cos^2(H)\right)H'' + \sin(2H)\left(l^2 - \lambda(H')^2\right) = 0\,, \tag{4.53}$$

while for the electromagnetic potential components we get from relations (4.28) and (4.29)

$$b_1(r) \;=\; \pm\,\frac{1-4b_3(r)}{2\sqrt{2}l}\,, \qquad b_2(r) \;=\; -b_3(r)\,, \tag{4.54}$$

with $b_3(r)$ being determined by the differential equation

$$b_3'' + \frac{K}{2}(1-4b_3)\sin^2(H)\left(2l^2 + 2\lambda(H')^2 + \lambda\cos^2(H)\right) = 0\,. \tag{4.55}$$

Let us now consider the corresponding time crystal equations, where−in order to avoid

[3]This extended electromagnetic duality does not correspond to the electromagnetic duality in vacuum because not all the components of the electric and magnetic fields hold the relation

$$\vec{E} \longrightarrow \vec{B}\,, \tag{4.52a}$$
$$\vec{B} \longrightarrow -\vec{E}\,. \tag{4.52b}$$

confusion−we denote the potential as $A_\mu = (a_1(r),\, 0,\, a_2(r),\, a_3(r))$ (namely, we label differently the components). The profile equation is, of course, given by Eq. (4.48) with the potential components related as

$$a_2(r) \,=\, \pm\frac{1}{4}l\sqrt{1-l^2\omega^2}(\omega - 4a_1(r))\,, \qquad a_3(r) \,=\, l^2\omega a_1(r) - \frac{(l^2\omega^2+1)}{4}\,, \tag{4.56}$$

and with a_1 determined by the following equation

$$a_1'' - \frac{K}{8}(\omega - 4a_1)\sin^2(H)\left(l^2(\lambda\omega^2-8) - \lambda + \lambda\left(l^2\omega^2-1\right)\cos(2H) - 8\lambda(H')^2\right) = 0\,. \tag{4.57}$$

These are just equations (4.45), (4.46) and (4.47) with the new labeling of the components.

An immediate observation is that profile equations (4.48) and (4.53) become identical if we set $\omega = -\frac{i}{l}$. Then, it is an easy task to see that (4.57) and (4.56) are mapped to (4.55) and (4.54) under the linear transformation

$$a_1(r) \,=\, \frac{i}{l}b_2(r)\,, \qquad a_2(r) \,=\, i\,l\,b_1(r)\,, \qquad a_3(r) \,=\, -b_3(r)\,. \tag{4.58}$$

The appearance of the imaginary units is not alarming since one also needs a suitable imaginary scaling in the relevant coordinates to map one space-time metric to the other. Notice that the imaginary part of the transformation involves only the γ and z components of A_μ. Hence, after performing such a transformation, the result is a real electromagnetic tensor of the Skyrmion case. We have to note, however, that transformation (4.58) is not unique. Other linear transformations map the two sets of equations to each other by mixing the electric and magnetic components. However, (4.58) belongs to a smaller class of transformations that associates the electric component of the time crystal potential a_1 with the purely magnetic components of the Skyrmion, namely b_2 and b_3. In particular, this property is respected by any linear transformation of the form $a_i = L_{ij} b_j$ as long as the following set conditions hold

$$L_{13} = L_{12} - \frac{i}{l}\,, \qquad L_{21} = i\,l\,, \qquad L_{23} = L_{22}\,, \qquad L_{11} = L_{31} = 0\,, \qquad L_{33} = L_{32} - 1\,. \quad (4.59)$$

Of course, the free parameters appearing in the above transformation must be chosen each time in such a way so that the result is strictly real. In the following table, we can see how the gauged Skyrmion and the time crystal (TC) components of the electromagnetic field are mapped into each other as well as H, A and G (in relations (4.12) and (4.13)) that are involved in the generalized hedgehog ansatz. So, we can see that the two configurations correspond to an interchange between the electric and one of the magnetic components that looks like a duality relation as seen in the plane formed by the x^1 and x^3 components. We shall call this transformation an *extended* or *generalized duality*.

gauged Skyrmion $\longrightarrow$	gauged TC
E_1	$-B_3$
B_2	$-B_2$
B_3	E_1
(H, A, G)	(H, A, G)

It is a surprising result that a duality symmetry exists, which maps the gauged Skyrmion into the gauged time crystal. Thus, if such extended duality transformations discussed here would have been known in advance, one could have found that time-crystals exist just by applying such transformations to the gauged Skyrmion. Moreover, the plots in Fig. 4.5 and Fig. 4.10 clearly show that, as the magnetic field is suppressed in the gauged Skyrmion core, the electric field is suppressed in the gauged time-crystal core. Thus, as gauged Skyrmions have some features in common with superconductor, gauged time-crystals have some features in common with dual superconductor.

4.4 External periodic fields

In this section, we will discuss an approximation which can be of practical importance in many applications from nuclear physics to astrophysics.

We have been able to construct analytically two different types of topologically non-trivial configurations of the full $(3+1)-$dimensional $U(1)$ gauged Skyrme model (which are dual to each other in the electromagnetic sense). Thus, it is natural to ask why we should analyze approximated solutions as we have the exact ones. The obvious reason is that, in this way, we will be able to discuss electromagnetic fields more general than the ones leading to the exact solutions discussed in the previous sections. In particular, it is interesting to discuss the physical effects of time$-$periodic electromagnetic fields (which do not belong to the class leading to the above exact solutions).

Here it will be considered the case in which the Skyrme configuration is fixed and not affected by the electromagnetic field (as in [32]) which is slowly turned on to get a tiny time-periodic electromagnetic field in these Skyrme background solutions. In this case, the Skyrme background plays the role of an effective medium for the Maxwell equations. Very interesting is the situation in which the background is a time-crystal as the reaction of the time$-$dependent Maxwell perturbation to the presence of the time-crystal critically depends on the ratio between the frequency of the perturbation and the frequency of the time-crystal.

4.4.1 Tiny time periodic fields in Skyrme background

Let us consider the approximate situation where we introduce a small enough electromagnetic field with the purpose not to consider its effect on the profile equations. Additionally, we demand that it is periodic in time in one of its components

$$A_\mu = \ (b_1(r),\, 0,\, b_2(r)\cos(\Omega\gamma),\, b_3(r))\ . \tag{4.60}$$

The electromagnetic current is conserved $\partial_\mu J^\mu = 0$, while the Maxwell equations constitute a compatible system of differential equations. For example, the one that corresponds to $b_2(r)$ is given by

$$\frac{b_2''}{b_2} = \frac{K}{2}\left[\lambda\left(8H'^2 + 2\left(1 - l^2\omega^2\right)\cos^2(H)\right) + 8l^2\right]\sin^2(H) - l^2\Omega^2\,, \tag{4.61}$$

and by considering the approximation $b_2 << 1$ we are led to the single profile equation

$$H'' - \frac{l^2\lambda\omega^2\sin(4H)}{4\left(l^2\left(\lambda\omega^2 - 4\right) - \lambda\right)} = 0\,, \tag{4.62}$$

therefore with a first integral given by

$$\frac{l^2\lambda\omega^2\cos(4H)}{16\left(l^2\left(\lambda\omega^2 - 4\right) - \lambda\right)} + \frac{1}{2}(H')^2 = I_0 = cte\,. \tag{4.63}$$

With the help of the latter and using the change of variable $x = \cos(H(r))$, we can express (4.61) as

$$\begin{aligned}\frac{b_2''}{b_2} = \frac{K}{2}\left(x^2 - 1\right)\Bigg(&\frac{2l^4\left(\lambda\omega^2 - 4\right)\left(\lambda x^2\omega^2 - 4\right) + l^2\lambda\left(\lambda\left(8x^4 - 10x^2 + 1\right)\omega^2 + 8\right)}{l^2\left(\lambda\omega^2 - 4\right) - \lambda} \\ &- 2\lambda\left(8I_0 + x^2\right)\Bigg) - l^2\Omega^2\,.\end{aligned} \tag{4.64}$$

From the form of (4.64) we can deduce that the nature of the solution strongly depends on the sign of the right hand side. If the sign is negative one expects a periodic type of behavior. On the other hand, if it is positive, we rather expect an exponential kind of behavior. Clearly, the appearance of these two possibilities has to do with the value of the frequency Ω of the field and its relation to the rest of the parameters of the model.

In general, one can consider the function

$$f(x)=\left(x^2-1\right)\left(\frac{2l^4\left(\lambda\omega^2-4\right)\left(\lambda x^2\omega^2-4\right)+l^2\lambda\left(\lambda\left(8x^4-10x^2+1\right)\omega^2+8\right)}{l^2\left(\lambda\omega^2-4\right)-\lambda}-2\lambda\left(8I_0+x^2\right)\right), \tag{4.65}$$

which at most possesses five extrema. The value $x=0$ is always a global extremum, for the rest of the values of x in $[-1,1]$ one may have from none up to four extrema depending on the parameters λ, l, ω and I_0. For example, when $l=\omega=\lambda=1$, $I_0=-1/2$ one gets five extrema in the region $x\in[-1,1]$, of which, $x=0$ is a global maximum; on the other hand, when $l=\omega=\lambda=1$, $I_0=-1$ one gets only one extremum, $x=0$, which now is a minimum.

In any case, it is possible to arrange the external field frequency Ω so that the right hand side of (4.64) has a clearly positive or negative sign. The critical value for this is $\Omega_{cr}=\frac{K}{2l^2}f(0)$, where

$$f(0)\;=\;\frac{\lambda\left(4l^2+\lambda\right)}{l^2\left(4-\lambda\omega^2\right)+\lambda}+8l^2+\lambda\left(16I_0-1\right)\;. \tag{4.66}$$

If $f(0)$ is a maximum, we need to have $\Omega>\Omega_{cr}$ in order to obtain a periodic type of behaviour. Alternatively, if $f(0)$ is a minimum, the condition $\Omega<\Omega_{cr}$ leads to an exponential type behavior.

This simple analysis shows that the reaction of a time periodic Maxwell perturbation to the presence of a time-crystal strongly depends on the relations between the frequency of the Maxwell perturbation and the parameters characterizing the time-crystal.

Let us conclude with an important remark. In this chapter, we found a suitable choice of variables that enables us to decouple the field equations partially. It is worth noticing that it can be successfully extended to analyze the gravitating Abelian-Higgs model [78]. Here, the strategy is the same, but now the decouple is between the Maxwell field from the Einstein field equations by demanding

$$A_\mu A^\mu=\nabla_\mu A^\mu=A_\mu\nabla^\mu\psi=0\,, \tag{4.67}$$

with ψ being the scalar field. With this requirement, three exact solutions characterized by a non-vanishing superconducting current were studied: *pp*-waves, AdS waves, and Kundt spaces for which both the Maxwell field and the gradient of the phase of the scalar are aligned with the null direction defining these spaces. One of the most interesting case was the Kundt family. Here, the geometry of the two-dimensional surfaces orthogonal to the superconducting current is determined by the solutions of the two-dimensional Liouville equation, and in consequence, these surfaces are of constant curvature, as it occurs in a vacuum. The solution to the Liouville equation also acts as a potential for the Maxwell field, which we integrate into a closed-form. Using these results, we show that the combined effects of the gravitational and scalar interactions can confine the electromagnetic field within a bounded region in the surfaces transverse to the current.

Chapter 5

Conclusion of Part I

In this first part of this thesis, we began by reviewing the main properties of the Skyrme model, particularly how it can be thought of as an approximate, low-energy effective field theory for QCD. Then, we stress the task of finding solutions to the Skyrme field equations by considering suitable ansatz for the Skyrme field. The background chapter concluded with a generalization of the hedgehog ansatz, which helped develop the next chapter.

In the second chapter by using the generalized hedgehog approach we constructed the first analytic and topologically non-trivial solutions of the $U(1)$ gauged Skyrme model in $(3+1)$-dimensional flat space-times at finite volume. There are two types of gauged solitons. Firstly, gauged Skyrmions living within a finite volume appear as the natural generalization of the usual Skyrmions living within a finite volume. The second type of gauged solitons corresponds to gauged time-crystals. These are smooth solutions of the $U(1)$ gauged Skyrme model whose periodic time-dependence is protected by a topological conservation law. Interestingly enough, electromagnetic duality can be extended to include these two types of solitons. Gauged Skyrmions manifest very interesting similarities with superconductors while gauged time-crystals with dual superconductors.

Due to the relations of the Skyrme model with low energy limit of QCD, the present results can be useful in many situations in which the back reaction of baryons on Maxwell field (and *viceversa*) cannot be neglected (this is especially true in plasma physics and astrophysics).

It is a very interesting issue to understand the relevance of the present results in Yang-Mills theory. From the technical point of view, the tools which allowed the construction of the present gauged configurations have been extended to the Yang-Mills case as well (see [79–81] and references therein). Thus, it is natural to wonder whether time-crystal can be defined in these theories. The present analysis suggests that this construction could shed some light on the dual superconductor picture.

Part II

Chapter 6

Introduction to Part II

6.1 Symmetries in Physics

As we observe the world around us, we see patterns and symmetrical combinations in the objects of nature. For instance, the crystals found in rocks exhibit unique symmetry patterns, which often allows us to reveal the structure of solids. The study of symmetry for objects spans hundreds of years; for example, the Greeks and others civilizations were intrigued by the symmetries in things and believed that these would be mirrored in the structure of nature. However, our perspective of analyzing symmetry had a twist. More than studying the well-proportioned objects, we care more about the symmetry of the *fundamental laws of nature* themselves.

In Physics, before Einstein's advance in 1905, the people thought of symmetry as a consequence of the dynamical laws of a physical system. By analyzing Maxwell's equations, Einstein changed the attitude to deal with symmetry by considering it a main feature of nature that constrains the dynamical laws. The invariance under Lorentz and gauge transformations of Electromagnetism were not derived from Maxwell's equations but were consequences of a symmetry principle. Fascinated with the geometry of space and time, Einstein thought of the "principle of equivalence" as a geometrization of symmetry which allowed him to understand the dynamics of gravity. Ten years later, it culminated in his famous General Relativity theory of gravitation.

With the progress in physics, symmetry principles have played an important role on the understanding and structure of the fundamental laws of nature. The simple fact of being able to carry out an experiment in different places at different times is a consequence of space-time translations.

We now know that symmetry principles constraint the form of the fundamental laws of nature. It can be understood already in the classical description of a mechanical system in the following way. For example, let us describe the position of a point particle in space by a dynamical variable $\vec{x}(t)$. The actual motion of the particle from $\vec{x}(t_1)$ to $\vec{x}(t_2)$ happens when the local functional$-$the action$-S[\vec{x}(t)]$ is extremal. This is known as a principle of least action and the classical equations of motion for $\widetilde{\vec{x}}(t)$ follow from this principle.

One says that there is a *symmetry* in the classical system when on the dynamical variable $\widetilde{\vec{x}}(t)$ we act with a transformation, $\widetilde{\vec{x}}(t) \to \mathcal{R}[\widetilde{\vec{x}}(t)]$, that leaves the action unchanged. If the classical equations of motion are also unchanged under that transformation, and $\mathcal{R}$ produces a symmetry of the action, then, $\mathcal{R}[\widetilde{\vec{x}}(t)]$ is also an extremum. Therefore, the symmetry can then be used to derive new solutions. This classic example reaffirms the fact that symmetry can constrain the form of the equations of motion.

Figure 6.1: Emmy Noether [1882-1935].

Probably, the main consequence of symmetry in physics is the existence of conservation laws. It can already be noted in the conservation of energy and momentum in Newton's law of mechanics due to time and spatial translation invariance, respectively. The connection between every *global* continuous symmetry−set of parameters varied continuously−and conservation laws was fully understood by Emmy Noether in 1918 [82]. In this paper, Noether proved two different theorems. The first theorem deals with symmetries generated by finite Lie groups and states that those global symmetries lead to conserved charges. It is important to emphasize that the transformations must be generated by continuous transformations. Discrete symmetries do not lead to conserved quantities.

The second theorem applies for infinite dimensional Lie groups and show that *gauge symmetries* containing arbitrary functions of space-time lead to (off-shell) relations among the equations of motion, the so-called *Noether identities*.

6.2 Gauge symmetries and trivial currents

Usually, the symmetries appearing in physics correspond to global symmetries.[1] They are formulated in terms of physical events; namely, after a symmetry transformation, we find a new physical situation, but *all* the observations are unchanged under that transformation. For example, spatial translations on a laboratory translate either the observer and the apparatus, but

[1] In terms of the Lagrangian approach, those that preserve the Lagrangian up to a boundary term.

observations will remain undisturbed. However, for gauge symmetries, the situation is different. Gauge symmetries or local symmetries are by definition a transformation in fields parametrized by one or more *arbitrary* functions of space-time. The main difference with global symmetries is that gauge symmetries only change the description of the same physical situation but do not lead to a different physical situation.

Despite its character, gauge theories have assumed a central role in describing fundamental theories of nature. The first time that gauge symmetry appeared was in Maxwell's electrodynamics. In this theory the observable fields $\vec{E}(t,\vec{x})$ and $\vec{B}(t,\vec{x})$ could be written in terms of a vector potential $\vec{A}$ to simplify the equations. In this description one can notice that there exists a transformation−gauge transformation− $\vec{A} \to \vec{A} + \vec{\nabla}\chi(x)$ that remains unchanged the values of $\vec{E}$ and $\vec{B}$.

From Noether's point of view, a natural question that follows from here is: Is there any connection between charges and global symmetries in the case of gauge theories? To understand this point, let us be more precise in certain concepts. A gauge theory is a Lagrangian theory such that Euler-Lagrange equations of motion admit non-trivial Noether identities and that admits non-trivial gauge transformations. If gauge transformations vanish on-shell, we would call them trivial gauge transformations.

Now, returning to the question previously posed, we find certain issues and ambiguities when first Noether theorem applies for gauge symmetries. The reason is that the would-be Noether current is trivially conserved, *i.e.* it is conserved without requiring the equations of motion (without the on-shell condition). As we will show, this is in fact a consequence of the second Noether theorem. Then, the would-be Noether charge is ambiguous as it has the arbitrary gauge parameter on it. To stress this point, let us put in action the First Noether Theorem for the gauge symmetry of pure Maxwell's electrodynamics theory:

In differential form language, Maxwell's electrodynamics theory is described by the Lagrangian is $\boldsymbol{L}[\mathbf{A}] = \mathbf{F} \star \mathbf{F}$, where the two-form field strength is the exterior derivative of the one-form connection, $\mathbf{F} = d\mathbf{A}$, and $\star$ is the Hodge operator[2]. The theory has a $U(1)$ gauge symmetry on the dynamical field $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + d\Lambda$, with the infinitesimal version $\delta_\lambda \mathbf{A} = -d\lambda$, with λ, a space-time function, corresponding to an infinitesimal parameter of $\Lambda \approx 1 + \lambda$. The general variation of the Lagrangian is given by

$$\delta \boldsymbol{L} \;=\; -2(d \star \mathbf{F})\delta\mathbf{A} + 2d(\delta\mathbf{A} \star \mathbf{F})\,. \tag{6.1}$$

If we now restrict this general variation to an infinitesimal gauge symmetry, say $\delta \to \delta_\lambda$, such that $\delta_\lambda \boldsymbol{L} = 0$, then the variation (6.1) reduces, *off-shell*, to

$$0 = 2(d \star \mathbf{F})d\lambda - 2d(d\lambda \star \mathbf{F}) = -2d((d \star \mathbf{F})\lambda + d\lambda \star \mathbf{F})\,, \tag{6.2}$$

where in the second equality we used $-2d(d \star \mathbf{F})\lambda = 0$ because $d^2 = 0$. We may be tempted to interpret the previous equation as a conservation law, say $d\boldsymbol{J}_\lambda = 0$, for the current

$$\boldsymbol{J}_\lambda \equiv -2(d \star \mathbf{F})\lambda - 2d\lambda \star \mathbf{F} \;=\; -2d(\lambda \star \mathbf{F})\,, \tag{6.3}$$

however, the second equality tells us that the current is trivially conserved, even without imposing the equation of motion. In the differential form parlance, a current built from a gauge symmetry is always (locally) an exact form and, therefore, a closed-form. This is another form of the old Second Noether Theorem. Therefore, the would-be Noether current $\mathbf{J}_\lambda$ is trivial and thus is physically meaningless to define a charge with it. In fact (6.2) is an off-shell identity by virtue of the Noether identity $\mathbf{N}_\lambda \equiv -2d(d\star\mathbf{F})\lambda \equiv 0$ which here, in the case of electromagnetism written in differential forms language, is a mere consequence of $d^2 = 0$ and not of whether the on-shell condition holds. This analysis is in agreement with the expected well-known result that gauge symmetries *do not* produce charges.

[2] All the operators and conventions are detailed in Appendix A.6 and 1.

Now, a subset of gauge symmetries in special cases may become global symmetries: the so-called *exact symmetries*. They are the non-trivial subset of gauge transformations generated by ϵ, for which $\delta_\epsilon \Phi = 0$, with Φ a generic solution to the field equations of a gauge theory. This type of symmetries do not move us in the solution space. If we consider diffeomorphisms, the exact symmetries are the isometries of space-time which are generated by Killing vector fields. In the presence of gauge fields, the exact symmetries are not limited to Killing vectors, and there could be a subset of internal gauge transformation which does not change a given solution.
In the case of pure electromagnetism this stands for solving $\delta_\lambda \mathbf{A} = -d\lambda = 0$ which has the solution (the global part of the $U(1)$ gauge transformations) $\lambda = \lambda_0 = cte.$[3] The previous analysis will still not change its triviality because it was wholly general, and this is just a particular choice of λ. This is important to stress because a naive use of $\boldsymbol{J}_{\lambda_0}$ as defined before will produce here the right formula for the electric charge. However, the logic is misleading, and that mistake will hit back in other gauge theories.

To get a sensitive charge from the exact symmetry in gauge theories, $\delta_{\lambda_0} \boldsymbol{A} = 0$, we should not follow Noether's approach but use another strategy.

6.3 Charges in General Relativity

Like in Electromagnetism, a similar situation happens in gravitational physics, mainly in General Relativity (GR). From the early days of GR, many proposals for computing conserved charges associated with exact isometries of space-time, Killing vector fields, have been considered. The first covariant formula for conserved charges associated with Killing vectors was given by A. Komar [83], which was followed by a Hamiltonian formulation by Arnowitt, Deser, and

[3] Just for electromagnetism this solution does not depend on the fields and thus the rest of the analysis is quite general. In the case of general relativity the analogous equation is the Killing equation which is a property of certain symmetric space-time, or for the case of Yang-Mills theory $\delta_\lambda \mathbf{A}^a = -d_{\mathbf{A}} \lambda^a = 0$ also depends on the fields. Therefore, there is no a general solution.

Misner (ADM) in [84] by defining charges associated with symmetries of asymptotic flat space on constant time slices. Also it is worth mentioning papers by Bondi, van der Burg, Metzner, and Sachs, in the same asymptotic context, but now in asymptotic flat isometries at null infinity [85,86]. Of course, there have also been different approaches in computing charges in gravity. The most popular are, by name of authors, Abbott-Deser-Tekin [87,88], Regge-Teitelboim [89], Lee-Wald [90], Iyer-Wald [91], Barnich-Brandt [92,93], and Torre-Anderson [94,95] method. Their developments were sometimes independent and cross inseminated. Nevertheless, all of them have common features. One is their relying, directly or indirectly, on the structure of the phase space, more specifically on the symplectic structure. Another common feature is that in all of them the obtained charges are expressed as closed surface integrals (or $(D-2)$-surface integrals for D-dimensional space-times). The equivalence and connection among some of them have been established in the literature.[4] However, a systematic study of their connection is still missing. Although these methods have many advantages in their ease of working, they have their shortcomings. For example, they are not covariant enough or crucially depend on the form of Einstein-Hilbert action or the asymptotic behavior of fields.

The study of the computation of charges in Lagrangian gauge theories constitutes the core of the second part of this thesis. With this context in mind, the present thesis is a first step to explore, from a different perspective, the study of physical and gauge symmetries of the symplectic structure when boundary conditions are imposed. To do so, we focus on "surface charges" as the principal quantity that encodes the physical information related to symmetries in the context of gauge theories.

One may ask for a "covariant" Hamiltonian formulation of generally invariant theories. A key object in this formulation is the construction of covariant phase space and its symplectic struc-

[4] See for instance a recent review about the equivalence among the Iyer-Wald and Barnich-Brandt procedure [96]. Analogously, in [97] was shown that the of-shell Abbot-Deser-Tekin formalism is related to the Iyer-Wald and Barnich-Brandt-Compère formalism.

ture. We follow in general terms the Lee-Wald symplectic method [90] but being well aware about the close equivalent Barnich-Brandt symplectic method [93], and also, being well aware that the method differs from the original Noether proposal. One main difference of the approach presented here is the emphasis on the quasi-local nature, explicit coordinate independence, and gauge invariance of the formulas for charges (but of course, they might be used at space-time asymptotic regions too).[5] To achieve clarity our terminology and notation slightly differ from the original Lee-Wald treatment but the core logic is the same. For a complementary approach see the lecture [98].

The procedure to compute charges for a given solution within a gauge theory can be ordered in five simple steps 1) Identify the fields infinitesimal gauge transformations, 2) Obtain the general surface charge density formula for the theory, 3) Identify the parameters solving the exact symmetry condition (*e.g.* generalized Killing equation), 4) Compute the surface charge integral with those parameters (to get a differential charge), and 5) Integrate the differential charge on phase space. The present part of the thesis extensively explores these steps for different gravity theories coupled to matter fields by considering three families of solutions of the field equations.

This part of this thesis is organized as follows. In Chapter 7 we make a brief review on symplectic mechanics, the covariant phase space method and its connection with the surface charges. In Section 7.2 we explicitly derive surface charges for a general gauge theory. In Sections 7.4 and 7.5 we progressively establish the explicit formulae for the surface charges for the theory of gravity in the presence of cosmological constant, gravity coupled to Maxwell and Skyrme fields, and spinors. In Subsection 7.5.3 we show that boundary terms in the Lagrangian

[5] It is worth to note here that the quasi-local treatment for the charge conservation ensures, given a space-time with exact symmetries, the independence of the radius. This contrast with some asymptotic approaches where $r \to \infty$ is required to get rid of terms appearing on the specific computation of charge formula even if the solution is everywhere specified, *e.g.* a black hole solution.

have no effects on surface charges. In Subsection 7.5.5 it is demonstrated that torsion, for two particular examples, disappears from the formulas. Finally, in Section 8 we perform a test of the reliability of the formalism by recovering the standard first law of black hole mechanics in a quasi-local way.

Chapter 7

Surface Charges in Gauge Theories

It is well-known for theoretical gravitational physicists that Noether's theorem does not work correctly in gauge theories [93, 95, 99]. This latter is borne out by the fact that the *would be-* Noether current is conserved without requiring the equations of motion of the theory, *i.e.* a trivial conservation law. As an alternative to overcome this problem, this chapter shows explicit derivations of charges through the so-called *surface charge method* with focus on GR coupled to matter fields. To get a better understanding of the mathematical background on phase space, we first show the main ingredients for the formulation of the *covariant phase space*.

7.1 Brief review of phase space

A symplectic manifold (usually referred as phase space) is a manifold $\mathbb{M}$ which is equipped with a two-form Ω, called *symplectic form*. Because the symplectic form Ω has degeneracy directions, the manifold $\mathbb{M}$ considered here is already the reduced symplectic phase space, namely without those redundancy directions. See an exhaustive analysis in gauge theories in [102].[1]

[1]We use the $\mathbb{BLACKBOARD}$ $\mathbb{BOLD}$ notation to refer to the phase space $\mathbb{M}$ and then avoiding confusion with space-time $\mathcal{M}$.

If the manifold $\mathbb{M}$ is parametrized by the coordinates X^A, inducing basis for the cotangent space of the manifold as δX^A, then $\Omega = \Omega_{AB}\delta X^A \wedge \delta X^B$, has the following properties:

- $\Omega_{AB} = -\,\Omega_{BA}$.
- $\delta\Omega = 0\,.$
- $\Omega_{AB}V^B = 0 \Longleftrightarrow V^B = 0\,,$

where V^A is a vector in the tangent space of the manifold. We use CAPITAL letters coordinates in phase space X^A, with $A, B = 1, ..., 2N$ where N are the degrees of freedom of the theory, and δ (instead of d) to denote the exterior derivative in phase space. In particular, in the canonical formalism one chooses a Darboux chart by introducing introducing momenta p_i and coordinates q^i, $i = 1, ..., N$, in which the symplectic structures takes the form

$$\Omega = \delta p_i \wedge \delta q^i\,. \tag{7.1}$$

However, it is convenient to combine the set $\{p_i, q^i\}$ in a variable, which we denote X^A. Here $X^i = \Omega^{ij}p_j$ for $i \leq N$ and $Q^i = q^{i-N}$ for $i > N$. The symplectic structure Ω can be thought as a two-form in phase space with the antisymmetric components Ω_{AB} being a $2N \times 2N$ matrix with non-zero elements $\Omega_{i,i+N} = -\Omega_{i+N,i} = 1$.

The first condition tells us simply that the symplectic structure is anti-symmetric. The second condition is read as Ω is closed, and the third one as Ω is non-degenerate. Due to this last condition, Ω is invertible and one can define its inverse $\Omega^{AB} = (\Omega^{-1})_{AB}$. In phase space, the matrices Ω_{AB} and Ω^{AB} are responsible for lowering and raising the indices, respectively.

By considering a differentiable function $f = f(X^A)$ on the phase space, it is possible to associate a vector field to it as

$$V_f \equiv \delta f = \partial_A f \delta X^A \,, \tag{7.2}$$

where $\partial_A \equiv \partial/\partial X^A$. In reverse, given a vector field $V^A(X^B)$, it is also possible to associate a function H_V (usually called Hamiltonian generator) to it such that

$$\delta H_V = V = \Omega_{AB} \delta X^A V^B \,, \tag{7.3}$$

in which $\delta H_V = \partial_A H_V \delta X^A$. The set of conditions for the existence of H_V are called *integrability conditions*. If V is not integrable, we should use the notation $\not\delta$ instead of δ, namely $V = \not\delta H_V$. In other words, there is no any function H_V such that its exterior derivative on all phase space would be equal to V.

With this formalism, the Poisson bracket of two functions f and g is defined as

$$\{f, g\} \equiv \Omega^{AB} (V_f)_A (V_g)_B = (V_f)^B \partial_B g = \mathscr{L}_{V_f} g \,, \tag{7.4}$$

where $\mathscr{L}_V$ denotes the Lie derivative in direction V on the phase space.[2] By anti-symmetry on Ω, the Eq. (7.4) is also equal to $-\mathscr{L}_{V_g} f$.

Example 1: *One-dimensional motion of a particle.*

The simplest example to apply this approach is the one-dimensional motion of a particle, with position q and momentum p. In our notation, $i, j, ... = 1$, then $A, B, ... = 1, 2$, and by setting $X^1 \equiv q$ and $X^2 \equiv p$, we have

[2]Remember that the Lie derivative, exterior derivative, interior product, etc are defined independent of the metric, therefore they apply in the phase space in the usual manner.

$$\Omega_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \{q,p\} = 1\,, \qquad \{q,q\} = \{p,p\} = 0\,. \tag{7.5}$$

The basis for the tangent and cotangent spaces are ∂_{X^A} and δX^A, respectively. They are related to the matrix Ω given in (7.5) as $\partial_{X^A} = \Omega_{AB}\delta X^B$, for instance in this case we have $\partial_q = \delta p$. Notice that it is also possible to read the symplectic two-form from the surface term in the variation of the Lagrangian. For this example, the dynamics of this particle is described by the Lagrangian functional $\mathcal{L} = \mathcal{L}(q,\dot{q})$, whose general variation is given by

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial q} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\right)\delta q + \frac{d}{dt}(p\delta q)\,. \tag{7.6}$$

A second variation of the on-shell action will then becomes $d(\delta p\wedge\delta q)/dt$ which is nothing but the time derivative of the symplectic structure two-form computed over the tangent space of the phase space parametrized by δp and δq. From here, we get

$$\Omega = \delta(p\wedge\delta q) = \delta p\wedge\delta q\,, \tag{7.7}$$

where in the second equality we used the nilpotency of the operator δ. In particular, we can choose the function f to be the momentum or Hamiltonian H. Then,

$$f = p \qquad \longrightarrow \qquad V_p = \delta p = \partial_q\,, \tag{7.8}$$

$$f = H \qquad \longrightarrow \qquad V_H = \delta H = -\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p = \dot{p}\partial_p + \dot{q}\partial_q = \partial_t\,, \tag{7.9}$$

where we used the Hamiltonian equations of motion. These relations correspond to the well-known results "momentum is generator of translations in space" and "Hamiltonian is generator of translations in time". While momentum or Hamiltonian are generators of evolution in spatial and time, respectively, any symmetry direction of the symplectic form is produced by a generator.

The symplectic structure Ω leads to the notion of symplectic symmetries over the phase space. A vector V is called a *symplectic symmetry* over the phase space if

$$\mathscr{L}_V \Omega = V \cdot d\Omega + d(V \cdot \Omega) = d(V \cdot \Omega) = 0 \,. \tag{7.10}$$

By using the Poincaré's lemma on the one form $V \cdot \Omega$, which says that it can be written locally as an exact form, we find[3]

$$V \cdot \Omega = dH_V \,, \tag{7.11}$$

or in index notation $\Omega_{AB} V^B = \partial_A H_V$. Multiplying this by the inverse Ω^{CA} we find

$$V^A = \Omega^{AB} \partial_B H_V \,. \tag{7.12}$$

Therefore, H_V is the generator of evolution along the symmetry vector field V through the Poisson bracket. It is worth noticing that symplectic symmetries form an algebra through the Lie bracket. Let us assume V, W are two symplectic symmetries, $\mathscr{L}_V \Omega = 0 = \mathscr{L}_W \Omega$. Then, the Lie bracket $[V, W] \equiv \mathscr{L}_V W$ is also a symplectic symmetry, explicitly

$$\mathscr{L}_{[V,W]} \Omega = (\mathscr{L}_V \mathscr{L}_W - \mathscr{L}_W \mathscr{L}_V) \Omega = 0 \,. \tag{7.13}$$

Example 2: *Free Scalar Field in Two-Dimensional Space-Time.*

Let us consider a massive scalar field $\Phi(t, x)$ with mass m living in a two-dimensional space-time $\mathcal{M}$ endowed with the Minkowski metric $\eta = \text{diag}(-1, 1)$. We will choose a particular foliation with time-like hyper-surfaces Σ_t. The action is given by

[3] If $\boldsymbol{\alpha}$ is a p-form which can be written in terms of a $(p-1)$-form $\boldsymbol{\beta}$ as $d\boldsymbol{\alpha} = \boldsymbol{\beta}$, we say that $\boldsymbol{\alpha}$ is an *exact form*. If now the p-form $\boldsymbol{\alpha}$ fulfills $d\boldsymbol{\alpha} = 0$, we say that $\boldsymbol{\alpha}$ is a *closed form*.

$$S[\Phi] = \int_{\mathcal{M}} L[\Phi, \partial\Phi] dt dx \,, \qquad L[\Phi, \partial\Phi] = \frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2\right) \,, \tag{7.14}$$

where $\mathcal{M} = [t_i, t_f] \times V$, with V a spatial region. Varying this action for $t_i < t < t_f$, we get

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} dt dx \left(\partial_\mu \delta\Phi \partial^\mu \Phi - m^2 \Phi \delta\Phi\right) \,, \\ &= \int_{\partial\mathcal{M}} dx (\partial^0 \Phi \delta\Phi) - \int_{\mathcal{M}} dt dx (\partial_\mu \partial^\mu \Phi + m^2 \Phi) \delta\Phi \,. \end{aligned} \tag{7.15}$$

By boundary conditions, the surface term at the spatial boundary vanishes, and the remaining surface terms specify the canonical one-form in phase space

$$\Theta(t) = \int_{\Sigma_t} dx \partial^0 \Phi(t,x) \delta\Phi(t,x) \,, \tag{7.16}$$

whose variational exterior derivative gives the canonical two-form symplectic structure

$$\Omega(t) = \int\int_{\Sigma_t} dx \delta\partial^0 \Phi(t,x) \wedge \delta\Phi(t,x) \,, \tag{7.17}$$

$$= \int\int_{\Sigma_t} dx dx' \delta(x - x') \delta\partial^0 \Phi(t,x) \wedge \delta\Phi(t,x') \,. \tag{7.18}$$

The only non-vanishing momentum p^μ is $p^0 = \partial^0 \Phi$. This is the usual conjugate variable computed as $\Pi \equiv \delta L / \delta(\partial_0 \Phi)$. Thus, $\Omega_{AB}(x,x')$ and its inverse $(\Omega^{-1})_{AB}(x,x')$ are given by

$$\Omega_{AB} = \begin{pmatrix} 0 & \delta(x-x') \\ -\delta(x-x') & 0 \end{pmatrix} \,, \qquad (\Omega^{-1})_{AB}(x,x') = \begin{pmatrix} 0 & -\delta(x-x') \\ \delta(x-x') & 0 \end{pmatrix} \,, \tag{7.19}$$

so that the Poisson brackets become

$$\{\Phi(t,x), p^\mu(t,x')\} = \delta^{\mu,0} \delta(x-x') \,, \qquad \{\Phi(t,x), \Phi(t,x')\} = \{p^\mu(t,x), p^\nu(t,x')\} = 0 \,. \tag{7.20}$$

It is worth mentioning that choosing this specific foliation (time foliation) breaks the theory's covariance. To have a covariant phase space, we should proceed differently. This result gives rise to the *covariant phase space method* which is a covariant way of defining phase space and symplectic structure for field theories and gauge theories.

7.2 Derivation of Surfaces Charges

7.2.1 Covariant Phase Space

In this section, we provide a brief introduction to the *covariant phase space method* (CPSM). The CPSM provides a systematic way of calculating variations of conserved charges in generic theories with local gauge symmetries, in particular generally covariant theories. This method was primarily developed in the papers of Iyer, Lee and Wald [90,91] (see also Crnkovic and Witten [100]). In this approach, the invariant symplectic form Ω is derived from the action. However, there exists an alternative way of formulating the CPSM, developed by Barnich, Brandt and Compère, based on equations of motion instead of the action [93,101]. We will see that for exact symmetries, these two formalisms become equivalent.

This section discusses a less appreciated application of the CPSM for computing the conserved charges associated with exact symmetries. First, we introduce the Lee-Wald symplectic structure, which provides a covariantly way of building a manifold equipped with a covariant symplectic two-form [90]. Then, by evaluating the symplectic structure on exact symmetries, we derive a lower degree conservation law that allows us to define the so-called *surface charges*. We will carry out this derivation in parallel both at geometric language as differential form language. To highlight the difference, we will use **BOLD** letters for differential forms and ordinary letters for geometric language (see also Appendix A.6 and 1 for conventions).

Let us start with a Lagrangian theory in a D-dimensional space-time $\mathcal{M}$

$$S[\Phi] = \int_{\mathcal{M}} \sqrt{-g}\, d^D x\, L[\Phi, \partial_\mu \Phi, \, ...]\,, \qquad \left[S[\Phi] \;=\; \int_{\mathcal{M}} \boldsymbol{L}[\Phi, \partial_\mu \Phi, \, ...]\right], \tag{7.21}$$

where $\boldsymbol{L}[\Phi, \partial_\mu \Phi, ...] = L\, d^D x$ is the diffeomorphism-invariant Lagrangian depending on the collection of dynamical field configuration $\Phi(x)$ and its first derivatives, with $\Phi = \Phi^i$.[4] A field configuration $\Phi(x)$ in the space-time $\mathcal{M}$ corresponds to a point in the phase space $\mathbb{M}$, to say Φ. Those field configurations satisfying the equations of motion of the theory form a sub-space in the phase space denoted by $\overline{\mathbb{M}}$.

An arbitrary variation of the Lagrangian is given by

$$\delta L = E[\Phi]\delta\Phi + \partial_\mu \Theta^\mu[\Phi, \delta\Phi]\,, \qquad \left[\delta \boldsymbol{L} \;=\; \boldsymbol{E}[\Phi]\delta\Phi + d\boldsymbol{\Theta}[\Phi, \delta\Phi]\right]\,, \tag{7.22}$$

where $\boldsymbol{E}[\Phi]$ are the equations of motion of the Lagrangian theory and they locally depend on the dynamical fields and their derivatives, while $\boldsymbol{\Theta}[\Phi, \delta\Phi]$ locally depends on the dynamical fields Φ, their variations $\delta\Phi$ and derivatives. The letter d stands for exterior derivative in space-time. In agreement with the previous section, an infinitesimal field perturbation $\delta\Phi(x)$ over a configuration $\Phi(x)$ then corresponds to a vector tangent to the phase space at Φ. We denote this vector by $\delta\Phi$[5].

By excluding theories with higher derivatives in the fields, the boundary term $\boldsymbol{\Theta}[\Phi, \delta\Phi]$ is linear in the variations $\delta\Phi$, therefore is a one-form in phase space. But, it is a $(D-1)-$form in space-time and is called the *symplectic potential form*. It suffers from two types of ambiguities: The first ambiguity arises from adding an exact $D-$form to the Lagrangian top$-$form, namely

[4] "$\dots$" in $L[\Phi]$ means for higher derivatives of the fields. We neglect theories with higher derivatives in the dynamical fields.

[5] Following the terminology of the previous section, we should write $[\delta\Phi]^A$ instead of $\delta\Phi$. However, for clarity in the notation we follow the latter.

$$L[\Phi,\partial\Phi]\to L[\Phi,\partial\Phi]+\partial_\mu A^\mu[\Phi]\,,\qquad [\boldsymbol{L}[\Phi,\partial\Phi]\to \boldsymbol{L}[\Phi,\partial\Phi]+d\boldsymbol{A}[\Phi]]\,, \tag{7.23}$$

which shifts the symplectic potential form by an exact form

$$\Theta^\mu[\Phi,\delta\Phi]\to\Theta^\mu[\Phi,\delta\Phi]+\delta A^\mu[\Phi]\,,\qquad [\boldsymbol{\Theta}[\Phi,\delta\Phi]\to\boldsymbol{\Theta}[\Phi,\delta\Phi]+\delta\boldsymbol{A}[\Phi]]\,. \tag{7.24}$$

The second ambiguity arises from adding an exact $(D-1)-$form to $\boldsymbol{\Theta}$ as follows

$$\Theta^\mu[\Phi,\delta\Phi]\to\Theta^\mu[\Phi,\delta\Phi]-\partial_\nu Y^{\mu\nu}[\Phi,\delta\Phi]\,,\qquad [\boldsymbol{\Theta}[\Phi,\delta\Phi]\to\boldsymbol{\Theta}[\Phi,\delta\Phi]-d\boldsymbol{Y}[\Phi,\delta\Phi]]\,, \tag{7.25}$$

where $\boldsymbol{Y}[\Phi,\delta\Phi]$ is a $(D-2)-$form space-time. As we will show in a moment, these freedoms of defining $\boldsymbol{\Theta}$ may cease to exist for exact symmetries. We will return to this point later.

The Lagrangian has a symmetry if for certain infinitesimal variations over the configuration space it becomes at most an exact form

$$\delta_\epsilon L=\partial_\mu M^\mu_\epsilon\,,\qquad [\delta_\epsilon\boldsymbol{L}\;=\;d\boldsymbol{M}_\epsilon]\;, \tag{7.26}$$

where ϵ are the collection of parameters that generate the infinitesimal symmetry, and denote the infinitesimal transformation generated over any quantity, in particular acting over the fields as $\delta_\epsilon\Phi$. Here $\boldsymbol{M}_\epsilon$ is a $(D-1)-$form in space-time. Comparing this last result with (7.22), we have

$$\partial_\mu M^\mu_\epsilon=E[\Phi]\delta_\epsilon\Phi+\partial_\mu\Theta^\mu[\Phi,\delta_\epsilon\Phi]\,,\qquad [d\boldsymbol{M}_\epsilon\;=\;\boldsymbol{E}[\Phi]\delta_\epsilon\Phi+d\boldsymbol{\Theta}[\Phi,\delta_\epsilon\Phi]]\;. \tag{7.27}$$

Now, let us consider that the transformations $\delta_\epsilon\Phi$ are linear in the symmetry parameters ϵ. This allows to remove the derivatives over all symmetry parameters by integrating by parts and formally decompose

$$E[\Phi]\delta_\epsilon \Phi = \partial_\mu S^\mu_\epsilon - N_\epsilon\,, \qquad [\boldsymbol{E}[\Phi]\delta_\epsilon\Phi = d\boldsymbol{S}_\epsilon - \boldsymbol{N}_\epsilon]\,, \tag{7.28}$$

such that in the quantity $\boldsymbol{N}_\epsilon$ the symmetry parameters appear only as factors. Using the new expression for $\boldsymbol{E}[\Phi]\delta_\epsilon\Phi$ we get

$$\partial_\mu\left(\Theta^\mu[\Phi,\delta_\epsilon\Phi] - M^\mu_\epsilon + S^\mu_\epsilon\right) = N_\epsilon\,, \qquad [d\left(\boldsymbol{\Theta}[\Phi,\delta_\epsilon\Phi] - \boldsymbol{M}_\epsilon + \boldsymbol{S}_\epsilon\right) = \boldsymbol{N}_\epsilon]\,. \tag{7.29}$$

Restricting to gauge symmetries, the very structure of the last expression implies

$$N_\epsilon \equiv 0\,. \tag{7.30}$$

These are called *Noether identities* and there is one of them for each independent gauge parameter. Notice that these identities are satisfied off-shell (see, for instance, Section 7.4.1 for the derivation of these identities in Einstein-Hilbert theory). They also are the usual constraints due to the redundancy of using gauge variables to deal with the theory and why the First Noether theorem does not apply appropriately for non-trivial gauge symmetries.[6] Considering this latter, it is natural to define the form

$$J^\mu_\epsilon \;\equiv\; \Theta^\mu[\Phi,\delta_\epsilon\Phi] - M^\mu_\epsilon + S^\mu_\epsilon\,, \qquad [\boldsymbol{J}_\epsilon \equiv \boldsymbol{\Theta}_\epsilon[\Phi,\delta_\epsilon\Phi] - \boldsymbol{M}_\epsilon + \boldsymbol{S}_\epsilon]\,, \tag{7.32}$$

such that by virtue of the Noether identities (7.30) satisfies

$$\partial_\mu J^\mu_\epsilon \;=\; 0\,, \qquad\qquad [d\boldsymbol{J}_\epsilon = 0]\,, \tag{7.33}$$

[6]A gauge transformation of the form $\delta_\epsilon\Phi^i = \epsilon^{[ij]}\delta S/\delta\Phi^j$ leaves the action $S[\Phi^i]$ invariant, but no matter what ϵ^{ij} are, namely

$$\delta S[\Phi^i] = \frac{\delta S}{\delta\Phi^i}\frac{\delta S}{\delta\Phi^j}\epsilon^{[ij]} \equiv 0\,. \tag{7.31}$$

In Hamiltonian formalism, these gauge symmetries vanish when the equations of motion hold (so called-"on-shell condition") because they are not generated by a constraint. This type of transformation are named trivial-gauge transformations [102].

which is identically conserved, *i.e.* even without the use of the equations of motion. Notice also that although δ_ϵ is a gauge symmetry, the current defined above has an off-shell conservation law. In this sense, one says that $\boldsymbol{J}_\epsilon$ is trivially conserved. When the equations of motion are used, $\boldsymbol{J}_\epsilon$ becomes the sometimes called Noether current $\boldsymbol{J}_\epsilon \approx \boldsymbol{\Theta}_\epsilon - \boldsymbol{M}_\epsilon$, which due to the previous analysis is also trivially conserved.

At this stage, we can evoke the Poincaré's lemma for the conservation law of the current $\boldsymbol{J}_\epsilon$, *i.e.* it says that locally exist a co-dimension two-form $\widetilde{\boldsymbol{Q}}_\epsilon$ such that

$$J^\mu_\epsilon = \partial_\nu \widetilde{Q}^{\mu\nu}_\epsilon \,, \qquad \left[\boldsymbol{J}_\epsilon = d\widetilde{\boldsymbol{Q}}_\epsilon\right] \,, \tag{7.34}$$

where $\widetilde{\boldsymbol{Q}}_\epsilon$ is a $(D-2)$-form in space-time called the *Noether-Wald potential*. We observe that $\widetilde{\boldsymbol{Q}}_\epsilon$ suffers from an ambiguity since is defined up to a closed co-dimension two-form.

Because of this trivial conservation, *charges* should not be defined using $\boldsymbol{J}_\epsilon$. A natural alternative is to consider lower-degree conservation laws [93–95] (all of these approaches can be thought as a generalization of Noether's theory of conserved currents to differential forms of any degree).

Now, let us return to our previous discussion. To go further let us assume that ϵ contains diffeomorphisms. More precisely, suppose the collection of gauge parameters can be split as $\epsilon = (\xi, \lambda)$, such that

$$\delta_\epsilon \Phi = \delta_\xi \Phi + \delta_\lambda \Phi \,, \tag{7.35}$$

where ξ is a vector field generating infinitesimal diffeomorphisms and δ_λ denotes the rest of the infinitesimal gauge symmetry transformations.

In order to define charges for gauge theories one has to rely on the symplectic structures of the theory. A way to define the symplectic structure current is [90]

$$\Omega^\mu[\Phi,\delta_1\Phi,\delta_2\Phi] \equiv \delta_1\Theta^\mu[\Phi,\delta_2\Phi] - \delta_2\Theta^\mu[\Phi,\delta_1\Phi] - \Theta^\mu[\,\Phi,[\delta_1,\delta_2]\Phi]\,]\,. \tag{7.36}$$

The last term ensures linearity on the variations, and where δ_1 and δ_2 are, for the moment, two arbitrary variations. Note that $\Omega^\mu[\Phi,\delta_1\Phi,\delta_2\Phi]$ is a double variation in the phase space and a $(D-1)$-form in space-time. The double variation can also be understood as a two-form in the phase space.

Now, let us consider the arbitrary (off-shell) variation of the current J^μ_ϵ in (7.32), we get

$$\delta J^\mu_\epsilon = \delta\Theta^\mu[\Phi,\delta_\epsilon\Phi] - \delta M^\mu_\epsilon + \delta S^\mu_\epsilon = \partial_\nu \delta\widetilde{Q}^{\mu\nu}_\epsilon\,, \qquad \left[\delta \boldsymbol{J}_\epsilon = d(\delta\widetilde{\boldsymbol{Q}}_\epsilon)\right]. \tag{7.37}$$

From here, we find

$$\delta\Theta^\mu[\Phi,\delta_\epsilon\Phi] = \delta M^\mu_\epsilon - \delta S^\mu_\epsilon + \partial_\nu\delta\widetilde{Q}^{\mu\nu}_\epsilon\,, \qquad [\delta\boldsymbol{\Theta}[\Phi,\delta_\epsilon\Phi] = \delta\boldsymbol{M}_\epsilon - \delta\boldsymbol{S}_\epsilon + d\delta\widetilde{\boldsymbol{Q}}_\epsilon]\,. \tag{7.38}$$

Evaluating the symplectic current (7.36) on the gauge symmetry, $\delta_2 \to \delta_\epsilon$ and $\delta_1 \to \delta$, one obtains

$$\Omega^\mu[\delta\Phi,\delta_\epsilon\Phi] = -\delta_\epsilon\Theta^\mu[\Phi,\delta\Phi] - \Theta^\mu[\Phi,\,[\delta,\delta_\epsilon]\Phi] + \delta M^\mu_\epsilon - \delta S^\mu_\epsilon + \partial_\nu\delta\widetilde{Q}^{\mu\nu}_\epsilon\,. \tag{7.39}$$

We will also assume that the Lagrangian theory is left invariant under transformations generated by λ, namely $\delta_\lambda L = 0$.[7] As a top form in the space-time manifold, the Lagrangian satisfies $\delta_\epsilon L = \delta_\xi L = \mathscr{L}_\xi L = d(i_\xi L)$, then, $M^\mu_\epsilon = \xi^\mu L$. Here and in the following we assume $\delta\xi = 0$. Then, the Eq. (7.37) yields

$$\delta J^\mu_\epsilon = \delta\Theta^\mu[\Phi,\delta_\epsilon\Phi] - \xi^\mu\delta L + \delta S^\mu_\epsilon = \partial_\nu\delta\widetilde{Q}^{\mu\nu}_\epsilon\,, \qquad \left[\delta\boldsymbol{J}_\epsilon = d(\delta\widetilde{\boldsymbol{Q}}_\epsilon)\right]. \tag{7.40}$$

[7] Notice that if λ is a gauge symmetry parameter for a Chern-Simons theory, this assumption is not longer valid [103, 104].

With this assumption, the symplectic current (7.39) takes the form[8]

$$\begin{aligned}
\Omega^\mu[\delta\Phi,\delta_\epsilon\Phi] &= \delta\Theta^\mu[\Phi,\delta_\xi\Phi]-\delta_\epsilon\Theta^\mu[\Phi,\delta\Phi]\,,\\
&= \xi^\mu\left(E\,\delta\Phi+\partial_\nu\Theta^\nu[\Phi,\delta\Phi]\right)-\delta S^\mu_\epsilon+\partial_\nu\delta\widetilde{Q}^{\mu\nu}_\epsilon-2\partial_\nu\left(\xi^{[\nu}\Theta^{\mu]}[\Phi,\delta\Phi]\right)-\xi^\mu\partial_\nu\Theta^\nu[\Phi,\delta\Phi]\,,\\
&= \xi^\mu E\,\delta\Phi-\delta S^\mu_\epsilon+\partial_\nu\left(\delta\widetilde{Q}^{\mu\nu}_\epsilon+2\xi^{[\mu}\Theta^{\nu]}[\Phi,\delta\Phi]\right),
\end{aligned}\tag{7.42}$$

in second line we replaced $\delta\Theta^\mu[\Phi,\delta_\xi\Phi]$ from Eq. (7.40), we expressed the variation of the Lagrangian δL, and we used the Lie derivative of the tensor density Θ^μ as $\delta_\xi\Theta^\mu=\mathscr{L}_\xi\Theta^\mu=2\partial_\nu(\xi^{[\nu}\Theta^{\mu]})+\xi^\mu\partial_\nu\Theta^\nu$.

To be consistent, with differential forms we have

$$\begin{aligned}
\mathbf{\Omega}[\delta\Phi,\delta_\xi\Phi] &= \delta\mathbf{\Theta}[\Phi,\delta_\xi\Phi]-\delta_\xi\mathbf{\Theta}[\Phi,\delta\Phi]\,,\\
&= i_\xi\left(\mathbf{E}\delta\Phi+d\mathbf{\Theta}[\Phi,\delta\Phi]\right)-\delta\mathbf{S}_\xi+d(\delta\widetilde{\boldsymbol{Q}}_\xi)-d(i_\xi\mathbf{\Theta}[\Phi,\delta\Phi])-i_\xi(d\mathbf{\Theta}[\Phi,\delta\Phi])\,,\\
&= i_\xi\mathbf{E}\delta\Phi-\delta\mathbf{S}_\epsilon+d\left(\delta\widetilde{\boldsymbol{Q}}_\epsilon-i_\xi\mathbf{\Theta}[\Phi,\delta\Phi]\right).
\end{aligned}\tag{7.43}$$

In the next section, we will implement a set of conditions on the Lee-Wald symplectic structure (7.42) to find the desired lower degree conservation law we are looking for.

7.2.2 Surface Charges

Let us consider a field configuration $\bar{\Phi}$ satisfying the equations of motion

[8] In the following, we are assuming that variations commute, *i.e.* $[\delta,\delta_\xi]\Phi=0$ and, therefore, $\mathbf{\Theta}(\Phi,[\delta,\delta_\xi]\Phi)=0$. This is true if ξ is assumed to be fix on the phase space as we do here. But if besides ξ the theory has more gauge symmetry parameters involved, *e.g.* a collection $\epsilon=(\xi,\lambda^{ab},\lambda^i,\dots)$, then the term $\mathbf{\Theta}(\Phi,[\delta,\delta_\epsilon]\Phi)$ can always be decomposed as (in analogy to Eq. (7.28))

$$\mathbf{\Theta}[\Phi,[\delta,\delta_\epsilon]\Phi]=d\boldsymbol{B}_{\delta\epsilon}+\mathbf{C}_{\delta\epsilon}\,,\tag{7.41}$$

where $\mathbf{C}_{\delta\epsilon}$ collects all terms linear in the variation of the symmetry parameter and it vanishes on-shell, $\mathbf{C}_{\delta\epsilon}\approx0$. The term $\boldsymbol{B}_{\delta\epsilon}$ will affect the surface charge density as far as some of the symmetry parameter could be field dependent, $\delta\epsilon=(\delta\xi,\delta\lambda^{ab},\delta\lambda^i,\dots)\neq0$, still with $\delta\xi=0$.

$$\boldsymbol{E}[\bar{\Phi}] = 0\,. \tag{7.44}$$

Given $\bar{\Phi}$, it is also possible to study perturbations around that exact solution at linear order. These kind of perturbations are called *linearized equations of motion*, and are found at first order expansion of the equations of motion as

$$\delta\boldsymbol{E}[\bar{\Phi}] = 0\,. \tag{7.45}$$

For example, for the Einstein-Hilbert theory in vacuum, the equations of motion are given by $G_{\mu\nu} = 0$. Here the dynamical field is the metric $g_{\mu\nu}$. The linearized equations of motion are given as $G^{(l.e.o.m)}_{\mu\nu} \equiv \frac{\delta G_{\mu\nu}}{\delta g_{\alpha\beta}} \mid_{\bar{g}} \delta g_{\alpha\beta} = 0$. We can think $\frac{\delta G_{\mu\nu}}{\delta g_{\alpha\beta}} \mid_{\bar{g}}$ as a differential operator built from the background metric $\bar{g}$ which acts linearly on $g_{\alpha\beta}$. See Section 7.4.1 for the explicit form of $G^{(l.e.o.m)}_{\mu\nu}$.

Therefore, when the equations of motion and linearized equations of motion are satisfied, the symplectic structure (7.42) takes the form

$$\Omega^\mu[\bar{\Phi}, \delta\bar{\Phi}, \delta_\epsilon\bar{\Phi}] \approx \partial_\nu \left(\delta\widetilde{Q}^{\mu\nu}_\epsilon + 2\xi^{[\mu}\Theta^{\nu]}[\bar{\Phi}, \delta\bar{\Phi}]\right)\,, \qquad \left[\boldsymbol{\Omega}[\bar{\Phi}, \delta\bar{\Phi}, \delta_\xi\bar{\Phi}] \approx d\left(\delta\widetilde{\mathbf{Q}}_\epsilon - i_\xi\boldsymbol{\Theta}[\bar{\Phi}, \delta\bar{\Phi}]\right)\right]\,. \tag{7.46}$$

Remember that $\approx$ denotes the on-shell condition. The final step consists in defining, up to a total derivative, the *surface charge density*

$$k^{\mu\nu}_\epsilon[\bar{\Phi}, \delta\bar{\Phi}] \equiv \delta\widetilde{Q}^{\mu\nu}_\epsilon + 2\xi^{[\mu}\Theta^{\nu]}[\bar{\Phi}, \delta\bar{\Phi}], \qquad \left[\boldsymbol{k}_\epsilon[\bar{\Phi}, \delta\bar{\Phi}] \equiv \delta\widetilde{\boldsymbol{Q}}_\epsilon - i_\xi\boldsymbol{\Theta}[\bar{\Phi}, \delta\bar{\Phi}]\right]\,. \tag{7.47}$$

In the special case of an exact symmetry, *i.e.* specific ϵ function satisfying $\delta_\epsilon\Phi = 0$, it turns out that the surface charge density is conserved on-shell

$$\partial_\mu k^{\mu\nu}_\epsilon \approx 0\,, \qquad [d\boldsymbol{k}_\epsilon \approx 0]\,. \tag{7.48}$$

The exactness condition of the surface charge density guarantees that the value of its integration on a closed surface is independent of the choice of the surface. By integrating it over any closed surface S (*e.g.* a $(D-2)$-dimensional sphere), one obtains the *surface charge*

$$\not{\delta} Q_\epsilon[\bar{\Phi}, \delta\bar{\Phi}] = \frac{1}{2(D-2)!}\oint_S k_\epsilon^{\mu\nu}\varepsilon_{\mu\nu\alpha_3\ldots\alpha_D} dx^{\alpha_3}\wedge\cdots\wedge dx^{\alpha_D} = \oint_S \boldsymbol{k}_\epsilon[\bar{\Phi}, \delta\bar{\Phi}]\,, \tag{7.49}$$

naturally on both languages one obtains the same value for the surface charge. Note that this quantity is a differential or one-form on the phase space. The symbol $\not{\delta}$ emphasizes that the surface charge is not necessarily an exact differential on the phase space of the solutions. In other words the function Q_ϵ may not exists. A sufficient condition for the integrability of the surface charge to become a finite charge is $\delta\left(\oint_S \boldsymbol{k}_\epsilon\right) = 0$. If the condition holds, then, after an integration on the phase space it is possible to obtain the finite surface charge Q_ϵ.
It is worth to emphasize that this quantity is a *charge* because it is *conserved* only because the exactness of the symmetry guarantee the conservation law (7.48).

Properties of Surfaces Charges:

1. The surface charge density exhibits linearity in the vector field generating diffeomorphism and in all the gauge parameters, *i.e*

$$\alpha_1\boldsymbol{k}_{\epsilon_1} + \alpha_2\boldsymbol{k}_{\epsilon_2} = \boldsymbol{k}_{\alpha_1\epsilon_1+\alpha_2\epsilon_2}\,, \tag{7.50}$$

where α_1 and α_2 are arbitrary functions defined on the phase space. Thus, if $\boldsymbol{k}_{\epsilon_1}$ and $\boldsymbol{k}_{\epsilon_2}$ are closed forms for exact symmetries generated by ϵ_1 and ϵ_2, then $\boldsymbol{k}_{\epsilon_3} \equiv \boldsymbol{k}_{\alpha_1\epsilon_1+\alpha_2\epsilon_2}$ is also a closed form for the exact symmetry generated by ϵ_3 with the precise identification $\epsilon_3 = \alpha_1\epsilon_1 + \alpha_2\epsilon_2$. This linearity property of surface charges will be exploited in the examples of Section 8.

2. The charges have to be *finite*. Usually, in the literature it is common to carry out an expansion, say in r, in the symmetry parameters or make the integration *"close to infinity"*, $r \to \infty$, on the $(D-2)$-surface ∂S. For both scenarios one could find divergences in the charges. However, in the case of exact solutions and exact symmetries thereof, the theory of charges developed above does not depend on asymptotic properties of the fields near some boundary.

3. The charges have to be *integrable*. In some situations, the function $Q_{\bar{\epsilon}}$ such that its variation on the phase space satisfies $\delta Q_{\bar{\epsilon}} = \not{\delta} Q_{\bar{\epsilon}}$ may not exist. A sufficient condition for its existence is

$$\delta\left(\not{\delta} Q_{\bar{\epsilon}}\right) = 0\,, \tag{7.51}$$

or

$$\delta_1 \oint_S \boldsymbol{k}_\epsilon[\Phi, \delta_2\Phi] - \delta_2 \oint_S \boldsymbol{k}_\epsilon[\Phi, \delta_1\Phi] = 0\,, \tag{7.52}$$

for all $\delta_i\Phi, i = 1, 2$. This is the condition of integrability for the surface charge to become integrable and therefore finite. In this context, it is worth mentioning that in the context of asymptotic symmetries, integrability implies that charges form a representation of the asymptotic symmetry algebra, up to a central extension [101].

4. The charges have to be conserved in time. This corresponds to the particular and usual case when one performs the integration of $k_\epsilon^{\mu\nu}$ on a space-like $(D-2)$-dimensional surface ∂S. Note that no mention on the asymptotic regions is realized.

7.3 Equivalence between Lee-Wald and Barnich-Brandt procedures for exact symmetries

To make contact with other approaches, in this section we introduce a different definition for surface charges used in [101], and further import the comparison with the prescription presented in the previous section. We elaborate this section in differential forms language. The key of this different definition is the direct use of the quantity S_ϵ introduced in (7.28). Namely, the particular equations of motion combined with the gauge parameters that result from the use of Noether identities. In other words, the only term appearing in the trivially conserved current, $\boldsymbol{J}_\epsilon = \boldsymbol{\Theta}[\Phi, \delta_\epsilon \Phi] - \xi \lrcorner \boldsymbol{L} + \boldsymbol{S}_\epsilon$, that does not depend directly of the Lagrangian boundary term. In this approach, the surface charge integrand is expressed as [101]

$$\boldsymbol{k}'_\epsilon \equiv I_{\delta\Phi} \boldsymbol{S}_\epsilon \,, \tag{7.53}$$

where $I_{\delta\Phi}$ is called the *homotopy operator* (see Appendix A for its conventions and definitions). The homotopy operator is an efficient way to get a sensible $(p-1)$-form from an exact p-form. In particular, it can be used to select the boundary term in the Lagrangian variation

$$\delta \boldsymbol{L} = \boldsymbol{E}\delta\Phi + d\left[\boldsymbol{\Theta}'[\Phi, \delta\Phi] + d\boldsymbol{Y}\right] = \boldsymbol{E}\delta\Phi + d\left[I_{\delta\Phi}\boldsymbol{L}\right]. \tag{7.54}$$

With the risk of keeping the discussion rather abstract while brief, we just pick up the properties that allow us to understand the comparison (see [101] for a detailed definition of the homotopy operator). The defining property of the homotopy operator is its relation with a variation of fields in the space of configuration

$$\delta' \equiv dI_{\delta\Phi} + I_{\delta\Phi} d \,, \tag{7.55}$$

where d is the exterior derivative. In fact, the homotopy operator provides a prescription to define a variation on the phase space. Therefore we called it δ' to distinguish it from our previous treatment. Note the analogy with the expression of the space-time Lie derivative (7.129).

Already with this property we can prove

$$\begin{aligned} d\boldsymbol{k}'_\epsilon &= -I_{\delta\Phi} d\boldsymbol{S}_\epsilon + \delta' \boldsymbol{S}_\epsilon \,, && (7.56)\\ &= -I_{\delta\Phi}[\boldsymbol{E}\delta_\epsilon \Phi] + \delta' \boldsymbol{S}_\epsilon \,, && (7.57)\\ &= -I_{\delta\Phi}[\boldsymbol{E}]\delta_\epsilon\Phi - (-1)^{p_E}\boldsymbol{E}\ I_{\delta\Phi}[\delta_\epsilon\Phi] + \delta'\boldsymbol{S}_\epsilon \,, && (7.58)\end{aligned}$$

where we used the Noether identities $\boldsymbol{E}\delta_\epsilon\Phi = d\boldsymbol{S}_\epsilon - N_\epsilon = d\boldsymbol{S}_\epsilon$, and p_E is the form degree of $\boldsymbol{E}$, *i.e.* $I_{\delta\Phi}\boldsymbol{E} = (-1)^{p_E}\boldsymbol{E}\ I_{\delta\Phi}$. Therefore, it is shown that $\boldsymbol{k}'_\epsilon$ is closed if the equations of motion, the linearized equations of motion, and the exactness condition hold, *i.e.* $\boldsymbol{E} = 0$, $\delta\boldsymbol{E} = 0$, and $\delta_\epsilon\Phi = 0$. These conditions are exactly the ones required for the surface charge integrand defined in (7.48) to be closed! In the previous calculation we made use of the so-called *invariant* pre-symplectic structure density

$$\boldsymbol{\Omega}'[\delta_1\Phi, \delta_2\Phi] \equiv I_{\delta_{[1}\Phi}(\boldsymbol{E}\delta_{2]}\Phi) \,. \tag{7.59}$$

It differs from the pre-symplectic structure density introduced before

$$\boldsymbol{\Omega}[\delta_1\Phi, \delta_2\Phi] = \delta_1\boldsymbol{\Theta}[\Phi, \delta_2\Phi] - \delta_2\boldsymbol{\Theta}[\Phi, \delta_1\Phi] - \boldsymbol{\Theta}[\Phi, [\delta_1, \delta_2]\Phi] \,. \tag{7.60}$$

Both prescription are in general inequivalent as it is shown in the following.

The boundary term $\boldsymbol{\Theta}[\Phi, \delta\Phi]$ has an intrinsic ambiguity that can be selected with the homotopy operator (7.54), we use it to fix the ambiguity of the pre-symplectic structure density

$$\boldsymbol{\Omega}[\delta_1\Phi, \delta_2\Phi] = \delta_1'(I_{\delta_2\Phi}\boldsymbol{L}) - \delta_2'(I_{\delta_1\Phi}\boldsymbol{L})\,. \tag{7.61}$$

The use of $\delta_{1,2}'$ as defined by (7.55) ensure linearity in the variations, then to introduce the commutator term is unnecessary. Although we have selected the boundary term, there is still another intrinsic ambiguity if the Lagrangian is allowed to change by an exact form, $\boldsymbol{L} \to \boldsymbol{L} + d\boldsymbol{\alpha}$, it is in this sense that this prescription for the symplectic structure density is not *invariant*. The comparison of both pre-symplectic structure densities goes as

$$\begin{aligned} \boldsymbol{\Omega}'[\delta_1\Phi, \delta_2\Phi] &= I_{\delta_{[1}\Phi}(\boldsymbol{E}\delta_{2]}\Phi)\,, && (7.62)\\ &= I_{\delta_{[1}\Phi}(\delta_{2]}'\boldsymbol{L} - dI_{\delta_{2]}\Phi}\boldsymbol{L})\,, && (7.63)\\ &= \delta_{[2}'I_{\delta_{1]}\Phi}\boldsymbol{L} + \delta_{[1}'I_{\delta_{2]}\Phi}\boldsymbol{L} - d\left(I_{\delta_{[1}\Phi}I_{\delta_{2]}\Phi}\boldsymbol{L}\right)\,, && (7.64)\\ &= \boldsymbol{\Omega}[\delta_1\Phi, \delta_2\Phi] - d\widetilde{\boldsymbol{E}}_{1,2}\,, && (7.65) \end{aligned}$$

where we used that the *homotopy operator* satisfies $I_{\delta_1}\delta_2' = \delta_2' I_{\delta_1}$,[9] and $\widetilde{\boldsymbol{E}}_{1,2} \equiv I_{\delta_{[1}\Phi}I_{\delta_{2]}\Phi}\boldsymbol{L}$. Thus, in the case we have exact symmetries (when δ_1 or δ_2 is a gauge symmetry), $\widetilde{\boldsymbol{E}}$ vanishes and there is a match in both prescriptions. The Lee-Wald and Barnich-Brandt procedures are equivalent for exact symmetries.

It is worth to point out the differences in the prescription: $\boldsymbol{k}_\epsilon'$ and $\boldsymbol{\Omega}'[\delta_1\Phi, \delta_2\Phi]$ depend directly on the equations of motion and it is insensitive to the intrinsic ambiguities of the variational principle. On the other hand, $\boldsymbol{k}_\epsilon$ and $\boldsymbol{\Omega}[\delta_1\Phi, \delta_2\Phi]$ can be computed from standard procedures without introducing the homotopy operator. As a final remark, we note that in (7.53) we exhibited and explicit formula for the homotopy operator written for a gravity theory in tetrad-connection variables.

[9] In this notation, this property is the equivalent of equation $[d_V, I_{d_V}] = 0$ where d_V denotes *vertical derivatives* in the jet-bundle approach (see A.5 in [101]).

Example: We can illustrate this comparison for GR theory defined in (7.69). The $(D-2)$-space-time form $\widetilde{\boldsymbol{E}}[\Phi, \delta_1\Phi, \delta_2\Phi]$ for this case is given, in components, by

$$\widetilde{E}^{\mu\nu}_{1,2} = \frac{\kappa}{4}\sqrt{-g}(\delta_1 g)^{[\mu}{}_{\sigma} \wedge (\delta_2 g)^{|\sigma|\nu]}\,, \tag{7.66}$$

which taking $\delta_1 \to \delta_\xi$ and $\delta_2 \to \delta$, we get

$$\widetilde{E}^{\mu\nu}_{1,2} \longrightarrow \widetilde{E}^{\mu\nu}[g;\delta g, \delta_\xi g] = -\frac{\kappa}{2}\sqrt{-g}\left(\nabla^{[\mu}\xi_\sigma + \nabla_\sigma \xi^{[\mu}\right) \wedge (\delta g)^{|\sigma|\nu]}\,, \tag{7.67}$$

where we have used $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$. Notice that this term vanishes for an exact symmetry, $i.e$ when ξ^μ is a Killing vector.

7.4 Gravity theories with metric variables

In this section, we study the surface charges for gravity theories in the metric formalism language whose action reads

$$S[g_{\mu\nu}, \Phi] \;=\; S^{(EH)}[g_{\mu\nu}] + S^{(matter)}[g_{\mu\nu}, \Phi]\,, \tag{7.68}$$

where $S^{(EH)}[g_{\mu\nu}]$ is the Einstein-Hilbert (EH) action describing GR, and $S^{(matter)}[g_{\mu\nu}, \Phi]$ is the action describing the dynamics of any matter fields Φ.

Particularly, we focus on the derivation of surface charges in GR by studying the known family of solutions and its comparison with standard procedures of the derivation of charges.

7.4.1 Einstein-Hilbert-Λ theory

Let us consider the Einstein-Hilbert (EH) action in D-dimensional space-time with an additional cosmological constant term Λ

$$S^{(EH)}[g_{\mu\nu}] \;=\; \frac{\kappa}{2}\int_{\mathcal{M}} d^D x\sqrt{-g}\,(R-2\Lambda)\;, \tag{7.69}$$

where $\kappa = 1/(8\pi G)$ with G the Newton's constant. The variation of the metric field under an infinitesimal diffeomorphism amounts for its Lie derivative generated by the vector field ξ

$$\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \,. \tag{7.70}$$

This corresponds to the gauge symmetry of GR. We take the perspective of GR as a gauge theory in the sense that for arbitrary vector field the Lie derivative of the metric is both 1) a local transformation and 2) a symmetry of the action (*i.e.* the varied action after replacement of the symmetry becomes, at most, a boundary term). Therefore, as we know, to compute charges we should use the surface charge method.

The ingredients that we need are

$$\begin{aligned}
L &= \frac{\kappa}{2}\sqrt{-g}\,(R-2\Lambda)\,, && (7.71)\\
E^{\mu\nu} &= \frac{\kappa}{2}\sqrt{-g}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R+\Lambda g^{\mu\nu}\right), && (7.72)\\
E^{\mu\nu}_{(l.e.o.m)} &= 2\bar\nabla_\alpha\bar\nabla_{(\nu}h^\alpha{}_{\mu)}-\bar\Box h_{\mu\nu}-\bar\nabla_\nu\bar\nabla_\mu h-\bar g_{\mu\nu}[\bar\nabla_\alpha\bar\nabla_\beta h^{\alpha\beta}-\bar\Box h]+\Lambda h_{\mu\nu}\,, && (7.73)\\
\Theta^\mu(g,\delta_\xi g) &= \kappa\sqrt{-g}\nabla^{[\alpha}(g^{\mu]\beta}\delta_\xi g_{\alpha\beta})\,,\\
&= \kappa\sqrt{-g}\left(\nabla_\alpha\nabla^{(\alpha}\xi^{\mu)}-\nabla^\mu\nabla^\alpha\xi_\alpha\right),\\
&= \kappa\sqrt{-g}\left(\nabla_\alpha\nabla^{[\alpha}\xi^{\mu]}+[\nabla_\alpha,\nabla^\mu]\xi^\alpha\right),\\
&= \kappa\sqrt{-g}\left(\nabla_\alpha\nabla^{[\alpha}\xi^{\mu]}+R^{\mu\alpha}\xi_\alpha\right), && (7.74)
\end{aligned}$$

where we used the identity $[\nabla_\alpha, \nabla^\mu]\xi^\alpha = R^{\mu\alpha}\xi_\alpha$, and we defined $h_{\mu\nu} \equiv \delta g_{\mu\nu}$ and $h \equiv h^\mu{}_\mu$, and $\bar\Box \equiv \bar\nabla_\mu\bar\nabla^\mu$. Notice that we used bar notation in the linearized equations of motion $E^{\mu\nu}_{(l.e.o.m)}$ to indicate that $\bar g_{\mu\nu}$ is the dynamical field satisfying the equations of motion $E^{\mu\nu} = 0$, and where the covariant derivative $\bar\nabla_\mu$ is built using $\bar g_{\mu\nu}$. Now, we first need to compute the variation of the Lagrangian generated by an arbitrary vector field $\xi = \xi^\mu\partial_\mu$. One has, from Eq. (7.27), that

$$
\begin{aligned}
\partial_\mu(\xi^\mu L) &= E_{\mu\nu}\delta_\xi g^{\mu\nu} + \nabla_\mu \Theta^\mu(g,\delta_\xi g)\,, \\
&= -2E_{\mu\nu}\nabla^{(\mu}\xi^{\nu)} + \nabla_\mu \Theta^\mu(g,\delta_\xi g)\,, \\
&= -\nabla^{(\mu}\left(2E_{\mu\nu}\xi^{\nu)}\right) + 2\xi^{(\nu}\nabla^{\mu)}E_{\mu\nu} + \nabla_\mu \Theta^\mu(g,\delta_\xi g)\,, \\
&= -\nabla^\mu\left(2E_{\mu\nu}\xi^\nu\right) + \nabla_\mu \Theta^\mu(g,\delta_\xi g)\,, \\
&= \nabla^\mu\left(-2E_{\mu\nu}\xi^\nu + \Theta_\mu(g,\delta_\xi g)\right)\,.
\end{aligned} \tag{7.75}
$$

In the second line we replaced[10] $\delta_\xi g^{\mu\nu} = -\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu = -2\nabla^{(\mu}\xi^{\nu)}$, in the third line we made use of the Leibniz's rule, and in the fourth line we used the symmetry of $E_{\mu\nu}$ and the Noether (Bianchi) identity $\nabla^\mu E_{\mu\nu} = 0$.

As seen in the general case, Eq. (7.29), and using Eqs. (7.71)-(7.72)-(7.74), one has the trivially conserved current

$$
\begin{aligned}
J^\mu_\xi &= \Theta^\mu(g,\delta_\xi g) - \xi^\mu L - 2\xi_\nu E^{\mu\nu}\,, \\
&= \kappa\sqrt{-g}\nabla_\nu\nabla^{[\nu}\xi^{\mu]}\,, \\
&= \kappa\,\partial_\nu\left(\sqrt{-g}\nabla^{[\nu}\xi^{\mu]}\right) = \kappa\,\partial_\nu\left(-\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\right)\,.
\end{aligned} \tag{7.76}
$$

Then, the Noether potential reads

$$
\widetilde{Q}^{\mu\nu}_\xi = -\kappa\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\,, \tag{7.77}
$$

because its anti-symmetry the current is trivially conserved, $\partial_\mu\partial_\nu\widetilde{Q}^{\mu\nu}_\xi = 0$, without using the equations of motion.

Finally, the surface charge density is taking the form

[10]Notice that from the variation of the metric, $\delta_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$, and from the identity $0 = \delta(\delta^\mu_\nu) = \delta g^{\mu\alpha}g_{\alpha\nu} + g^{\mu\alpha}\delta g_{\alpha\nu}$, the variation of the inverse metric gets a minus sign.

$$
\begin{aligned}
k_\xi^{\mu\nu} &= \delta \widetilde{Q}_\xi^{\mu\nu} + 2\xi^{[\mu}\Theta^{\nu]}(g,\delta g)\,, \\
&= -\kappa\delta\left(\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\right) + 2\kappa\sqrt{-g}\xi^{[\mu}\nabla^{[\alpha}\left(g^{\nu]]\beta}\delta g_{\alpha\beta}\right)\,. \qquad (7.78)
\end{aligned}
$$

To compute the variation of the first term we use

$$
\delta(\nabla_\alpha \xi^\nu) = \delta\Gamma^\nu{}_{\alpha\gamma}\xi^\gamma = \frac{1}{2}g^{\nu\lambda}\left(\nabla_\gamma \delta g_{\alpha\lambda} + \nabla_\alpha \delta g_{\gamma\lambda} - \nabla_\lambda \delta g_{\alpha\gamma}\right)\xi^\gamma\,, \qquad (7.79)
$$

we insert this in the first term of (7.78) as

$$
\begin{aligned}
\delta\left(\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\right) &= \delta\left(\sqrt{-g}g^{[\mu|\alpha}\nabla_\alpha\xi^{\nu]}\right)\,, \qquad (7.80) \\
&= \sqrt{-g}\left(-\frac{1}{2}\delta g\nabla^{[\mu}\xi^{\nu]} + \delta g^{\sigma[\mu}\nabla_\sigma\xi^{\nu]} - \xi_\sigma\nabla^{[\mu}\delta g^{\nu]\sigma}\right)\,, \qquad (7.81)
\end{aligned}
$$

with $\delta g \equiv g_{\mu\nu}\delta g^{\alpha\beta}$, then we replace the result in (7.78). Then, the surface charge density for GR is given by the following expression[11]

$$
\mathring{k}_\xi^{\mu\nu} = \sqrt{-g}\kappa\left(\xi^{[\nu}\nabla_\sigma\delta g^{\mu]\sigma} - \xi^{[\nu}\nabla^{\mu]}\delta g + \xi_\sigma\nabla^{[\mu}\delta g^{\nu]\sigma} - \frac{1}{2}\delta g\nabla^{[\nu}\xi^{\mu]} + \delta g^{\sigma[\nu}\nabla_\sigma\xi^{\mu]}\right)\,, \qquad (7.82)
$$

where $\delta g = g_{\alpha\beta}\delta g^{\alpha\beta}$. The surface charge density does not depend on the cosmological constant term. This last is because the cosmological term in the EH action does not produce a boundary term when one integrates by parts in the derivation of the pre-symplectic structure $\Theta(g,\delta g)$. Moreover, it is linear in the symmetry vector field ξ and in the variation of the metric field $\delta g^{\mu\nu}$. Note also that $\mathring{k}_\xi^{\mu\nu}$ is anti-symmetric: An indication that the differential form language may be appropriated here.

If we assume the condition that ξ obeys an exact symmetry for the metric field, *i.e.* ξ is a Killing vector

[11] See also [98] or [105]. Note the overall sign difference. It is due to our conventions on the variation symbol δ. We assume it respects $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$.

$$\mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,, \tag{7.83}$$

then, the surface charge density satisfies a conservation law, $\partial_\mu \mathring{k}_\xi^{\mu\nu} = 0$, and therefore it is suitable to define a surface charge through the integral expression (7.49).

7.4.2 Einstein-Hilbert-Maxwell action

Maxwell theory is considered to be a fundamental theory with a domain of validity at a much larger energy scale than other effective models. In the gravity context, Maxwell potential is the simplest matter field that, coupled to GR, allows for black hole solutions [106, 107]. As we will see next in the examples, it introduces an additional term of the form $\Phi\delta Q$ in the first law of black hole mechanics that takes into account the variation in the mass of the black hole when its electric charge Q varies. In this term, Φ corresponds to the electric potential on the horizon.

The Einstein-Hilbert-Maxwell theory is described by the following action

$$S[g_{\mu\nu}, A_\mu] = \int_{\mathcal{M}} d^D x \sqrt{-g} \left(\frac{\kappa}{2} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)\,, \tag{7.84}$$

where A_μ is the electromagnetic gauge potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength.

The infinitesimal gauge transformations for this theory are

$$\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\,, \tag{7.85}$$

$$\delta_\epsilon A_\mu = \delta_{(\xi,\lambda)} A_\mu = \mathscr{L}_\xi A_\mu - \nabla_\mu \lambda' = \xi^\nu F_{\nu\mu} - \nabla_\mu \lambda\,, \tag{7.86}$$

where in addition to the diffeomorphisms on the A_μ field we must consider the $U(1)$ gauge symmetry of the electromagnetism. Note that we use $\lambda = \lambda' - \xi^\mu A_\mu$, which is the prescription for the so-called "improved gauge transformations". It will be explained better in Section 7.5.2.

We use $\epsilon = (\xi, \lambda)$ to pack all gauge symmetry parameters. This prescription is useful each time there is a gauge transformation acting on connections. The advantage is that the transformations become explicitly invariant as far as the gauge parameter λ transforms in an invariant way. To undo the prescription, a simple replacement of λ in the final formulas is enough.

The surface charge density for Einstein-Hilbert-Maxwell theory is[12] (see Appendix E.1)

$$k_\epsilon^{\mu\nu} = \mathring{k}_\xi^{\mu\nu} + \sqrt{-g}\left[\lambda\left(\delta F^{\mu\nu} - \frac{1}{2}\delta g F^{\mu\nu}\right) - \delta A_\alpha \left(\xi^\alpha F^{\mu\nu} + 2\xi^{[\mu}F^{\nu]\alpha}\right)\right] , \tag{7.87}$$

where again $\delta g \equiv g_{\mu\nu}\delta g^{\mu\nu}$.

In order to define a conserved surface charge the exact symmetry conditions must be satisfied. The conditions stand for equating the infinitesimal gauge transformations (7.85) and (7.86) to zero and solve for the parameters $\epsilon = (\xi, \lambda)$. As we know for pure gravity this is the Killing condition on ξ, but here we also need to solve λ in terms of ξ. The standard way to solve this system is first consider a (given) Killing field ξ, and then introduce it in (7.86) to solve $\lambda = \lambda(\xi)$. Note that even if there is no Killing field at all, still $\xi = 0$ and $\lambda = \lambda_0$ is a general solution. This is the origin of the electric charge in curved space-times discussed in the Introduction. This formula will be used when we study the example of the electrically charged and rotating either $(2+1)$ and $(3+1)$-dimensional black hole in the examples in Chapter 8.

7.4.3 Einstein-Hilbert-Skyrme action

As we explained in the first part of this thesis, the Skyrme theory is one of the most useful nuclear and particle physics models due to its close relationship to low energy QCD. With its various developments in particle physics, it has been interested to apply this theory to GR and astrophysics. For example, through numerical computations, the existence of spherically

[12]This formula should be compared with the results in [99], Eq. (4.22) in [108], Eq. (4.13) in [109], or recently Eq. (51) in [97]. In the explicit formulas of [108] and [109], an extra term $2F^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}$ appears, due to their different definition of the variation of the field strength with the indices up: $\delta F^{\mu\nu} \equiv g^{\mu\sigma}g^{\nu\rho}\delta F_{\sigma\rho}$.

symmetric black solutions with a non-trivial Skyrme field has been found in [8, 110], and sectors with non-vanishing topological charge where the Skyrme model has interesting consequences [111–113].

Here we consider gravity coupled to a Skyrme field $U(x^\mu)$, which is a $SU(2)$ group valued field on space-time. The Einstein-Hilbert-Skyrme action is

$$S[g_{\mu\nu}, U] = \int_{\mathcal{M}} d^D x \sqrt{-g} \left(\frac{\kappa}{2} (R - \Lambda) + \frac{K}{4} \left\langle L_\mu L^\mu + \frac{\lambda}{8} [L_\mu, L_\nu] [L^\mu, L^\nu] \right\rangle \right), \tag{7.88}$$

with L_μ defined in (3.3), and K and λ are positive coupling constants. Here we use $\langle \cdot \rangle$ to denote the trace on the $\mathfrak{su}(2)$ algebra elements.

Analogously with the previous theory there is no new gauge symmetry. Then, we must consider just the infinitesimal diffeomorphism transformations on the fields

$$\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \,, \tag{7.89}$$

$$\delta_\xi U = \mathscr{L}_\xi U = \xi^\mu \partial_\mu U \,. \tag{7.90}$$

In Appendix E.2, it is worked out the derivation of the surface charge density. The result is

$$k_\xi^{\mu\nu} = \mathring{k}_\xi^{\mu\nu} + K\sqrt{-g}\xi^{[\mu} \left\langle \left(L^{\nu]} + \frac{\lambda}{4} \left[L_\sigma, \left[L^{\nu]}, L^\sigma \right] \right] \right) U^{-1} \delta U \right\rangle . \tag{7.91}$$

Again, to have the conservation law, $\partial_\mu k_\xi^{\mu\nu} \approx 0$, the exactness symmetry condition must hold. That is, equating Eqs. (7.89) and (7.90) to zero and solving for ξ (have a Killing vector).

As an application, this is the formula one should use, within this formalism, to compute the mass/energy of a spherically symmetric black holes in the presence of a Skyrme field.

7.5 Gravity Theories in Differential Form Language

Theories of gravity are mainly studied in metric formalism, where the metric tensor $g_{\mu\nu}$ is the dynamical variable describing the gravitational field. It comes from Einstein's cornerstone idea that gravity is geometry. When Einstein was formulating his theory of gravity, the only geometry available to him was the Riemannian geometry of metric described by the tensor calculus of Ricci and Levi-Civita. However, at the same time, Élie Cartan was developing a very different type of geometry [114, 115]. The geometry can also be recast using the language of differential forms. This alternative approach is nowadays known as the *Cartan formulation* of gravity, and the metric tensor is traded for tetrads and spin connections as dynamical variables in the action principle.

The Cartan formulation of gravity, though less preferred in the literature,[13] has undoubted advantages to provide an explicit coordinate-invariant description, describe the coupling with fermionic matter fields, and study dynamical and non-dynamical torsion.

In the following, we start by presenting theories of gravity in tetrad formalism. The reader might appreciate comparing the two formalisms mentioned above and the benefits and disadvantages in either formulation. We aim at presenting the surface charge formulas for the Einstein-Cartan theory coupled to electromagnetic and matter fields. For the equivalent theories, the formulas in this section are equivalent to those derived in the previous section with the metric formalism. We comment on some features highlighted by using differential forms by deepening the direct consequences of the formulas.

In the case of a pure gravity theory and for asymptotically (anti)-de Sitter space-times, we show that the corresponding surface charge evaluated in the asymptotic region gets a very compact form, as recently noted in [117, 118]. Finally, a new result is about torsional gravity

[13] One recent exception is the work [116], where boundary charges for the Holst Lagrangian in tetrad-connection variables are analyzed.

theories: We show two relevant examples where the charges are unaffected by the presence of non-dynamical torsion fields.

7.5.1 Einstein-Cartan formalism

In the early 1900s, Einstein noted that under the assumptions of locality in space and short periods, one should not feel the effect of gravity in a freely falling laboratory. This *Gendanken-experiment* tells us at least two things: 1) the experiments carried out in gravity, under these assumptions, will be indistinguishable from those in Minkowski space.; 2) in a local neighborhood, the space-time will be ruled by Lorentz invariance. As we know, Lorentz invariance is reached by performing a coordinate transformation to a freely falling reference system. However, in the absence of gravity, one can consider an accelerated laboratory where gravitation and acceleration look the same in a small space-time region. This latter is known as *the equivalence principle.*

From the point of view of differential geometry, to have *locally inertial frames* moving in straight lines corresponds to have a flat tangent space at every point of a differentiable manifold. Here we introduce the basics of the Einstein-Cartan formulation of gravity. For a complete review we suggest [119, 120].

Let us consider the space-time as a differentiable D-dimensional manifold $\mathcal{M}$, *i.e.* one can considers the differentiable manifold $\mathcal{M}$ as smoothly connected local infinitesimal patches of $\mathbb{R}^D$. At every point $x \in \mathcal{M}$ there is a $D-$dimensional flat tangent space T_x describing, in the neighborhood of x, the manifold $\mathcal{M}$. In other words, one can define the open bijective map $\Phi : V \to U$, where $V \subset \mathbb{R}^D$ and $U \subset \mathcal{M}$. Physically, this map corresponds to a change of reference frame to that of a freely falling observer (change in space-time coordinates).

In Einstein's theory of special relativity, an observer at $x \in \mathcal{M}$ relies upon the ability of measuring lengths and angles locally. In order to do this, we must define a scalar product of vectors $\vec{v}, \vec{u} \in T_x$ such that lengths and angles are well-defined. In gravity, to this end, we need an extra ingredient: the metric. Let us define a metric $g(x)$ on $\mathcal{M}$ defined at each point in $x \in \mathcal{M}$ such that

$$\langle \vec{v}(x), \vec{u}(x) \rangle \equiv g(x)[\vec{v}(x), \vec{u}(x)] = v^{\mu}(x) g_{\mu\nu}(x) u^{\nu}(x) \,. \tag{7.92}$$

g defines a scalar product $\langle \, , \, \rangle$ between two vectors. The invariance under coordinate transformation, *e.g.* Lorentz transformation $\Lambda \in SO(1,3)$ requires that $g(x)$ transforms as

$$g_{\mu'\nu'}(x) = \Lambda^{\mu}{}_{\mu'}(x) \, \Lambda^{\nu}{}_{\nu'}(x) \, g_{\mu\nu}(x) \,. \tag{7.93}$$

Then, lengths and angles are well-defined in the standard way in terms of the scalar product, and the infinitesimal element is defined by

$$ds^2(x) = g_{\mu\nu}(x) dx^{\mu}(x) dx^{\nu}(x) \,. \tag{7.94}$$

The isomorphism between $\mathcal{M}$ and the collection of tangent vectors $\{T_x\}$ can be constructed as a coordinate transformation between a local coordinates $\{x^{\mu}\}$ in an open neighborhood in $\mathcal{M}$ and an orthonormal frame in the Minkowski space T_x with coordinates x^a. The Jacobian matrix

$$e^{a}{}_{\mu}(x) \equiv \frac{\partial x^a}{\partial x^{\mu}} \,, \tag{7.95}$$

is such that allows us construct a relation between tensors on $\mathcal{M}$ and tensors in T_x, namely if $\mathcal{T}$ is a tensor in $\mathcal{M}$, then the corresponding tensor in the tangent space T_x is given by

$$P^{a_1 \cdots a_n}(x) = e^{a_1}{}_{\mu_1}(x) \cdots e^{a_n}{}_{\mu_n}(x) \mathcal{T}^{\mu_1 \cdots \mu_n}(x) \,. \tag{7.96}$$

In particular, for the Minkowski metric $g(x) = \eta$, this isomorphism induces a metric on $\mathcal{M}$ as follows: by considering a coordinate separation dx^μ between two infinitesimally close points on $\mathcal{M}$, the corresponding separation x^a in T_x is

$$dx^a = e^a{}_\mu dx^\mu \,. \tag{7.97}$$

Then, the length in T_x given in (7.94) can also be expressed as $\eta_{ab}e^a{}_\mu(x)e^b{}_\nu(x)dx^\mu dx^\nu$, where the metric in $\mathcal{M}$ is identified as

$$g_{\mu\nu}(x) \;=\; e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab} \,. \tag{7.98}$$

Therefore, this relation tells us that all metric properties of space-time are contained in the so-called *veilbein* $e^a{}_\mu(x)$. Notice that there are many choices of veilbein that give the same metric $g_{\mu\nu}$ because there exists Lorentz invariance.
Under coordinate transformation $e^a{}_\mu(x)$ transforms as a covector

$$e^a{}_{\mu'}(x) = (J^{-1})_{\mu'}{}^\mu(x)e^a{}_\mu(x) \,, \tag{7.99}$$

while for a Lorentz transformation, the veilbein $e^a{}_\mu(x)$ transforms as a vector

$$e^{a'}{}_\mu(x) = \Lambda^{a'}{}_b(x)e^b{}_\mu(x) \,, \tag{7.100}$$

$\Lambda^a{}_b(x) \in SO(D-1,1)$. The veilbein has inverse, and is defined by

$$e_a{}^\mu(x)\, e^a{}_\nu(x) \;=\; \delta^\mu_\nu \,. \tag{7.101}$$

Now, let us introduce another important ingredient to describe Cartan geometry. In standard Riemannian geometry, the notion of parallelism of a vector is codified by the connection $\Gamma^\lambda_{\mu\nu}$, also known as *Christoffel symbol*, which roughly speaking does not belong to space-time manifold but rather it belongs to a bigger manifold called fiber bundle. We want to implement the same idea on an orthonormal basis defined by the veilbein. The problem one faces is that the derivative of the components of a vector of $SO(D-1,1)$ is *not* a vector of $SO(D-1,1)$.

As an attempt, suppose $V^a(x)$ to be a field that transforms like a vector under the Lorentz group $SO(D-1,1)$, and we will take the following ansatz for denoting its covariant derivative

$$D_\mu V^a(x) \;=\; \partial_\mu V^a(x) + \omega^a{}_{b\mu}(x) V^b(x)\,, \tag{7.102}$$

where $\omega^a{}_b(x) = \omega^a{}_{b\mu}(x)dx^\mu$ is a one-form playing the analogous role of $\Gamma^\lambda_{\mu\nu}$ in the basis manifold but now in the tangent space T_x. Under a $SO(D-1,1)$ rotation, $\omega^a{}_b(x)$ transforms as

$$\omega^a{}_{b\mu}(x) \;=\; \Lambda^a{}_c(x)\Lambda_b{}^d(x)\omega^c{}_{d\mu}(x) + \Lambda^a{}_c(x)\partial_\mu\Lambda_b{}^c(x)\,, \tag{7.103}$$

where $\Lambda_b{}^d = \eta_{ab}\eta^{cd}\Lambda^a{}_c$ is the inverse transpose of $\Lambda^b{}_d$. Due to its presence in the covariant derivative, $\omega_{ab}(x)$ is called the *Lorentz connection* and define the parallel transport of Lorentz tensors in the tangent space between T_x and T_{x+dx}. For instance, the parallel transport of a vector $V^a(x)$ from x to $x+dx$ is a vector $V^a_{||}(x)$ defined as

$$V^a_{||}(x) \;\equiv\; V^a(x) + dx^\mu D_\mu V^a(x)\,, \tag{7.104}$$

or,

$$dx^\mu D_\mu V^a(x) \;=\; V^a_{||}(x) - V^a(x)\,. \tag{7.105}$$

So far, we see that all the affine properties are encoded in the arbitrary and metric-independent connection $\omega^a{}_{b\mu}(x)$. On the other hand, we would like to find a relation between the two covariant derivative ∇_μ and D_μ. To this end, let us consider an arbitrary vector field

$$X^a = e^a{}_\mu X^\mu\,, \tag{7.106}$$

satisfying

$$D_\mu X^a = \partial_\mu X^a + \omega^a{}_{b\mu} X^b \,, \qquad\qquad \nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda \,. \tag{7.107}$$

As we see, these two derivatives have only in common the ordinary partial derivative ∂ and it seems there is no further relation to be established. To go further, let us define a full covariant derivative of a hybrid vector

$$\mathcal{D}_\mu(\cdot) \equiv \partial_\mu(\cdot) + \omega^a{}_{b\mu}\,(\cdot)^b{}_a + \Gamma^\nu_{\mu\lambda}\,(\cdot)_\nu \,. \tag{7.108}$$

Now, applying $\mathcal{D}_\mu$ on the following hybrid object $X^a = e^a{}_\mu X^\lambda$, we get

$$\mathcal{D}_\mu X^a = (\mathcal{D}_\mu e^a{}_\lambda) X^\lambda + e^a{}_\lambda \mathcal{D}_\mu X^\lambda \,. \tag{7.109}$$

For each term, the full covariant derivative $\mathcal{D}_\mu$ reduces to the covariant derivative of each factor, D or ∇, when the connection in the other factor vanishes. With this in mind, we find the relation

$$\left(\mathcal{D}_\mu e^a{}_\lambda - \partial_\mu e^a{}_\lambda - \omega^a{}_{b\mu} e^b{}_\lambda + \Gamma^\rho_{\mu\lambda} e^a{}_\rho\right) X^\lambda = 0 \,. \tag{7.110}$$

Because we supposed X to be an arbitrary vector, from this equation we find the full covariant derivative of the veilbein

$$\mathcal{D}_\mu e^a{}_\lambda = \partial_\mu e^a{}_\lambda + \omega^a{}_{b\mu} e^b{}_\lambda - \Gamma^\rho_{\mu\lambda} e^a{}_\rho \,. \tag{7.111}$$

Now, if one adds the assumption of that the metric (and its inverse) should be invariant under parallel transport, *i.e*

$$\nabla_\lambda g_{\mu\nu} = 0 \,, \qquad \nabla_\lambda g^{\mu\nu} = 0 \,, \tag{7.112}$$

using (7.98), we find

$$2\eta_{ab} (\mathcal{D}_\lambda e^a{}_\mu) e^b{}_\nu = 0 \,, \tag{7.113}$$

and because the veilbein has inverse, multiplying by ${e_c}^{\nu}$, we get

$$\mathcal{D}_{\lambda}{e^a}_{\mu} = \partial_{\mu}{e^a}_{\lambda} + {\omega^a}_{b\mu}{e^b}_{\lambda} - \Gamma^{\rho}_{\mu\lambda}{e^a}_{\rho} \;=\; 0\,. \tag{7.114}$$

This expression is so-called *veilbein postulate.* It is worth mentioning that this equation is not an extra hypothesis on the notion of parallel transport, but rather a simple relation between the connections $\Gamma^{\lambda}_{\mu\nu}$ and ${\omega^a}_{b\mu}$.
An important consequence of this postulate is that together with the metric compatibility condition (7.112) restricts to ω_{ab} to be anti-symmetric

$$\omega_{ab} \;=\; -\omega_{ba}\,. \tag{7.115}$$

These two fundamental fields: the veilbein e^a and the Lorentz connection ω_{ab} define the so-called *Cartan formalism* or *first order formalism.*

Lorentz curvature and Torsion

One of the fundamental properties of the ordinary derivative operator is that $[\partial_{\mu}, \partial_{\nu}] = 0$, *i.e.* the partial derivatives commute. However, the covariant derivative D does not hold this. In effect, it is an easy check to see

$$D^2V^a = D \wedge (dV^a + {\omega^a}_b \wedge V^b) = (d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b) \wedge V^b\,. \tag{7.116}$$

As we see, it does not involve any derivative of V^a! The term in angle brackets in the rightest expression is named the *two-form Lorentz curvature*, and is defined as

$$R^a{}_b \equiv d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b\,. \tag{7.117}$$

This verifies that both $\omega^a{}_b(x)$ and the gauge potential $A = A_\mu(x)dx^\mu$ in Yang-Mills like-theories share very similarities from the fiber bundle perspective because they are connections of a gauge group [119].

Another two-form that one can define solely involving derivatives of the veilbein is

$$De^a \;=\; de^a + \omega^a{}_b \wedge e^b \;\equiv\; T^a\,, \tag{7.118}$$

which is called *torsion*. In an explicit coordinate basis, reads

$$\partial_\mu e^a{}_\nu + \omega^a{}_{\mu b}\, e^b{}_\nu - \partial_\nu e^a{}_\mu - \omega^a{}_{\nu b} \wedge e^b{}_\mu = T^a_{\mu\nu}\,, \tag{7.119}$$

where the two-form torsion is given by $T^a = (1/2)T^a_{\mu\nu}dx^\mu \wedge dx^\nu$. The last equation can be rewritten as

$$\partial_\mu e^a{}_\nu + \omega^a{}_{\mu b}\, e^b{}_\nu = \frac{1}{2}T^a_{\mu\nu} + \frac{1}{2}P^a_{\mu\nu} = \frac{1}{2}\left(T^\lambda_{\mu\nu} + P^\lambda_{\mu\nu}\right) e^a{}_\lambda\,, \tag{7.120}$$

where $P^a_{\mu\nu}$ is an arbitrary symmetric tensor. By comparing this result with veilbein postulate (7.114) we find

$$\Gamma^\lambda_{\mu\nu} \;=\; T^\lambda_{\mu\nu} + P^\lambda_{\mu\nu}\,, \tag{7.121}$$

where by Eq. (7.118) $T^\lambda_{\mu\nu}$ is the torsion tensor, and $P^\lambda_{\mu\nu}$ is still non-defined. But, if we evoke the metric compatibility condition $\nabla_\lambda g_{\mu\nu} = 0$, or equivalently $D\eta_{ab} = 0$, we find

$$\frac{1}{2}P^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\,. \tag{7.122}$$

Let us conclude this review section with a final comment. At the end of the fifties, the seminal paper by Yang and Mills [121] generated a great interest in describing gravity as a gauge theory. In 1956, Ryoyu Utiyama showed that Einstein theory would be rewritten as a gauge theory with a gauge group: the Lorentz group [122]. This latter raised the question of whether one would be able of replicating this idea for a bigger gauge group including translations, *i.e.* the Poincaré group. However, the main problem one faces is how the Poincaré symmetry, as a gauge symmetry, looks for the dynamical fields e, ω and $g_{\mu\nu}$. In spite that gravity in four space-time dimensions cannot be developed as a gauge theory using the standard fiber bundle machinery, this does not prevent rewriting this theory elegantly in the language of differential forms.

7.5.2 Einstein-Cartan-Λ

The EH action with cosmological constant (7.69) is equivalent to the so-called Einstein-Cartan action with cosmological constant

$$S[e^a, \omega^{ab}] = \kappa' \int_{\mathcal{M}} \varepsilon_{abcd} \left(R^{ab} e^c e^d \pm \frac{1}{2\ell^2} e^a e^b e^c e^d \right), \tag{7.123}$$

where the wedge product among forms is left implicit, for instance $e^a e^b = e^a \wedge e^b = \frac{1}{2} e^a{}_\nu e^b{}_\mu dx^\nu \wedge dx^\mu = -\frac{1}{2} e^a{}_\mu e^b{}_\nu dx^\nu \wedge dx^\mu = -e^b \wedge e^a$. The coupling constants are related to the old ones by $\kappa' = \kappa/4 = 1/(32\pi G)$ and $\ell^2 = \frac{3}{|\Lambda|}$. The $\pm$ signs correspond to negative and positive cosmological constants, respectively. As expected from the metric analysis, the surface charges will not depend on the cosmological constant in the Einstein-Cartan formalism either.

In metric formalism, space-time symmetries or isometries are encoded in the Killing equation. The general wording used to refer to it on arbitrary fields is the *exact symmetry condition*. As we showed before, it is not always just a Lie derivative because when a local gauge symmetry is present, it quickly spoils the symmetry condition. The infinitesimal transformations of the fundamental fields in this formalism, the vielbein and connection, by the local Lorentz group is[14]

[14] Throughout this section we use d_ω instead of D for the covariant derivative defined by the Lorentz connection ω.

$$\delta_\lambda e^a = \lambda^a{}_b e^b\,, \tag{7.124}$$

$$\delta_\lambda \omega^a{}_b = -(d_\omega \lambda)^a{}_b = -d\lambda^a{}_b - \omega^a{}_c\lambda^c{}_b + \omega_b{}^c\lambda^a{}_c\,, \tag{7.125}$$

where $\lambda^{ab} = -\lambda^{ba}$ are the parameters of the infinitesimal Lorentz transformation $\Lambda \approx \delta^a{}_b + \lambda^a{}_b$. Remember that it is a gauge symmetry; the group elements take different values at different points on the manifold $\mathcal{M}$.

Improved gauge transformations

The infinitesimal transformations of the fields due to diffeomorphisms are normally assumed to be generated by an arbitrary vector field

$$\widetilde{\delta}_\xi e = \mathscr{L}_\xi e = d(\xi \lrcorner e) + \xi \lrcorner (de)\,, \tag{7.126}$$

$$\widetilde{\delta}_\xi \omega = \mathscr{L}_\xi \omega = d(\xi \lrcorner \omega) + \xi \lrcorner (d\omega)\,, \tag{7.127}$$

where in the second equality we use the Cartan's formula: $\mathscr{L}_\xi e^a = d(\xi \lrcorner e^a) + \xi \lrcorner (de^a)$. However, note that they are not homogeneous under local Lorentz transformation due to the presence of exterior derivatives. The intuitive interpretation of $\widetilde{\delta}_\xi e$ and $\widetilde{\delta}_\xi \omega$ as infinitesimal variation require them to be homogeneous under the action of the local Lorentz group. More precisely, if we attach ourselves to the intuitive idea of variations as a comparison of fields in a neighborhood, $\delta e \approx e' - e$, we expect them to have a covariant transformation under the local Lorentz group. This criterion is not satisfied by the infinitesimal diffeomorphism transformation presented before, and therefore we correct Eqs. (7.126) and (7.127) by eliminating the non-homogeneous part. This can be done by adding an infinitesimal Lorentz transformation with a parameter $\xi \lrcorner \omega$. This corrects the non-homogeneous part of both transformations at once, and we get

$$\delta_\xi e = \mathscr{L}_\xi e + \delta_{\xi \lrcorner \omega} e = d_\omega(\xi \lrcorner e) + \xi \lrcorner (d_\omega e)\,, \tag{7.128}$$

$$\delta_\xi \omega = \mathscr{L}_\xi \omega + \delta_{\xi \lrcorner \omega} \omega = \xi \lrcorner R\,. \tag{7.129}$$

Another way to think about this, is that in the transformation of the tetrad, the exterior derivative d is promoted to a covariant exterior derivative d_ω, while in the transformation of the Lorentz connection, because of the identity $d(\xi \lrcorner \omega) + \xi \lrcorner d\omega = d_\omega(\xi \lrcorner \omega) + \xi \lrcorner R$, the ill-transforming part, $d_\omega(\xi \lrcorner \omega)$, is subtracted. This kind of Lie derivatives are also called *the Lie-Lorentz derivative* which implements the adequate compensating local Lorentz for each general coordinate transformation [123–125]. In simple terms, it is just a Lorentz-covariant Lie derivative.

By grouping the infinitesimal parameters as $\epsilon = (\xi, \lambda)$, such that $\delta_\epsilon \equiv \mathscr{L}_\xi + \delta_{\lambda + \xi \lrcorner \omega}$, the exact symmetry condition that is gauge invariant is

$$\delta_\epsilon e^a = d_\omega(\xi \lrcorner e^a) + \xi \lrcorner (d_\omega e^a) + \lambda^a{}_b e^b = 0\,, \tag{7.130}$$

$$\delta_\epsilon \omega^{ab} = \xi \lrcorner R^{ab} - d_\omega \lambda^{ab} = 0\,, \tag{7.131}$$

which in fact can be understood as a Lie derivative on forms plus a specific infinitesimal Lorentz transformation,[15] $\delta_{\lambda'} e^a = \lambda'^a{}_b e^b$ generated by the field dependent parameter $\lambda'^{ab} = \lambda^{ab} + \xi \lrcorner \omega^{ab}$.

This combination of infinitesimal transformation is just a convenient prescription, sometimes called *improved transformation*, and it has the advantage of being homogeneous under local Lorentz transformations, $\delta_\epsilon(\Lambda^a{}_b e^b) = \Lambda^a{}_b\, \delta_\epsilon e^b$, which is crucial to keep the local Lorentz gauge symmetry explicitly free while imposing the exact symmetry. If we do not do this, a local Lorentz transformation will change the Killing equation, and one has to keep track of the extra piece in all formulas. A simple analysis shows that the exact symmetry condition $\delta_\epsilon e^a = 0$, an on-shell condition, implies the usual Killing equation on the metric field

[15] We use the notation $\xi \lrcorner = i_\xi$ for the interior product, for instance $\xi \lrcorner e^a = \xi^\mu e^a{}_\mu$.

$$\mathscr{L}_\xi g = \mathscr{L}_\xi e^a \otimes e_a + e^a \otimes \mathscr{L}_\xi e_a = \delta_\epsilon e^a \otimes e_a + e^a \otimes \delta_\epsilon e_a = 0\,, \tag{7.132}$$

where we used that $g = g_{\mu\nu}(dx^\mu \otimes dx^\nu) = \eta_{ab}(e^a \otimes e^b)$ with $\otimes$ the symmetric tensorial product, and the anti-symmetry of λ^{ab} makes that $\delta_\lambda e^a \otimes e_a = \lambda^{ab}\, e_b \otimes e_a = 0$. Note also that if the connection can be expressed in terms of the vielbein, $\omega^{ab}(e)$, the condition $\delta_\epsilon \omega^{ab} = 0$ is a trivial consequence of $\delta_\epsilon e^a = 0$. And note also that the equation is linear and thus it is straightforward to solve for the parameter λ^{ab}. In fact, we find

$$\lambda^{ab} = e^a \lrcorner (d_\omega(\xi \lrcorner e^b))\,, \tag{7.133}$$

which is equivalent to

$$\lambda^{ab} = e^{a\mu} e^{b\nu} \nabla_{[\mu}\xi_{\nu]}\,, \tag{7.134}$$

where the anti-symmetry in the Greek indexes is an explicit consequence of the Killing equation $\nabla_{(\mu}\xi_{\nu)} = 0$.

Now, let us return to the derivation of the surface charge density. For the details we refer to the Appendix E.3, where the surface charge density for this theory is worked out step−by−step. The final result is simply (see [126, 127])

$$\mathring{k}_\epsilon = -\kappa' \varepsilon_{abcd} \left(\lambda^{ab} \delta(e^c \wedge e^d) - \delta\omega^{ab} \xi \lrcorner (e^c \wedge e^d) \right)\,, \tag{7.135}$$

or equivalently

$$\mathring{k}_\epsilon = -2\kappa' \varepsilon_{abcd} \left(\lambda^{ab} \delta e^c - \delta\omega^{ab} \xi \lrcorner e^c \right) \wedge e^d\,. \tag{7.136}$$

Thus, as mentioned earlier, it does not depend explicitly on the cosmological constant. In particular, if there is no cosmological constant term in the action, the surface charge density formula is the same. We explicitly put the wedge product among differential forms again to emphasize that this expression is a two-form in space-time.

The previous formula can be used to define charges in wherever space-time region the exact symmetry condition holds! Suppose the space-time is assumed to have an exact symmetry with parameters ϵ defined on a limited region in that region. In that case, the surface charge density will satisfy $d\mathring{k}_\epsilon \approx 0$, and therefore, a surface charge may be defined. In particular, for exact solutions with exact symmetries, like simple black hole solutions, the surface charge defines charges quasi-locally: an asymptotic analysis *is not* needed, and the charge is defined at any two-surface enclosing the black hole. This property makes surface charge a useful tool to compute the charges in space-times with complicated asymptotic structures.

As a final comment, let us mention again that the cosmological constant term does not affect the surface charge. This fact is just because it does not contribute to the pre-symplectic structure. A similar phenomenon, but not equal, happens for any addition of boundary terms in the Lagrangian and topological terms. This result will be analyzed in detail in the next section.

7.5.3 Topological terms effect on surface charges

As we mentioned in Section 7.2, any boundary term added to the Lagrangian does not affect the surface charge formula for exact symmetries. This is a remarkable property that is in high contrast with usual Noether procedures to compute charges (see, for instance, the recent review [128]). To see how this happens let us add a boundary term to the Lagrangian: $L \to L + d\alpha$, with the assumption that α is gauge invariant, $\delta_\epsilon \alpha = \mathscr{L}_\xi \alpha$. Now we can repeat the procedure of Section 7.2 by keeping track of this boundary term. We have $J_\epsilon \to J_\epsilon + d(\xi \lrcorner \alpha)$, and the surface charge density acquires the additional terms

$$k_\epsilon \quad \longrightarrow \quad k_\epsilon + \delta(\xi \lrcorner \alpha) - \xi \lrcorner \delta\alpha \,. \tag{7.137}$$

If we use that $\delta\xi = 0$, thus, the terms proportional to α in the last expression vanish, and therefore no change at all for the surface charges.

The previous case is quite general. Let us take a few examples. For example in $D = 4$ the Nieh-Yan topological term, given by

$$\chi^{(4)}_{NY} = \int_{M^{(4)}} \Big(T_a \wedge T^a - R^{ab} \wedge e_a \wedge e_b \Big) \,, \tag{7.138}$$

is in this category. However, there are examples where α is not gauge invariant. This is the case for the Euler or the Pontryaguin topological terms, whose densities are, respectively, given by

$$\chi^{(4)}_{E} = \int_{M^{(4)}} \epsilon_{abcd}\, R^{ab} \wedge R^{cd} \,, \qquad \chi^{(4)}_{P} = \int_{M^{(4)}} R^{a}{}_{b} \wedge R^{b}{}_{a} \,. \tag{7.139}$$

Both are exterior derivatives of Chern-Simons Lagrangian which are gauge quasi-invariant forms [120]. The contribution of the Euler term to the surface charge density expression explicitly yields

$$k^{(E)}_{\epsilon} = -2\kappa \left(\lambda \star \delta R - \delta\omega \star \xi \lrcorner R \right) = -2\kappa \, \left(d(\lambda \star \delta\omega) + \delta_{\epsilon}\omega \star \delta\omega \right) \,. \tag{7.140}$$

The first term vanishes because is an exact form and disappears when the density is integrated on a smooth boundary $\partial\Sigma = S$ of a manifold, and the second factor vanishes by the exactness symmetry condition on $\delta_{\epsilon}\omega = 0$. A similar computation for the Pontryaguin $\chi^{(4)}_{P}$ yields

$$k^{(P)}_{\epsilon} = -2\kappa \left(\lambda \delta R - \delta\omega \xi \lrcorner R \right) = -2\kappa \left(d(\lambda\delta\omega) + \delta_{\epsilon}\omega\delta\omega \right) \,, \tag{7.141}$$

where in the rightest side of the last expression: the first term does not contribute to the conservation law $dk_{\epsilon} \approx 0$ (because $d^2 = 0$); and the second one vanishes due to the exact symmetry condition $\delta_{\epsilon}\omega = 0$. Then, surface charges are blind to the Pontryaguin topological term too.

Finally, in $D = 4$ we may also be interested in using the Holst term density

$$\chi_H^{(4)} = \int_{M^{(4)}} e_a \wedge e_b \wedge R^{ab}\,, \tag{7.142}$$

inside the gravity action. This is not a topological term by itself but a part of the Nieh-Yan term, and it does not affect the equations of motion either. To deal with it note that $e_a \wedge e_b \wedge R^{ab} = T^a \wedge T_a - d(e^a \wedge T_a)$. The second term was already studied, then, it is enough to keep track of $T^a \wedge T_a$ in the computation of surface charges potential. This term also does not produce any changes because already at the level of the pre-symplectic structure density, $\Omega(\delta, \delta_\epsilon)$, the contributions are all proportional to the torsion T^a and therefore vanish when the on-shell condition holds.

Then, boundary terms, and in particular topological terms, do not affect the surface charges. Note that this is already explicit for surface charges computed through the contracting homotopy operator (7.53) because it depends only of $S_{\bar{\epsilon}}$ and not of the Lagrangian. In this sense here we have stressed what is already indirectly known due to the fact that surface charges obtained through both methods are equivalent (see section 7.3).

7.5.4 Einstein-Cartan-Maxwell

The electromagnetic field is described by the one-form potential A and the field strength is simply the exterior derivative of the potential, $F = dA$. The Einstein-Cartan action coupled to the electromagnetic field is

$$S[e^a, \omega^{ab}, A] = \int_{\mathcal{M}} \left(\kappa' \varepsilon_{abcd} R^{ab} e^c e^d + \alpha\, F \star F\right), \tag{7.143}$$

with $\alpha = -1/2$. The coupling with the vielbein field in the second term is through the Hodge operator $\star$. Explicit components on the frame field are $F = \frac{1}{2} F_{ab} e^a e^b$, and the Hodge dual is $\star F = \frac{1}{4} \varepsilon_{abcd} F^{ab} e^c e^d$ (see Appendix 1 for conventions).

To impose the exact symmetry conditions we still should impose the equations $\delta_\epsilon e^a = 0$

and $\delta_\epsilon \omega^{ab} = 0$ as in (7.130) and (7.131), but now we have the field A with its own extra gauge symmetry. The corresponding infinitesimal gauge transformation is $\delta_{\lambda'} A = -d\lambda'$. Therefore, besides the two previous conditions, we should add a third exact symmetry condition directly on the potential, namely

$$\delta_\epsilon A = \xi \lrcorner F - d\lambda = 0 \, . \tag{7.144}$$

Once more for the symmetry condition, we use the improved transformation or the Lie-Maxwell derivative that is the composition of a Lie derivative and a particular $U(1)$ gauge transformation with parameter $\lambda' = \lambda + \xi \lrcorner A$; explicitly

$$\delta_\epsilon A = \mathscr{L}_\xi A + \delta_{(\lambda + \xi \lrcorner A)} A = d(\xi \lrcorner A) + \xi \lrcorner dA - d(\lambda + \xi \lrcorner A) \, . \tag{7.145}$$

In this set up we group together all the parameters as $\epsilon = (\xi, \lambda^{ab}, \lambda)$. Notice that A does not transform with local Lorentz, *i.e* $\delta_{\lambda^{ab}} A = 0$. Likewise, e^a or ω^{ab} do not change with $U(1)$ gauge transformation.

Then, the surface charge density is the sum of $\mathring{k}_\epsilon$, from (7.135), and an extra electromagnetic piece

$$k_\epsilon^{ECM} = \mathring{k}_\epsilon - 2\alpha \left(\lambda \, \delta \star F - \delta A \, \xi \lrcorner \star F \right) \, . \tag{7.146}$$

The derivation is presented in Appendix E.4 for a general Yang-Mills theory. The surface charge for gravity coupled with extra fields is then given by the surface charge of the pure gravity plus the contributions from the additional fields. This is because these additional fields enter the boundary term $\Theta(\delta)$ in a linear way. An exception to this structure is for non-linear couplings to gravity (see, for instance, the conformally coupled scalar field studied in [123]).

Particularly, we mention that the exact symmetry condition $\delta_\lambda A = 0$ is solved for $\lambda = \lambda_0 = cte$, such that the gauge symmetry turns into a rigid symmetry. Note that here the exact condition is independent of the fields and admits a general solution. Then, we have

$$dk_{\lambda_0} \approx 0\,, \tag{7.147}$$

which can be integrated in a three-dimensional space-like surface Σ enclosed by a two-dimensional space-like surface S

$$\int_{\Sigma} dk_{\lambda_0} = \oiint_{S} k_{\lambda_0} = 0\,, \tag{7.148}$$

where we have used the Gauss' theorem in the first equality. It defines

$$\not{\delta} Q_{\lambda_0} = \frac{\lambda_0}{4\pi} \oiint_{S} \star\, \delta F\,, \tag{7.149}$$

where we have restored the value of α, and where the overall factor $1/4\pi$ is due to that S has been considered a sphere. For simplicity, the parameter λ_0 is chosen to be a constant in the phase space. Then, the variation can be trivially removed by integration on phase space. The integration constant coming from this operation can be fixed to zero by demanding that the fields vanish at infinity. Then, we obtain the definition of the electric charge

$$Q_{\lambda_0} = \frac{\lambda_0}{4\pi} \oiint_{S} \star F\,, \tag{7.150}$$

The conservation $dk_{\lambda_0} \approx 0$ ensures that for any other surface, say S', obtained by a continuous deformation of S, the electric charge is the same. If there are no sources S can be contracted to a point and all charges are zero. This finishes the analysis of surface charges for Einstein-Cartan-Maxwell theory.

7.5.5 Einstein-Cartan with Torsion: Two Examples

In the Einstein-Cartan theory, as a first-order formulation, the connection ω^{ab} is an independent variable. Therefore, in this frame, the torsion does not vanish in general. It is enough to have a source in the corresponding field equation to turn on the torsion field. While being a natural option to consider for the geometry of real space-times, so far, the torsion seems an elusive feature of physical space-time, and there is no experimental indication for it at the moment. Still, it is not ruled out, and thus for the sake of generality, it is worth studying.

Here, in particular, we wonder in the following questions:

How is that surface charges get modified with the presence of torsion in space-time? How the torsion affects the space-time charges?

We do not have a general answer. However, by studying two particular and quite different theories, the conclusion for both of them is that the torsion field seems *not* to affect the general formula for the charges.

The role of torsion in the following charge formulas is analogous with the role played by the cosmological constant term: Although present in gravity theories, explicitly modifying the field equations and their respective solutions, it does not appear as a direct contributing term into surface charges.

The first theory we consider is a simple, pure gravity example in $(2+1)$-space-time where torsion is sourced without adding extra fields. The term, first introduced by Mielke and Baekler in [129], is built from the same gravity fields. In a second example, now in $(3+1)$-dimensional space-time, we consider the well-known Einstein-Cartan-Dirac theory where Dirac spinors source torsion.

For both theories, we find the remarkable fact that torsion is explicitly absent from the charge formula (this result may be contrasted with the recent work [130]). It remains an open problem to specify under which conditions the torsion does not affect the charges.

Einstein-Cartan in $(2+1)$-dimensions plus a Torsion Term

Consider the gravitational action in three dimensions

$$S[e^a, \omega^{ab}] = \int_{\mathcal{M}} \left(\varepsilon_{abc} e^a \wedge R^{bc} + \beta\, e_a \wedge T^a \right) , \qquad (7.151)$$

for simplicity we set the overall parameter to one and introduce β as the coupling constant for the new term, where $T^a = d_\omega e^a$. As a matter of fact, $e_a \wedge T^a$ cannot be written as a boundary term. Furthermore, it produces a source for torsion, as can be checked in the equations of motion (E.43). This action can be seen as a sector of the more general Mielke-Baekler model [129] which contains two more terms: A cosmological constant term and a Chern-Simons term built with ω^{ab}. A general analysis of the surface charges for the Mielke-Baekler model is straightforward, but this additional torsional term is enough for our purposes. An even more elegant perspective can be done by writing the full Mielke-Baekler model as a Chern-Simons theory and use the results of the Chern-Simons Section 7.5.6 below.

Because the theory depends only on vielbein and spin connection fields, the corresponding exact symmetry conditions are just the same as before, (7.130) and (7.131), which are valid for any dimension. The computation of the surface charge density is straightforward and is done in full detail in Appendix E.5. It reads

$$k_\epsilon \;=\; -\varepsilon_{abc}(\lambda^{ab}\delta e^c - \delta\omega^{ab}\xi \lrcorner e^c) + 2\beta \xi \lrcorner e^a \delta e_a \,. \qquad (7.152)$$

Note that the first two terms have a similar structure than the four dimensional case Eq. (7.135), this is not casual (see a discussion about the general formula in arbitrary dimensions, for Lovelock-Cartan theories in [123]).

As explained in Appendix E.5 we can go further and split $\omega^{ab} = \tilde{\omega}^{ab} + \bar{\omega}^{ab}$, such that the torsionless part of the connection satisfies $d_{\tilde{\omega}} e^a = de^a + \tilde{\omega}^a{}_b e^b = 0$, and we can use the equations of motion to solve algebraically the contorsion $\bar{\omega}^{ab}$. With this, the surface charge formula simplifies to

$$k_\epsilon = -\varepsilon_{abc}(\tilde{\lambda}^{ab}\delta e^c - \delta\tilde{\omega}^{ab}\xi \lrcorner e^c) \,, \qquad (7.153)$$

where $\tilde{\lambda}^{ab} = e^a \lrcorner (d_{\tilde{\omega}}(\xi \lrcorner e^b))$ is the parameter that solves the exact symmetry condition with just the torsionless part of the connection, (E.52). Then, (7.153) shows that the contorsion is absent from the seed formula for the charges, $i.e$ before computing it for any specific solution and even without use any kind of symmetry. This is not a fact due to exact symmetry. As we will see in a moment, this is also true in a completely different theory.

Einstein-Cartan-Dirac

The action for the massless Einstein-Cartan-Dirac theory in four space-time dimensions is

$$S[e^a, \omega^{ab}, \psi] = \int_{\mathcal{M}} \varepsilon_{abcd}\, e^a \wedge e^b \wedge \left[\kappa' R^{cd} - \frac{i}{3}\alpha_\psi\, e^c \wedge \left(\bar{\psi}\gamma^d\gamma_5 d_\omega\psi + \overline{d_\omega\psi}\gamma^d\gamma_5\psi\right)\right], \tag{7.154}$$

with α_ψ the coupling parameter and the γ-matrices satisfying $\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$. The covariant derivative acting on spinors is

$$d_\omega\psi = d\psi + \frac{1}{2}\omega_{ab}\gamma^{ab}\psi\,, \qquad \overline{d_\omega\psi} = d\overline{\psi} - \frac{1}{2}\omega_{ab}\gamma^{ab}\overline{\psi}\,, \tag{7.155}$$

with $\gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b]$. The special matrix $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ satisfies $\gamma_5\gamma_a = -\gamma_a\gamma_5$, and we use the bar to denote the complex conjugate.

Besides the exact symmetry condition on e^a and ω^{ab} given by (7.130) and (7.131), we need to impose the exact symmetry condition directly on the spinor field

$$\delta_\epsilon\psi = \mathscr{L}_\xi\psi + \lambda'\psi = \xi \lrcorner d_\omega\psi + \lambda\psi = 0\,, \tag{7.156}$$

where again we used an improved version with the algebra valued parameter $\lambda' = \frac{1}{2}\lambda'^{ab}\gamma_{ab} = \frac{1}{2}(\lambda^{ab} + \xi \lrcorner \omega^{ab})\gamma_{ab}$. Remember that the gauge symmetry for spinors is the local Lorentz symmetry, then $\epsilon = (\xi, \lambda^{ab})$. Spinors change under an infinitesimal Lorentz transformation as $\delta_{\lambda'}\psi = \lambda'\psi$, with the algebra valued parameter $\lambda' = \frac{1}{2}\lambda'^{ab}\gamma_{ab}$. Hence, in this section λ' without indices is a matrix.

For the surface charge density the calculations are long, details are in Appendix E.7. The result is

$$k_\epsilon = \mathring{k}_\epsilon - i\alpha_\psi\, \varepsilon_{abcd}\, \xi\lrcorner e^a e^b e^c \delta\left(\bar{\psi}\gamma^d\gamma_5\psi\right)\,, \tag{7.157}$$

which is again a simple modification of the Einstein-Cartan surface charge density produced by the spinor field. The new term comes directly from the spinor contribution to the boundary term in the varied action.

As we saw in the previous section we can go further if we consider the splitting $\omega^{ab} = \tilde{\omega}^{ab} + \bar{\omega}^{ab}$ such that $\tilde{\omega}^{ab}$ is the torsionless part of the connection and the contorsion field $\bar{\omega}^{ab}$ is solved from the equations of motion. Replacing this back we find a cancellation to simply get (see Appendix E.7)

$$k_\epsilon \;=\; -\kappa'\varepsilon_{abcd}\left(\widetilde{\lambda}^{ab}\delta(e^c e^d) - \delta\tilde{\omega}^{ab}\xi\lrcorner(e^c e^d)\right)\,, \tag{7.158}$$

where again $\widetilde{\lambda}^{ab} = e^a\lrcorner(d_{\widetilde{\omega}}(\xi\lrcorner e^b))$. This surface charge density is exactly $\mathring{k}_\epsilon$ but using on it the Levi-Civita connection, $\widetilde{\omega}^{ab}(e)$, instead of the general connection ω^{ab}. Therefore, the conclusion for the Einstein-Cartan-Dirac theory is the same, contorsion leaves *no* trace on charges.

7.5.6 Chern-Simons action

In this section we discuss the surface charge formula for the Chern-Simons (CS) theory in $(2+1)$-space-time dimensions. CS theories are quasi gauge invariant theories only defined in odd dimensions which allows describe elegantly gravitational actions with promising quantum versions.

In $D = 3$ space-time dimensions the CS action is given by

$$S[A] \;=\; \kappa_{CS}\int_{\mathcal{M}}\left\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right\rangle, \tag{7.159}$$

the one-form gauge connection A is valued on the Lie algebra defining the theory, $\langle\cdot\rangle$ denotes a group invariant symmetric polynomial here of rank $r = 2$, which in this dimension is also named the bilinear form; and κ_{CS} the *level* of the theory which is not relevant for the classical analysis. Under a gauge transformation the CS action (7.159) is not invariant but quasi-invariant because it produces a boundary term, this is of course still a gauge symmetry of the theory as far as the equations of motion are concerned. In the following, we consider diffeomorphisms and gauge symmetries and group them in $\epsilon = (\xi, \lambda)$, with ξ a vector field and λ a Lie algebra valued gauge parameter. The general infinitesimal symmetry transformation reads

$$\delta_\epsilon A = \mathscr{L}_\xi A - d_A \lambda' = \xi \lrcorner F - d_A \lambda \,, \tag{7.160}$$

notice that we use the exterior covariant derivative $d_A(\cdot) \equiv d(\cdot) + [A, (\cdot)]$ and define a displaced parameter as $\lambda = \lambda' - \xi \lrcorner A$ to work directly with the improved general transformation. Now, the variation of the action produces the equation of motion $F = dA + A \wedge A = 0$ which holds only if one gets rid of the boundary term given by the potential $\Theta(\delta A) = \kappa_{CS} \langle A \wedge \delta A \rangle$. The symplectic structure density is simply $\Omega(\delta_1, \delta_2) = 2\kappa_{CS} \langle \delta_1 A \wedge \delta_2 A \rangle$, which we evaluate with one of its entries on the gauge symmetry transformation (7.160)

$$\Omega(\delta, \delta_\epsilon) = 2\kappa_{CS} \langle \delta A \wedge (\xi \lrcorner F - d_A \lambda) \rangle = 2\kappa_{CS}\, d \langle \delta A \lambda \rangle \,, \tag{7.161}$$

where to get second equality we used the equation of motion, $F = 0$ and the linearized equation of motion $\delta F = d_A \delta A = 0$. Hence, for the CS theory in $D = 3$ we have the surface charge density

$$k_\epsilon^{CS} = 2\kappa_{CS} \langle \lambda \delta A \rangle \,. \tag{7.162}$$

This simple formula covers all CS theories in $D = 3$ dimensions in the sense that the algebra of the theory is not specified yet.[16] In particular we can choose the Poincaré or (anti-)de Sitter group to obtain the surface charge formula for general relativity in $(2+1)$-dimensions (as we do in the first torsion example).

The previous derivation is a particular case of the more general derivation for CS theory in $D = 2n + 1$ dimensions. The details of the general calculation are explained in Appendix E.8. The general result for the surface charge density is

$$k_{\epsilon}^{(2n+1)} = n(n+1)\kappa_{CS} \left\langle \lambda \delta A \wedge F^{n-1} \right\rangle , \qquad (7.163)$$

which could probably had been guessed, in fact we note there is also a very direct computation to get this result by using the contracting homotopy operator, see Appendix E.8.1. The infinitesimal symmetry transformation for the connection are the same (7.160) and the actions for these theories are compactly written in equation (E.92).

We conclude this chapter with a summary table of all formulas found for the surface charge densities corresponding to each theory described previously and additional ones studied in [123]. Moreover, we choose three family solutions as examples to show the complete scheme in full detail for three different gravity formulations.

[16]This formula coincides with [131] when diffeomorphisms are considered only, *i.e.* setting $\lambda = -\xi \lrcorner A$ in Eq. (7.162).

Theory	**Action**	**Surface charge density**
Einstein-Hilbert (EH)	$S^{(EH)}=\frac{\kappa}{2}\int_{\mathcal{M}} d^Dx\sqrt{-g}\,(R-2\Lambda)$	$\mathring{k}^{\mu\nu}_{\xi}=\sqrt{-g}\kappa\left(\xi^{[\nu}\nabla_{\sigma}\delta g^{\mu]\sigma}-\xi^{[\nu}\nabla^{\mu]}\delta g+\xi_{\sigma}\nabla^{[\mu}\delta g^{\nu]\sigma}-\frac{1}{2}\delta g\nabla^{[\nu}\xi^{\mu]}+\delta g^{\sigma[\nu}\nabla_{\sigma}\xi^{\mu]}\right)$
EH-Maxwell	$S^{(EH)}-\frac{1}{4}\int_{\mathcal{M}} d^Dx\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$	$\mathring{k}^{\mu\nu}_{\xi}+\sqrt{-g}\left[\lambda\left(\delta F^{\mu\nu}-\frac{1}{2}\delta gF^{\mu\nu}\right)-\delta A_{\alpha}\left(\xi^{\alpha}F^{\mu\nu}+2\xi^{[\mu}F^{\nu]\alpha}\right)\right]$
EH-Φ	$S^{(EH)}-\frac{1}{2}\int_{\mathcal{M}} d^Dx\sqrt{-g}\left(\partial^{\mu}\Phi\partial_{\mu}\Phi-\zeta_{D}R\Phi^2\right)$	$\mathring{k}^{\mu\nu}_{\xi}\left(1+\frac{\zeta_D}{\kappa}\Phi^2\right)+\zeta_D\sqrt{-g}\xi^{[\mu}\left(\delta g^{\nu]\alpha}\nabla_{\alpha}\Phi^2-\nabla^{\nu]}\delta\Phi^2-\frac{2}{\zeta_D}\delta\Phi\nabla^{\nu]}\Phi\right)$
EH-Skyrme	$S^{(EH)}+\frac{K}{4}\int_{\mathcal{M}} d^Dx\sqrt{-g}\left\langle R^{\mu}R_{\mu}+\frac{\lambda}{8}F_{\mu\nu}F^{\mu\nu}\right\rangle$	$\mathring{k}^{\mu\nu}_{\xi}+K\sqrt{-g}\xi^{[\mu}\left\langle\left(R^{\nu]}+\frac{\lambda}{4}[R_{\sigma},F^{\nu]\sigma}]\right)U^{-1}\delta U\right\rangle$
Lanczos-Lovelock	$\int_{\mathcal{M}} d^Dx\sqrt{-g}\sum_{m=0}^{[(D-1)/2]}c_mL_m$	$\sqrt{-g}\left(\delta gP^{\mu\nu\alpha}{}_{\beta}\nabla_{\alpha}\xi^{\beta}-2\delta P^{\mu\nu\alpha}{}_{\beta}\nabla_{\alpha}\xi^{\beta}+2P^{\mu\nu\alpha}{}_{\beta}\xi_{\gamma}\nabla_{\alpha}\delta g^{\gamma\beta}-4\xi^{[\mu}P^{\nu]}{}_{\alpha\beta}{}^{\gamma}\nabla_{\gamma}\delta g^{\alpha\beta}\right)$
Einstein-Cartan (EC)	$S^{(EC)}=\kappa'\int_{\mathcal{M}}\varepsilon_{abcd}\left(R^{ab}e^ce^d\pm\frac{1}{2\ell^2}e^ae^be^ce^d\right)$	$\mathring{k}_{\epsilon}=-\kappa'\varepsilon_{abcd}\left(\lambda^{ab}\delta(e^ce^d)-\delta\omega^{ab}\xi\lrcorner(e^ce^d)\right)$
EC-Maxwell	$S^{(EC)}+\alpha\int_{\mathcal{M}}F\star F$	$\mathring{k}_{\epsilon}-2\alpha\left(\lambda\delta\star F-\delta A\xi\lrcorner\star F\right)$
EC-Yang-Mills	$S^{(EC)}+\alpha_{YM}\int_{\mathcal{M}}\langle F\star F\rangle$	$\mathring{k}_{\epsilon}-2\alpha_{YM}\left(\lambda^i\delta\star F^i-\delta A^i\xi\lrcorner\star F^i\right)$
Torsional $(2+1)$-EC	$\int_{\mathcal{M}}\left(\varepsilon_{abc}e^aR^{bc}+\beta e_aT^a\right)$	$-\varepsilon_{abc}(\tilde{\lambda}^{ab}\delta e^c-\delta\tilde{\omega}^{ab}\xi\lrcorner e^c)$
EC-Dirac	$S^{(EC)}-\frac{i}{3}\alpha_{\psi}\int_{\mathcal{M}}\varepsilon_{abcd}e^ae^be^c\left(\bar{\psi}\gamma^d\gamma_5d_{\omega}\psi+\overline{d_{\omega}\psi}\gamma^d\gamma_5\psi\right)$	$-\kappa'\varepsilon_{abcd}\left(\tilde{\lambda}^{ab}\delta(e^ce^d)-\delta\tilde{\omega}^{ab}\xi\lrcorner(e^ce^d)\right)$
Lovelock-Cartan	$\int_{\mathcal{M}}\sum_{p=0}^{[D/2]}\kappa_p^D\varepsilon_{a_1\cdots a_D}R^{a_1a_2}\cdots R^{a_{2p-1}a_{2p}}e^{a_{2p+1}}\cdots e^{a_D}$	$-\sum_{p=1}^{[D/2]}p\kappa_p^D\,\varepsilon_{a_1\cdots a_D}\left(\lambda^{a_1a_2}\delta-\delta\omega^{a_1a_2}\xi\lrcorner\right)R^{a_3a_4}\cdots R^{a_{2p-1}a_{2p}}e^{a_{2p+1}}\cdots e^{a_D}$
$(2+1)$-Chern-Simons	$\kappa_{CS}\int_{\mathcal{M}}\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$	$2\kappa_{CS}\mathrm{Tr}\left(\lambda\delta A\right)$
$(2n+1)$-Chern-Simons	$\kappa_n\int_{\mathcal{M}}\int_0^1 dt\,\langle A\wedge F_t^n\rangle$	$n\kappa_n\left\langle\lambda\delta A\wedge F^{n-1}\right\rangle$
BF Gravity	$\int_{\mathcal{M}}\left(B^iF^i\pm\frac{1}{2\ell^2}B^iB^i+\frac{1}{2}\chi^{ij}B^iB^j\right)$	$-\delta B^i\lambda^i+\delta A^i\xi\lrcorner B^i$
BF-like Jackiw-Teitelboim	$\int_{\mathcal{M}}\left(B_a(de^a+\varepsilon^a{}_b\omega e^b)+\tilde{B}(d\omega+\varepsilon_{ab}e^ae^b)\right)$	$-\delta\tilde{B}\lambda-\delta B^a\lambda_a$

Chapter 8

Examples: Surface Charges *in Action*

In this chapter, we exhibit the surface charge method to compute charges in three different examples: (i) The rotating charged BTZ black hole, (ii) The anti-de Sitter Kerr-Newman family, and (iii) The Lorentzian rotating Taub-NUT solution in four dimensions. For each of them, we compute step-by-step the charges associated with the exact symmetries of the solution. Some of them have an explicit replicable calculation in *Mathematica* notebooks available at [sites.google.com/view/surfacechargetoolkit/h].

8.0.1 Charged and rotating black hole

As an example, we apply the result to a particular black hole family. We show that surface charges are compatible with the ones obtained through the standard asymptotic analysis. Then, we offer how the quasi-local nature of surface charge allows having the first law of black hole mechanics without relying on the asymptotic structure of space-time [132]. Note that this quasi-local perspective is the best that can be done when the black hole is embedded in an asymptotically de Sitter space-time.

We consider a black hole solution family which is electrically charged, rotating, and satisfies the asymptotically constant curvature boundary conditions

$$R^{ab} \pm \frac{1}{\ell^2} e^a e^b = 0 \qquad \text{in} \quad \partial\mathcal{M}\,. \tag{8.1}$$

This is known as the (anti-)de Sitter Kerr-Newman family. Its horizon is homeomorphic to a sphere, and its metric, which is axisymmetric, reads in Boyer-Lindquist-type coordinates as [133, 134]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left[dt - \frac{a\sin^2\theta}{\Xi}\,d\phi\right]^2 + \frac{\rho^2}{\Delta_r}\,dr^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{\Xi}\,d\phi\right]^2, \tag{8.2}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad\qquad \Xi = 1 - \frac{a^2}{\ell^2}\,, \tag{8.3}$$

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2mr + z^2\,, \qquad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta\,, \tag{8.4}$$

with z defined by $z^2 = q^2 + q_m^2$, q and q_m being the electric and magnetic charge parameters, respectively. For simplicity, we set $q_m = 0$. m is associated with the source's mass, and a is called the rotation parameter and is responsible for measuring the twist of the repeated principal null congruence. Note also that the constant Ξ in the metric (8.2) can be removed by applying a rescaling to the angular coordinate ϕ, as $d\phi/\Xi \to d\phi$. However, this constant is included to have a metric with a well-behaved axis both at $\theta = 0$ and $\theta = \pi$ with $\phi \in [0, 2\pi[$.[1]

The electromagnetic potential and a possible tetrad describing the solution are[2] [135]

[1] A metric without this rescaling implies the appearance of a singularity along the axial axis. In this last case, one should take into account those singularities by changing the region of integration for the charges. Something similar will happen in the Taub-NUT example.

[2] We use $e^a = e^a{}_\mu dx^\mu$ and $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where the flat Minkowski metric is $\eta = \text{diag}(-1, +1, +1, +1)$.

$$
\begin{aligned}
e^0 &= \frac{\sqrt{\Delta_r}}{\rho}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right), \qquad\qquad e^1 = \frac{\rho}{\sqrt{\Delta_r}}dr\,, &(8.5)\\
e^2 &= -\frac{\rho}{\sqrt{\Delta_\theta}}d\theta\,, \qquad e^3 = \frac{\sqrt{\Delta_\theta}\sin\theta}{\rho}\left(adt - \frac{a^2+r^2}{\Xi}d\phi\right), &(8.6)\\
A &= -\frac{qr}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right). &(8.7)
\end{aligned}
$$

We stress that it is possible to use another set of variables related to a gauge transformation. Still, as the procedure is explicitly gauge invariant it will not have any impact on the results. In particular, rotating e^I by an arbitrary Lorentz transformation or adding a term of the form $d\tilde{\lambda}$ to A has no effect. From the equation $d_\omega e^I = 0$ we solve the connection and compute: $\delta\omega^{IJ}$, $\delta\bar{R}^{IJ}$, δA, and $\delta\star F$. At this level, we have reduced the phase space to the particular family solution spanned by the parameters (m, a, q). Thus, the variation δ acts only on functions of those parameters.

In metric formalism, ∂_t and ∂_ϕ are two independent Killing fields. Through the solution of the exactness conditions for e^I, Eq. (7.128), we get λ_t^{IJ} and λ_ϕ^{IJ}, respectively. Similarly through the exactness conditions on A, Eq. (7.144), we obtain the corresponding λ_t and λ_ϕ. Now we have the ingredients to compute surface charges. Plugging all the quantities in (7.146) we get the associated integrands k_t and k_ϕ, one for each symmetry. The space-time described by e^I has non-contractible spheres due to the singularity. The integration can be performed over *any* two-surface enclosing the singularity. The surface charges associated to the exact symmetries generated by $\epsilon_t = (\partial_t, \lambda_t^{IJ}, \lambda_t)$ and $\epsilon_\phi = (\partial_\phi, \lambda_\phi^{IJ}, \lambda_\phi)$ are

$$
\begin{aligned}
\not\delta Q_t &= \oiint_S k_t = \frac{\delta m}{\Xi} \pm \frac{3am\delta a}{\ell^2\Xi^2}\,, &(8.8)\\
\not\delta Q_\phi &= \oiint_S k_\phi = -\frac{a\delta m}{\Xi^2} + \left(\frac{3}{\Xi^2} - \frac{4}{\Xi^3}\right)m\delta a\,, &(8.9)
\end{aligned}
$$

where S is a two-sphere at r and t constant, then we must integrate in θ and ϕ. The exactness condition $\delta_\epsilon A = 0$ has a further independent solution for a constant λ_0 such that $\delta_{\lambda_0} A = -d\lambda_0 =$

0. The corresponding exact symmetry parameter is $\epsilon_{\lambda_0} = (0, 0, \lambda_0)$ and the surface charge is

$$\oint Q_{\lambda_0} \;=\; \oiint_S k_{\lambda_0} \;=\; \frac{\lambda_0}{4\pi}\oiint_S \delta(\star F) \;=\; -\lambda_0\left(\frac{\delta q}{\Xi} \pm \frac{2aq\delta a}{\ell^2\Xi^2}\right). \tag{8.10}$$

To proceed now, we have two strategies: Fit the scheme in the results from the asymptotic picture or insist on a quasi-local approach. We sketch both.

Asymptotic strategy: In order to fit with the asymptotic picture we can exploit the linearity of each surface charge (7.50), and to adjust the freedom of the gauge parameters in the phase space to obtain the standard integrated charges (see for instance [136])

$$M \;\equiv\; Q_{\xi=\partial_t \mp (a/\ell^2)\partial_\phi} = \frac{m}{\Xi^2}\,, \tag{8.11}$$

$$J \;\equiv\; Q_{-\phi} = \frac{am}{\Xi^2}\,, \tag{8.12}$$

$$Q \;\equiv\; Q_{\lambda_0=-1} = \frac{q}{\Xi}\,. \tag{8.13}$$

The surface charge associated to ∂_t is not integrable. However, the linearity property (7.50) allows us to choose a different combination of the symmetry parameter $\xi \equiv \partial_t \mp \frac{a}{\ell^2}\partial_\phi$ that in fact produces an integrable charge. Note that ξ is phase space dependent: $\delta\xi \neq 0$.

The charges satisfy the equation known as the black hole fundamental thermodynamics relation

$$M^2 = \frac{S}{4\pi}\left(1 \pm \frac{S}{\pi\ell^2}\right)^2 + J^2\left(\frac{\pi}{S} \pm \frac{1}{\ell^2}\right) + \frac{Q^2}{2}\left(1 \pm \frac{S}{\pi\ell^2} + \frac{\pi Q^2}{2S}\right), \tag{8.14}$$

which can be obtained by rewriting the condition $\Delta_r = 0$ in terms of the integrated charges plus $S \equiv A/4$ with the area of the horizon $A = 4\pi(r_+^2 + a^2)$. The horizon is at $r = r_+$ with r_+ the largest solution of $\Delta_r = 0$. From the last equation it follows

$$\delta M \;=\; T\delta S + \Omega\delta J + \Phi\delta Q\,, \tag{8.15}$$

where the parametrization of the phase space is done with the integrable charges S, J, and Q such that $M = M(S, J, Q)$. Then, the quantities T, Ω, and Φ have the usual physical interpretation: $T \equiv \frac{\partial M}{\partial S}$ coincides with the Hawking's temperature, $\Omega \equiv \frac{\partial M}{\partial J}$ is the horizon angular velocity, and $\Phi \equiv \frac{\partial M}{\partial Q}$ the electric potential at the horizon.

The drawback of this logic line is that it relies on previous results. Ultimately, it depends upon a choice of asymptotic tailing of the field components, which admits an asymptotic time symmetry and allow us to make sense of a general asymptotic mass definition. We fixed the gauge parameters in practice to obtain a known mass expression obtained with the asymptotic method. That, for the case of anti-de Sitter space-times, indeed relies on asymptotic analysis. However, in the cases of asymptotically de Sitter space-times, there is no notion of time symmetry in the asymptotic region and not a physical argument to define a standard mass,[3] we just kept the $\pm$ in the formulae because it is consistent. Thus, given the quasi-local construction just developed, a pertinent question is: Is there a way to derive the first law of black hole mechanics based just on a quasi-local data?

Quasi-local strategy: To use the area of the black hole horizon as a starting point *is* a possibility. The horizon area is a well−defined quasi-local quantity which is also a finite function of the parameters of the solution. The variation of $A(m, a, q)$ on the phase space can be expressed as a combination of all the surface charges

$$\begin{aligned} \delta A = \oint k_\epsilon &= \alpha(m,a,q)\oint Q_t + \beta(m,a,q)\oint Q_\phi + \gamma(m,a,q)\oint Q_{\lambda_0}\,, && (8.16)\\ &= \alpha'(m,a,q)\oint k_\xi + \beta'(m,a,q)\oint k_{-\phi} + \gamma'(m,a,q)\oint k_{\lambda_0=-1}\,, && (8.17)\\ &= \frac{4}{T}\delta M - \frac{4\Omega}{T}\delta J - \frac{4\Phi}{T}\delta Q\,, && (8.18) \end{aligned}$$

[3]Remember that the boundary of asymptotically de Sitter space-times are two disconnected three-dimensional spacelike regions, one for the infinite past and one for the infinite future, and therefore none of them have a notion of time symmetry.

we expressed the freedom of the gauge parameter on the phase space explicitly. On the second line, we expanded in a linear combination of integrable quantities. The problem reduces to find the coefficient accompanying the integrated charges. Indeed, we already know that the result, expressed in the third line, is a rearrangement of the first law presented just before. However, we stress the difference in the logic; in this approach, the mass appears as an integrable charge computed in a quasi-local way *without* the need of any asymptotic structure or physical interpretation to define it. This quantity coincides with the mass obtained by an asymptotic definition when such definition is at its disposal. Still, it is more general because it requires just a quasi-local description of the space-time.

Note that the two closed two-surfaces where the integration of k_ϵ is performed, enclosing the singularity, are arbitrary. For a matter of physical interpretation, the one of $\oint\!\!\oint k_\epsilon$ can be chosen to be a section of the horizon, thus being associated with the area. In contrast, for each of the other integrals, it can be chosen at convenience, producing the same value of the charges for each of them. This freedom plus the gauge invariance of k_ϵ can be exploited to compute the quantities quickly. For instance, when a bifurcated horizon is at disposal, the pullback of a particular combination of the Killing fields vanishes on it, and the surface charge formula simplifies considerably.

Summarizing, from this second perspective the first law of black hole mechanics is a consequence of the expansion of $\delta A = \oint\!\!\oint k_\epsilon$ into independent integrable quantities. One for each independent exact symmetry ϵ_i. To accomplish integrability the symmetry parameters should satisfy the condition $\delta \oint\!\!\oint k_{\epsilon_i} = 0$ in each case, where the variation δ becomes an exterior derivative on the reduced phase space. Certainly, to have a true first law much more should be said, and it has been said, regarding the physical interpretation of each term, but the stress here is that the quantity sometimes playing the role of the mass can be relegated and be indirectly defined, in particular when the asymptotic time translation symmetry is not present or is difficult to identify.[4] To decide the true thermodynamic value of the quasi-local first law relation obtained we would need to figure out a thermodynamics processes that allows us to change the value of the integrated charges. That is, a physical exchange of the amount of charges to flow in a description outside the reduced phase space, even when the usual far away of the black hole notion is not available.

8.0.2 $(2+1)-$Black Hole with Rotation and Electric Charge

In $(2+1)-$space-time dimensions the equations of motion for gravity coupled to electromagnetism (E.5) admits a black hole solution with rotation and electric charge [137]. The solution is a generalization of the black hole solution known as Bañados-Teitelboim-Zanelli (BTZ) [138]. Part of the solution is given by the metric field $g_{\mu\nu}$. The line element is

$$ds^2 = -\frac{r^2}{R^2}F^2dt^2 + \frac{dr^2}{F^2} + R^2(N^\phi dt + d\phi)^2\,, \tag{8.19}$$

with r-dependent functions [139]

$$R^2 = r^2 + \left(\frac{\omega^2}{1-\omega^2}\right)r_+^2 + \frac{2}{\pi}(q\omega\ell)^2\ln\left(\frac{r}{r_+}\right)\,, \tag{8.20}$$

$$F^2 = \frac{r^2}{\ell^2} - \frac{r_+^2}{\ell^2} - \frac{2}{\pi}q^2(1-\omega^2)\ln\left(\frac{r}{r_+}\right)\,, \tag{8.21}$$

$$N^\phi = -\frac{\ell}{R^2}\left(\frac{\omega}{1-\omega^2}\right)\left(\frac{r^2}{\ell^2} - F^2\right)\,, \tag{8.22}$$

where the negative cosmological constant $\Lambda = -1/\ell^2$, the constant r_+ is defined by $F^2(r_+) = 0$, $\omega^2 < 1$, and q associated to the electric charge. Another part of the solution is the electromagnetic field

[4] For instance, this is the strategy used in [140], where the embedding of a charged and rotating black hole in a magnetic field makes subtle the selection of a preferred asymptotic time-like Killing vector field to define the space-time mass.

$$A = A_\mu dx^\mu \,, \qquad A_t = -\frac{q}{2\pi} \ln\left(\frac{r}{\ell}\right) \,, \qquad A_\phi = \frac{q\,\ell\,\omega}{2\pi} \ln\left(\frac{r}{\ell}\right) \,. \tag{8.23}$$

A curiosity of this solution is the logarithmic dependence on the radius brought in by the electric charge of the black hole. Not present in other dimensions, this particular dependence has put some trouble to the usual asymptotic analysis because standard asymptotic tailings do not admit it—the charge formulas for the standard asymptotic blow up when is computed for this black hole.

In [139, 141] the asymptotic analysis has been widened to correctly work out the asymptotic charges. However, it is possible to take another perspective and use the surface charge formula corresponding to the whole theory. Because we have an explicit solution with exact symmetries defined everywhere, we do not need to rely on asymptotic analysis and perform the calculation quasi-local.

To illustrate this example, we choose the formula for surface charge density $k_\xi^{\mu\nu}$ in metric variables (7.87). Indeed, the same can be done by using the surface charge formula in differential form language for Einstein-Cartan-Maxwell theory in $(2+1)$-dimensions.[5]

To compute charges, we first identify the exact symmetries of the solution and the parameters generating them, (ξ, λ). They should solve (7.85) and (7.86). Then, we replace the parameters and the field solutions with the corresponding surface charge formula. This has to be done for each independent exact symmetry, obtaining thus a surface charge for each of them.

For the solution (8.19)-(8.23) we have three exact symmetries (remember the use of improved prescription $\lambda = \lambda' - \xi^\mu A_\mu$)

There is a surface charge density for each of them: $k_{(t)}^{\mu\nu}$, $k_{(\phi)}^{\mu\nu}$, and $k_{(e)}^{\mu\nu}$. For the time symmetry

[5]A Mathematica notebook with the surface charge formula implemented for gravity coupled to electromagnetism in $(2+1)$-dimensions can be found at [sites.google.com/view/surfacechargetoolkit/h].

Exact Symmetries for BTZ black hole		
Type of Symmetry	Diffeomorphism parameter	Gauge symmetry parameter
Temporal	$\xi^{\mu}_{(\phi)} = (1,0,0)$	$\lambda_t = \frac{q}{2\pi}\ln\left(\frac{r}{\ell}\right)$
Axial	$\xi^{\mu}_{(\phi)} = (0,0,-1)$	$\lambda_\phi = \frac{\ell\omega q}{2\pi}\ln\left(\frac{r}{\ell}\right)$
$U(1)$-rigid	$\xi^{\mu}_{(e)} = (0,0,0)$	$\lambda_e = -1$

the non-vanishing components are $k^{tr}_{(t)} = -k^{rt}_{(t)}$ and $k^{r\phi}_{(t)} = -k^{\phi r}_{(t)}$. Explicitly

$$
\begin{aligned}
k^{tr}_{(t)} = \frac{1}{8\pi^2\ell^2 r_+ (1-\omega^2)^2} \Big[& (1-\omega^4)(\pi r_+^2 - \ell^2 q^2(1-\omega^2))\delta r_+ \\
& - 2\ell^2 q r_+ (1-\omega^2)^2 \left(\omega^2 + (1+\omega^2)\ln\left(\frac{r_+}{\ell}\right)\right)\delta q \\
& + 2 r_+ \omega \left(\pi r_+^2 - \ell^2 q^2 (1-\omega^2)^2 \left(1 + \ln\left(\frac{r_+}{\ell}\right)\right)\right)\delta\omega \Big], \qquad (8.24)
\end{aligned}
$$

and

$$
\begin{aligned}
k^{r\phi}_{(t)} = \frac{1}{8\pi^2\ell^3 r_+ (1-\omega^2)^2} \Big[& -2\omega(1-\omega^2)(\pi r_+^2 - \ell^2 q^2(1-\omega^2))\delta r_+ \\
& + 2\ell^2 q r_+ \omega (1-\omega^2)^2 \left(1 + 2\ln\left(\frac{r_+}{\ell}\right)\right)\delta q \\
& - r_+ \left(\pi r_+^2 (1+\omega^2) - \ell^2 q^2 (1-\omega^2)^2 \left(1 + 2\ln\left(\frac{r_+}{\ell}\right)\right)\right)\delta\omega \Big]. \qquad (8.25)
\end{aligned}
$$

It is not always the case, but for this example, all surface charge densities $k^{\mu\nu}_{\xi}$ are coordinate independent. Usually, coordinate independence is achieved only after space-time integration (see the following example to contrast this point). In particular, notice that there is no risk in evaluating them at asymptotic regions, say $r \to \infty$, because the value is simply independent of r. By construction ($\partial_\mu k^{\mu\nu}_{\xi} = 0$), the integration can be done over any loop $\mathcal{C}$ around the origin $r = 0$ at $t = cte$, and because coordinate independence holds, it is trivially performed

$$\not\delta M \equiv \frac{1}{2}\oint_{\mathcal{C}} k^{\mu\nu}_{(t)}\varepsilon_{\mu\nu\rho}dx^\rho = \int_0^{2\pi} k^{tr}_{(t)}d\phi = 2\pi k^{tr}_{(t)}\,. \tag{8.26}$$

We remember that $\not\delta$ means *a priori* non-integrability in phase space. Note also that $k^{r\phi}_{(t)}$ does not play a role.

A non-trivial check is the integrability on the reduced phase space $\delta(\not\delta M) = 2\pi\delta k^{tr}_{(t)} = 0$, where the variation δ acts on r_+, q, and ω. This guarantees the existence of the finite charge $M(r_+, q, \omega)$. If the integrability condition was not satisfied one may try to define a combination of Killing symmetries that produce integrable charges, as we did in the Kerr-Newman-de Sitter black hole solution [127]. Here, by integrating the differential surface charge and setting the integration constant to zero we get the finite charge

$$M = \frac{r_+^2}{8\ell^2}\left(\frac{1+\omega^2}{1-\omega^2}\right) - \frac{q^2}{4\pi}\left(\omega^2 + (1+\omega^2)\ln\left(\frac{r_+}{\ell}\right)\right)\,. \tag{8.27}$$

Notice that leaving implicit the integration on time as $\Delta_t \equiv \int_{-\infty}^{\infty} dt$, we can compute a second differential charge as

$$\not\delta \widetilde{M} = \frac{1}{2}\oint k^{\mu\nu}_{(t)}\varepsilon_{\mu\nu\rho}dx^\rho = \int_{-\infty}^{\infty} k^{r\phi}_{(t)}dt = \Delta_t k^{r\phi}_{(t)}\,, \tag{8.28}$$

where we identified future with past asymptotic regions in order to have a closed integration path. Fortunately, $\not\delta\widetilde{M}$ is an integrable charge, with its integrated version

$$\widetilde{M} = -\frac{\omega}{\ell}\left(\frac{r_+^2}{4\ell^2(1-\omega^2)} - \frac{q^2}{4\pi}\left(1 + 2\ln\left(\frac{r_+}{\ell}\right)\right)\right)\frac{\Delta_t}{2\pi}\,. \tag{8.29}$$

The interpretation of this charge is not entirely clear, but we advance that it may has a precise meaning for the Euclidean BTZ black hole (for an Euclidean time Δ_t becomes finite).

For the axial symmetry we find that the surface charge density is related with the previous one, (8.24) and (8.25), as

$$k^{tr}_{(\phi)} \;=\; -\ell^2 k^{r\phi}_{(t)}\,, \tag{8.30}$$

$$k^{r\phi}_{(\phi)} \;=\; -k^{tr}_{(t)} + \delta\left(\frac{q^2}{8\pi^2}(1-\omega^2)\right)\,. \tag{8.31}$$

After integration on any loop $\mathcal{C}$ of constant time the associated surface charge differential is

$$\not\delta J \;\equiv\; \frac{1}{2}\oint_{\mathcal{C}} k^{\mu\nu}_{(\Phi)}\varepsilon_{\mu\nu\rho}dx^\rho \;=\; \int_0^{2\pi} k^{tr}_{(\phi)}d\phi \;=\; 2\pi k^{tr}_{(\phi)}\,. \tag{8.32}$$

This quantity can be integrated on phase space and produces a charge identified with the angular momentum[6]

$$J \;=\; \frac{\omega}{4\ell}\left(\frac{r_+^2}{1-\omega^2} - \frac{q^2\ell^2}{\pi}\left(1+2\ln\left(\frac{r_+}{\ell}\right)\right)\right)\,. \tag{8.33}$$

Finally, for the $U(1)$-rigid symmetry

$$k^{tr}_{(e)} = \frac{\delta q}{2\pi}\,, \qquad \text{and} \qquad k^{r\phi}_{(e)} = -\frac{1}{2\pi\ell}\left(q\delta\omega + \omega\delta q\right)\,, \tag{8.34}$$

by integrating both in space-time (loop around the origin) and in phase space we obtain the (electric) charge[7]

$$Q_e \;=\; q\,. \tag{8.35}$$

Now, we can establish the first law of black hole mechanics for this family of black hole solutions spanned by the parameter r_+, ω, and q. With all the charges at hand and defining the entropy

$$S \;=\; \frac{L}{4} = \frac{\pi}{2}R(r_+) = \frac{\pi r_+}{2\sqrt{1-\omega^2}}\,, \tag{8.36}$$

with L the perimeter of the black hole horizon and $R(r_+)$ is given by (8.20), we have

[6]Notice the interesting relation $J = -\ell^2\frac{2\pi}{\Delta_t}\widetilde{M}$. A similar relation is obtained from (8.31). The relations among tilded and untilded quantities is a clear signal of a duality for the Euclidean BTZ charges not reported so far.

[7]There is also the second charge, $\widetilde{Q}_e = -q\frac{\omega}{\ell}\frac{\Delta_t}{2\pi}$, we can obtain identifying past and future and integrating over an (infinite) loop in time, as we did for $\widetilde{M}$ in (8.28).

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q_e\,. \tag{8.37}$$

The previous first law, expressed in terms of r_+, ω, and q, is a linear system exactly solved by

$$T = \frac{\sqrt{1-\omega^2}}{2\pi\ell^2}\left(1 - \frac{q^2\ell^2}{\pi r_+^2}(1-\omega^2)\right)\,, \tag{8.38}$$

$$\Omega = \frac{\omega}{\ell}\,, \tag{8.39}$$

$$\Phi = -\frac{q}{2\pi}(1-\omega^2)\ln\left(\frac{r_+}{\ell}\right)\,, \tag{8.40}$$

these quantities might be further identified with standard physical quantities: A temperature (Hawking thermal radiation), angular velocity of the horizon, and electrostatic potential, respectively.

Notice that one may get a non-physical negative temperature by increasing the electric charge. This problem is related to the possibility of having a negative mass (8.27). This result is a non-desirable possibility if one expects to interpret the mass as the energy of the system. The negativity of this mass expression has been addressed from a Hamiltonian analysis of charges and an asymptotic perspective in [139, 141]. There, the authors introduce an *improper gauge transformation* for the electromagnetic field and then impose the so-called holographic asymptotic boundary conditions on the fields to fix it. The result within this frame is a consistently positive mass: A better candidate for the energy of the system. From the surface charge perspective, it can also be reached by adding from the very beginning an arbitrary function of the phase space in the Maxwell potential A (called ν in [139, 141]) such that it gives the well thermodynamic behaviour between the mass and the electric charge. In our context, the arbitrariness of this function is completely determined by the integrability condition at the end. It is worth noticing again that no mention of the asymptotic region is made.
Now, we give another family of solutions that breaks the standard relation between the entropy and horizon area of the black hole.

8.0.3 Lorentzian Rotating Taub-NUT space-time

The properties of black holes are intimately related with gravitational thermodynamics [1,2,142]. As we observe in the previous examples, the area A of the event horizon of a black hole can be identified with 4 times its entropy S, and its surface gravity is proportional to its temperature β^{-1}. Usually, the understanding of black hole entropy comes from the use of Euclidean space-times. In this Euclidean regime, the event horizon continue being and obstruction to foliate the space-time with surface of constant periodic time τ. Such obstructions give rise to entropy by leading to

$$S = H\Delta\tau - I\,, \tag{8.41}$$

where I corresponds to the Euclidean on-shell action, H the Hamiltonian and $\Delta\tau$ the period of the Euclidean time.

As the event horizon in the black holes, there exists a similar defect in Taub-NUT (TN) space-times obstructing the foliation: the Misner string [143–145]. At the Euclidean regime, Hawking and Hunter [146] showed that the Misner strings could contribute to the entropy of these space-times, by leading that the entropy was not just a quarter the are, as it is for usual black holes [2]. For a long time, it has been considered as a interesting property of gravitation which reveals some features of the geometric entropy.

Usually, from a Euclidean signature perspective, the interpretation of thermodynamic quantities in TN geometries has been a work in progress in the literature. Although this analytical continuation in time does not produce significant problems in the Euclidean regime, in Lorentzian signature, it implies closed time-like curves (CTCs) and then cast doubt on the physical relevance of these singularities.

Very recently, the study of Lorentzian versions of TN space-times with and without cosmological constant has attracted enough attention. For example, in [147–149] the authors consider, based on the observation of Lorentzian TN space-times, the motion of free-falling observers in the presence of Misner strings. One of the main results is that the Misner string is entirely invisible for their geodesics hitting it and the TN space-times turn out to be geodesically complete.

In this example, we face whether the Misner strings have relevance in establishing a First Law for TN space-times. In particular, we will focus on TN space-times with rotation. We do not adopt periodicity in time, and we work directly with them at the Lorentzian regime. The non-trivial contributions to the mass and angular momentum of these solutions (supported by the presence of the NUT charge) can be understood with a suitable integration in space-time. With this quantities, what would-be an entropy can be obtained naturally from a conservation law's perspective.

The rotating TN space-time is described by the metric [150, 151]

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}\left(dt+(2n\cos\theta+2Cn-a\sin^2\theta)d\phi\right)^2+\frac{\Sigma}{\Delta}dr^2+\Sigma\, d\theta^2 \\ &\quad +\frac{\sin^2\theta}{\Sigma}\left(adt-(r^2+a^2+n^2-2anC)d\phi\right)^2 , \end{aligned} \tag{8.42}$$

where the metric functions are given by

$$\Delta = r^2+a^2-2mr-n^2 , \tag{8.43a}$$

$$\Sigma = r^2+(n+a\cos\theta)^2 , \tag{8.43b}$$

with m the integration constant associated to the mass, n the so-called NUT parameter, and a the rotation parameter.

The two Killing symmetries produce two independent varied surface charges. We check that our corrected formula delivers the expected result. Let us consider the surface charge $\boldsymbol{k}_{\vec{\xi}}$, being $\vec{\xi}$ a time-like Killing vector $\vec{\xi}=(1,0,0,0)$ or axial-like Killing vector $\vec{\xi}=(0,0,0,1)$, integrated on the volume limited by two spherical shells, say S_1 and S_2, at $t=cte$ and $r=cte$, with radius R_1 and R_2, respectively, such that $R_2>R_1$, and two auxiliary tubes (actually cones) T_S and T_N, at $t=cte$ and $\theta=\pi+\epsilon$, with $\epsilon\ll 1$, opening towards the south pole and passing to close the wire (See Fig. 8.1). The Misner strings correspond to string/wire singularities stretching from the $r=0$ to the infinity. They are labelled by the discrete integer C which is related to the location of the singularity: for $C=+1$ the south pole axis is regular, for $C=-1$ the north one is regular, and for $C=0$ both wires/strings are symmetrically present.

The integration on the volume enclosed by these surfaces can be translated to the boundary by using the Gauss' theorem, we have

$$0=\int_\Sigma d\boldsymbol{k}_\xi=\int_{\partial\Sigma}\boldsymbol{k}_\xi=\int_{S_1}\boldsymbol{k}_\xi-\int_{S_2}\boldsymbol{k}_\xi+\int_{T_S}\boldsymbol{k}+\int_{T_N}\boldsymbol{k}_\xi\,, \tag{8.44}$$

where the radial integration over each cone decouples in two terms associated to the circles found as the intersection of the cone with the shells, we have

$$\int_{T_S}\boldsymbol{k} = K_\xi\Big|_{r=R_1;\,\theta=\pi-\epsilon}-K_\xi\Big|_{r=R_2;\,\theta=\pi-\epsilon}\,, \tag{8.45a}$$

$$\int_{T_N}\boldsymbol{k} = K_\xi\Big|_{r=R_1;\,\theta=\epsilon}-K_\xi\Big|_{r=R_2;\,\theta=\epsilon}\,, \tag{8.45b}$$

where K_ξ denotes the primitive of $\boldsymbol{k}_\xi$ in the south or north cone. We refer to $\theta=0$ as south pole, and $\theta=0$ as north. Then, we take the limit $\epsilon\to 0$ and reorder (8.44) with terms associated to each shell to get an expression for the surface charge that is defined at a sphere S and is truly conserved, *i.e.* its value does not depend on the specific surface S

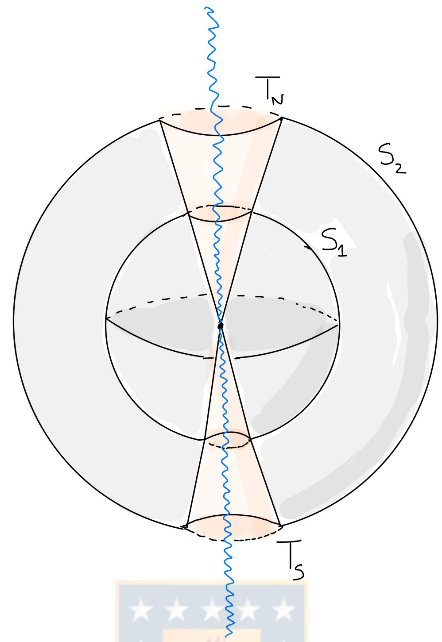

Figure 8.1: Integration region: a volume delimited by two spheres and two cones. The wavy blue lines indicate the Misner strings symmetrically present ($C = 0$).

$$\oint Q_\xi = \int_S \boldsymbol{k}_\xi + K_\xi \bigg|_{S;south} + K_\xi \bigg|_{S;north} . \tag{8.46}$$

This charge corresponds to the mass or angular momentum depending on whether the Killing vector $\vec{\xi}$ is time-like or axial-like, respectively.

Although we formally use a specific parametrization for the geometrical objects, the result does not depend on them. We can deform either the shells or the cones before shrinking them and still have the same resulting surface charge. In particular, it does not depends on the specific radius we compute it. Therefore, we have sketched how the K contribution solves the problem of naively computing (7.49) when defects cross the surface S. For space-times, without strings-like defects, the K terms automatically vanish. Still, we note that even for regular space-times, we can introduce artificial string-like defects. This breaks down the charge conservation unless terms similar to the K are considered (an example of this is the conical defect appearing in the axial axis for some versions of the Kerr-(A)dS metric solution).

We emphasize that to get (8.46) we rely on the conservation law and its correct integrated version, which requires the identification of the cones to compute the K-es correctly. We do not know another way to identify how to correct (7.49) relying just on the surface S, even knowing that string-like defects cross it.

Mass and angular momenta

The two Killing symmetries produces two independent varied surface charges, we check that our corrected formula delivers the expected result. In fact, for the time translations symmetry generator, as we expect the mass/energy let us rename $\delta M = \oint Q_{\partial_t}$, then (8.46) reproduces

$$\delta M = \delta m\,, \tag{8.47}$$

which is indeed integrable, $\delta^2 M = 0$. We set the integration constant equal to zero to obtain the finite charge

$$M = m\,, \tag{8.48}$$

which means that the parameter m can still be identified with the total mass/energy of rotating TN space-time.

Analogously, for the axial symmetry generator we expect the angular momentum, thus we rename $\delta J = \oint Q_{-\partial_\Phi}$. We find an integrable charge $\delta^2 J = 0$. The variation of the angular momentum is

$$\delta J = (\delta a - 3C\delta n)m + (a - 3Cn)\delta m\,, \tag{8.49}$$

with a finite version given by

$$J = (a - 3Cn)m\,. \tag{8.50}$$

Note that as expected the TN space-time with $a = 0$ still carries angular momentum but just when then strings are not symmetrically distributed along the axis.

Let us conclude here with a few remarks:

- the charges do not present divergences of any kind (*e.g.* r-divergences when we evaluate it in asymptotic regions). While each integral in (8.46) is r-dependent and therefore susceptible to divergences, it is only the sum of the three that produces r-independent charges which are insensitive to asymptotic limits.
- No asymptotic region is required. We do not demand a particular location for the sphere S. In that sense, we shall say that the charges computed above are quasi-local.
- The charges are trivially integrable. The procedure also works with the presence of the integer parameter C.

Entropy and The First Law

We now turn to the thermodynamics of rotating TN space-time. We constructively present the first law to determine the entropy of this space-time. For simplicity, we set $C = 0$ in the following formulas.

The temperature is

$$T=\frac{\sqrt{-a^2+m^2+n^2}}{4\pi\left(m\left(\sqrt{-a^2+m^2+n^2}+m\right)+n^2\right)}\,, \tag{8.51}$$

and the angular velocity, of the $r=r_+$ surface as measured from infinity, is

$$\Omega=\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r_+}=-\frac{a}{2\left(m\left(\sqrt{-a^2+m^2+n^2}+m\right)+n^2\right)}\,, \tag{8.52}$$

with r_+ read from $\Delta(r_+)=0$. The expectation is that the temperature multiplies the entropy variation, we call it $\not\delta S$, and the angular velocity multiplies the angular momentum variation. We isolate $\not\delta S$ to get

$$\not\delta S=\frac{1}{T}\left(\delta M-\Omega\delta J\right)=\frac{2\pi m}{\sqrt{-a^2+m^2+n^2}}\left[-a\delta a+\delta m\left(2\left(\sqrt{-a^2+m^2+n^2}+m\right)+\frac{2n^2-a^2}{m}\right)\right]\,. \tag{8.53}$$

Note that for $n=0$ the previous expression is integrable, and it is the variation of the usual black hole entropy of the Kerr solution. For TN space-time, $n\neq 0$, the expression is not integrable, $\delta^2 S\neq 0$, which is a problem to have a well-defined entropy charge. Note that, as presented, the entropy variation is indeed conserved in the sense that it can be directly computed from (8.46) with the Killing vector $\xi=\frac{1}{T}\left(\partial_t+\Omega\partial_\phi\right)$, that is, it is a proper varied surface charge as it does not depend on the surface one choose to compute it. Therefore, this expression should be a good seed to compute an entropy as a Noether charge as we did with the mass or the angular momentum (we keep the *Noether charge* name, understanding that surface charge method to compute charges generalizes the Noether procedure).

In the literature, it has been explored the possibility of adding extra terms to the first law associated with the Misner strings. At this level, this strategy is equivalent to the splitting of (8.53) in different terms and providing interpretation to each one. Here we choose another path.

We stick ourselves to the criteria of having an entropy which is a conserved, a Noether charge. Therefore the expression in (8.53) can not be split; otherwise, we can not write the entropy variation as (8.46) with an arbitrary closed surface S.

Discussion parenthesis: We do not have formal proof that there is no splitting simultaneously producing conserved and integrable charges. We have just checked the natural splitting of surface and tubes terms in (8.46). As explained before, separately surfaces and tube terms are not conserved quantities and neither they are integrable. One may be tempted to fix the arbitrary closed surface and choose the $r = r_+$ surface in an analogy with black holes. In this case, the surface integral in (8.46) indeed becomes the variation of one-fourth of the area. However, we are not computing a conserved charge if we have to fix the surface where it is defined. For completeness, insisting in this approach the terms associated with the tubes are not integrable and another *temperature* should be introduced to achieve their integrability. This path would be similar to the one presented in [151].

A way to have, simultaneously, the entropy as a charge and integrable is to reduce the parameter space. Let us assume that m, a, and n are dependent quantities and solve this dependence such that the integrability condition is satisfied. Consider $m(n, a)$, because the way these parameters enter in the metric components it is easy to see that, under scaling of the metric, all the tree parameters should scale with the same factor, then, the allowed relation among them is simpler: $m = n\, h(a/n)$, with $h(a/n)$ a function to be determined by the integrability condition $\delta(\not\oint S) = 0$. The differential equation is solved by

$$h\left(\frac{a}{n}\right) = \alpha\,, \tag{8.54}$$

with α an integration constant. Note that this relation among parameter is simpler rewritten in terms of m, indeed: $m = \alpha n$.

Thus, with this relation we have reduced the parameter space of our solution to two parameters. Replacing this in the equation for the varied surface charge (8.53) we can integrate it to obtain a finite entropy

$$S = 2\pi\alpha n^2 \left(\alpha + \sqrt{1+\alpha^2 - \frac{a^2}{n^2}} \right) . \tag{8.55}$$

where we have set the integration constant to zero. Our method do not fix α and in what follows it can be left untouched. However, we can fix the constant α if we require our result to be compatible with the non-rotating TN entropy obtained through standard Euclidean methods in [146], $S_{HH} = \pi n^2$, then, with $\alpha = \frac{1}{2\sqrt{2}}$ we get

$$S = \frac{1}{4}\pi n^2 \left(1 + \sqrt{9 - \frac{8a^2}{n^2}} \right) . \tag{8.56}$$

as the generalization of the entropy for the rotating TN solution. This entropy fulfills a standard first law

$$\delta M = T\delta S + \Omega \delta J \, , \tag{8.57}$$

where $M = \frac{n}{2\sqrt{2}}$ and $J = \frac{an}{2\sqrt{2}}$. In contrast with the usual black hole solutions, the entropy is not one-fourth of the area, already noted in [146] for non-rotating TN. This first law is standard as to be fulfilled. It does not require additional terms not justified in terms of charges. It may be seen as not standard as we had reduced the parameter space to achieve integrability. But as explained before, this is a consequence of having well-defined charges M, J, and S to compose the first law. Then we conclude that there is a consistent thermodynamic description only for a subset of the parameter space of the rotating TN.

The entropy (8.56) is compatible with the Euclidean method. We use the grand canonical ensemble computed in [151] as a limit of a regularized Euclidean (A)dS action

$$G(T,\Omega) = \frac{m}{2} = \frac{1}{8\pi T\left(1+3\sqrt{1+\frac{\Omega^2}{4\pi^2T^2}}\right)}\,, \tag{8.58}$$

where in the second equality we have replaced m in terms of T and Ω following (8.51) and (8.52) and the relation among parameters $m = nh(a/n)$ with the fix constant $\alpha = 1/(2\sqrt{2})$. The entropy of (8.56) is recovered from the standard equation

$$S = -\frac{\partial G}{\partial T}\,. \tag{8.59}$$

Thus, we have shown that our covariant symplectic method to compute charges and the first law is fully compatible with the Euclidean approach.

Chapter 9

Conclusion to Part II

In this second part of this thesis, we have analyzed a powerful tool to deal with physical symmetries in gauge theories. In the Introduction 6 we have shown the importance of symmetries in physics and how they may carve a physical theory. We examined Emmy Noether's work about the deep relationship between global symmetries and conservation laws with this motivation. As a particular consequence, she taught us that the celebrated principle of energy conservation in a given theory is a consequence of the symmetry under time translations of that theory. Her general result is known as the First Noether Theorem. We discussed another Noether's result now applying to gauge theories: For each independent gauge symmetry, a relation among field equations holds (off-shell relations), the so-called Noether identities (*e.g.* Bianchi identity for gravity). This is the content of the Second Noether Theorem.

Motivated by the fact that the First Noether theorem does not apply appropriately in gauge theories, we have presented the method, known as the surface charge method, to compute charges for gauge theories in Chapter 7.

We stress the main difference with the First Noether Theorem is that in the surface charge method the construction is over a one-dimension lower sub-manifold. That is, for theories without gauge symmetries, the usual Noether current J^{μ}_{ϵ} corresponds to a $(D-1)$-form and defines a charge over a $(D-1)$-dimensional space-time slice Σ as $Q_{\epsilon} = \int_{\Sigma} J^{\mu}_{\epsilon} d\Sigma_{\mu}$. In Σ contrast, for gauge theories, we have the surface charge density $k^{\mu\nu}_{\epsilon}$, playing the role of a *current*, that corresponds to a $(D-2)$-form and defines a (differential) charge over a closed $(D-2)$-dimensional surface S as $\not\delta Q_{\epsilon} = \oint_{S} k^{\mu\nu}_{\epsilon} dS_{\mu\nu}$. Thus, a second difference is that in the surface charge method, instead of finite charges, one defines differential charges on phase space. They require further phase space integration.

One of the aims of Chapter 7 and the next is to fill a key gap by obtaining the formulae of the surface charges from the usual canonical symplectic approach (7.46). Another aim is to show the relation with the so called invariant symplectic approach based on the contracting homotopy operators [101]. We do so because a part of the community is unaware of the powerful results related with the surface charges. And specifically, unaware of the way they help to solve the problems of using Noether currents in gauge theories.
Along the derivation of surface charges we put attention on asymptotic charges, which are a consequence of asymptotic symmetries, a lot of work is focused on providing the conditions to have integrable asymptotic charges. In contrast insufficient effort is put in providing an explanation of what those non-conserved asymptotic charges mean. Through these notes it is clear that to have a conservation law an exact symmetry equation is needed: An equation that by definition asymptotic symmetries do not satisfy.

In the hope to easy the work of gravity physicist and specially to those who decide to use alternative variables, in Chapters 7.4 and 7.5 we systematically wrote the surface charge density formulas for many gravity theories, presented all of them in a Table, and furthermore we also performed all the step-by-step calculation in the appendices.

A key field where the results here are of direct use is in analyzing black holes, more specifically, their thermodynamic properties. The surface charge method allows us to have good control of the space of exact symmetries and the space of differential charges at once. In the case of black holes this analysis can lead to a well-posed first law of black hole mechanics. In the first of the three examples, Chapter 8, this was worked in detail for a complicated enough black hole family and showed that a consistent first law holds.

As a final comment, and to put this part of this thesis in a wider perspective, we should stress that all the methods to compute charges for gauge theories are defined in a classical regime. Then, it remains as a challenge to understand how the surface charge method works within a quantum gravity theory. It certainly depends on the quantization procedure one applies to the classical phase space structure and the different quantities defined there. Still, nowadays there is no clear path to achieve this quantization. We hope that future research will provide it.

Appendix A

Three differential form operations

Let us consider a p-form expressed in a coordinate basis

$$\boldsymbol{\alpha} = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{A.1}$$

The operator exterior derivative on differential forms defined as

$$d : \Omega^p \longmapsto \Omega^{p+1}\,, \tag{A.2}$$

has the explicit action

$$d\boldsymbol{\alpha} = \frac{1}{p!}\partial_{\mu_0}\alpha_{\mu_1\ldots\mu_p}dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{A.3}$$

This operator satisfies the following Leibniz's rule $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta$, and it is nilpotent, that is, $d^2 = 0$.

The Hodge dual operator

$$\star : \Omega^p \longmapsto \Omega^{D-p}\,, \tag{A.4}$$

acts on a p-form as

$$\star\,\boldsymbol{\alpha} = \frac{1}{(D-p)!p!}\alpha^{a_1\ldots a_p}\varepsilon_{a_1\ldots a_p b_1\ldots b_{D-p}}e^{b_1}\wedge\cdots\wedge e^{b_{D-p}}\,, \tag{A.5}$$

or with the differential form expressed in a coordinate basis

$$\star\,\boldsymbol{\alpha} = \frac{1}{(D-p)!p!}\sqrt{|g|}\alpha^{\mu_1\ldots\mu_p}\varepsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_{D-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{D-p}}\,. \tag{A.6}$$

For a vector field $\xi = \vec{\xi} = \xi^a e_a = \xi^\mu\partial_\mu$, the interior product on forms is either denoted by i_ξ or also with the alternative notation $\xi\lrcorner$. This operation lows by one the form degree

$$i_\xi \equiv \xi\lrcorner : \Omega^p \longmapsto \Omega^{p-1}\,. \tag{A.7}$$

Explicitly, on a p-form it has the action

$$\xi\lrcorner\boldsymbol{\alpha} = \frac{1}{(p-1)!}\xi^\nu\alpha_{\nu\mu_2\ldots\mu_p}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{A.8}$$

The homotopy operator or Anderson's homotopy operator $I^p_{\delta\Phi}$ is defined as the map

$$I^p_{\delta\Phi} : \Omega^p \longmapsto \Omega^{p-1}\,, \tag{A.9}$$

and acts as

$$I^p_{\delta\Phi}\boldsymbol{\alpha} = \sum_{k\geq 0}\frac{k+1}{n-p+k+1}\partial_{\mu_1}\cdots\partial_{\mu_k}\left(\delta\Phi^i\frac{\delta}{\delta\Phi^i_{\mu_1\ldots\mu_k\nu}}\frac{\partial\boldsymbol{\alpha}}{\partial dx^\nu}\right)\,. \tag{A.10}$$

This operator satisfies the following relations

$$\delta = \delta\Phi^i\frac{\delta}{\delta\Phi^i} - dI^n_{\delta\Phi}\,, \qquad \text{when acting on space-time } n\text{-forms}\,, \tag{A.11}$$

$$\delta = I^{p+1}_{\delta\Phi}d - dI^p_{\delta\Phi}\,, \qquad \text{when acting on space-time } p\text{-forms } (p<n)\,. \tag{A.12}$$

and,

$$\delta I^p_{\delta\Phi} = I^p_{\delta\Phi}\delta\,. \tag{A.13}$$

For example, expanding the sum in (A.10), we get [98]

$$I^n_{\delta\Phi} = \left[\delta\Phi^i \frac{\partial}{\partial\partial_\mu\Phi^i} - \delta\Phi^i\partial_\nu \frac{\partial}{\partial\partial_\mu\partial_\nu\Phi^i} + \partial_\nu\delta\Phi^i \frac{\partial}{\partial\partial_\mu\partial_\nu\Phi^i} + \cdots\right]\frac{\partial}{\partial dx^\mu}\,, \tag{A.14}$$

$$I^{n-1}_{\delta\Phi} = \left[\frac{1}{2}\delta\Phi^i \frac{\partial}{\partial\partial_\mu\Phi^i} - \frac{1}{3}\delta\Phi^i\partial_\nu \frac{\partial}{\partial\partial_\mu\partial_\nu\Phi^i} + \frac{2}{3}\partial_\nu\delta\Phi^i \frac{\partial}{\partial\partial_\mu\partial_\nu\Phi^i} + \cdots\right]\frac{\partial}{\partial dx^\mu}\,, \tag{A.15}$$

where dots mean for higher derivatives depending on the nature of the theory.

Appendix B

Derivation of the Skyrme field equations

In this appendix, we make an explicit derivation of the Skyrme field equations (3.11). Let us start by applying a general variation to the Lagrangian density associated to the Skyrme action (3.7)

$$\begin{aligned}\delta\mathcal{L}_{Skyrme}[U] &= \frac{K}{4}\,\delta\,\mathrm{Tr}\left(L^{\mu}L_{\mu}+\frac{\lambda}{8}[L_{\mu},L_{\nu}][L^{\mu},L^{\nu}]\right),\\ &= \frac{K}{2}\,\mathrm{Tr}\left(L^{\mu}\delta L_{\mu}+\frac{\lambda}{8}[L^{\mu},L^{\nu}]\,\delta\left([L_{\mu},L_{\nu}]\right)\right),\end{aligned} \tag{B.1}$$

The variation of the tensor L_{μ} is given by

$$\begin{aligned}\delta L_{\mu} &= \delta(U^{-1}\partial_{\mu}U)\,,\\ &= \delta(U^{-1})\partial_{\mu}U+U^{-1}\delta(\partial_{\mu}U)\,,\\ &= -U^{-1}\delta U L_{\mu}+L_{\mu}U^{-1}\delta U+\partial_{\mu}(U^{-1}\delta U)\,,\end{aligned} \tag{B.2}$$

where we have used the identity $\delta U^{-1} = -U^{-1}\delta U U^{-1}$ deduced from the variation of the invertibility condition $U^{-1}U = 1$. With this, we get

$$\mathrm{Tr}(L^\mu \delta L_\mu) = -\mathrm{Tr}\left[\partial_\mu L^\mu U^{-1}\delta U + \partial_\mu \left(U^{-1}\delta U L^\mu\right)\right] . \tag{B.3}$$

To compute the second term in (B.1) we notice that

$$\begin{aligned} [L_\mu, L_\nu] &= [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U] , \\ &= U^{-1}\partial_\mu U U^{-1}\partial_\nu U - U^{-1}\partial_\nu U U^{-1}\partial_\mu U , \\ &= -\partial_\mu(U^{-1}\partial_\nu U) + \partial_\nu(U^{-1}\partial_\mu U) , \\ &= -\partial_\mu L_\nu + \partial_\nu L_\mu . \end{aligned} \tag{B.4}$$

Now, we take a variation on this last term

$$\begin{aligned} \delta([L_\mu, L_\nu]) &= -\partial_\mu \delta L_\nu + \partial_\nu \delta L_\mu , \\ &= -\partial_\mu(-U^{-1}\delta U L_\nu + L_\nu U^{-1}\delta U) + \partial_\nu(-U^{-1}\delta U L_\mu + L_\mu U^{-1}\delta U) . \end{aligned} \tag{B.5}$$

Therefore, the second term in (B.1) reads

$$\begin{aligned} \mathrm{Tr}\left([L^\mu, L^\nu]\delta([L_\mu, L_\nu])\right) &= 2\mathrm{Tr}\left([L^\mu, L^\nu]\partial_\nu(-U^{-1}\delta U L_\mu + L_\mu U^{-1}\delta U)\right) , \\ &= 2\mathrm{Tr}\left(\partial_\mu(U^{-1}\delta U[L_\nu, [L^\mu, L^\nu]]) - U^{-1}\delta U[L_\nu, \partial_\mu[L^\mu, L^\nu]]\right) . \end{aligned} \tag{B.6}$$

Then, plugging back Eqs. (B.3) and (B.6) into Eq. (B.1), we find

$$\delta\mathcal{L} = -\frac{K}{2}\,\mathrm{Tr}\left((\partial_\mu L^\mu + \frac{\lambda}{4}[L_\nu, \partial_\mu[L^\mu, L^\nu]])U^{-1}\delta U\right) , \tag{B.7}$$

but $[L_\nu, \partial_\mu[L^\mu, L^\nu]] = \partial_\mu[L_\nu, [L^\mu, L^\nu]]$, then

$$\delta\mathcal{L} = -\frac{K}{2}\,\mathrm{Tr}\,\left(\partial_\mu(L^\mu + \frac{\lambda}{4}[L_\nu,[L^\mu,L^\nu]])U^{-1}\delta U\right)\,, \tag{B.8}$$

for an arbitrary group element $U \neq 0$, we arrive to the Skyrme field equations

$$\partial_\mu\left(L^\mu + \frac{\lambda}{4}[L_\nu,[L^\mu,L^\nu]]\right) = 0\,. \tag{B.9}$$

Appendix C

$SU(2)$ group

In this appendix we provide details of the parametrization of the $SU(2)$ group. The $SU(2)$ group is the set of all traceless anti-Hermitian 2×2 complex matrices $M_2(\mathbb{C})$ of det $= +1$ that satisfies $A^\dagger A = \mathbb{I}$, namely

$$SU(2) = \left\{ A \in M_2(\mathbb{C}) \,\middle|\, A^\dagger = A^{-1}\,, \det(A) = 1 \right\} , \tag{C.1}$$

where $A^\dagger$ denotes the Hermitian conjugate of A, namely, conjugate transpose of the matrix A, explicitly $A^\dagger = (A^T)^*$. The Lie algebra associated to this group over the field $\mathbb{C}$ is given by

$$\mathfrak{su}(2) = \left\{ A \in M_2(\mathbb{C}) \,\middle|\, (e^{tA})^\dagger = (e^{tA})^{-1}, \det(e^{tA}) = 1, \forall t \in \mathbb{R} \right\} . \tag{C.2}$$

Since $(e^{tA})^\dagger = e^{(tA)^\dagger} = e^{tA^\dagger}$ and $\det(e^{tA}) = e^{\mathrm{tr}(tA)} = e^{t\,\mathrm{tr}(A)}$, the conditions translate to

$$A^\dagger = A^{-1} \qquad \text{and} \qquad \mathrm{tr}(A) = 0\,. \tag{C.3}$$

An arbitrary representation of the group $SU(2)$ is given by the set of the three generators T_i which satisfy the Lie algebra

$$[T_i, T_j] \; = \; -2\epsilon_{ijk} T_k \,, \qquad \epsilon_{012} = +1 \,. \tag{C.4}$$

An element of this group is given by the matrix

$$U \; = \; e^{T^k \omega_k} \,, \tag{C.5}$$

where the vector $\vec{\omega}$ has components ω_i in a given coordinate system, and in the fundamental representation $T_k = \tau_k$, with $\tau_k = i\,\sigma_k$, $k = 1, 2, 3$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{C.6}$$

the standard Pauli matrices, which satisfy the relation

$$\tau_i \, \tau_j \; = \; -\delta_{ij} - \epsilon_{ijk} \tau_k \,. \tag{C.7}$$

It implies

$$[\tau_i, \tau_j] = -2\epsilon_{ijk} \tau_k \,, \qquad \{\tau_i, \tau_j\} = -2\delta_{ij} \,. \tag{C.8}$$

Geometrically, the matrices U are generators of spinor rotations in three dimensional space $\mathbb{R}^3$ and the components of the vector ω_i are the corresponding angles of rotation. For $SU(2)$ group, there exists a parametrization for the elements U called the Euler parametrization and is defined in terms of three angles θ, φ and ψ as

$$U(\theta,\varphi,\psi) = U_z(\varphi)U_y(\theta)U_z(\psi) = e^{\tau_3\varphi}\, e^{\tau_2\theta}\, e^{\tau_3\psi}\,, \tag{C.9}$$

$$= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}, \tag{C.10}$$

$$= \begin{pmatrix} \cos\theta\, e^{i(\psi+\varphi)} & \sin\theta\, e^{-i(\psi-\varphi)} \\ -\sin\theta\, e^{i(\psi-\varphi)} & \cos\theta\, e^{-i(\psi+\varphi)} \end{pmatrix}, \tag{C.11}$$

with $0 \leq \theta \leq \pi\,,\ 0 \leq \varphi \leq 2\pi\,,\ 0 \leq \psi \leq 4\pi$. Then, the $SU(2)$ group manifold is isomorphic to three-sphere S^3.[1]

Using the matrices (C.9) it is possible to define the so-called left and right invariant one-forms on the group $SU(2)$: the Maurer-Cartan one-forms[2]

$$L = U^{-1}dU = L^k\,\tau_k\,, \qquad R = dUU^{-1} = R^k\,\tau_k\,. \tag{C.12}$$

Because $\det(U) = 1$ then $\mathrm{tr}(L) = \mathrm{tr}(R) = 0$. In term of the Euler angles, the Maurer-Cartan forms in the Pauli basis are given by

$$L_1 = -\sin(2\psi)\,d\theta + \cos(2\psi)\,\sin(2\theta)\,d\varphi\,, \quad R_1 = \sin(2\varphi)\,d\theta - \cos(2\varphi)\sin(2\theta)\,d\psi\,, \tag{C.13}$$

$$L_2 = \cos(2\psi)\,d\theta + \sin(2\psi)\,\sin(2\theta)\,d\varphi\,, \quad R_2 = \cos(2\varphi)\,d\theta + \sin(2\varphi)\sin(2\theta)\,d\psi\,, \tag{C.14}$$

$$L_3 = d\psi + \cos(2\theta)\,d\varphi\,, \quad R_3 = d\varphi + \cos(2\theta)\,d\psi\,, \tag{C.15}$$

satisfying the Maurer-Cartan equations

[1]The reduction to the parametric space $SO(3)$ is reached through $\psi \sim \psi + 2\pi$.

[2]In the literature these quantities are defined in the opposite way, *i.e* $\tilde{R} := U^{-1}dU$ and $\tilde{L} := dUU^{-1}$. Because in much of the Part I of the thesis we work with $\tilde{R}$, in order to avoid confusion with Part II of the thesis, we prefer to use the conventions of Eq. (C.12).

$$dR_i = \epsilon_{ijk} R_j \wedge R_k \,, \qquad dL_i = -\epsilon_{ijk} L_j \wedge L_k \,. \tag{C.16}$$

Appendix D

Derivation of the reduced system of equations (4.23), (4.26) and (4.43)

In this appendix we demonstrate how equations (4.23), (4.26) and (4.43) are obtained from the general field equations (4.7) and (4.7b); thanks to the generalized Hedgehog ansatz [27, 28, 113, 152], which remarkably enough still holds when the Skyrme field is coupled to Maxwell theory.

By considering the order of the space-time coordinates in the gauged Skyrmion case as $x^\mu = (z, r, \gamma, \phi)$, it can be easily seen that, under the choice (4.21) for the Euler angles and (4.22) for the electromagnetic potential, the three components of $L_\mu = L^i_\mu t_i$, $i = 1, 2, 3$ read

$$L^1_\mu = \left(b_1 \cos(q\phi)\sin(2H), -\sin(q\phi)H', \left(\frac{p}{2} + b_2\right)\cos(q\Phi)\sin(2H), b_3\cos(q\phi)\sin(2H)\right) \, , \quad \text{(D.1a)}$$

$$L^2_\mu = \left(b_1 \sin(q\phi)\sin(2H), \cos(q\phi)H', \left(\frac{p}{2} + b_2\right)\sin(q\Phi)\sin(2H), b_3\sin(q\phi)\sin(2H)\right) \, , \quad \text{(D.1b)}$$

$$L^3_\mu = \left(-2b_1\sin^2(H), 0, \frac{p}{2}\cos(2H) - 2b_2\sin^2(H), \frac{q}{2} - 2b_3\sin^2(H)\right) \, . \quad \text{(D.1c)}$$

With the help of the latter, the electromagnetic current vector can be computed through (4.8) to be

$$J^\mu = \frac{K}{2l^2}\left(M_{1I}b_I + N_1, 0, M_{2I}b_J + N_2, M_{3I}b_J + N_3\right), \quad I, J = 1, 2, 3\,, \tag{D.2}$$

with the expressions M_{IJ} and N_J being given by (4.24) and (4.25), respectively. It can be easily verified that $\nabla_\mu J^\mu = 0$ holds as an identity for the previous expression.

By using (D.1), in the three gauged Skyrme equations

$$D^\mu\left(L^i_\mu + \frac{\lambda}{4}\left[L^\nu, [L_\mu, L_\nu]\right]^i\right)\tau_i \;\equiv\; E^i\tau_i \;=\; 0\,, \tag{D.3}$$

the latter become

$$E^1 = -\frac{\sin(q\Phi)}{16l^4}\mathcal{A}(r) = 0\,, \tag{D.4a}$$

$$E^2 = \frac{\cos(q\Phi)}{16l^4}\mathcal{A}(r) = 0\,, \tag{D.4b}$$

$$E^3 \equiv 0\,, \tag{D.4c}$$

where

$$\begin{aligned}\mathcal{A}(r) =& \, 4\left(8\lambda\sin^2(H)\left(-2l^2b_1^2 + b_2(2b_2+p) - b_3(q-2b_3)\right) + 4l^2 + \lambda\left(p^2+q^2\right)\right)H'' \\ & - 16\lambda\sin(2H)\left(2l^2b_1^2 - b_2(2b_2+p) + b_3(q-2b_3)\right)(H')^2 \\ & - 32\lambda\sin^2(H)\left(4l^2b_1b_1' - (4b_2+p)b_2' + (q-4b_3)b_3'\right)H' \\ & + \lambda\left(4l^2b_1^2\left(p^2+q^2\right) - (2qb_2 + p(q-2b_3))^2\right)\sin(4H) \\ & + 16l^2\left(2l^2b_1^2 - b_2(2b_2+p) + b_3(q-2b_3)\right)\sin(2H)\,.\end{aligned} \tag{D.5}$$

We can see that the τ_3 component becomes identically zero, while the other two are proportional after the substitution of all the involved quantities. The remaining ϕ variable is decoupled from r and the system is reduced to the single equation, $\mathcal{A} = 0$, for $H(r)$, which we have expressed in a more compact form in (4.26). At the same time, the current J^μ, as given by (D.2), is only

r-dependent and leads to the Maxwell set of equations (4.23).

The exact same thing can be repeated for the profile equation of the gauged time crystal (4.43). This time we have to consider (4.39) for the Euler angles, with the help of which the three L_μ components are written as (remember that now $x^\mu = (\gamma, r, z, \phi)$)

$$L^1_\mu = \sin(2H)\left(b_1\cos(\omega\gamma),\, -\frac{\sin(\omega\gamma)}{\sin(2H)}H',\, b_2\cos(\omega\gamma)\,,\, \left(\frac{1}{2}+b_3\right)\cos(\omega\gamma)\right)\,, \tag{D.6a}$$

$$L^2_\mu = \sin(2H)\,\left(b_1\sin(\omega\gamma),\, \frac{\cos(\omega\gamma)}{\sin(2H)}H'\,,\, b_2\sin(\omega\gamma)\,,\, \left(\frac{1}{2}+b_3\right)\sin(\omega\gamma)\right)\,, \tag{D.6b}$$

$$L^3_\mu = \left(\frac{\omega}{2} - 2b_1\sin^2(H)\,,\, 0,\, -2b_2\sin^2(H)\,,\, \frac{1}{2}\cos(2H) - 2b_3\sin^2(H)\right)\,. \tag{D.6c}$$

In the same manner the variables are decoupled in the three profile equations (D.3) and the system once more reduces to the single equation. The τ_3 component is identically zero, while the rest two are proportional to each other leading to a single equation for $H(r)$, which is given by (4.43). In particular, we obtain

$$E^1 = -\frac{\sin(\omega\gamma)}{16l^4}\mathcal{B}(r) = 0\,, \tag{D.7a}$$

$$E^2 = \frac{\cos(\omega\gamma)}{16l^4}\mathcal{B}(r) = 0\,, \tag{D.7b}$$

$$E^3 \equiv 0\,, \tag{D.7c}$$

with

$$\begin{aligned}\mathcal{B}(r) = {}& 4\left(8\lambda\sin^2(H)\left(l^2b_1(\omega-2b_1)+2b_2^2+b_3(1+2b_3)\right)+l^2\left(4-\lambda\omega^2\right)+\lambda\right)H'' \\ &+16\lambda\sin(2H)\left(l^2b_1(\omega-2b_1)+2b_2^2+b_3(1+2b_3)\right)(H')^2 \\ &+32\lambda\sin^2(H)\left(l^2(\omega-4b_1)b_1'+4b_2b_2'+(4b_3+1)b_3'\right)H' \\ &+\lambda\left(l^2\omega^2+4l^2b_1(b_1-\omega)+4b_2^2\left(l^2\omega^2-1\right)+4l^2\omega b_3(-2b_1+\omega b_3+\omega)\right)\sin(4H) \\ &-16l^2\left(l^2b_1(\omega-2b_1)+2b_2^2+2b_3^2+b_3\right)\sin(2H)\,.\end{aligned} \tag{D.8}$$

It is easy to verify that $\mathcal{B} = 0$ is equivalent to (4.43). Of course the same considerations are also true for the Maxwell equations and relation (D.2) still holds for the current, where now the M_{IJ} and N_I are given by the expressions (4.41) and (4.42), respectively.

Appendix E

Derivation of Surfaces Charges in Metric Formalism

E.1 Einstein-Hilbert-Maxwell action

The variation of the Lagrangian generated by an arbitrary vector field $\xi = \xi^\mu \partial_\mu$ and gauge transformation $\lambda' = \lambda + \xi^\mu A_\mu$ (with $\delta_\epsilon = \delta_\xi + \delta_{\lambda'}$) is given by

$$\begin{aligned}
\delta_\xi L &= E_{\mu\nu}\delta_\xi g^{\mu\nu} + E_\mu \delta_\epsilon A^\mu + \partial_\mu \Theta^\mu(\delta_\epsilon)\,, && \text{(E.1)}\\
\partial_\mu(\xi^\mu L) &= -E_{\mu\nu}(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) + E_\mu(\xi_\nu F^{\nu\mu} - \nabla^\mu \lambda) + \partial_\mu \Theta^\mu(\delta_\epsilon)\,, && \text{(E.2)}\\
\partial_\mu(\xi^\mu L) &= \partial_\mu\left[-2(\xi_\nu E^{\mu\nu}) - E^\mu \lambda + \Theta^\mu(\delta_\epsilon)\right]\,, && \text{(E.3)}
\end{aligned}$$

where we used the corresponding Noether identities $F^\mu{}_\nu E^\nu - 2\nabla_\nu E^{\mu\nu} = 0$ and $\nabla_\mu E^\mu = 0$. The explicit functions in (E.3) are

$$
\begin{aligned}
L &= \sqrt{-g}\left(\frac{\kappa}{2}(R-2\Lambda)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), && \text{(E.4)}\\
E^{\mu\nu} &= \sqrt{-g}\left[\frac{\kappa}{2}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R+\Lambda g^{\mu\nu}\right)-\frac{1}{2}\left(g_{\alpha\beta}F^{\alpha\mu}F^{\beta\nu}-\frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\right], && \text{(E.5)}\\
E^{\mu} &= -\sqrt{-g}\nabla_\nu F^{\mu\nu}, && \text{(E.6)}\\
\Theta^\mu(\delta_\epsilon) &= \sqrt{-g}\left(\kappa\nabla^{[\alpha}(g^{\mu]\beta}\delta_\xi g_{\alpha\beta})-\delta_\epsilon A_\nu F^{\mu\nu}\right), && \text{(E.7)}\\
&= \sqrt{-g}\left[\kappa\left(\nabla_\alpha\nabla^{(\alpha}\xi^{\mu)}-\nabla^\mu\nabla^\alpha\xi_\alpha\right)-\left(\xi^\alpha F_{\alpha\nu}-\nabla_\nu\lambda\right)F^{\mu\nu}\right].
\end{aligned}
$$

Replacing them we note that the three terms inside the total derivative in (E.3) can be rewritten as a total derivative too

$$
J^\mu_\epsilon \equiv \Theta^\mu(\delta_\epsilon)-\xi^\mu L-2\xi_\nu E^{\mu\nu}-\lambda E^\mu = \partial_\nu\left[-\sqrt{-g}\left(\kappa\nabla^{[\mu}\xi^{\nu]}-\lambda F^{\mu\nu}\right)\right], \tag{E.8}
$$

where we used $[\nabla^\mu,\nabla^\nu]\xi_\mu = R^{\mu\nu}\xi_\mu$. The term inside the total derivative is $\widetilde{Q}^{\mu\nu}_\xi$. Then, we obtain the surface charge density

$$
\begin{aligned}
k^{\mu\nu}_\xi &= \delta\widetilde{Q}^{\mu\nu}_\xi + 2\xi^{[\mu}\Theta^{\nu]}(\delta), && \text{(E.9)}\\
&= -\delta\left[\sqrt{-g}\left(\kappa\nabla^{[\mu}\xi^{\nu]}-\lambda F^{\mu\nu}\right)\right]+2\sqrt{-g}\xi^{[\mu}\left(\kappa\nabla^{[\alpha}(g^{\nu]]\beta}\delta g_{\alpha\beta})-\delta A_\alpha F^{\nu]\alpha}\right).
\end{aligned}
$$

Note that when using improved transformations we have $\delta\lambda = \delta(\lambda'-\xi^\mu A_\mu) = -\xi^\mu\delta A_\mu$.[1] Expanding all we obtain the surface charge density for this theory (7.87).

E.2 Einstein-Hilbert-Skyrme action

The variation of the Lagrangian (7.88) generated by an arbitrary vector field $\xi = \xi^\mu\partial_\mu$ is

[1] Equivalently, we can forget this and note that in the symplectic structure density the variations do not commute, thus we should include a term $\Theta([\delta,\delta_\epsilon])$ which will contribute to the surface charge. The specific non-commutation in this case is $[\delta,\delta_\epsilon]A_\mu = \delta_{(\xi^\nu\delta A_\nu)}A_\mu = -\partial_\mu(\xi^\nu\delta A_\nu)$.

$$\delta_\xi L = E_{\mu\nu}\delta_\xi g^{\mu\nu} + \langle E_U \delta_\xi U\rangle + \partial_\mu \Theta^\mu(\delta_\xi)\,, \tag{E.10}$$

$$\partial_\mu(\xi^\mu L) = -E_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) + \langle E_U \xi_\mu \nabla^\mu U\rangle + \partial_\mu \Theta^\mu(\delta_\xi)\,, \tag{E.11}$$

$$\partial_\mu(\xi^\mu L) = \partial_\mu\left[-2\xi_\nu E^{\mu\nu} + \Theta^\mu(\delta_\xi)\right]\,, \tag{E.12}$$

where we used the Noether identity $2\nabla^\mu E_{\mu\nu} + \langle E_U \nabla_\nu U\rangle = 0$. The explicit functions in (E.12) are

$$L = \sqrt{-g}\left[\frac{\kappa}{2}(R-2\Lambda) + \frac{K}{4}\left\langle R^\mu R_\mu + \frac{\lambda}{8}F_{\mu\nu}F^{\mu\nu}\right\rangle\right] \tag{E.13}$$

$$E^{\mu\nu} = \sqrt{-g}\left[\frac{\kappa}{2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \Lambda g^{\mu\nu}\right)\,,\right.$$
$$\left. + \frac{K}{4}\left\langle R^\mu R^\nu - \frac{1}{2}g^{\mu\nu}R^\alpha R_\alpha + \frac{\lambda}{4}\left(F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)\right\rangle\right]\,, \tag{E.14}$$

$$E_U = -\frac{K}{2}\sqrt{-g}\left\langle \nabla_\mu\left(R^\mu + \frac{\lambda}{4}[R_\nu, F^{\mu\nu}]\right)U^{-1}\right\rangle\,, \tag{E.15}$$

$$\Theta^\mu(\delta_\xi) = \sqrt{-g}\left[\kappa\nabla^{[\alpha}(g^{\mu]\beta}\delta_\xi g_{\alpha\beta}) + \frac{K}{2}\left\langle\left(R^\mu + \frac{\lambda}{4}[R_\nu, F^{\mu\nu}]\right)U^{-1}\delta_\xi U\right\rangle\right]\,. \tag{E.16}$$

and again, after cancellations, we simply have

$$J^\mu_\epsilon \equiv \Theta^\mu(\delta_\xi) - \xi^\mu L - 2\xi_\nu E^{\mu\nu} = \partial_\nu\left[-\kappa\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\right]\,,$$

where the cyclic property of the trace was used to show that $\langle [R_\nu, F^{\mu\nu}]\xi^\alpha R_\alpha - \xi_\nu F^{\mu\alpha}F^\nu{}_\alpha\rangle = 0$. Therefore, $\widetilde{Q}^{\mu\nu}_\xi = -\kappa\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}$, and we use the surface charge formula $k^{\mu\nu}_\xi = \delta\widetilde{Q}^{\mu\nu}_\xi + 2\xi^{[\mu}\Theta^{\nu]}(\delta_\xi)$ to get (7.91).

E.3 Einstein-Cartan-Λ

Consider four-dimensional General Relativity with a cosmological term in the differential form language[2]

[2]We will omit the wedge product $\wedge$ among differential forms throughout these Appendix.

$$S[e^a,\omega^{ab}] = \kappa' \int_{\mathcal{M}} \varepsilon_{abcd} \left(R^{ab} e^c e^d \pm \frac{1}{2\ell^2} e^a e^b e^c e^d \right) , \quad \ell^2 = \frac{3}{|\Lambda|} , \tag{E.17}$$

with $a, b = 0, 1, 2, 3$. The general variation of the Lagrangian density is

$$\delta L = E_a \delta e^a + E_{ab} \delta\omega^{ab} + d\Theta(\delta\omega) . \tag{E.18}$$

The equations of motion and boundary term are given by

$$E_a = 2\kappa' \varepsilon_{abcd} \left(R^{bc} \pm \frac{1}{\ell^2} e^b e^c \right) e^d = 0 , \tag{E.19}$$

$$E_{ab} = 2\kappa' \varepsilon_{abcd} T^c e^d = 0 , \tag{E.20}$$

$$\Theta(\delta\omega) = \kappa' \varepsilon_{abcd} \delta\omega^{ab} e^c e^d . \tag{E.21}$$

The action (E.17) is invariant under diffeomorphisms and local Lorentz transformations. Infinitesimal generators of these symmetries are a vector field ξ and the set of parameters λ^{ab}, we group them in $\epsilon = (\xi, \lambda^{ab})$. Both can be combined such that the dynamical fields transform infinitesimally as

$$\delta_\epsilon e^a = d_\omega(\xi \lrcorner e^a) + \xi \lrcorner (d_\omega e^a) + \lambda^a{}_b e^b , \tag{E.22}$$

$$\delta_\epsilon \omega^{ab} = \xi \lrcorner R^{ab} - d_\omega \lambda^{ab} . \tag{E.23}$$

The symplectic structure density computed with these local symmetries as one of its entries is

$$\Omega(\delta,\delta_\epsilon) = \delta\Theta(\delta_\epsilon\omega) - \delta_\epsilon\Theta(\delta\omega) - \Theta([\delta,\delta_\epsilon]\omega) , \tag{E.24}$$

$$= \kappa' \varepsilon_{abcd} \left(\delta_\epsilon \omega^{ab} \delta(e^c e^d) - \delta\omega^{ab} \delta_\epsilon(e^c e^d) \right) , \tag{E.25}$$

$$= 2\kappa' \varepsilon_{abcd} \left([\xi \lrcorner R^{ab} - d_\omega \lambda^{ab}] e^c \delta e^d \; - \; \delta\omega^{ab} [d_\omega \xi \lrcorner e^c + \xi \lrcorner d_\omega e^c + \lambda^c{}_f e^f] e^d \right) , \tag{E.26}$$

$$= dk_\epsilon , \tag{E.27}$$

note a subtle point, the term $\Theta([\delta,\delta_\epsilon]\omega)$ should be formally included to guarantee the bilinearity on δ and δ_ϵ because the variations do not commute in general. The surface charge density is

$$k_\epsilon = -\kappa' \varepsilon_{abcd} \left(\lambda^{ab} \delta(e^c e^d) - \delta\omega^{ab} \xi \lrcorner (e^c e^d) \right) . \tag{E.28}$$

In the last step of (E.27) we used both, the equations of motion and the linearized equations of motion too. Knowing the result of this kind of calculation the strategy is always to rearrange the exterior derivatives on the parameters ξ and λ^{ab}, second and third terms in (E.26), to complete exact differential forms and then check that all the remaining terms vanish due to the equations of motion and the linearized equations of motion, for instance $T^a = de^a + \omega^a{}_b e^b = 0$ and $\delta T^a = d\delta e^a + \delta\omega^a{}_b e^b + \omega^a{}_b \delta e^b = 0$, as a general rule all of them have to be explicitly used.

E.4 Einstein-Cartan-Yang-Mills

The four-dimensional Einstein-Cartan gravity coupled to a non-Abelian field reads

$$S[e^a, \omega^{ab}, A] = \int_{\mathcal{M}} \left(\kappa' \varepsilon_{abcd} R^{ab} e^c e^d + \alpha_{YM} \langle F \star F \rangle \right) , \tag{E.29}$$

where $\langle \cdot \rangle$ denotes an invariant bilinear form on the Lie algebra of the non-Abelian Lie group $SU(N)$, α_{YM} is the coupling constant, the two-form F is defined as $F = dA + A \wedge A = \frac{1}{2} F^i{}_{ab} e^a e^b \tau_i$, the Hodge operator acts as $\star F = \frac{1}{2} \varepsilon_{abcd} F^{iab} e^c e^d \tau_i$, with τ_i the $SU(N)$ generators.

The variation of the Lagrangian is

$$\delta L = E_a \delta e^a + E_{ab} \delta\omega^{ab} + \langle E_A \delta A \rangle + d\Theta(\delta\omega, \delta A) , \tag{E.30}$$

with the equations of motion and boundary term given by

$$
\begin{aligned}
E_a &= \kappa' \varepsilon_{abcd} R^{bc} e^d - \alpha_{_{YM}} \langle e_a \lrcorner F \star F - F e_a \lrcorner \star F \rangle = 0\,, &\text{(E.31)}\\
E_{ab} &= 2\kappa' \varepsilon_{abcd} T^c e^d = 0\,, &\text{(E.32)}\\
E_A &= -2\alpha_{_{YM}}\, d_A \star F = 0\,, &\text{(E.33)}\\
\Theta(\delta\omega, \delta A) &= \kappa' \varepsilon_{abcd} \delta\omega^{ab} e^c e^d + 2\alpha_{_{YM}} \langle \delta A \star F \rangle\ , &\text{(E.34)}
\end{aligned}
$$

with the covariant derivative defined by $d_A(\cdot) = d(\cdot) + [A, \cdot]$. Remember the operation of the interior product $e_a \lrcorner F = \frac{1}{2} F_{bc} e_a \lrcorner (e^b e^c) = \frac{1}{2} F_{bc}(\delta^b_a e^c - e^b \delta^c_a) = F_{ac} e^c$.

The gauge symmetries are diffeomorphisms, local Lorentz transformations, and $SU(N)$ acting on A. The parameters of the infinitesimal symmetries are grouped in $\epsilon = (\xi, \lambda^{ab}, \lambda^i)$, with λ^i the components of the algebra valued gauge parameter $\lambda = \lambda^i \tau_i$. The improved exact symmetry conditions are

$$
\begin{aligned}
\delta_\epsilon e^a &= d_\omega(\xi \lrcorner e^a) + \xi \lrcorner (d_\omega e^a) + \lambda^a{}_b e^b = 0\,, &\text{(E.35)}\\
\delta_\epsilon \omega^{ab} &= \xi \lrcorner R^{ab} - d_\omega \lambda^{ab} = 0\,, &\text{(E.36)}\\
\delta_\epsilon A^i &= \xi \lrcorner F^i - d_A \lambda^i = 0\,. &\text{(E.37)}
\end{aligned}
$$

As showed in the general case for the differential form language, the surface charge density is the sum of three terms

$$
k_\epsilon = \delta \widetilde{Q}_\epsilon - \xi \lrcorner \Theta(\delta) - B_{\delta\epsilon}\,. \tag{E.38}
$$

We already have the boundary term, (E.34). Evaluating on an infinitesimal gauge symmetry $E_a \delta_\epsilon e^a + E_{ab} \delta_\epsilon \omega^{ab} + \langle E_A \delta_\epsilon A \rangle = dS_\epsilon + N_\epsilon$, with the Noether identities $N_\epsilon = 0$, we obtain S_ϵ. Then, as usual, the would-be Noether charge $J_\epsilon = \Theta(\delta_\epsilon) - \xi \lrcorner L + S_\epsilon = d\widetilde{Q}_\epsilon$ is an exact form with

$$
\widetilde{Q}_\epsilon = -\kappa' \varepsilon_{abcd} \lambda^{ab} e^c e^d - 2\alpha_{_{YM}} \langle \lambda \star F \rangle\ . \tag{E.39}
$$

Now, we use that $[\delta, \delta_\epsilon] = [\delta, \mathscr{L}_\xi + \delta_{\lambda^{ab} + \xi \lrcorner \omega^{ab}} + \delta_{\lambda + \xi \lrcorner A}] = \delta_{\delta\lambda^{ab} + \xi \lrcorner \delta\omega^{ab}} + \delta_{\delta\lambda + \xi \lrcorner \delta A}$ because the vector field ξ is assumed fixed on the phase space. Then, we have $\Theta([\delta, \delta_\epsilon]) = dB_{\delta\epsilon} + C_\epsilon$ such that on-shell $C_\epsilon \approx 0$. Thus, we obtain

$$B_{\delta\epsilon} = -\kappa' \varepsilon_{abcd}(\delta\lambda^{ab} + \xi\lrcorner\delta\omega^{ab})e^c e^d - 2\alpha_{YM}\left\langle(\delta\lambda + \xi\lrcorner\delta A) \star F\right\rangle \,. \tag{E.40}$$

Replacing all back in the general expression (E.38) we get

$$k_\epsilon = -\kappa' \varepsilon_{abcd}\left(\lambda^{ab}\delta(e^c e^d) - \delta\omega^{ab}\xi\lrcorner(e^c e^d)\right) - 2\alpha_{YM}\left\langle\lambda\delta \star F - \delta A\xi\lrcorner \star F\right\rangle \,, \tag{E.41}$$

with the first two terms just the surface charge density of pure gravity (E.28). Thus, roughly, to consider the extension to a general YM theory from a pure electromagnetic field one should include the $\langle\cdot\rangle$ brackets to deal with the algebra valued fields.

E.5 Einstein-Cartan in $(2+1)$-dimensions plus a Torsional Term

In this appendix, we compute the surface charge density explicitly. Because is faster and equivalent, we use the contracting homotopy operator method to do it. Consider the Lagrangian for $(2+1)$-space-time dimensions

$$L = \varepsilon_{abc}e^a R^{bc} + \beta e_a T^a \,, \tag{E.42}$$

where $T^a = de^a + \omega^a{}_b e^b$, and the term proportional to β is the one that would produce torsion. The variation of the Lagrangian is

$$\delta L = \delta e^a(\varepsilon_{abc}R^{bc} + 2\beta T_a) + \delta\omega^{ab}(\varepsilon_{abc}T^c - \beta e_a e_b) - d(\varepsilon_{abc}e^a\delta\omega^{bc} + \beta e^a \delta e_a) \,, \tag{E.43}$$

the second equation of motion tells us that torsion does not vanish, and because of the first one, we can advance that in fact it plays the role of a cosmological constant term.

With the improved infinitesimal gauge transformation for the fields, $\delta_\epsilon e^a$ and $\delta_\epsilon\omega^{ab}$ from

(7.130) and (7.131), respectively, we rearrange the combination $E_a \delta_\epsilon e^a + E_{ab} \delta_\epsilon \omega^{ab} = dS_\epsilon + N_\epsilon$, such that

$$S_\epsilon = \xi \lrcorner e^a (\varepsilon_{abc} R^{bc} + 2\beta T_a) - \lambda^{ab} (\varepsilon_{abc} T^c - \beta e_a e_b), \tag{E.44}$$

and $N_\epsilon = 0$ are the Noether identities. The S_ϵ is what we need to compute the surface charge density using the contracting homotopy operator. For the Einstein-Cartan theory the operator is (see Eq. (3.29) in [127])

$$I_{\delta e, \delta\omega} \equiv \delta e^a \frac{\partial}{\partial T^a} + \delta\omega^{ab} \frac{\partial}{\partial R^{ab}} , \tag{E.45}$$

because apart from e^a and ω^{ab} there are no extra fields in the phase space, this is the full operator for this theory. Then, the surface charge density, $k_\epsilon \equiv I_{\delta e, \delta\omega} S_\epsilon$, becomes

$$k_\epsilon = -\varepsilon_{abc} (\lambda^{ab} \delta e^c - \delta\omega^{ab} \xi \lrcorner e^c) + 2\beta \xi \lrcorner e^a \delta e_a \,. \tag{E.46}$$

By following the prescription that uses the symplectic structure density, $\Omega(\delta, \delta_\epsilon) = dk_\epsilon$, we arrive at exactly the same expression.

We remember that in general, boundary terms (exact forms) at the level of the Lagrangian do not contribute to the surface charges, or at most they contribute as exact forms in the k_ϵ formula and therefore can be neglected. However, notice that the term used here can not be written as an exact form, the tentative term one would try vanishes identically $d(e^a e_a) = T^a e_a - e^a T_a = 0$. Then, at this level $\beta e_a T^a$ is a genuine term that produces torsion. In particular, it contributes to the surface charge density as is explicit in (E.46) with the last term, and again, this contribution is not an exact form at this level either.

Now, we make a step further to analyze the surface charge density by performing a split of the connection in torsionless and contorsion parts. That is,

$$\omega^{ab} = \tilde{\omega}^{ab} + \bar{\omega}^{ab} \,, \tag{E.47}$$

with $\widetilde{\omega}^{ab}(e)$ solving $de^a + \widetilde{\omega}^a{}_b e^b = 0$. Then, the torsion is simply $T^a = \bar{\omega}^a{}_b e^b$. From the equation of motion, $\varepsilon_{abc}T^a = \beta e_b e_c$, we solve

$$\bar{\omega}^{ab} = \frac{\beta}{2}\varepsilon^{abc}e_c\,, \tag{E.48}$$

where we used $\varepsilon_{abc}\varepsilon^{dbc} = -\varepsilon_{abc}\tilde{\varepsilon}^{dbc} = -2\delta^d_a$.[3] The split of the connection induces also a split of the parameter, λ^{ab}, that solves the exact symmetry condition. In general the condition $\delta_\epsilon e^a = 0$ is solved by

$$\begin{aligned}
\lambda^{ab} &= e^a\lrcorner(d_\omega(\xi\lrcorner e^b) + \xi\lrcorner d_\omega e^b)\,, && \text{(E.49)}\\
&= e^a\lrcorner(d(\xi\lrcorner e^b) + \tilde{\omega}^b{}_c\xi\lrcorner e^c + \bar{\omega}^b{}_c\xi\lrcorner e^c + \xi\lrcorner(\bar{\omega}^b{}_c e^c))\,, && \text{(E.50)}\\
&= e^a\lrcorner(d_{\tilde{\omega}}(\xi\lrcorner e^b)) - \xi\lrcorner\bar{\omega}^{ab}\,, && \text{(E.51)}\\
&= \tilde{\lambda}^{ab} - \xi\lrcorner\bar{\omega}^{ab}\,, && \text{(E.52)}
\end{aligned}$$

where we used $d_{\widetilde{\omega}}e^b = 0$, $e^a\lrcorner e^c = \eta^{ac}$, and introduced $\tilde{\lambda}^{ab} \equiv e^a\lrcorner(d_{\tilde{\omega}}(\xi\lrcorner e^b))$, the torsionless part of the parameter. An equivalent way to define this parameter is to use the exact symmetry condition but improving the transformation only with the torsionless connection

$$\delta_\epsilon e^a = \xi\lrcorner(d_{\widetilde{\omega}}e^a) + d_{\widetilde{\omega}}(\xi\lrcorner e^a) + \tilde{\lambda}^a{}_b e^b = 0\,, \tag{E.53}$$

note that the first term vanishes by construction. Collecting all, we can now use the split of the connection and the λ^{ab} parameter on the surface charge density formula to show that

$$\begin{aligned}
k_\epsilon &= -\varepsilon_{abc}(\tilde{\lambda}^{ab}\delta e^c - \delta\tilde{\omega}^{ab}\xi\lrcorner e^c) + \varepsilon_{abc}(\xi\lrcorner\bar{\omega}^{ab}\delta e^c + \delta\bar{\omega}^{ab}\xi\lrcorner e^c) + 2\beta\xi\lrcorner e_a\delta e^a\,, && \text{(E.54)}\\
&= \tilde{k}_\epsilon + \frac{\beta}{2}\varepsilon_{abc}\varepsilon^{abd}(\xi\lrcorner e_d\delta e^c + \delta e_d\xi\lrcorner e^c) + 2\beta\xi\lrcorner e_a\delta e^a\,, && \text{(E.55)}\\
&= \tilde{k}_\epsilon\,, && \text{(E.56)}
\end{aligned}$$

[3] Remember $\varepsilon^{dbc} = \eta^{dd'}\eta^{bb'}\eta^{cc'}\varepsilon_{d'b'c'} = -\varepsilon_{dbc} = -\tilde{\varepsilon}^{dbc}$ we put a twiddle to the Levi-Civita symbol that do not carry information about the flat metric even if it has upstairs indices. Check this prescription in (1.7).

in the second line we defined $\tilde{k}_\epsilon \equiv -\varepsilon_{abc}(\tilde{\lambda}^{ab}\delta e^c - \delta\widetilde{\omega}^{ab}\xi\lrcorner e^c)$ and used the explicit expression for $\bar{\omega}^{ab}$ computed at (E.48). To reach the third line remember that $\varepsilon^{abd} = -\tilde{\varepsilon}^{abd}$, as in (1.7), then $\varepsilon_{abc}\varepsilon^{abd} = -2\delta^d_c$. The remarkable fact is the third line. At the level of surface charge density, the torsion contribution to the connection cancels exactly with the extra source term $2\beta\xi\lrcorner e_a\delta e^a$. In other words, the surface charge can be computed with the usual expression if one uses the Levi-Civita, or torsionless, connection $\tilde{\omega}^{ab}(e)$.

The equation $k_\epsilon = \tilde{k}_\epsilon$ tells us that contorsion will never contribute to the charges in this theory. For this simple theory this result could have been expected as from the beginning we knew that the torsional term in the action is equivalent to a cosmological constant term. And we already know, at least in four dimensions but for any dimensions is the same, that a cosmological term do not enter in the surface charge formula.[4]

E.6 From a Chern-Simons perspective

The previous result can be understood from a Chern-Simons (CS) perspective too. In fact, it is well-known that three-dimensional General Relativity with negative cosmological constant can be written as a topological Chern-Simons theory of a one-form gauge connection $\widetilde{A} = e^a\widetilde{P}_a + \omega^a\widetilde{J}_a$ valued on the anti-de Sitter algebra in three space-time dimensions (AdS_3), $\mathfrak{so}(2,1)$. For the spin connection we use $\omega_a = \frac{1}{2}\varepsilon_{abc}\omega^{cb}$. The algebra reads

$$[\widetilde{P}_a, \widetilde{P}_b] = \Lambda\varepsilon_{abc}\widetilde{J}^c\,, \qquad [\widetilde{J}_a, \widetilde{P}_b] = \varepsilon_{abc}\widetilde{P}^c\,, \qquad [\widetilde{J}_a, \widetilde{J}_b] = \varepsilon_{abc}\widetilde{J}^c\,, \tag{E.57}$$

[4] As an extra comment, we contrast our results with the charges computed in [153]. To compare, all contributions coming from the action term, $\omega d\omega + \frac{2}{3}\omega^3$, in [153], shall be set to zero. Still, due to the e_aT^a term in the action, it is found that the theory admits a so-called *BTZ solution with torsion*. The formulas for the mass and angular momentum presented in [153] have a direct torsion contribution, and not only through the *effective cosmological constant* parameter, as can be appreciated in Eqs. (20) and (21) there. This is in tension with our results because, as we just checked, torsion disappears from our charge formulas. Another curiosity is that in [153] the proposed quasi-local charge expressions depend on the r coordinate. This dependence is avoided there, in their final formula, by taking the usual $r \to \infty$. From the surface charges density perspective we adopt here this can not happen simply because of the conservation law, $dk_\epsilon = 0$, guarantee independence of the radius.

with Λ the cosmological constant. The algebra (E.57) admits a non-degenerate and invariant bilinear form

$$\langle \widetilde{J}_a, \widetilde{P}_b \rangle = \eta_{ab} \,. \tag{E.58}$$

These two ingredients provide a CS construction for three-dimensional General Relativity as follows

$$\begin{aligned} L_{CS} &= \left\langle \widetilde{A} \wedge d\widetilde{A} + \frac{1}{3}\widetilde{A} \wedge [\widetilde{A}, \widetilde{A}] \right\rangle \,, \\ &= 2e^a R_a(\omega) + \frac{\Lambda}{3}\varepsilon_{abc} e^a e^b e^c + d(e^a \omega_a) \,, \end{aligned} \tag{E.59}$$

with the curvature $R_a(\omega) = d\omega_a + \frac{1}{2}\varepsilon_{abc}\omega^b\omega^c$. Note that the equivalence is up to a boundary term. Now, the physics does not depend on the chosen algebra basis. Let us introduce a different basis for the algebra generators

$$P_a = \widetilde{P}_a + \frac{\beta}{2}\widetilde{J}_a \,, \tag{E.60}$$

$$J_a = \widetilde{J}_a \,, \tag{E.61}$$

with β a constant. Then, the AdS_3 algebra commutators (E.57) in this basis are

$$\begin{aligned} [P_a, P_b] &= \left(\Lambda - \frac{\beta^2}{4} \right) \varepsilon_{abc} J^c + \beta \varepsilon_{abc} P^c \,, \\ [J_a, P_b] &= \varepsilon_{abc} P^c \,, \\ [J_a, J_b] &= \varepsilon_{abc} J^c \,. \end{aligned} \tag{E.62}$$

The invariant and non-degenerate bilinear form associated to (E.62) is now

$$\langle J_a, P_b \rangle = \eta_{ab} \,, \qquad \langle P_a, P_b \rangle = \beta \eta_{ab} \,. \tag{E.63}$$

Thus, the equivalent CS Lagrangian for the one-form gauge connection $A = e^a P_a + \omega^a J_a$ valued on the algebra (E.57) and associated to the bilinear form (E.63) is

$$L_{CS} \;=\; 2e^a R_a(\omega) + \beta e^a T_a + \frac{\Lambda_{eff}}{3}\varepsilon_{abc}e^a e^b e^c + d(e^a\omega_a)\,, \tag{E.64}$$

with the torsion $T_a = de_a + \varepsilon_{abc}\omega^b e^c$, and the effective cosmological constant $\Lambda_{eff} \equiv \Lambda - \frac{3}{4}\beta^2$. We can choose the parameter $\beta^2 = \frac{4}{3}\Lambda$ and the last Lagrangian becomes exactly the Lagrangian (E.42) considered previously. Therefore, we conclude that the torsional Lagrangian (E.42) is just equivalent to the Einstein-Cartan Lagrangian (E.59) with a specific value for the cosmological constant $\Lambda = \frac{3}{4}\beta^2$.

E.7 Einstein-Cartan-Dirac

Consider the gravity contribution to the surface charge density in four space-time dimensions

$$\mathring{k}_\epsilon = -\kappa'\varepsilon_{abcd}\left[\lambda^{ab}\delta(e^c e^d) - \delta\omega^{ab}\xi\lrcorner(e^c e^d)\right]\,. \tag{E.65}$$

As a preliminary we will rearrange this formula. First note that the spin connection can have a torsion part, named the contorsion, we want to isolate its contribution into the formula. Exactly as in the previous appendix, (E.52), we perform a split of the spin connection, $\omega^{ab} = \tilde{\omega}^{ab}(e) + \bar{\omega}^{ab}$, such that $d_{\tilde{\omega}}e^a = 0$. The exact symmetry condition, $\delta_\epsilon e^a = 0$, is solved by the parameter

$$\lambda^{ab} \;=\; e^{[a}\lrcorner(\xi\lrcorner d_\omega e^{b]}) + e^{[a}\lrcorner(d_\omega \xi\lrcorner e^{b]}) \;=\; e^{[a}\lrcorner(d_{\tilde{\omega}}\xi\lrcorner e^{b]}) - \xi\lrcorner\bar{\omega}^{ab}\,, \tag{E.66}$$

thus, the split of the connection is translated in a split of the parameter $\lambda^{ab} = \tilde{\lambda}^{ab} + \bar{\lambda}^{ab}$. Then, the gravity contribution to the surface charge density has two parts

$$\mathring{k}_\epsilon = \tilde{k}_\epsilon + \bar{k}_\epsilon\,, \tag{E.67}$$

the torsionless part

$$\tilde{k}_\epsilon = -\kappa' \varepsilon_{abcd} \left[\tilde{\lambda}^{ab} \delta(e^c e^d) - \delta\tilde{\omega}^{ab} \xi \lrcorner (e^c e^d) \right] , \tag{E.68}$$

where $\tilde{\lambda}^{ab} = e^a \lrcorner (d_{\tilde{\omega}}(\xi \lrcorner e^b))$, and the contorsion part that we wanted to isolate

$$\bar{k}_\epsilon = -\kappa' \varepsilon_{abcd} \left[\bar{\lambda}^{ab} \delta(e^c e^d) - \delta\bar{\omega}^{ab} \xi \lrcorner (e^c e^d) \right] , \tag{E.69}$$

where $\bar{\lambda}^{ab} = -\xi \lrcorner \bar{\omega}^{ab}$. If we further express the contorsion one-form in frame components, $\bar{\omega}^{ab} = \bar{\omega}^{ab}{}_f e^f$, we can write

$$\bar{k}_\epsilon = 2\kappa' \varepsilon_{abcd} \left[\bar{\omega}^{ab}{}_f e^c (\xi \lrcorner e^f \delta e^d + \xi \lrcorner e^d \delta e^f) - \delta\bar{\omega}^{ab}{}_f e^f e^c \xi \lrcorner e^d \right] . \tag{E.70}$$

Now, we compute the whole surface charge density. Consider the Einstein-Cartan-Dirac action

$$S[e^a, \omega^{ab}, \psi] = \int_{\mathcal{M}} \varepsilon_{abcd} e^a e^b \left[\kappa' R^{cd} - \frac{i}{3} \alpha_\psi \, e^c \left(\bar{\psi} \gamma^d \gamma_5 d_\omega \psi + \overline{d_\omega \psi} \gamma^d \gamma_5 \psi \right) \right] , \tag{E.71}$$

with $d_\omega \psi = d\psi + \frac{1}{2} \omega_{ab} \gamma^{ab} \psi$ and $\gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b]$ satisfying the Lorentz algebra. The special matrix $\gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3$ satisfies $\gamma_5 \gamma_a = -\gamma_a \gamma_5$. The following computation of surface charge is very sensitive to the coefficients, therefore we make a *scriptsize* detour to be self-contained and to check the consistence of our conventions.

The γ-matrices satisfy the Clifford algebra $\{\gamma_a, \gamma_b\} = \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$. Then, we have also $[\gamma_a, (\gamma_b \gamma_c - \gamma_c \gamma_b)] = 4(\eta_{ab} \gamma_c - \eta_{ac} \gamma_b)$. If we define $\gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b]$ we can check that $[\gamma_a, \gamma_{bc}] = \eta_{ab} \gamma_c - \eta_{ac} \gamma_b$, and the matrices γ_{ab} satisfy the Lorentz algebra, namely

$$[\gamma_{ab}, \gamma_{cd}] = \eta_{bd} \gamma_{ca} - \eta_{ad} \gamma_{cb} + \eta_{bc} \gamma_{ad} - \eta_{ac} \gamma_{bd} , \tag{E.72}$$

this fix the 1/4 normalization of the γ_{ab} definition. The coefficient multiplying the connection in the covariant derivative acting on a spinor is fixed by the defining equation of the covariant derivatives on spinors

$$d_{\omega'}(\Lambda \psi) = \Lambda d_\omega \psi . \tag{E.73}$$

The Lorentz transformation is $\Lambda = \exp\left(\frac{1}{2}\lambda^{ab}\gamma_{ab}\right)$, thus we define the algebra valued coefficient $\lambda = \frac{1}{2}\lambda^{ab}\gamma_{ab}$. We check that the 1/2 is consistent with (E.73). If we expand to first order in the Lorentz transformation both sides of (E.73), we get

$$d\lambda\psi + \lambda d\psi - \frac{1}{2}(d_\omega\lambda_{ab})\gamma^{ab}\psi + \frac{1}{2}\omega_{ab}\gamma^{ab}\lambda\psi = \lambda d\psi + \frac{1}{2}\lambda\omega_{ab}\gamma^{ab}\psi\,, \tag{E.74}$$

where we used $\omega'^{ab} = \omega^{ab} + \delta_\lambda\omega^{ab}$ with $\delta_\lambda\omega_{ab} = -d_\omega\lambda_{ab} = -d\lambda_{ab} + \lambda_{ac}\omega^c{}_b - \lambda^c{}_b\omega_{ac}$. The last expression brings into the formula the convention for the infinitesimal transformation λ_{ab} used in the other variables ($\delta_\lambda e^a = \lambda^a{}_b e^b$ or equivalently $\delta_\lambda\omega_{ab} = -d_\omega\lambda_{ab}$). Then, we check that (E.74) is in fact an identity because our conventions are correct such that $d\lambda = \frac{1}{2}d\lambda_{ab}\gamma^{ab}$, and

$$\omega_{ab}\left[\lambda, \gamma^{ab}\right] = \frac{1}{2}\omega_{ab}\lambda_{cd}\left[\gamma^{cd}, \gamma^{ab}\right] = -\frac{1}{2}\omega_{ab}\lambda_{cd}\left[\gamma^{ab}, \gamma^{cd}\right]\,, \tag{E.75}$$

$$= \left(\lambda_{ac}\omega^c{}_b - \lambda^c{}_b\omega_{ac}\right)\gamma^{ab}\,. \tag{E.76}$$

Those checks set correctly the three coefficients in $\gamma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, $\lambda = \frac{1}{2}\lambda_{ab}\gamma^{ab}$, and $d_\omega\psi = d\psi + \frac{1}{2}\omega_{ab}\gamma^{ab}\psi$.

Now, besides the usual exact symmetry conditions on the gravity fields, (7.130) and (7.131), we should impose the exact symmetry condition on the spinor field. Spinor field transform under an infinitesimal local Lorentz transformation as $\delta_{\lambda'}\psi = \lambda'\psi$. Therefore, the correct exact symmetry condition is

$$\delta_\epsilon\psi = \mathscr{L}_\xi\psi + \lambda'\psi = \xi\lrcorner d_\omega\psi + \lambda\psi = 0\,, \tag{E.77}$$

with, again, the improved prescription given by $\lambda = \frac{1}{2}\gamma_{ab}(\lambda'^{ab} - \xi\lrcorner\omega^{ab})$ (remember $\lambda_{ab} = \lambda'_{ab} - \xi\lrcorner\omega_{ab}$).

A general formula for the surface charge density in differential form language is (see Eq. (2.19) in [127])

$$k_\epsilon = \delta\widetilde{Q}_\epsilon - \xi\lrcorner\Theta(\delta) - B_{\delta\epsilon}\,. \tag{E.78}$$

The variation of the Lagrangian is

$$\delta L = E_a\delta e^a + E_{ab}\omega^{ab} + E_\psi\delta\psi + +E_{\bar\psi}\delta\bar\psi + d\Theta(\delta)\,, \tag{E.79}$$

with the boundary term

$$\Theta(\delta) = \varepsilon_{abcd} e^a e^b \left(\kappa' \delta\omega^{cd} + \frac{i}{3} \alpha_\psi \, e^c \delta \left(\bar{\psi} \gamma^d \gamma_5 \psi \right) \right). \tag{E.80}$$

This is the middle term we need in (E.78). For the first term we compute the *trivial current* $J_\epsilon = \Theta(\delta_\epsilon) - \xi \lrcorner L + S_\epsilon = d\widetilde{Q}_\epsilon$, and after cancellations we get the usual $\widetilde{Q}_\epsilon = -\kappa' \varepsilon_{abcd} e^a e^b \lambda^{cd}$ we find for pure Einstein-Cartan theory. For the third term in (E.78), the prescription tells us that the symplectic potential term $\Theta([\delta, \delta_\epsilon]) = dB_{\delta\epsilon} + C_{\delta\epsilon}$ with $C_{\delta\epsilon} \approx 0$, thus we use the commutation of variations $[\delta, \delta_\epsilon] = [\delta, \mathscr{L}_\xi + \delta_{\lambda + \xi \lrcorner \omega}] = \delta_{\delta\lambda + \xi \lrcorner \delta\omega}$, and we get $B_{\delta\epsilon} = -\kappa' \varepsilon_{abcd} e^a e^b (\delta\lambda^{cd} + \xi \lrcorner \delta\omega^{cd})$. Thus the spinor field does not contribute through $B_{\delta\epsilon}$ nor $\widetilde{Q}_\epsilon$ in the general formula (E.78), it only enters through the extra boundary term in (E.80). The complete surface charge density for the Einstein-Cartan-Dirac theory is

$$k_\epsilon = -\varepsilon_{abcd} \left(\kappa' \left(\lambda^{ab} \delta(e^c e^d) - \delta\omega^{ab} \xi \lrcorner (e^c e^d) \right) + i\alpha_\psi \, \xi \lrcorner e^a e^b e^c \delta \left(\bar{\psi} \gamma^d \gamma_5 \psi \right) \right). \tag{E.81}$$

We stress that the addition of a spinorial mass term in the action does not change the surface charge formula. Then, this result is already useful enough to compute charges for this theory, its massive spinor equivalent, or even with an additional cosmological constant term. But we can go further. Let us split the gravity terms as we did at the beginning, (E.67), then

$$k_\epsilon \; = \; \tilde{k}_\epsilon + \bar{k}_\epsilon + k_\epsilon^\psi \,, \tag{E.82}$$

with

$$k_\epsilon^\psi = -i\alpha_\psi \, \varepsilon_{abcd} \xi \lrcorner e^a e^b e^c \delta (\bar{\psi} \gamma^d \gamma_5 \psi) \,. \tag{E.83}$$

We can compute $\bar{k}_\epsilon$ explicitly, as given by (E.70), by solving the contorsion $\bar{\omega}^{ab}$ from the torsion equation of motion. We do it step by step. The equation we need is

$$\varepsilon_{abcd} T^c e^d = \frac{i}{12} \alpha_\psi \, \varepsilon_{cdmn} e^c e^d e^n \bar{\psi} \left(\delta^m_a \gamma_b - \delta^m_b \gamma_a \right) \gamma_5 \psi \,, \tag{E.84}$$

with $T^c = d_\omega e^c = \bar{\omega}^c{}_f e^f = \bar{\omega}^a{}_{fg} e^g e^f$, we have

$$\varepsilon_{abcd}\bar{\omega}^a{}_{fg}e^g e^f e^d = \frac{i}{12}\alpha_\psi\, \varepsilon_{cdmn}e^c e^d e^n \bar{\psi}\, (\delta^m_a \gamma_b - \delta^m_b \gamma_a)\, \gamma_5 \psi\,, \tag{E.85}$$

or its dual equation

$$\varepsilon_{abcd}\varepsilon^{gfdh}\bar{\omega}^c{}_{fg} \;=\; \frac{i}{12}\alpha_\psi\, \varepsilon_{cdmn}\varepsilon^{cdnh}\bar{\psi}\, (\delta^m_a \gamma_b - \delta^m_b \gamma_a)\, \gamma_5 \psi\,. \tag{E.86}$$

Note that in the left hand side $\varepsilon_{abcd}\varepsilon^{gfdh} = -(-3!\delta^g_{[a}\delta^f_b\delta^h_{c]}) = 3!\delta^g_{[a}\delta^f_b\delta^h_{c]}$, where we have to be careful with the extra minus sign because we raise indices with the flat metric η^{ab}. We also use in the right hand side $\varepsilon_{cdmn}\varepsilon^{cdnh} = -\varepsilon_{cdnm}\varepsilon^{cdnh} = -(-6\delta^h_m) = 6\delta^h_m$. Therefore

$$2\bar{\omega}^h{}_{ba} + \delta^h_a\bar{\omega}^c{}_{cb} - \delta^h_b\bar{\omega}^c{}_{ca} = \frac{i}{2}\alpha_\psi\, \bar{\psi}\, (\delta^m_a \gamma_b - \delta^m_b \gamma_a)\, \gamma_5 \psi\,. \tag{E.87}$$

To completely solve this equation we have to contract it and then replace the result in itself. Contracting $h = b$ we get $\bar{\omega}^c{}_{ca} = \frac{3i}{2}\alpha_\psi\, \bar{\psi}\gamma_a\gamma_5\psi$. Then

$$\bar{\omega}^h{}_{ba} = \frac{i}{2}\alpha_\psi\, \bar{\psi}\, \left(\delta^h_b \gamma_a - \delta^h_a \gamma_b\right) \gamma_5 \psi\,, \tag{E.88}$$

or equivalently

$$\bar{\omega}^{ab}{}_f = \frac{i}{2}\alpha_\psi\, \bar{\psi}\, \left(\eta^{ab}\gamma_f - \delta^a_f \gamma^b\right) \gamma_5 \psi\,. \tag{E.89}$$

We are ready to replace this into the expression for $\bar{k}_\epsilon$, (E.70). We do it by parts. First note that the following combination simply vanishes

$$\varepsilon_{abcd}\bar{\omega}^{ab}{}_f e^c(\xi\lrcorner e^f\delta e^d + \xi\lrcorner e^d\delta e^f) = \frac{i}{2}\alpha_\psi\, \varepsilon_{abcd}\bar{\psi}\left(\eta^{ab}\gamma_f - \delta^a_f\gamma^b\right)\gamma_5\psi e^c(\xi\lrcorner e^f\delta e^d + \xi\lrcorner e^d\delta e^f) = 0\,.$$

Then

$$\bar{k}_\epsilon = -2\varepsilon_{abcd}\delta\bar{\omega}^{ab}{}_f e^f e^c \xi \lrcorner e^d = i\alpha_\psi\, \varepsilon_{abcd} e^a e^b \xi \lrcorner e^c \delta(\bar{\psi}\gamma^d\gamma_5\psi)\,, \tag{E.90}$$

this term is exactly $-k_\epsilon^\psi$ as in (E.83). Therefore, as it might have been suspected, for the surface charge density, we have an exact cancellation of all the terms concerning the spinor field

$$\bar{k}_\epsilon + k_\epsilon^\psi = 0\,, \tag{E.91}$$

or equivalently, this means that the full surface charge density for the Einstein-Cartan-Dirac theory is simply $k_\epsilon = \tilde{k}_\epsilon$ as in equation (E.68). In particular, this implies that in a space-time with a spinor field living on it, as far as exact symmetries are satisfied, it is not needed to have the explicit solution for the spinor field to compute charges.

E.8 D-dimensional Chern-Simons form

A CS Lagrangian in $D = 2n+1$ dimensions is a local function of a one-form gauge connection, A, valued on a Lie algebra. That is $A = A^i\tau_i = A^i_a\tau_i\, e^a$ with e^a the one-form frame field, τ_i the generators of the algebra, $[\tau_i, \tau_j] = f_{ij}{}^k\tau_k$, and $f_{ij}{}^k$ the algebra structure constants. The full CS Lagrangian can be expressed in a very compact form using the trick of an integral over an auxiliary variable t [154]

$$L^{(2n+1)}[A] = \kappa_n \int_0^1 dt \langle A F_t^n \rangle\,, \tag{E.92}$$

where $F_t \equiv dA_t + A_t \wedge A_t$, $A_t = tA$, and $\kappa_n = \kappa_{CS}(n+1)$ with κ_{CS} the CS level. Notice that the one-form nature of A induces the algebra commutator on the A_t^2 term, explicitly $A_t^2 = (tA)\wedge(tA) = t^2 A\wedge A = \frac{1}{2}t^2\, A^i \wedge A^j[\tau_i, \tau_j]$. The angled bracket $\langle\, \cdot \,\rangle$ denotes the symmetric invariant polynomial on the algebra such that for any two algebra valued forms, says the p-form P and the q-form Q, the usual commutation properties are respected, namely

$$\big\langle \cdots PQ \cdots \big\rangle = (-1)^{pq} \big\langle \cdots QP \cdots \big\rangle\,. \tag{E.93}$$

The CS theory possesses two symmetries, in fact the Lagrangian is invariant under diffeomorphisms and also *quasi-invariant* (up to a boundary term/exact form) under gauge symmetries, these are the key gauge symmetries that push us to compute surface charges. The trick to have a compact expression for the Lagrangian allows us to perform all calculations directly, we show them in detail. Let us start with a general variation of (E.92), we have

$$\begin{aligned}
\delta L^{(2n+1)}[A] &= \kappa_n \int_0^1 dt \langle \delta A F_t^n + nA\delta F_t F_t^{n-1} \rangle \,, \\
&= \kappa_n \int_0^1 dt \langle \delta A F_t^n + nA\, d_{A_t}(\delta A_t) F_t^{n-1} \rangle \,, \\
&= \kappa_n \int_0^1 dt \langle \delta A F_t^n + \frac{d}{dt}\delta A_t F_t^n - \delta A F_t^n - nd(A\delta A_t F_t^{n-1}) \rangle \,, \\
&= \kappa_n \langle \delta A F^n \rangle - d\Theta(\delta A) \,,
\end{aligned} \tag{E.94}$$

where we used $\delta F_t = d\delta A_t + [A_t, \delta A_t] = d_{A_t}\delta A_t$ with the notation d_{A_t} for the exterior covariant derivative for the connection A_t, we used also the Leibniz's rule for d_{A_t}, that $d_{A_t}F_t = 0$, the identity $\frac{d}{dt}F_t = d_{A_t}A$, integration by parts in the variable t, that $\frac{d}{dt}\delta A_t = \delta A$, and the invariance property of the symmetric polynomial $\langle \cdot \rangle$. Then, the equations of motion and the boundary term are

$$\langle F^n \rangle = 0 \,, \tag{E.95}$$

$$\Theta(\delta A) = -n\kappa_n \int_0^1 dt \langle \delta A_t A F_t^{n-1} \rangle \,. \tag{E.96}$$

Notice that we defined the boundary term with an overall minus sign, this convention save us of carrying a minus sing in the following calculations. This is conventional, remember that surface charge densities are defined up to overall factors.

Later we will also need the linearized equation of motion, namely

$$\delta\langle F^n \rangle = n\langle (\delta F) F^{n-1} \rangle = n\langle d_A(\delta A) F^{n-1} \rangle = 0 \,, \tag{E.97}$$

where $d_A(\cdot) \equiv d(\cdot) + [A, \cdot\,]$ denotes the covariant exterior derivative for the connection A.

To obtain the surface charge density we first compute the symplectic structure density using the boundary term. With two independent general variations on the phase space, say δ_1 and δ_2, the symplectic structure density reads

$$\begin{aligned}\Omega(\delta_1,\delta_2) &= \delta_1\Theta(\delta_2 A) - \delta_2\Theta(\delta_1 A) - \Theta([\delta_1,\delta_2]A)\,,\\ &= -n\kappa_n\int_0^1 dt\big\langle 2\delta_2 A_t\delta_1 A F_t^{n-1} + \delta_2 A_t A\delta_1 F_t^{n-1} - \delta_1 A_t A\delta_2 F_t^{n-1}\big\rangle\,. \end{aligned} \tag{E.98}$$

Now, the key to get a more tractable expression is to rewrite the second term as

$$\begin{aligned}&\big\langle\delta_2 A_t A\delta_1 F_t^{n-1}\big\rangle\\ &= (n-1)\big\langle\delta_2 A_t A\, d_{A_t}(\delta_1 A_t)F_t^{n-2}\big\rangle\,,\\ &= (n-1)\big\langle d\left(\delta_2 A_t A\delta_1 A_t F_t^{n-2}\right) - d_{A_t}(\delta_2 A_t)A\delta_1 A_t F_t^{n-2} + \delta_2 A_t\, d_{A_t}A\delta_1 A_t F_t^{n-2}\big\rangle\,,\\ &= (n-1)\big\langle d(\delta_2 A_t A\delta_1 A_t F_t^{n-2}) - \delta_2 F_t A\delta_1 A_t F_t^{n-2} + \delta_2 A_t\frac{d}{dt}(F_t)\delta_1 A_t F_t^{n-2}\big\rangle\,,\\ &= (n-1)\big\langle d(\delta_2 A_t A\delta_1 A_t F_t^{n-2}) - \delta_2 F_t A\delta_1 A_t F_t^{n-2}\big\rangle + \frac{d}{dt}\big\langle\delta_2 A_t\delta_1 A_t F_t^{n-1}\big\rangle - 2\big\langle\delta_2 A_t\delta_1 A F_t^{n-1}\big\rangle\,,\end{aligned}$$

where we used in the second line the Leibniz's rule for the covariant derivative d_{A_t} and the identity $d_{A_t}F_t^{n-2} = 0$. In the third line, $d_{A_t}(\delta_2 A_t) = \delta_2 F_t$ and $\frac{d}{dt}F_t = d_{A_t}A$. In the fourth line, we introduce a total derivative in t, we use that all the expression is inside the bracket $\langle\cdot\rangle$ to perform commutations of the algebra valued forms, and used also that $\frac{d}{dt}\delta A_t = \delta A$.

Now, replacing back, the second and fourth terms of (E.99) cancel exactly the first and third terms of (E.98), respectively. We are left with a total derivative in t which we can integrate trivially, and also an exact form. Then, the result is a symplectic structure density composed by a piece that could have been expected plus another piece which is an exact form

$$\Omega^{(2n+1)}(\delta_1,\delta_2) = n\kappa_n\big\langle\delta_1 A\delta_2 A F^{n-1}\big\rangle - n(n+1)\kappa_n\, d\left(\int_0^1 dt\big\langle\delta_2 A_t A\delta_1 A_t F_t^{n-2}\big\rangle\right)\,. \tag{E.99}$$

We observe that unlike other theories, the symplectic structure density for CS is not gauge invariant due to the last term, this is expected because the Lagrangian as well as the boundary term $\Theta(\delta A)$ are not gauge invariant forms. Remember that the theory is just quasi-invariant.

Now, we combine infinitesimal diffeomorphisms and gauge transformations for the connection to write an improved general infinitesimal symmetry transformation as

$$\delta_\epsilon A = \mathscr{L}_\xi A - d_A \lambda' = \xi \lrcorner F - d_A \lambda \,, \tag{E.100}$$

where as usual we select the parameter as $\lambda' = \lambda + \xi \lrcorner A$ in order to define an overall homogeneous infinitesimal transformation. Remember that $\lambda = \lambda^i \tau_i = (\lambda'^i - \xi^\mu A^i_\mu)\tau_i$ is an algebra valued gauge parameter which is also field dependent.

Then, we evaluate the symplectic structure density, (E.99), such that one of its entries is an improved symmetry transformation, $\delta_2 A \to \delta_\epsilon A$ as in (E.100) (and $\delta_1 A \to \delta A$). Using the equation of motion (E.95), and also the linearized equation of motion (E.97) (varied equation of motion on phase space), it is straightforward to show that the first term becomes also an exact form

$$\Omega^{(2n+1)}(\delta, \delta_\epsilon) \quad = \quad n\kappa_n d\big\langle \lambda \delta A F^{n-1} \big\rangle - n(n+1)\kappa_n d \left(\int_0^1 dt \big\langle \delta_\epsilon A A_t \delta A_t F_t^{n-2} \big\rangle \right) \,. \tag{E.101}$$

When the exact symmetry condition is satisfied: $\delta_\epsilon A = 0$ the symplectic structure density simply vanishes and it also vanishes the integral second term of the last expression. Therefore, we conclude that the surface charge density for a D-dimensional Chern-Simons theory, that satisfies the conservation law (*i.e.* is closed $dk_\epsilon = 0$), is

$$k_\epsilon^{(2n+1)} = n\kappa_n \big\langle \lambda \delta A F^{n-1} \big\rangle \,. \tag{E.102}$$

This is the main result of this appendix and it could have been expected by symmetry considerations. In fact, with only a connection at disposal there are no other ways to write a $(D-2)$-form which is also a variation (or one-form in field space) and at the same time a gauge invariant expression.

On the other hand, in contrast to [155] and [156] we observe the advantage to group the improved gauge parameter λ and the Killing vector ξ in the symmetry parameter ϵ which allows us to define an unique charge.

Note that for $n = 1$ we recover the standard $D = 3$ dimensional surface charge for a CS theory computed in the main text.

We remark that in the last step we needed to invoke the exact symmetry condition to get rid of the integral term in the symplectic structure density, (E.101). This is not usually the case. For all other theories worked out through these notes the surface charge density is read directly once we replace the symmetry transformation as one of the entries of the symplectic structure. Instead, here there is this extra exact form, expressed as an integral in t. As stressed before this is related with the quasi-invariance of the CS theory and that our prescription to define the symplectic structure relies on the Lagrangian.[5] Having said that, at this stage it should be already clear through our discussions that it is only for those cases, when the exact symmetry condition holds, that the surface charge density is closed and therefore it becomes a meaningful formula to compute true charges.

E.8.1 D-CS surface charge from the contracting homotopy operator

As a final remark of this appendix we note that the surface charge density formula for CS in $D = 2n + 1$ could had been easily obtained using the corresponding contracting homotopy operator. For a CS theory we can sketch the operator as

[5] In the method based on the contracting homotopy operator this is not the case and the symplectic structure is defined directly form the equations of motion. Because of this reason this alternative prescription is sometimes called *invariant* symplectic structure [101].

$$I_{\delta A} \equiv \delta A \frac{\partial}{\partial F}\,. \tag{E.103}$$

Now the Noether identity implies that $-\langle \kappa_n \delta_\epsilon A F^n \rangle = dS_\epsilon + \cancelto{0}{N_\epsilon}$ with $S_\epsilon = \kappa_n \langle \lambda F^n \rangle$. Thus, for the surface charge density

$$k_\epsilon^{(2n+1)} \equiv I_{\delta A} S_\epsilon = n\kappa_n \left\langle \lambda \delta A F^{n-1} \right\rangle , \tag{E.104}$$

and we directly recover (E.102). This short calculation shows the power of the contracting homotopy operator approach. Of course the procedure to obtain (E.103) as a rigorous expression for the operator is the missing part here but, as we checked, its naive application it is certainly powerful enough.

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