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# GEOMETRIZATION OF INTEGRABLE SYSTEMS WITH APPLICATIONS TO HOLOGRAPHY

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*Dedicada a mi Flor.*

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*"If the doors of perception*
*were cleansed,*
*every thing would appear to man*
*as it is: Infinite."*

William Blake

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# Abstract

The holographic principle originates from the observation that black hole entropy is proportional to the horizon area and not, as expected, to the volume. This principle has found a concrete manifestation in the Anti-de Sitter/Conformal Field Theory ($AdS_D/CFT_{D-1}$) correspondence. In lower dimensions, there is also a deep link between $CFT_2$ and the Korteweg-de Vries (KdV) hierarchy of integrable systems. Therefore, it is natural to think that there is a connection between the asymptotic structure of gravity in $3D$ and $2D$ integrable systems. Indeed, this is precisely what the "geometrization of integrable systems" performs. In a nutshell, this method consists on identifying the Lagrange multipliers of a set of boundary conditions for gravity with the polynomials that span a hierarchy of integrable systems at the boundary. In this thesis we extend the discussion to include thermal stability and phase transitions. We also construct a hierarchy of integrable systems in $2D$ whose Poisson structure corresponds to the Bondi-Metzner-Sachs algebra in three dimensions ($BMS_3$). Then, we extend the geometrization of integrable systems method to describe our new hierarchy in terms of the Riemannian geometry of three dimensional locally flat spacetimes. Remarkably, we make use of this sort of flat holography to understand the entropy of the cosmological spacetime in $3D$ as the microscopic counting of states of a dual field theory with consistent but non-standard modular and scaling properties.

# Introduction

## The puzzle of black holes

Black holes are undoubtedly some of the most intriguing objects in the universe, its extreme properties have attracted the attention of the expert community and all the general public. In September 2015, Ligo detected the gravitational waves produced by the collision of two black holes [1]. This discovery, was the first time that gravitational waves were detected, but also it definitively corroborated the existence of these astrophysical objects. As if that were not enough, in April of this year, the Event Horizon Telescope (EHT) collaboration managed to get a real picture of the event horizon of a supermassive black hole at the center of the galaxy M87 [2]. It is precisely this surface, the event horizon, the property of black holes that most puzzles our intuition, since in simple words, its the surface beyond which gravity is so strong that nothing that crosses it, even the light, can come out again.

In the early seventies, Bekenstein, Hawking and his collaborators demonstrated that black holes behave like thermodynamic objects [3, 4, 5]. Consequently, challenging its own classical definition, the event horizon radiate at a certain finite temperature and has an entropy associated with it, which is proportional to the area of the horizon

$$S = \frac{k_B c^3}{\hbar} \frac{A}{4G}\,, \tag{1}$$

where $k_B$, $\hbar$, $c$ and $G$ are the Boltzmann constant, the Planck constant, the speed of light and the Newton constant, respectively[1]. This is a very surprising result for several reasons and without any doubt, the one that most disquiet physicists is the following: Since the black hole is a thermodynamic object, its entropy should also be understood in terms of a suitable microscopic counting of states [6]. However, despite a large number of efforts by the community, the underlying nature of these microstates remains elusive.

## Holography

A significant result that shed some light about the origin of the black hole entropy, was proposed by Strominger and Vafa [7]. By a microscopic counting they derived the Bekenstein-Hawking area law (1) for a class of $5D$ extreme black holes in string theory. However, the tools that the authors use in that derivation, are very constrained by supersymmetry and string theory, so they can not be applied to more realistic black hole solutions in 4D, like the Schwarzschild solution, which is not extreme and devoid of supersymmetry.

Around the early nineties [8], 't Hooft pushed forward the radical suggestion that gravity in 4D must effectively become three dimensional at the Planck scale. Independently [9], Susskind relied on string theory to reach a similar conclusion, explaining the law of the quarter of the area (1) as a manifestation of the "holographic principle". One of the strongest realizations of the holographic principle is the celebrated AdS/CFT correspondence [10, 11, 12], which conjectures a one-to-one duality between gravity on AdS in $D$ spacetime dimensions and a $(D-1)$-dimensional Conformal Field Theory (CFT) at the boundary. The first example in this regard was in the context of string theory, where asymptotically $\mathrm{AdS}_5 \times S^5$ geometries were shown to be dual to a supersymmetric $SU(N)$ Yang-Mills theory in $4D$. Nowadays, this idea has been applied to many ways and for many different contexts. The full impact of gauge/gravity dualities is beyond

[1]Hereafter, we will work in units where $k_B$, $\hbar$ and $c$ are set to 1.

the purpose of this introduction. However, as we will see below, the duality does offer several useful insights into black hole thermodynamics.

More than a decade before Maldacena's proposal, Brown and Henneaux shown that the asymptotic symmetries of General Relativity in three dimensions with negative cosmological constant correspond to the conformal algebra in two dimensions with a classical central charge given by $c = 3\ell/2G$, where $\ell$ is the AdS radius and $G$ the Newton constant [13]. This result naturally suggested that a quantum theory of gravity in three dimensions could be described by a two-dimensional conformal field theory ($\mathrm{CFT}_2$). Based on this result, Strominger [14] showed that the semi-classical entropy of the Bañados-Teitelboim-Zanelli (BTZ) black hole [15, 16] can be recovered by a microscopic counting of states by means of the Cardy formula [17],

$$S = 2\pi\sqrt{\frac{c_L}{6}E_L} + 2\pi\sqrt{\frac{c_R}{6}E_R}\ , \tag{2}$$

where the left and right central charges are equal and correspond to the classical Brown-Henneaux central charge, and the left and right energies are given in terms of the global charges of the BTZ metric. This simple example (devoid of supersymmetry and string theory) gave rise to an active field of research regarding the thermodynamic properties of lower dimensional black holes and how they could be holographically related to a dual field theory that describes the much sought after quantum gravity.

## Geometrization of integrable systems

According to the seminal result of Strominger, the leading role of the asymptotic symmetry (embodied by the presence of the central charge) seemed unquestionable in this context. Even beyond that, the study of the non-trivial exact symmetries at the asymptotic region or boundary has gained great relevance lately, since it constitutes one of the vertices of the so-called "infrared triangle" (along with soft theorems and the memory effect), which is an equivalence relation that governs the infrared dynamics of physical theories with massless particles [18]. For our interest, this surprising relationship could shed some light on resolving the black hole information paradox [19, 20].

At this point, the relationship between boundary conditions and two-dimensional conformal symmetry is well understood, and hundreds of papers have been written where this has been realized. However, there is also a deep relationship between $\mathrm{CFT}_2$ and the Korteweg-de Vries (KdV) hierarchy of integrable systems [21, 22]. Therefore, it is natural to think that there is also a connection between boundary conditions for gravity in $3D$ and $2D$ integrable systems. Indeed, this is precisely what was demonstrated in [23], where it has been shown that General Relativity on $\mathrm{AdS}_3$ can be endowed with a suitable set of boundary conditions, so that the Einstein equations on the reduced phase space precisely reduce to the ones of the KdV hierarchy in two spacetime dimensions. In turn, the dynamics of the KdV hierarchy can then be understood in terms of the geometry of spacelike surfaces that evolve within a three-dimensional spacetime of negative constant curvature. The authors also show how this new set of boundary conditions leads to a generalized description of the thermodynamic properties of the BTZ black hole. Remarkably, they also manage to reconstruct the entropy by means of the microscopic counting of quantum states of a two-dimensional dual theory with anisotropic scaling properties. Recently, the relationship between the conserved charges of the KdV equation and the stress tensor of a $\mathrm{CFT}_2$ has also been explored in the context of "Generalized Gibbs ensembles" [24, 25, 26, 27].

## The Bondi-Metzner-Sachs algebra

As aforementioned, the importance of AdS/CFT correspondence is invaluable, notwithstanding, there are important cases where it does not apply, as is the case of black holes such as Schwarzschild or Kerr solutions, which correspond to astrophysical objects in vacuum. In this case, as shown in the seminal work of Bondi, van der Burg, Metzner and Sachs in the early sixties [28, 29, 30], the algebra of asymptotically flat spacetimes at null infinity turns out to be an infinite-dimensional extension of the Poincaré algebra, the so-called Bondi-Metzner-Sachs (BMS) algebra.

Regarding three-dimensional gravity, let us begin considering a Galilean conformal algebra which can be understood as the nonrelativistic limit of the algebra of the conformal group (see, e.g. [31, 32, 33]). In two spacetime dimensions, the Galilean conformal algebra ($GCA_2$) is then obtained from a suitable Inönü-Wigner contraction of two copies of the Virasoro algebra, where the parameter of the contraction is the speed of light ($c \to \infty$). Remarkably, $GCA_2$ is isomorphic to the Bondi-Metzner-Sachs algebra in three spacetime dimensions ($BMS_3$), which spans the diffeomorphisms that preserve the asymptotic form of the metric for General Relativity [34, 35, 36], possibly endowed with parity-odd terms [37]. The Poisson bracket algebra is given by

$$\begin{aligned} i\left\{\mathcal{J}_m,\mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n}+\frac{c_{\mathcal{J}}}{12}m^3\delta_{m+n,0}\ , \\ i\left\{\mathcal{J}_m,\mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n}+\frac{c_{\mathcal{P}}}{12}m^3\delta_{m+n,0}\ , \\ i\left\{\mathcal{P}_m,\mathcal{P}_n\right\} &= 0\ . \end{aligned} \tag{3}$$

with $m$ and $n$ arbitrary integers. The central extensions $c_{\mathcal{P}}$ and $c_{\mathcal{J}}$ are related to the Newton constant and to the coupling of the parity-odd terms, respectively. The $BMS_3$ algebra (3) is then described by the semi-direct sum of a Virasoro algebra, spanned by $\mathcal{J}_m$, with the Abelian ideal generated by $\mathcal{P}_m$. Note that the Poincaré algebra in three dimensions is manifestly seen as the subalgebra of (3) spanned by the subset of generators with $m$, $n = -1,0,1$ (after suitable trivial shifts of $\mathcal{J}_0$ and $\mathcal{P}_0$).

The $BMS_3$ algebra also naturally arises in diverse contexts of physical interest. For instance, it describes the worldsheet symmetries of the bosonic sector in the tensionless limit of closed string theory [38, 39, 40, 41, 42, 43, 44, 45], and it is then expected to be relevant for the description of interacting higher spin fields [46, 47, 48, 49, 50] (for a review, see e.g. [51]). In two spacetime dimensions, the $BMS_3$ algebra also describes the symmetries of a "flat analog" of Liouville theory [52, 53], while on Minkowski spacetime in 3D, in the absence of central extensions, the algebra (3) manifests itself through nonlocal symmetries of a free massless Klein-Gordon field [54]. The algebra (3) also plays a key role in nonrelativistic holography [32, 36, 55]. Furthermore, by virtue of a Sugawara-like construction, their generators have been recently seen to emerge as composite operators of the affine currents that describe the asymptotic symmetries of the "soft hairy" boundary conditions in [56, 57, 58]. Similar results also hold for the analysis of the near horizon symmetries of non-extremal black holes, so that (twisted) warped conformal algebras also lead to $BMS_3$ [59, 60, 61]. The minimal supersymmetric extension of the $BMS_3$ algebra has been shown to generate the asymptotic symmetries of $\mathcal{N} = 1$ supergravity in 3D [62, 63], for a suitable set of boundary conditions [37, 64], and it is then isomorphic to the minimal supersymmetric extension of $GCA_2$ [65, 66] (see also [67]). Supersymmetric extensions of $BMS_3$ with $\mathcal{N} > 1$ have also been discussed along diverse lines in [41, 42, 68, 69, 66, 70, 71, 72]. Interestingly, nonlinear extensions of the $BMS_3$ algebra are known to exist when higher spin bosonic or fermionic generators are included, see [73, 74, 75, 76, 77, 78]. Further generalizations of the $BMS_3$ algebra can also be found from suitable expansions of the Virasoro algebra [79, 80, 81, 82, 83].

## Flat holography and open questions

The $BMS_3$ algebra also plays an important role in the problem of understanding the Bekenstein-Hawking entropy (1) through a microscopic counting of states. This goal was achieved by a flat version of the Cardy formula [84, 85],

$$S = 2\pi\sqrt{\frac{c_{\mathcal{P}}}{2\mathcal{P}}\mathcal{J}} \ . \tag{4}$$

Here, the $BMS_3$ algebra, represented by the central charge $c_{\mathcal{P}}$, takes the role played by the conformal symmetry in (2) and $\mathcal{P}$, $\mathcal{J}$ correspond to the energy and momentum of the system in the grand canonical ensemble, respectively. This formula successfully reproduces the entropy of the so-called Flat Space Cosmology (FSC) [84, 85], which is the flat analog of the BTZ black hole, i.e., is an asymptotically locally flat spacetime that is solution to General Relativity in 3D without cosmological constant and possesses a non-trivial temperature and entropy associated to its cosmological horizon. Therefore, in order to (4) matches with the semi-classical entropy of the comology, one has to replace its global charges and the classical central extension $c_{\mathcal{P}} = 3/G$, found in the $BMS_3$ algebra of the asymptotic symmetries of Einstein gravity without cosmological constant [35]. Recently, in [86, 87], this "flat Cardy formula" formula has been proposed in order to determining the entropy of black holes in any dimension, assuming that dimensional reduction is possible and imposing certain boundary conditions at the horizon.

In this way, flat holography opens a range of possibilities to extend the promising results of $AdS_3/CFT_2$ (see e.g., [88, 89, 90, 91, 92, 93, 94, 95, 96]). In particular, in the absence of cosmological constant, it is then natural to wonder whether General Relativity in three spacetime dimensions might also be linked with some sort of integrable systems, or possibly a complete hierarchy of them. Could this link lead to a general description of the thermodynamics of the FSC? If so, could its entropy continue to be understood through the flat Cardy formula (4)? These and further related questions turn out to be one of the main subjects to be addressed throughout this thesis.

## Plan of the thesis

The thesis is organized as follows. In Chapter 1 we will review in detail the results found in [23], about how the KdV hierarchy entered in the asymptotic structure of General Relativity in $AdS_3$, and how it reflects the thermodynamics of the BTZ black hole and its corresponding holographic description. At the end of the chapter, we also include some original results [97] that show that KdV-type boundary conditions lead to a generalized version of the Smarr formula and how they could affect the thermodynamic stability of the different phases present in the theory. In Chapter 2 we construct a new hierarchy of integrable systems whose Poisson structure corresponds to the $BMS_3$ algebra, and then discuss its description in terms of the Riemannian geometry of locally flat spacetimes in three dimensions. We also discuss how the thermodynamics of the FSC behaves within this generalized description of the asymptotic structure. All the material covered in this chapter corresponds to original results, some of them already published in [98]. Chapter 3 is devoted to the development of a new formalism from which we can obtain through a simple set-up, the standard and flat Cardy-like formulas and their corresponding anisotropic generalizations in a unified way. In particular, we will show that the flat anisotropic version of the Cardy formula reproduces the entropy of the FSC endowed with boundary conditions defined trough the family of integrable systems described in the previous chapter. Finally, we conclude with a discussion of the main results of this thesis and its possible extensions. Appendices A, B, C and D are devoted to some technical remarks.

## Chapter 1

# Boundary conditions for $AdS_3$ gravity, geometrization of integrable systems and its connection with black hole thermodynamics

As we mentioned in the introduction, several efforts have been made to generalize the gauge/gravity proposal for non AdS asymptotics (see e.g. [36, 84, 99, 100]). In this scenario, a lot of attention has been focus on on $2D$ dual theories with anisotropic scaling properties [101, 102, 103, 104], which are found in the context of non-relativistic condense matter physics [105]. Most of the work on this subject has been done along the lines of non-relativistic holography [106, 107, 108, 109, 110] and more specifically, Lifshitz holography, where the gravitational counterparts are given by asymptotically Lifshitz geometries (see e.g. [111] and references therein). However, this class of spacetimes are not free of controversies. In fact, the Lifshitz spacetime itself suffers from divergent tidal forces at its origin. Additionally, asymptotically Lifshitz black holes are not vacuum solutions to General Relativity, and therefore it is mandatory to include extra matter fields or extend the theory beyond Einstein's gravity.

Following [23], we will adopt an unconventional approach to this holographic realization for field theories possessing anisotropic scaling properties, where the spacetime anisotropy at the boundary emerges from a very special choice of boundary conditions for General Relativity on $AdS_3$, instead of Lifshitz asymptotics. This new set of boundary conditions is labeled by a nonnegative integer $n$, and is related with the Korteweg-de Vries (KdV) hierarchy of integrable systems[1].

The present Chapter is organized as follows. In Section 1.1 we provide a review of boundary conditions for General Relativity on $AdS_3$ in both metric and Chern-Simons formulations. In Section 1.2 we show that any of the infinite number of the charges of the KdV hierarchy allow to define a new family of boundary conditions for the theory through a particular choice of the Lagrange multipliers at infinity. Section 1.3 is devoted to the analysis of how KdV-type boundary conditions change the thermodynamics of the BTZ black hole, and we will briefly comment on how to reproduce its entropy through a suitable generalization of the Cardy formula. In Section 1.4 it is shown that an anisotropic Smarr formula emerges from the radially conserved charge associated with the anisotropic scale invariance of the reduced Einstein-Hilbert action endowed with KdV-type boundary conditions. Finally, the local and global thermal stability of the BTZ black hole with KdV-type boundary conditions is also analyzed in Section 1.5.

## 1.1 Chern-Simons formulation of General Relativity on $AdS_3$

General Relativity with negative cosmological constant in three dimensional spacetimes can be formulated as the difference of two Chern-Simons actions for gauge fields $A^{\pm}$, evaluated on two

[1] See [112] for an introduction to the KdV hierarchy and for integrable systems in general.

independent copies of the $sl(2,\mathbb{R})$ algebra [113, 114],

$$I_{EH} = I_{CS}[A^+] - I_{CS}[A^-]\,, \tag{1.1}$$

where the Chern-Simons action for a generic gauge field $A$ reads

$$I_{CS}[A] = \frac{k}{4\pi}\int \mathrm{tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]. \tag{1.2}$$

The action (1.1) corresponds precisely to General Relativity on AdS$_3$ only if both Chern-Simons levels are given by $k=\ell/4G$, where $\ell$ is the AdS radius and $G$ the Newton constant.

In order to describe boundary conditions, it is convenient to do the analysis using auxiliary fields $a^\pm$, which are defined by a precise gauge transformation on $A^\pm$ [115],

$$A^\pm = b_\pm^{-1}\left(a^\pm + d\right)b_\pm\,, \tag{1.3}$$

with $b_\pm = e^{\pm\log(r/\ell)L_0^\pm}$. So that, the radial component of $a^\pm$ vanishes, while the remain ones only depend on time and the angular coordinate. Proceeding as in [116, 117], the remaining components of the boundary condition[2] are given by

$$a_\phi^\pm = L_\pm - \frac{1}{4}\mathcal{L}_\pm L_\mp\,,\quad a_t^\pm = \pm\mu_\pm L_\pm - \partial_\phi\mu_\pm L_0 \pm \frac{1}{2}\left(\partial_\phi^2\mu_\pm - \frac{1}{2}\mu_\pm\mathcal{L}_\pm\right)L_\mp\,, \tag{1.4}$$

where $\mathcal{L}_\pm(t,\phi)$ stand for the dynamical fields, and $\mu_\pm(t,\phi)$ correspond to the values of the Lagrange multipliers at infinity.

In this formulation, the asymptotic form of the Einstein field equations,

$$f^\pm = da^\pm + a^\pm\wedge a^\pm = 0\,, \tag{1.5}$$

[2]From now on, we will adopt the nomenclature of [118], where we will distinguish between "boundary conditions" and "asymptotic conditions". For the former we mean the conditions that are held fixed at the boundary, while for the latter we refer to the fall-off of the dynamical fields at infinity, which is an open set.

reduce to,

$$\partial_t \mathcal{L}_\pm = \pm \mathcal{D}^\pm \mu_\pm \,, \quad \mathcal{D}^\pm := (\partial_\phi \mathcal{L}_\pm) + 2\mathcal{L}_\pm \partial_\phi - 2\partial_\phi^3 \,. \tag{1.6}$$

The gauge fields (1.4), must preserve their form under gauge transformations,

$$\delta a^\pm = d\lambda^\pm + [a^\pm, \lambda^\pm] \,, \tag{1.7}$$

where, $\lambda^\pm$ correspond to the $sl(2,\mathbb{R})$-valued parameters of the transformation. The preservation of component $a_\phi^\pm$ implies; gauge parameters of the form

$$\lambda^\pm[\varepsilon_\pm] = \varepsilon_\pm L_\pm \mp \partial_\phi \varepsilon_\pm L_0 + \frac{1}{2}\left(\partial_\phi^2 \varepsilon_\pm - \frac{1}{2}\varepsilon_\pm \mathcal{L}_\pm\right) L_\mp \,, \tag{1.8}$$

where $\varepsilon_\pm$ are arbitrary functions of $t$ and $\phi$, and the following transformation law for the dynamical fields

$$\delta \mathcal{L}_\pm = \mathcal{D}^\pm \varepsilon_\pm \,. \tag{1.9}$$

Then, the preservation of the component $a_t^\pm$ implies the following condition for the variation of the Lagrange multipliers at the boundary

$$\delta \mu_\pm = \pm \partial_t \varepsilon_\pm + \varepsilon_\pm \partial_\phi \mu_\pm - \mu_\pm \partial_\phi \varepsilon_\pm \,. \tag{1.10}$$

Accordant to the canonical approach [119], the variation of the generators of the symmetries spanned by (1.8), are readily found as

$$\delta Q_\pm[\varepsilon_\pm] = \frac{k}{2\pi}\int d\phi \,\mathrm{tr}\left[\lambda^\pm \delta a_\phi^\pm\right] = \frac{k}{8\pi}\int d\phi \,\varepsilon_\pm \delta \mathcal{L}_\pm \,, \tag{1.11}$$

therefore, according to (1.1), the variation of the canonical generator associated with the gravitational configuration is given by

$$\delta Q[\varepsilon_+,\varepsilon_-]=\delta Q_+[\varepsilon_+]-\delta Q_-[\varepsilon_-]=\frac{k}{8\pi}\int d\phi\,\left(\varepsilon_+\delta\mathcal{L}_+-\varepsilon_-\delta\mathcal{L}_-\right)\,. \tag{1.12}$$

As a cross-check, it is simple to verify that the variation of the canonical generators is conserved ($\delta\dot{Q}=0$) provided that the consistency conditions for the symmetry parameters in (1.10) are satisfied.

It is worth highlighting that the boundary conditions are fully specified once a precise form of the Lagrange multipliers at infinity is provided. In the standard approach of Brown and Henneaux [13], the Lagrange multipliers are set as $\mu_\pm=1/\ell$, and according to (1.6), the dynamical fields are chiral. Going one step further, one can generalize this analysis by choosing arbitrary functions of the coordinates, $\mu_\pm=\mu_\pm(t,\phi)$, which, in order to have a well-defined action principle, are held fixed at the boundary ($\delta\mu_\pm=0$) [116, 117]. However, even beyond that, if one allows that the Lagrange multipliers may depend on the dynamical fields and their spatial derivatives, one can still guaranted the integrability of the boundary term in the action.

In simpler case, where the Lagrange multipliers do not functionally depend on the dynamic fields, we can directly integrate (1.11) as

$$Q_\pm[\varepsilon_\pm]=\frac{k}{8\pi}\int d\phi\,\varepsilon_\pm\mathcal{L}_\pm\,, \tag{1.13}$$

Therefore, since the canonical generators must fulfill $\delta_\lambda Q[\gamma]=\{Q[\gamma],Q[\lambda]\}$, their algebra can be directly computed through (1.9). Expanding in Fourier modes,

$$\mathcal{L}_\pm(\phi)=\frac{1}{2\pi}\sum_m\mathcal{L}^\pm_m e^{im\phi}\,, \tag{1.14}$$

the algebra of the asymptotic symmetries reduces to two copies of the Virasoro algebra

$$
\begin{aligned}
i\{\mathcal{L}^{\pm}_m\,,\,\mathcal{L}^{\pm}_n\} &= (m-n)\mathcal{L}^{\pm}_{m+n}+\frac{c_\pm}{12}m^3\delta_{m+n\,,0}\,,\\
i\{\mathcal{L}^{\pm}_m\,,\,\mathcal{L}^{\mp}_n\} &= 0\,,
\end{aligned}
\tag{1.15}
$$

where both the central charges are equal and given by

$$
c_\pm = 6k = \frac{3\ell}{2G}\,. \tag{1.16}
$$

For an arbitrary choice of $\mu_\pm(t,\phi)$, the integrability of (1.11) requires that the allowed parameters $\varepsilon_\pm$ must correspond to the variation of some functional, since $\varepsilon_\pm = \delta Q_\pm/\delta\mathcal{L}_\pm$. Nevertheless, it must be emphasized that a explicit form of the conserved charges $Q$ is not so simple to find, because it amounts to know the general solution of the consistency condition for the parameters in (1.10). Indeed, in the most general case, although the consistency condition is linear in the parameters, the generic solution manifestly depends on the dynamical fields and their spatial derivatives that fulfill a nonlinear field equation. Thus, for a generic choice of the Lagrange multipliers, solving (1.10) is actually an extremely difficult task.

### 1.1.1 Metric formulation

In the metric formulation, the leading terms of the fall-off of the metric can be computed according to

$$
g_{\mu\nu} = \frac{\ell^2}{2}\mathrm{tr}\left[\left(A^+_\mu - A^-_\mu\right)\left(A^+_\nu - A^-_\nu\right)\right]\,. \tag{1.17}
$$

Thus, using (1.3) and (1.4), we obtain the following non-vanishing components

$$
\begin{aligned}
g_{tt} &= \frac{\ell^2}{4}\left(\mu_+^2\mathcal{L}_+ + \mu_-^2\mathcal{L}_-\right) - \mu_+\mu_-\left(\frac{\ell^4\mathcal{L}_+\mathcal{L}_-}{16r^2} + r^2\right) \\
&\quad + \frac{\ell^2}{2}\left(\frac{\ell^2\mathcal{L}_+\mu_-''}{4r^2} - \mu_+''\right)\mu_+ + \frac{\ell^2}{2}\left(\frac{\ell^2\mathcal{L}_-\mu_+''}{4r^2} - \mu_-''\right)\mu_- \\
&\quad - \frac{\ell^2}{4}\left(\mu_+' - \mu_-' + \frac{\ell^2}{r^2}\mu_+''\mu_-''\right) , \\
g_{rr} &= \frac{\ell^2}{r^2} , \\
g_{tr} &= -\frac{\ell^2}{2r^2}\left(\mu_+' - \mu_-'\right) , \\
g_{\phi\phi} &= \left(\frac{\ell^2\mathcal{L}_+}{4r} + r\right)\left(\frac{\ell^2\mathcal{L}_-}{4r} + r\right) , \\
g_{t\phi} &= \frac{\ell^2}{2}\left(\mu_+\mathcal{L}_+ - \mu_-\mathcal{L}_-\right) + \left(r^2 + \frac{\ell^4\mathcal{L}_+\mathcal{L}_-}{4r^2}\right)\left(\mu_+ - \mu_-\right) \\
&\quad - \frac{\ell^2}{2}\left(\mu_+'' - \mu_-''\right) + \frac{\ell^4}{8r^2}\left(\mathcal{L}_+\mu_-'' - \mathcal{L}_-\mu_+''\right) .
\end{aligned}
\tag{1.18}
$$

This metric corresponds to the most general solution (up to trivial diffeomorphisms) to the Einstein field equations with $\Lambda = -1/\ell^2$ [120], endowed with the asymptotic conditions associated to (1.4). It is worth to remembering that in a generic case, in (1.18) the Lagrange multipliers could be given by functions of the dynamical fields $\mathcal{L}_\pm$ and their spatial derivatives.

If the parameters of the gauge transformations (1.7), are given by

$$\lambda^\pm = \xi^\mu a_\mu^\pm , \tag{1.19}$$

it can be shown that

$$\delta_{(\lambda^\pm)} a_\mu^\pm = \mathfrak{L}_\xi a_\mu^\pm + \xi^\nu f_{\mu\nu}^\pm , \tag{1.20}$$

where $\mathfrak{L}_\xi$ correspond to the Lie-derivative with respect to the vector field $\xi$. This means that on-shell ($f^\pm = 0$) infinitesimal gauge invariance under transformations with parameter $\lambda^\pm$, are equivalent to isomorphisms spanned by the Killing vector $\xi^\mu\partial_\mu$.

According to the equivalence above, we can use (1.12) to compute the angular momentum and the variation of the mass of the gravitational systems described by the gauge fields (1.4). The former charge is associated with spatial translations at the boundary, i.e., $\xi = \partial_\phi$, which according to (1.19) and (1.8), corresponds to $\varepsilon_\pm = 1$. Thus, the angular momentum is given by

$$Q\left[\partial_\phi\right] = \frac{k}{8\pi}\int d\phi\,\left(\mathcal{L}_+ - \mathcal{L}_-\right)\,. \tag{1.21}$$

For time translations, $\xi = \partial_t$, the equivalence corresponds to $\varepsilon_\pm = \pm\mu_\pm^{(n)}$. Hence, the variation of the mass is given by

$$\delta Q\left[\partial_t\right] = \frac{k}{8\pi}\int d\phi\left(\mu_+\delta\mathcal{L}_+ + \mu_-\delta\mathcal{L}_-\right)\,. \tag{1.22}$$

## 1.2 KdV-type boundary conditions

As previously emphasized, the boundary conditions are completely fixed only once the Lagrange multipliers $\mu_\pm$ have been specified. In general they could also depend on the coordinates. Nevertheless, demanding a well defined action principle, constraint us to choose the Lagrange multipliers as functions of the dynamical fields and their spatial derivatives. We can achieve the latter by choosing the Lagrange multipliers as the variation with respect to $\mathcal{L}_\pm$ of some functionals of them.

Here, we will focus on the family of KdV-type boundary conditions[3], introduced in [23], which are labeled by a non negative integer $n$. In this context, the Lagrange multipliers are chosen to be given by the $n$-th Gelfand-Dikii polynomial evaluated on $\mathcal{L}_\pm$ and can be obtained

[3]Other examples of this relationship between 2D integrable systems and gravity in 2+1, have been also made for the cases of "flat" and "soft hairy" boundary conditions in Chapter 2 and [121], respectively.

by the functional derivative of the $n$-th Hamiltonian of the KdV hierarchy[4],

$$\mu_{\pm}^{(n)} = \frac{1}{\ell} \frac{\delta H_{\pm}^{(n)}}{\delta \mathcal{L}_{\pm}} \,, \tag{1.23}$$

where the following recursion relation is satisfied[5]

$$\partial_{\phi} \mu_{\pm}^{(n+1)} = \frac{n+1}{2n+1} \mathcal{D}^{\pm} \mu_{\pm}^{(n)} \,. \tag{1.24}$$

Thus, for the case $n = 0$, one recovers the Brown-Henneaux boundary conditions ($\mu_{\pm}^{(0)} = 1/\ell$). In the case $n = 1$, the Lagrange multipliers are given by $\mu_{\pm}^{(1)} = \mathcal{L}_{\pm}/\ell$, and then, the field equations reduce to two copies of the KdV equation, while for the remaining cases ($n > 1$) the field equations are given by the corresponding $n$-th member of the KdV hierarchy.

Concerning to the symmetries associated with these family of boundary conditions, it can be shown that, remarkably, the consistency condition (1.10) reduces to the corresponding $n$-th member of the KdV equation for the $\varepsilon_{\pm}$. Thus, since the KdV hierarchy corresponds to an integrable system, we know its general solution assuming that gauge parameters are local functions of $\mathcal{L}_{\pm}$ and their spatial derivatives. It is given by a linear combination of the Gelfand-Dikii polynomials,

$$\varepsilon_{\pm} = \sum_{n=0}^{\infty} \eta_{\pm}^{(n)} \mu_{\pm}^{(n)} \,. \tag{1.25}$$

Consequently, for an arbitrary symmetry spanned by (1.25), we can integrate the canonical generators (1.11) as

$$Q_{\pm}[\varepsilon_{\pm}] = \frac{k}{8\pi} \sum_{n=0}^{\infty} \eta_{\pm}^{(n)} H_{\pm}^{(n)} \,. \tag{1.26}$$

Therefore, the algebra of the canonical generators is then found to be an abelian one and devoid

[4]A list of the first Gelfand-Dikii polynomials and the corresponding Hamiltonians of the KdV hierarchy is shown in Appendix A.

[5]For later convenience, we have chosen the factor such that the polynomials become normalized according to $\ell \mu_{\pm}^{(n)} = \mathcal{L}_{\pm}^{n} + \cdots$, where the ellipsis refers to terms that depend on derivatives of $\mathcal{L}_{\pm}$.

of central charges, which goes by hand with the well-known fact that the conserved charges of an integrable system, as it is the case of KdV, are in involution.

Notice that, by virtue of (1.23), the energy of the gravitational system (1.22), in the case of KdV-type boundary conditions (1.23), integrates as

$$Q\left[\partial_t\right] = \frac{k}{8\pi}\left(H_+^{(n)} + H_-^{(n)}\right) . \tag{1.27}$$

it is worth emphasizing that, although these boundary conditions describe asymptotically locally $AdS_3$ spacetimes, the associated dual field theory at the boundary[6] possesses an anisotropic scaling of Lifshitz type,

$$t \to \lambda^z t\,, \quad \phi \to \lambda\phi\,, \tag{1.28}$$

provided that $\mathcal{L}_\pm \to \lambda^{-2}\mathcal{L}_\pm$, where the dynamical exponent $z$ is related to the KdV label $n$ by $z = 2n+1$. This feature, gives rise to a different alternative in order to study holography for non-relativistic field theories possessing anisotropic scaling invariance, where the gravitational counterpart now corresponds to General Relativity on $AdS_3$.

## 1.3 The BTZ black hole with KdV-type boundary conditions

For each allowed choice of $n$ (or equivalently $z$), the spectrum of solutions is quite different. Nonetheless, the BTZ black hole fits within every choice of boundary conditions of the KdV-type. Indeed, this class of configurations are described by the following state-dependent functions,

$$\mathcal{L}_\pm^{BTZ} = \frac{1}{\ell^2}\left(r_+ \pm r_-\right)^2 . \tag{1.29}$$

[6] As shown in [122], by performing the Hamiltonian reduction of KdV-type boundary conditions, the equations (1.6) actually corresponds to the conservation law of the energy-momentum tensor of the corresponding theory at the boundary. For the particular case $n = 0$, the field equations and the conservation law coincides.

These dynamical fields are constant, and since the Lagrange multipliers (1.23) depend functionally of them, all the spatial derivatives of $\mu_\pm$ in (1.18) vanish. Consequently, the boundary condition (1.18) writen in the Schwarzschild gauge aquire the following form

$$ds^2 = -\frac{\ell^2}{4}\left(\mu_+ + \mu_-\right)^2 \mathcal{F}\left(r\right)^2 dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + r^2\left(d\phi + \mathcal{N}^\phi\left(r\right)dt\right)^2 , \tag{1.30}$$

where

$$\begin{aligned} \mathcal{F}(r)^2 &= \frac{r^2}{\ell^2} - \frac{1}{2}\left(\mathcal{L}_+ + \mathcal{L}_-\right) + \frac{\ell^2}{16}\frac{\left(\mathcal{L}_+ - \mathcal{L}_-\right)^2}{r^2} , \\ \mathcal{N}^\phi(r) &= \frac{1}{2}\left(\mu_+ - \mu_-\right) + \frac{\ell}{2}\left(\mu_+ + \mu_-\right)\frac{\ell}{4}\frac{\left(\mathcal{L}_+ - \mathcal{L}_-\right)}{r^2} . \end{aligned} \tag{1.31}$$

Note that this class of configurations provides an exact solution of the field equation in (1.6) for all possible values of $n$.

### 1.3.1 Thermodynamics

In the standard formulation of black hole thermodynamics, the temperature and the chemical potential for the angular momentum do not enter explicitly in the metric. They appear when we demand regularity at the horizon in the Euclidean section of the black hole metric, which implies to identify the periods of the Euclidean time and the angle. This means that the periods of the coordinates are fixed but vary from one solution to another [116]. For (1.30), if we fix the $\mu_\pm$, then the periods of the identification will depend on the solution. If we demand fixed periods ($t_E \sim t_E + 1$, $\phi \sim \phi + 2\pi$), we must keep the Lagrange multipliers unfixed in the metric. Following this prescription, the Lagrange multipliers naturally relate to the chemical potentials of the black hole (temperature and angular velocity).

In the generic case where $\mathcal{L}_\pm$ are constants (which includes (1.29)), according to the normalization choice in (1.24), it is possible to show that the Lagrange multipliers generically acquire a remarkably simple form, namely, $\mu_\pm^{(n)} = \mathcal{L}_\pm^n N_\pm$. Here, we include the factors $N_\pm$ in order to keep track of the explicit presence of the chemical potentials in the metric, and therefore, fixing the periods of the Euclidean coordinates. Note that $\mu_\pm^{(0)} = N_\pm$, and the standard Brown-Henneaux analysis is recovered by setting $N_\pm = 1/\ell$. It is worth to noting that if we define $N_\pm = N/\ell \pm N^\phi$, then the Brown-Henneaux choice corresponds naturally to the canonical gauge fixing of the Lapse and Shift functions at infinity, i.e., $N = 1$ and $N^\phi = 0$. In what follows, we will use the dynamical exponent $z$, instead of the KdV-label $n$; thus, by $z = 2n + 1$, we can rewrite the KdV-type Lagrange multipliers as[7],

$$\mu_\pm^{(z)} = \mathcal{L}_\pm^{\frac{z-1}{2}} N_\pm[z] \,. \tag{1.32}$$

On the other hand, demanding regularity on the Euclidean continuation of (1.30), the Lagrange multipliers fix as

$$\mu_\pm^{(z)} = \frac{2\pi}{\sqrt{\mathcal{L}_\pm}} \,. \tag{1.33}$$

Therefore, according to (1.32), for a generic choice of $z$, the chemical potentials are given by

$$N_\pm[z] = 2\pi \mathcal{L}_\pm^{-\frac{z}{2}} \,, \tag{1.34}$$

thus, in the standar case ($z = 1$), for the BTZ black hole (1.29), the chemical potentials reduce to the well known left and right inverse temperatures

$$N_\pm[1] = \frac{2\pi\ell}{r_+ \pm r_-} \,, \tag{1.35}$$

which correspond to the Hawking temperature and angular velocity of the BTZ black hole trough

[7] In the context of AdS/CFT correspondence, the relationship between the chemical potentials and conserved charges is known as "multi-trace deformations" of the dual theory [123], see also [124].

$N_\pm = N/\ell \pm N^\phi$,

$$T = \frac{1}{N} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}\,, \quad \Omega = -\frac{N^\phi}{N} = \frac{r_-}{\ell r_+}\,. \tag{1.36}$$

Note that, from (1.34) we can deduce the following expression for the chemical potentials.

$$N_\pm[z] = (2\pi)^{1-z}\, N_\pm[1]^z\ . \tag{1.37}$$

For this latter expression we will return in Chapter 3.

According to the canonical approach, the energies of the left and right movers also take a simple form for a generic choice of $n$, namely $E_\pm^{(n)} = \frac{k}{8\pi} H_\pm^{(n)} = \frac{k}{4}\frac{1}{n+1}\mathcal{L}_\pm^{n+1} N_\pm$. Thus, in terms of the dynamical exponent we can rewrite them as

$$E_\pm^{(z)} = \frac{k}{2}\frac{1}{z+1}\mathcal{L}_\pm^{\frac{z+1}{2}} N_\pm\,. \tag{1.38}$$

Therfore, we can directly replace this expression in (1.27), to obtain the mass of the black hole for an arbitrary choice of the KdV-type boundary conditions

$$M\,[z] = \frac{1}{8G}\frac{\ell^{-(z+1)}}{z+1}\left[(r_+ + r_-)^{z+1} + (r_+ - r_-)^{z+1}\right]\,. \tag{1.39}$$

Thus, in the standard case, $z = 1$, we obtain the well-known mass of the BTZ black hole with Brown-Henneaux boundary conditions,

$$M = \frac{1}{8G}\frac{r_+^2 + r_-^2}{\ell^2}\,. \tag{1.40}$$

Replacing (1.29) on (1.21), we obtain the angular momentum of the BTZ black hole

$$J = \frac{1}{8G}\frac{2r_+ r_-}{\ell}\,. \tag{1.41}$$

It is worth mentioning that the angular momentum is independent of $z$, because the generic formula for $Q[\partial_\phi]$ does not depend on the Lagrange multipliers.

As a matter of fact, the entropy of the BTZ black hole satisfy the Bekenstein-Hawking area law, independently of the choice of boundary conditions

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G} \,, \tag{1.42}$$

which, through (1.29) and (1.38), can can be expressed in terms of the left and right energies $E_\pm$,

$$S = k\pi \left[ \left( \frac{2}{k}(z+1) E_+[z] \right)^{\frac{1}{z+1}} + \left( \frac{2}{k}(z+1) E_-[z] \right)^{\frac{1}{z+1}} \right] . \tag{1.43}$$

Remarkably, in [23] the authors showed that using a generalization of the S-modular transformation of the torus, it is possible to find an anisotropic version of the Cardy formula[8], given by

$$S = 2\pi\,(z+1) \left[ \left( \frac{|E_+^0[z^{-1}]|}{z} \right)^z E_+[z] \right]^{\frac{1}{z+1}} + 2\pi\,(z+1) \left[ \left( \frac{|E_-^0[z^{-1}]|}{z} \right)^z E_-[z] \right]^{\frac{1}{z+1}} . \tag{1.44}$$

This formula reproduces exactly the entropy of the BTZ black hole with arbitrary boundary conditions of KdV-type (1.43). In Chapter 3, we will discuss in more detail how to derive this formula through the saddle-point approximation aforementioned and how to identify the variables in order to recover the mentioned semi-classical entropy.

[8]As explained in [23], for odd values of $n = (z-1)/2$, Euclidean BTZ with KdV-type boundary conditions is diffeomorphic to thermal $AdS_3$, but with reversed orientation, and in consequence, there is a opposite sign between Euclidean and Lorentzian energies of the ground state. As it will be shown in Section 1.5, this leads to a local thermodynamic instability of the system for odd values of $n$. So, it is mandatory to adopt $E_\pm^0\left[z^{-1}\right] \to -\left|E_\pm^0\left[z^{-1}\right]\right|$, in the Lorentzian ground state energies of the anisotropic Cardy formula.

## 1.4 The anisotropic Smarr formula

It was shown that the reduced Einstein-Hilbert action coupled to a scalar field on $AdS_3$ is invariant under a set of scale transformations which leads to a radial conservation law by means of the Noether theorem [125]. When this conserved quantity is evaluated on a black hole solution of the theory, one obtains a Smarr relation [126]. This method has been successfully applied to several cases in the literature [127, 128, 129, 130, 131, 132, 133] for different theories. By following this procedure, we will show that a new anisotropic version of Smarr formula for the BTZ black hole naturally emerges as a consequence of the scale invariance of the reduced Einstein-Hilbert action, as long as we consider KdV-type boundary conditions.

By considering stationary and circularly symmetric spacetimes described by the following line element

$$ds^2 = -\mathcal{N}(r)^2 \mathcal{F}(r)^2 dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + r^2 \left(d\phi + \mathcal{N}^\phi(r)\, dt\right)^2 , \tag{1.45}$$

the reduced action principle of General Relativity in the canonical form, is given by

$$I = -2\pi (t_2 - t_1) \int dr \left(\mathcal{N}\mathcal{H} + \mathcal{N}^\phi \mathcal{H}_\phi\right) + B\,, \tag{1.46}$$

where $\mathcal{N}$, $\mathcal{N}^\phi$ stand for their corresponding Lagrange multipliers associated to the surface deformation generators,

$$\mathcal{H} = -\frac{r}{8\pi G\ell^2} + 32\pi G r (\pi^{r\phi})^2 + \frac{(\mathcal{F}^2)'}{16\pi G}\,, \tag{1.47}$$

$$\mathcal{H}_\phi = -2(r^2\pi^{r\phi})'\,, \tag{1.48}$$

where prime denotes derivative with respect to $r$. The only nonvanishing component of the momenta $\pi^{ij}$ is explicitly given by

$$\pi^{r\phi} = -\frac{(\mathcal{N}^\phi)' r}{32\pi G\mathcal{N}}\,, \tag{1.49}$$

The boundary term $B$ must be added in order to have a well-defined variational principle. Straighforwadly, we can see that the above action turns out to be invariant under the following set of scale transformations

$$\bar{r} = \xi r \,, \quad \bar{\mathcal{N}} = \xi^{-2}\mathcal{N} \,, \quad \bar{\mathcal{N}^{\phi}} = \xi^{-2}\mathcal{N}^{\phi} \,, \quad \bar{\mathcal{F}^2} = \xi^2\mathcal{F}^2 \,, \tag{1.50}$$

where $\xi$ is a positive constant. By applying the Noether theorem, we obtain a radially conserved charge associated with the aforementioned symmetries,

$$C(r) = \frac{1}{4G}\left[-\mathcal{N}\mathcal{F}^2 + \frac{r\mathcal{N}\left(\mathcal{F}^2\right)'}{2} - \frac{r^3\left(\mathcal{N}^{\phi}\right)'\mathcal{N}^{\phi}}{\mathcal{N}}\right] \,, \tag{1.51}$$

which means that $C' = 0$ on-shell. We will find a Smarr formula by exploiting the fact that this radial conserved charge must satisfy $C(r_+) = C(\infty)$, where left and right hand sides denotes evaluations on the event horizon of the black hole and the asymptotic region, respectively.

At the horizon, $\mathcal{F}^2(r_+) = 0$, and in order to the Euclidean configuration being smooth around this point, the metric functions must to fulfill the regularity conditions [134],

$$\mathcal{N}(r_+)\mathcal{F}^2(r_+)' = 4\pi \,, \quad \mathcal{N}^{\phi}(r_+) = 0 \,. \tag{1.52}$$

In consequence, the value of the radial charge at the event horizon corresponds exactly to the entropy of the BTZ black hole

$$C(r_+) = \frac{\pi r_+}{2G} = S \,. \tag{1.53}$$

In order to obtain the radial charge in the asymptotic region, we use the explicit form of the BTZ metric functions in (1.45), namely, the ones introduced in (1.30). Then we replace the Lagrange multipliers at infinity according to the ones fixed by KdV-type boundary conditions

(1.32),

$$C(\infty) = \frac{\ell}{8G}\left(N_+\mathcal{L}_+^{\frac{z+1}{2}} + N_-\mathcal{L}_-^{\frac{z+1}{2}}\right), \tag{1.54}$$

which in terms of the left and right energies (1.38), reads

$$C(\infty) = (z+1)N_+E_+ + (z+1)N_-E_- \,. \tag{1.55}$$

Therefore, equalizing both expressions we finally obtain an anisotropic version of the Smarr formula,

$$S = (z+1)N_+E_+ + (z+1)N_-E_- \,. \tag{1.56}$$

Identifying the Lagrange multipliers as the inverse of left and right temperatures $T_\pm = N_\pm^{-1}$, and then turning off the angular momentum, the above expression reduces to[9]

$$E = \frac{1}{(z+1)}TS\,, \tag{1.57}$$

which fits with the Smarr formula for Lifshitz black holes in 3D (see e.g. [135, 136]).

It is worth to point out that despite of the BTZ black hole corresponds to an asymptotically $AdS_3$ spacetime, the anisotropic nature of (1.56), also present in asymptotically Lifshitz black holes [137], in this case is induced by the scaling properties of KdV-type boundary conditions. This common anisotropic feature leads to an interesting consequence. Since (1.56) was derived from a stationary metric (BTZ with $\mathcal{L}_+ \neq \mathcal{L}_-$), the contribution due to the rotation naturally appears, even though the fact that there is no a rotating Lifshitz black hole in three dimensions.

It is also worth mentioning that in the limit $z \to 0$, (1.56) fits with the corresponding Smarr relation of soft hairy horizons in three spacetimes dimension [58], where the infinite-dimensional symmetries at the horizon span soft hair excitations in the sense of Hawking, Perry

[9]The left and right temperatures are related with the Hawking temperature through $T = \frac{2T_+T_-}{(T_++T_-)}$, so in the absence of rotation $T_+ = T_- = T$.

and Strominger [138, 139, 140].

### 1.4.1 Relationship with the anisotropic Cardy formula

As was mentioned at the end of Section 1.3, one can derive the anisotropic Cardy formula (1.44) by making use of the saddle-point approximation on the asymptotic growth of the number of states, provided of an anisotropic S-duality relation between high and low temperatures regime of the partition function (see Chapter 3),

$$\beta'_{\pm} = \frac{(2\pi)^{1+\frac{1}{z}}}{\beta_{\pm}^{\frac{1}{z}}} , \qquad z' = \frac{1}{z} . \tag{1.58}$$

This procedure throws a critical point $\beta^* = \{\beta_+, \beta_-\}$ in terms of $E^0_{\pm}$ and $E_{\pm}$. If we solve that relation for the ground state energies, we found that

$$\left|E^0_{\pm}[z^{-1}]\right| = zE_{\pm}[z] \left(\frac{\beta_{\pm}}{2\pi}\right)^{1+\frac{1}{z}} , \tag{1.59}$$

and then we replace the above in (1.44), we found that identifying $\beta_{\pm} \to N_{\pm}$, the entropy exactly matches with the anisotropic Smarr formula (1.56). Here we show explicitly that the relationship between Cardy and Smarr formulas is given by the critical point (1.59), which remarkably, is nothing else than the anisotropic version of the Stefan-Boltzmann law [23],

$$E_{\pm}\left[z\right] = \frac{1}{z} |E^0_{\pm}\left[z^{-1}\right]| \, (2\pi)^{1+\frac{1}{z}} \, T_{\pm}^{1+\frac{1}{z}} . \tag{1.60}$$

Interestingly enough, from equation (1.44), it is clear that the entropy written as a function of left and right energies scales as $(z+1)^{-1}$, hence it is reassuring to prove that the anisotropic Smarr formula (1.56) can be recovered by simply applying the Euler theorem for homogeneous functions.

The following section is devoted to the local and global thermal stability of the BTZ black hole endowed with KdV-type boundary conditions.

## 1.5 Thermodynamic stability and phase transitions

In concordance with the spirit of the previous sections, we will analyze the thermodynamic stability of the BTZ black hole in the KdV-type ensemble from a holographic perspective. That is, we will first study the behavior of the presumed dual field theory, described by two independent left and right movers, whose energies and entropy are given by the anisotropic Stefan-Boltzmann law and the anisotropic Smarr formula, respectively. Then we will compute the thermodynamic quantities associated with the gravitational system endowed with KdV-type boundary conditions and show that both descriptions match.

We analyze the thermodynamic stability at fixed chemical potentials. Local stability condition can be determined by demanding a negative defined Hessian matrix of the free energy of the system. Nonetheless, in the current ensemble it can equivalently be performed by the analysis of the left and right specific heats with fixed chemical potential. From the anisotropic Stefan-Boltzmann law (1.60) one finds that left and right specific heats are given by

$$C_{\pm}\left[z\right]=\frac{\partial E_{\pm}}{\partial T_{\pm}}=\frac{z+1}{z^{2}}\left|E_{\pm}^{0}\left[z^{-1}\right]\right|(2\pi)^{1+\frac{1}{z}}T_{\pm}^{\frac{1}{z}}\,. \tag{1.61}$$

We see that, for all possible values of $z$, the specific heats are continuous monotonically increasing functions of $T_{\pm}$, and always positive[10], which means that the system is at least locally stable. Since the specific heats are finite and positive regardless of the value of $z$, the BTZ black hole with generic KdV-type boundary conditions can always reach local thermal equilibrium with the heat bath at any temperature, as it is expected for a black hole solution in a Chern-Simons

[10]Strictly speaking, specific heats $C_{\pm}$ are always positive provided that $T_{\pm}>0$. In terms of the temperature and angular velocity of the black hole, the above is equivalent to the non-extremality condition; $0<T\,,\ -1<\Omega<1\,.$ Since in the present paper we are not dealing with the extremal case, we will consider that this condition is always fulfilled.

theory [141, 142, 143, 144, 145, 146]. It is important to remark that, as mentioned at footnote 8, if we had not warned on the correct sign of the ground state energies for odd $n$, the sign of the specific heats would have depended on $z$, and in consequence, for odd values of $n$ the black hole would be thermodynamically unstable.

Once local stability is assured, it makes sense to ask about the global stability of the system. Following the seminal paper of Hawking and Page [147], we use the free energies at fixed values of the chemical potentials in order to realize which of the phases present in the spectrum is thermodynamically preferred.

In the semiclassical approximation, the on-shell Euclidean action is proportional to the free energy of the system. Taking into account the contributions of the left and right movers, the action acquires the following form

$$I = N_+ E_+ + N_- E_- - S\,. \tag{1.62}$$

Assuming a non-degenerate ground state with zero entropy and whose left and right energies are equal and negative defined, $E_\pm \to -\left|E^0\left[z\right]\right|$, we see that the value of the action of the ground state is given by

$$I_0 = -|E^0\left[z\right]|\left(\frac{1}{T_+} + \frac{1}{T_-}\right)\,. \tag{1.63}$$

On the other hand, considering in (1.62) a system whose entropy is given by the formula (1.56), we obtain that

$$I = -z\left(N_+ E_+ + N_- E_-\right)\,, \tag{1.64}$$

hence, using the formulae for the energies in (1.60), the action then reads

$$I = -|E^0\left[z^{-1}\right]|\,(2\pi)^{\frac{z+1}{z}}\left(T_+^{\frac{1}{z}} + T_-^{\frac{1}{z}}\right)\,. \tag{1.65}$$

Therefore, it is straightforward to see that, regardless of the value of $z$, the partition function $Z =$

$e^{I+I_0}$, will be dominated by (1.63) at low temperatures, and by (1.65) at the high temperatures regime. Consistently, it can also be shown that the same ground state action can be found by making use of the anisotropic S-modular transformation (1.58) on (1.65).

In what follows, we will focus on the simplest case where the whole system is in equilibrium at a fixed temperature $T_\pm = T$. Then, the free energy of the system, $F = I/\beta$, at high and low temperatures will respectively given by

$$F = -2|E^0\left[z^{-1}\right]|\,(2\pi)^{1+\frac{1}{z}}\,T^{1+\frac{1}{z}}\,, \qquad F_0 = -2|E^0\left[z\right]|\,. \tag{1.66}$$

Finally, comparing them, we can obtain the self-dual temperature, at where both free energies coincide,

$$T_s\left[z\right] = \frac{1}{2\pi}\left|\frac{E^0\left[z\right]}{E^0\left[z^{-1}\right]}\right|^{\frac{z}{z+1}}\,, \tag{1.67}$$

which manifestly depend on the dynamical exponent. At this point, a highly non trivial detail is worth to be mentioned. The fact that the self-dual temperature $T_s$ depends on the specific choice of $z$, is because the S-modular transformation involves an inversion of the dynamical exponent between the high and low temperature regimes, namely, $z \to z^{-1}$. This is a defining property of the partition function of the theories that we are dealing with. If one does not take this detail into account, the self-dual temperature would be the same for all values of $z$.

From the gravitational hand, according to the formulas written in the Section 1.3.1 and the Smarr relation (1.57), the free energy, $F = E - TS$, of the static BTZ black hole and thermal $AdS_3$ spacetime with KdV-type boundary conditions are given by

$$F_{BTZ} = -\frac{\ell}{4G}\frac{z}{z+1}\left(2\pi T\right)^{\frac{z+1}{z}}\,, \qquad F_{AdS} = -\frac{\ell}{4G}\frac{1}{z+1}\,, \tag{1.68}$$

and then, the self-dual temperature for which the two phases are equally likely is

$$T_s\left[z\right] = \frac{1}{2\pi}\left(\frac{1}{z}\right)^{\frac{z}{z+1}} . \tag{1.69}$$

Therefore, it is reassuring to verify that if one identifies the ground state energy of the field theory with the one of the $AdS_3$ spacetime with KdV-type boundary conditions in (1.67), it exactly matches with the above gravitational self-dual temperature.

As it is shown in (1.69), for an arbitrary temperature below the self-dual temperature ($T < T_s$), the thermal $AdS_3$ phase has less free energy than the BTZ, and therefore the former one is the most probable configuration, while if $T > T_s$, the black hole phase dominates the partition function and hence is the preferred one. Note that for higher values of $z$, the self-dual temperature becomes lower. The latter point entails to a remarkable result. In the case of Brown-Henneaux boundary conditions ($z = 1$), one can deduce that in order for the black hole reach the equilibrium with a thermal bath at the self-dual point $T_s$, the event horizon must be of the size of the $AdS_3$ radius, i.e., $r_+ = \ell$. Nonetheless, for a generic choice of $z$, the horizon size has to be

$$r_+^s\left[z\right] = \ell\left(\frac{1}{z}\right)^{\frac{1}{z+1}} . \tag{1.70}$$

This means that the size of the black hole at the self-dual temperature decreases for higher values of $z$. In the same way, at $T_s$, the energy of the BTZ, $E_{BTZ} = E_+ + E_-$, endowed with generic KdV-type boundary conditions, acquires the following form

$$E_s\left[z\right] = \frac{\ell}{4G}\frac{1}{z\left(z+1\right)} , \tag{1.71}$$

and when compared to the $AdS_3$ spacetime energy,

$$\Delta E = E_{BTZ} - E_{AdS} = \frac{\ell}{4G}\frac{1}{z} , \tag{1.72}$$

we can observe that at the self-dual temperature there is an endothermic process, where the system absorbs energy from the surround thermal bath at a lower rate for higher values of $z$.

From these last points we can conclude that the global stability of the system is certainly sensitive to which KdV-type boundary condition is chosen, since the free energy of the possible phases of the system are explicitly $z$-dependent. Moreover, the temperature at which both phases have equal free energy, the size of the black hole horizon and the internal energy of the system at that temperature, decrease for higher values of $z$, giving rise to a qualitatively different behavior of the thermodynamic stability of the system, compared to the standard analysis defined by $z = 1$.

# Chapter 2

# Integrable systems with $\text{BMS}_3$ Poisson structure and the dynamics of locally flat spacetimes

One of the main purposes of this chapter is exploring whether the $\text{BMS}_3$ algebra could be further linked with some sort of integrable systems. There are some hints that can be borrowed from CFTs in two dimensions that suggest to look towards this direction. Indeed, it is known that CFTs in 2D admit an infinite set of conserved charges that commute between themselves, which can be constructed out from suitable nonlinear combinations of the generators of the Virasoro algebra and their derivatives (see e.g. [21, 22]). Remarkably, these composite operators turn out to be precisely the conserved charges of the KdV equation, which also correspond to the Hamiltonians of the KdV hierarchy. Therefore, since the $\text{BMS}_3$ algebra can be seen as a limiting case of the conformal algebra in 2D, it is natural to wonder about the possibility of performing a similar construction in that limit. Specifically, one would like to know about the existence of an infinite number of commuting conserved charges that could be suitably recovered from the

$\mathrm{BMS}_3$ generators, as well as its possible relation with some integrable system, or even to an entire hierarchy of them. Here we show that this is certainly the case.

Furthermore, and noteworthy, the dynamics of this class of integrable systems can be equivalently understood in terms of Riemannian geometry. Indeed, following similar strategy as the one in [23] (and reviewed in Chapter 1), here we show that General Relativity without cosmological constant in 3D can be endowed with a suitable set of boundary conditions, so that in the reduced phase space, the Einstein equations precisely agree with the ones of the hierarchy aforementioned. In other words, the dynamics of our class of integrable system can be fully geometrized, since it can be seen to emerge from the evolution of spacelike surfaces embedded in locally flat spacetimes. As a remarkable consequence, in the geometric picture, the symmetries of the integrable systems correspond to diffeomorphisms that maintain the asymptotic form of the spacetime metric, so that they manifestly become Noetherian. Hence, the infinite set of conserved charges can be readily obtained from the surface integrals associated to the asymptotic symmetries in the canonical approach.

In the next section we explicitly construct dynamical (Hamiltonian) systems whose Poisson structure corresponds to the $\mathrm{BMS}_3$ algebra. In order to analyze their symmetries and conserved charges, in Section 2.2 we show how the Drinfeld-Sokolov formulation can be adapted to our case, through the use of suitable flat connections for $isl(2,\mathbb{R})$. Section 2.3 is devoted to a thorough construction and the analysis of an entire bi-Hamiltonian hierarchy of integrable systems with $\mathrm{BMS}_3$ Poisson structure, labeled by a nonnegative integer $n$. We start with a very simple dynamical system ($n = 0$) from which the bi-Hamiltonian structure can be naturally unveiled. The case of $n = 1$ is described in Section 2.3.2, where the Abelian infinite-dimensional symmetries and conserved charges are explicitly identified in terms of a suitable generalization of the Gelfand-Dikii polynomials (see also appendices A and B). The hierarchy of integrable systems with $\mathrm{BMS}_3$ Poisson structure is then explicitly discussed in Section 2.3.3, where it is shown that the so-called "perturbed KdV equations" are included as a particular case. Section 2.3.4 is devoted to the construction of a wide interesting class of analytic solutions for a generic value of the label of the hierarchy $n$. In Section 2.4, we show how the dynamics of the hierarchy of integrable systems can be fully geometrized in terms of locally flat three-dimensional spacetimes. The deep link with General Relativity in 3D is explicitly addressed. We conclude in Section 2.5 with the analysis of the thermodynamics of the Flat Space Cosmology endowed with BMS-type boundary conditions.

## 2.1 Dynamical systems with $BMS_3$ Poisson structure

In order to construct dynamical systems whose Poisson structure is described by the $BMS_3$ algebra, let us consider two independent dynamical fields, $\mathcal{J} = \mathcal{J}(t, \phi)$ and $\mathcal{P} = \mathcal{P}(t, \phi)$, being defined on a cylinder whose coordinates range as $0 \leq \phi < 2\pi$, and $-\infty < t < \infty$. The Poisson structure we look for can then be defined in terms of the following operator

$$\mathcal{D}^{(2)} \equiv \begin{pmatrix} \mathcal{D}^{(\mathcal{J})} & \mathcal{D}^{(\mathcal{P})} \\ \mathcal{D}^{(\mathcal{P})} & 0 \end{pmatrix}, \tag{2.1}$$

where $\mathcal{D}^{(\mathcal{J})}$ and $\mathcal{D}^{(\mathcal{P})}$ stand for Schwarzian derivatives, given by

$$\begin{aligned} \mathcal{D}^{(\mathcal{J})} &= 2\mathcal{J}\partial_\phi + \partial_\phi \mathcal{J} - c_{\mathcal{J}}\partial_\phi^3 , \\ \mathcal{D}^{(\mathcal{P})} &= 2\mathcal{P}\partial_\phi + \partial_\phi \mathcal{P} - c_{\mathcal{P}}\partial_\phi^3 , \end{aligned} \tag{2.2}$$

with arbitrary constants $c_{\mathcal{J}}$ and $c_{\mathcal{P}}$.

The operator $\mathcal{D}^{(2)}$ in eq. (2.1) then allows to write the Poisson brackets of two arbitrary functionals of the dynamical fields, $F = F[\mathcal{J}, \mathcal{P}]$ and $G = G[\mathcal{J}, \mathcal{P}]$, according to

$$\{F, G\} \equiv \int d\phi \begin{pmatrix} \frac{\delta F}{\delta \mathcal{J}} & \frac{\delta F}{\delta \mathcal{P}} \end{pmatrix} \begin{pmatrix} \mathcal{D}^{(\mathcal{J})} & \mathcal{D}^{(\mathcal{P})} \\ \mathcal{D}^{(\mathcal{P})} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta G}{\delta \mathcal{J}} \\ \frac{\delta G}{\delta \mathcal{P}} \end{pmatrix} . \tag{2.3}$$

Therefore, the brackets of the dynamical fields read

$$\begin{aligned} \{\mathcal{J}(\phi), \mathcal{J}(\bar{\phi})\} &= \mathcal{D}^{(\mathcal{J})} \delta(\phi - \bar{\phi}) , \\ \{\mathcal{J}(\phi), \mathcal{P}(\bar{\phi})\} &= \mathcal{D}^{(\mathcal{P})} \delta(\phi - \bar{\phi}) , \\ \{\mathcal{P}(\phi), \mathcal{P}(\bar{\phi})\} &= 0 , \end{aligned} \tag{2.4}$$

so that expanding in Fourier modes as $X = \frac{1}{2\pi} \sum_n X_n e^{-in\phi}$, the algebra of the Poisson brackets in (2.4) precisely reduces to the $\mathrm{BMS}_3$ algebra in eq. (3).

The field equations for the class of dynamical systems we were searching for can then be readily defined as follows

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}} \\ \mu_{\mathcal{P}} \end{pmatrix} , \tag{2.5}$$

where dot denotes the derivative in time, while $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ stand for arbitrary functions of the dynamical fields and their derivatives along $\phi$. When these functions are defined in terms of the variation of a functional $H = H[\mathcal{J}, \mathcal{P}]$, so that

$$\mu_{\mathcal{J}} = \frac{\delta H}{\delta \mathcal{J}} \quad , \quad \mu_{\mathcal{P}} = \frac{\delta H}{\delta \mathcal{P}} , \tag{2.6}$$

the dynamical system is Hamiltonian; and hence, by virtue of (2.3), the field equations can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}} \\ \mu_{\mathcal{P}} \end{pmatrix} = \begin{pmatrix} \{\mathcal{J}, H\} \\ \{\mathcal{P}, H\} \end{pmatrix} . \tag{2.7}$$

Note that unwrapping the angular coordinate to range as $-\infty < \phi < \infty$, allows to extend this class of dynamical systems to $\mathbb{R}^2$, provided that the fall-off of the dynamical fields $\mathcal{J}$, $\mathcal{P}$ is sufficiently fast in order to get rid of boundary terms. Hereafter, for the sake of simplicity, we will assume that the dynamical systems are defined on a cylinder, with a single exception for an interesting particular solution that is described in section 2.3.4.

## 2.2 Zero-curvature formulation

In order to study the properties of the dynamical systems with $\text{BMS}_3$ Poisson structure that evolve according to eq. (2.5), including their symmetries and the corresponding conserved charges, it turns out to be useful to reformulate them in terms of a flat connection for a suitable Lie algebra (see e.g. [112, 148]). In the standard approach of Drinfeld and Sokolov [149], the Lie algebra is assumed to be semisimple. Here we slightly extend this approach in a sense explained right below.

For our purposes, the relevant Lie algebra to be considered corresponds to $isl(2,\mathbb{R})$, which is isomorphic to the Poincaré algebra in 3D. Their commutation relations can then be written as

$$[J_a, J_b] = \epsilon_{abc} J^c \quad , \quad [J_a, P_b] = \epsilon_{abc} P^c \quad , \quad [P_a, P_b] = 0 \, , \tag{2.8}$$

where $J_a$ stand for the generators of $sl(2,\mathbb{R}) \simeq so(2,1)$. It is worth emphasizing that the algebra (2.8) is not semisimple; but nonetheless, it admits an invariant bilinear metric whose nonvanishing components read

$$\langle J_a, J_b \rangle = c_{\mathcal{J}} \eta_{ab} \qquad , \qquad \langle J_a, P_b \rangle = c_{\mathcal{P}} \eta_{ab} \, , \tag{2.9}$$

where $c_{\mathcal{J}}$ and $c_{\mathcal{P}}$ are arbitrary constants. Note that the invariant bilinear metric is nondegenerate provided that $c_{\mathcal{P}} \neq 0$, which will be assumed afterwards.[1]

Hence, by virtue of (2.9), the analysis of the class of dynamical systems defined in the previous section can still be performed *à la* Drinfeld-Sokolov, provided that the field equations are able to be reproduced in terms of a flat connection for $isl(2,\mathbb{R})$.

We then propose that the spacelike component of the $isl(2,\mathbb{R})$-valued gauge field $a = a_\mu dx^\mu$ is given by

$$a_\phi = J_1 + \frac{1}{c_{\mathcal{P}}}\left[\left(\mathcal{J} - \frac{c_{\mathcal{J}}}{c_{\mathcal{P}}}\mathcal{P}\right)P_0 + \mathcal{P}J_0\right] , \tag{2.10}$$

with $\mathcal{J} = \mathcal{J}(t,\phi)$ and $\mathcal{P} = \mathcal{P}(t,\phi)$, while the timelike component reads

$$a_t = \Lambda\left(\mu_{\mathcal{J}}, \mu_{\mathcal{P}}\right) , \tag{2.11}$$

where

$$\begin{aligned}\Lambda\left(\mu_{\mathcal{J}}, \mu_{\mathcal{P}}\right) =& \mu_{\mathcal{P}}P_1 + \mu_{\mathcal{J}}J_1 - \mu_{\mathcal{P}}{}'P_2 - \mu_{\mathcal{J}}{}'J_2 \\ &+ \left(\frac{1}{c_{\mathcal{P}}}\mathcal{P}\mu_{\mathcal{J}} - \mu_{\mathcal{J}}{}''\right)J_0 + \left[\frac{1}{c_{\mathcal{P}}}\left(\mathcal{J} - \frac{c_{\mathcal{J}}}{c_{\mathcal{P}}}\mathcal{P}\right)\mu_{\mathcal{J}} + \frac{1}{c_{\mathcal{P}}}\mathcal{P}\mu_{\mathcal{P}} - \mu_{\mathcal{P}}{}''\right]P_0 \,. \end{aligned}\tag{2.12}$$

Here $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ can be assumed to be given by some arbitrary functions of $\mathcal{J}$, $\mathcal{P}$, and their derivatives along $\phi$ (being denoted by a prime here and afterwards). Therefore, requiring the field strength for the gauge field defined in (2.10) and (2.11) to vanish, i.e.,

$$f = da + a^2 = 0\,, \tag{2.13}$$

implies that the field equations for the dynamical system with $\text{BMS}_3$ Poisson structure in (2.5) hold. It is worth mentioning that $(a_t, a_\phi)$ can then be interpreted as the components of an

[1]We choose the orientation according to $\epsilon_{012} = 1$, while the Minkowski metric $\eta_{ab}$ is assumed to be non-diagonal, whose non-vanishing components are given by $\eta_{01} = \eta_{10} = \eta_{22} = 1$.

$isl(2,\mathbb{R})$-valued Lax pair.

### 2.2.1 Symmetries and conserved charges

One of the advantages of formulating the field equations in terms of a flat connection, is that the symmetries of the dynamical system in (2.5) turn out to correspond to gauge transformations, $\delta_\lambda a = d\lambda + [a,\lambda]$, that preserve the form of the gauge field defined through (2.10) and (2.11).

Hence, requiring the form of the spacelike component of the connection $a_\phi$ in (2.10) to be preserved under gauge transformations, restricts the Lie-algebra-valued parameter to be of the form

$$\lambda = \Lambda\left(\varepsilon_{\mathcal{J}}, \varepsilon_{\mathcal{P}}\right) , \tag{2.14}$$

where $\Lambda$ is precisely given by eq. (2.12), but now depends on two arbitrary functions $\varepsilon_{\mathcal{J}} = \varepsilon_{\mathcal{J}}(t,\phi)$ and $\varepsilon_{\mathcal{P}} = \varepsilon_{\mathcal{P}}(t,\phi)$, while the dynamical fields have to transform according to

$$\begin{pmatrix} \delta\mathcal{J} \\ \delta\mathcal{P} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} . \tag{2.15}$$

Analogously, preserving the timelike component of the gauge field $a_t$ in (2.11), implies that the transformation law of the functions $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ is given by

$$\delta\mu_{\mathcal{J}} = \dot{\varepsilon}_{\mathcal{J}} + \varepsilon_{\mathcal{J}}\mu_{\mathcal{J}}' - \mu_{\mathcal{J}}\varepsilon_{\mathcal{J}}' , \tag{2.16}$$

$$\delta\mu_{\mathcal{P}} = \dot{\varepsilon}_{\mathcal{P}} + \varepsilon_{\mathcal{J}}\mu_{\mathcal{P}}' + \varepsilon_{\mathcal{P}}\mu_{\mathcal{J}}' - \mu_{\mathcal{J}}\varepsilon_{\mathcal{P}}' - \mu_{\mathcal{P}}\varepsilon_{\mathcal{J}}' . \tag{2.17}$$

However, $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ generically depend on the dynamical fields and their spatial derivatives, which means that eqs. (2.16), (2.17) actually become a consistency condition to be fulfilled by the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries of the dynamical system.

In the case of Hamiltonian systems, $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ are determined by the corresponding functional variations of the Hamiltonian as in (2.6), and consequently, the consistency condition for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$, that arises from (2.16), (2.17), can be compactly written as

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}\left(t,\phi\right) \\ \dot{\varepsilon}_{\mathcal{P}}\left(t,\phi\right) \end{pmatrix} = -\begin{pmatrix} \frac{\delta}{\delta\mathcal{J}(t,\phi)} \\ \frac{\delta}{\delta\mathcal{P}(t,\phi)} \end{pmatrix} \int d\varphi \left(\mathcal{D}^{(2)}\begin{pmatrix} \mu_{\mathcal{J}} \\ \mu_{\mathcal{P}} \end{pmatrix}\right)^{T} \begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} . \tag{2.18}$$

In sum, the functions that parametrize the symmetries of the Hamiltonian system with $\text{BMS}_3$ Poisson structure must fulfill the consistency condition in (2.18), which for an arbitrary choice of Hamiltonian, implies that $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ generically acquire an explicit dependence on the dynamical fields and their spatial derivatives.

The variation of the canonical generators associated to the symmetries spanned by $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ can then be readily found by virtue of eqs. (2.9), (2.10) and (2.14), which reduces to the following simple expression

$$\delta Q\left[\varepsilon_{\mathcal{J}},\varepsilon_{\mathcal{P}}\right] = \int d\phi \left\langle \lambda\delta a_{\phi}\right\rangle = \int d\phi \left(\varepsilon_{\mathcal{J}}\delta\mathcal{J} + \varepsilon_{\mathcal{P}}\delta\mathcal{P}\right) . \tag{2.19}$$

As a cross-check, it is simple to verify that the variation of the canonical generators is conserved ($\delta\dot{Q} = 0$) provided that the consistency condition for the symmetry parameters in (2.18) is satisfied.

It is also worth emphasizing that the integrability conditions of (2.19) require that the allowed parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ must correspond to the variation of a functional, since $\varepsilon_{\mathcal{J}} = \frac{\delta Q}{\delta\mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = \frac{\delta Q}{\delta\mathcal{P}}$.

Nevertheless, it must be highlighted that finding the explicit form of the conserved charges $Q$ is not so simple, because it amounts to know the general solution of the consistency condition for the parameters in (2.18). Indeed, although the consistency condition is linear in the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$, the generic solution manifestly depends on the dynamical fields and their spatial derivatives that fulfill a nonlinear field equation. Thus, for a generic choice of the Hamiltonian, solving eq. (2.18) is actually an extremely difficult task.

However, for a generic Hamiltonian that is independent of the coordinates, the conserved charges associated to translations along space and time can be directly constructed. In fact, for a flat connection, diffeomorphisms spanned by $\xi = \xi^{\mu}\partial_{\mu}$ are equivalent to gauge transformations generated by $\lambda = \xi^{\mu}a_{\mu}$, since $\mathfrak{L}_{\xi}a = d\lambda + [a, \lambda]$. Therefore, for $\xi = \partial_{\phi}$, the linear momentum on the cylinder readily integrates as

$$Q[\partial_{\phi}] = Q[1,0] = \int d\phi \mathcal{J} \,. \tag{2.20}$$

Analogously, the variation of the energy is obtained for $\xi = \partial_t$, so that

$$\delta Q[\partial_t] = \delta Q[\mu_{\mathcal{J}}, \mu_{\mathcal{P}}] = \int d\phi \left(\mu_{\mathcal{J}}\delta\mathcal{J} + \mu_{\mathcal{P}}\delta\mathcal{P}\right) \,, \tag{2.21}$$

which by virtue of (2.6), integrates as expected, i.e.,

$$Q[\partial_t] = H \,. \tag{2.22}$$

Note that, generically, there might be additional nontrivial solutions of eq. (2.18) that would lead to further conserved charges.

As a closing remark of this section, it must be emphasized that in order to construct an integrable system with the $BMS_3$ Poisson structure, one should at least specify the precise form of the Hamiltonian, so that the general solution of the consistency condition for the parameters in (2.18) could be obtained. Explicit examples of integrable systems of this sort that actually belong to an infinite hierarchy of them are discussed in the next section.

## 2.3 Hierarchy of integrable systems with $BMS_3$ Poisson structure

In this section we introduce a bi-Hamiltonian hierarchy of integrable systems with $BMS_3$ Poisson structure in a constructive way. We start from an extremely simple case, which nonetheless, possesses the key ingredients in order to propose a precise nontrivial integrable system of this type, that can be extended to an entire hierarchy labeled by a nonnegative integer $n$. The contact with some known results in the literature for certain particular cases is also addressed. Furthermore, a wide class of analytic solutions are explicitly constructed for an arbitrary representative of the hierarchy, including a couple of simple and interesting particular examples.

### 2.3.1 Warming up with a simple dynamical system $(n = 0)$

Let us begin with one of the simplest possible examples of a dynamical system with $BMS_3$ Poisson structure. The field equations can be obtained from (2.7) with $\mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(0)}$ and $\mu_{\mathcal{P}} = \mu_{\mathcal{P}}^{(0)}$ constants, given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(0)} \\ \mu_{\mathcal{P}}^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}, \tag{2.23}$$

so that, according to (2.6), the Hamiltonian is given by $H = H^{(0)}$, with

$$H^{(0)} = \int d\phi \left(\mathcal{J} + a\mathcal{P}\right) . \tag{2.24}$$

The field equations then explicitly read

$$\begin{aligned} \dot{\mathcal{J}} &= \mathcal{J}' + a\mathcal{P}' , \\ \dot{\mathcal{P}} &= \mathcal{P}' , \end{aligned} \tag{2.25}$$

which are trivially integrable. Indeed, the general solution of (2.25) on the cylinder is described by left movers and it can be expressed in terms of periodic functions $\mathcal{M} = \mathcal{M}(t+\phi)$ and $\mathcal{N} = \mathcal{N}(t+\phi)$, so that it reads

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \,, \\ \mathcal{J} &= \mathcal{N} + at\mathcal{M}' \,. \end{aligned} \tag{2.26}$$

Besides, since the field equations are very simple in this case, the consistency condition for the parameters of their symmetries in (2.18) becomes independent of the dynamical fields and their spatial derivatives, which explicitly reduces to

$$\begin{aligned} \dot{\varepsilon}_{\mathcal{J}} &= \varepsilon_{\mathcal{J}}{}' \,, \\ \dot{\varepsilon}_{\mathcal{P}} &= \varepsilon_{\mathcal{P}}{}' + a\varepsilon_{\mathcal{J}}{}' \,. \end{aligned} \tag{2.27}$$

Note that (2.27) coincides with the field equations in (2.25) for $\varepsilon_{\mathcal{J}} = \mathcal{P}$ and $\varepsilon_{\mathcal{P}} = \mathcal{J}$, and hence, if one assumes that the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ depend only on the coordinates $t$, $\phi$, and not on the dynamical fields, the general solution of the consistency conditions for the parameters is also given by chiral (left mover) functions as in (2.26). Therefore, the variation of the canonical generators in (2.19) readily integrates as

$$Q\left[\varepsilon_{\mathcal{J}}, \varepsilon_{\mathcal{P}}\right] = \int d\phi \left(\varepsilon_{\mathcal{J}}\mathcal{J} + \varepsilon_{\mathcal{P}}\mathcal{P}\right) \,. \tag{2.28}$$

The algebra of the conserved charges (2.28) can then be directly obtained from their corresponding Poisson brackets. As a shortcut, by virtue of

$$\left\{Q\left[\varepsilon_1\right], Q\left[\varepsilon_2\right]\right\} = \delta_{\varepsilon_2} Q\left[\varepsilon_1\right] \,, \tag{2.29}$$

the algebra can also be read from the transformation law of the fields in (2.15), and it is then found to be given precisely by the $BMS_3$ algebra in (2.4), which once expanded in modes reads as in eq. (3).

It is worth highlighting that this simple dynamical system actually turns out to be bi-Hamiltonian. This is so because the field equations can be expressed in terms of two different Poisson structures, so that (2.25) can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(0)} \\ \mu_{\mathcal{P}}^{(0)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} , \tag{2.30}$$

where $\mathcal{D}^{(2)}$ is the $BMS_3$ one in (2.1), while $\mathcal{D}^{(1)}$ stands for the "canonical" Poisson structure, defined through the following differential operator

$$\mathcal{D}^{(1)} \equiv \begin{pmatrix} 0 & \partial_\phi \\ \partial_\phi & 0 \end{pmatrix} . \tag{2.31}$$

In (2.30) the functions $\mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}}^{(1)}$ are then given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} = \begin{pmatrix} \mathcal{P} \\ \mathcal{J} + a\mathcal{P} \end{pmatrix} , \tag{2.32}$$

and thus, according to (2.6), the canonical Poisson structure (2.31) is associated to the following Hamiltonian

$$H^{(1)} = \int d\phi \left( \mathcal{J}\mathcal{P} + \frac{a}{2}\mathcal{P}^2 \right) . \tag{2.33}$$

It is worth highlighting that the conserved charge $H^{(1)}$ can also be obtained from (2.19) provided that the parameters of the symmetries are given by $\varepsilon_{\mathcal{J}} = \mu_{\mathcal{J}}^{(1)} = \mathcal{P}$ and $\varepsilon_{\mathcal{P}} = \mu_{\mathcal{P}}^{(1)} = \mathcal{J} + a\mathcal{P}$. Indeed, if the parameters are allowed to depend only on the fields and their derivatives, but not explicitly on the coordinates $t$, $\phi$, one is able to construct an infinite set of independent commuting conserved charges of this sort. This is shown in section 2.3.3.

In sum, the analysis of this extremely simple dynamical system with $\text{BMS}_3$ Poisson structure $\mathcal{D}^{(2)}$, being trivially integrable, allows to unveil a naturally related Poisson structure given by $\mathcal{D}^{(1)}$. The presence of both Poisson structures turns out to be the key in order to proceed with the construction of nontrivial integrable systems as well as an entire hierarchy associated to them. This can be seen as follows. One begins verifying that both Poisson structures are "compatible" in the sense that any linear combination of $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$ also defines a Poisson structure. It is then enough proving that the operator $\mathcal{D}^{(3)} = \mathcal{D}^{(2)} + \mathcal{D}^{(1)}$ also defines a Poisson structure (see e.g. [150]). This is so because the Poisson bracket constructed out from $\mathcal{D}^{(3)}$ is clearly antisymmetric, and furthermore, since $\mathcal{D}^{(3)}$ also fulfills the $\text{BMS}_3$ algebra, but just being shifted by the zero modes of $\mathcal{P}_0 \to \mathcal{P}_0 + \pi$, the Jacobi identity also holds. Besides, by virtue of the fact that our Poisson structure $\mathcal{D}^{(1)}$ is nondegenerate, the hypotheses of a strong theorem for bi-Hamiltonian systems in [151], and further elaborated in [150], are satisfied, which guarantees the existence of the type of hierarchy of integrable systems that we are searching for.

Furthermore, and remarkably, the simple dynamical system described in this section can be seen to be equivalent to the Einstein equations for the reduced phase space that is obtained from a suitable set of boundary conditions for General Relativity in three spacetime dimensions, including its extension with purely geometrical parity-odd terms in the action. This is discussed in section 2.4. It is also worth pointing out that our field equations (2.26), in the case of $c_{\mathcal{J}} = a = 0$, can be interpreted as the ones of a compressible Euler fluid [152].

In the next subsection we carry out the explicit construction and the analysis of a simple, but nontrivial, integrable system with $\text{BMS}_3$ Poisson structure.

### 2.3.2 Integrable bi-Hamiltonian system $(n = 1)$

The first nontrivial integrable system of our hierarchy is obtained from (2.7) with $\mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}} = \mu_{\mathcal{P}}^{(1)}$, where $\mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}}^{(1)}$ are given by eq. (2.32), so that the Hamiltonian corresponds to $H^{(1)}$ in (2.33). The field equations are then explicitly given by

$$\begin{aligned}
\dot{\mathcal{J}} &= 3\mathcal{J}'\mathcal{P} + 3\mathcal{J}\mathcal{P}' - c_{\mathcal{P}}\mathcal{J}''' - c_{\mathcal{J}}\mathcal{P}''' + a\left(3\mathcal{P}'\mathcal{P} - c_{\mathcal{P}}\mathcal{P}'''\right) , \\
\dot{\mathcal{P}} &= 3\mathcal{P}'\mathcal{P} - c_{\mathcal{P}}\mathcal{P}''' .
\end{aligned} \tag{2.34}$$

Note that $\mathcal{P}$ evolves according to the KdV equation,[2] while the remaining equation is linear in $\mathcal{J}$, with an inhomogeneous source term that is entirely determined by $\mathcal{P}$ and their spatial derivatives.

The field equations in (2.34) can also be seen to arise from a bi-Hamiltonian system with the same $\text{BMS}_3$ and canonical Poisson structures given by (2.1) and (2.31), respectively. Indeed, they can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(2)} \\ \mu_{\mathcal{P}}^{(2)} \end{pmatrix} , \tag{2.35}$$

where the functions $\mu_{\mathcal{J}}^{(2)}$ and $\mu_{\mathcal{P}}^{(2)}$ are given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(2)} \\ \mu_{\mathcal{P}}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}'' \\ 3\mathcal{J}\mathcal{P} - c_{\mathcal{P}}\mathcal{J}'' - c_{\mathcal{J}}\mathcal{P}'' + a\left(\frac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}''\right) \end{pmatrix} , \tag{2.36}$$

which can be obtained from the functional derivatives of a different Hamiltonian, as in (2.6),

[2] This is a direct consequence of the choice of $\mu_{\mathcal{J}}$ and the fact that the off-diagonal terms in the Poisson structure $\mathcal{D}^{(2)}$ exclusively depend on $\mathcal{D}^{(\mathcal{P})}$. Besides, a common practice in the literature is rescaling the field and the coordinates so that the KdV equation does not depend on $c_{\mathcal{P}}$ (see, e.g., [150]). However, as explained in section 2.3.2, for our purposes, and for the sake of simplicity, keeping $c_{\mathcal{P}}$ explicitly in the field equations turns out to be very useful and convenient.

that reads

$$H^{(2)} = \int d\phi \left[\frac{3}{2}\mathcal{P}^2\mathcal{J} - c_{\mathcal{P}}\mathcal{P}''\mathcal{J} + \frac{c_{\mathcal{J}}}{2}\mathcal{P}'^2 + a\left(\frac{1}{2}\mathcal{P}^3 + \frac{c_{\mathcal{P}}}{2}\mathcal{P}'^2\right)\right] , \tag{2.37}$$

being clearly conserved.

### Symmetries

As explained in section 2.2.1, in order to find the remaining conserved quantities, it is necessary to find the general solution of the consistency conditions in (2.18) for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries of the field equations. In this case ($n = 1$), the consistency conditions in (2.18), with $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ given by eq. (2.32), explicitly reduce to

$$\begin{aligned}
\dot{\varepsilon}_{\mathcal{J}} &= 3\mathcal{P}\varepsilon'_{\mathcal{J}} - c_{\mathcal{P}}\varepsilon'''_{\mathcal{J}} , \\
\dot{\varepsilon}_{\mathcal{P}} &= 3\mathcal{J}\varepsilon'_{\mathcal{J}} + 3\mathcal{P}\varepsilon'_{\mathcal{P}} - c_{\mathcal{P}}\varepsilon'''_{\mathcal{P}} - c_{\mathcal{J}}\varepsilon'''_{\mathcal{J}} + a\left(3\mathcal{P}\varepsilon'_{\mathcal{J}} - c_{\mathcal{P}}\varepsilon'''_{\mathcal{J}}\right) .
\end{aligned} \tag{2.38}$$

Note that the equations in (2.38) are linear for the parameters $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$. However, finding their solution is not that simple because their coefficients depend on $\mathcal{J}$ and $\mathcal{P}$, who evolve according to the nonlinear field equations in (2.34). Nevertheless, if one assumes that the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ depend only on the dynamical fields and their spatial derivatives, but not explicitly on the coordinates $t$, $\phi$, and takes into account that the parameters must correspond to the variation of a functional, the theorem in [150, 151] then guarantees that the general solution of the consistency conditions (2.38) can be formally found. In our case, the explicit solution turns out to be given by a linear combination of two independent arrays, $K^{(j)}$ and $\tilde{K}^{(j)}$, that stand for a suitable generalization of the Gelfand-Dikii polynomials. The solution can then be written as

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = \sum_{j=0}^{\infty}\left[\eta_j K^{(j)} + \tilde{\eta}_j \tilde{K}^{(j)}\right] , \tag{2.39}$$

where $\eta_j$ and $\tilde{\eta}_j$ are arbitrary constants, and both generalized polynomials $K^{(j)}$ and $\tilde{K}^{(j)}$ fulfill the same recursive relationship, given by

$$\mathcal{D}^{(1)} K^{(i+1)} = \mathcal{D}^{(2)} K^{(i)} \,. \tag{2.40}$$

If the initial seeds of the independent arrays are chosen as

$$K^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \tilde{K}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{2.41}$$

the recursion relation (2.40) then implies that the remaining ones are given by

$$K^{(n)} = \begin{pmatrix} 0 \\ R^{(n)} \end{pmatrix} , \quad \tilde{K}^{(n)} = \begin{pmatrix} R^{(n)} \\ T^{(n)} \end{pmatrix} , \tag{2.42}$$

where $R^{(n)}$ stand for the standard Gelfand-Dikii polynomials, while $T^{(n)}$ correspond to a different set of polynomials that fulfill the following recursion relationships

$$\partial_\phi R^{(n+1)} = \mathcal{D}^{(\mathcal{P})} R^{(n)} , \tag{2.43}$$

$$\partial_\phi T^{(n+1)} = \mathcal{D}^{(\mathcal{P})} T^{(n)} + \mathcal{D}^{(\mathcal{J})} R^{(n)} \,. \tag{2.44}$$

Remarkably, both sets of polynomials can be obtained from the variation of two independent functionals, $H^{(n)}_{\text{KdV}}[\mathcal{P}]$ and $\tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]$, so that

$$R^{(n)} = \frac{\delta H^{(n)}_{\text{KdV}}[\mathcal{P}]}{\delta \mathcal{P}} = \frac{\delta \tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]}{\delta \mathcal{J}} , \tag{2.45}$$

$$T^{(n)} = \frac{\delta \tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]}{\delta \mathcal{P}} , \tag{2.46}$$

where $H^{(n)}_{\text{KdV}}$ stands for $n$-th conserved quantity of the KdV equation, while $\tilde{H}^{(n)}\left[\mathcal{P},\mathcal{J}\right]$ depends linearly on $\mathcal{J}$ and it is given by

$$\tilde{H}^{(n)}\left[\mathcal{P},\mathcal{J}\right]=c_{\mathcal{J}}\frac{\partial H^{(n)}_{\text{KdV}}\left[\mathcal{P}\right]}{\partial c_{\mathcal{P}}}+\int d\phi\mathcal{J}\frac{\delta H^{(n)}_{\text{KdV}}\left[\mathcal{P}\right]}{\delta\mathcal{P}}\,. \tag{2.47}$$

Therefore, the generalized polynomials can also be compactly defined as

$$K^{(n)}=\left(\begin{array}{c}\frac{\delta}{\delta\mathcal{J}}\\ \frac{\delta}{\delta\mathcal{P}}\end{array}\right)H^{(n)}_{\text{KdV}}\quad,\quad\tilde{K}^{(n)}=\left(\begin{array}{c}\frac{\delta}{\delta\mathcal{J}}\\ \frac{\delta}{\delta\mathcal{P}}\end{array}\right)\tilde{H}^{(n)}\,. \tag{2.48}$$

An explicit list of the first six polynomials $R^{(n)}$ and $T^{(n)}$, with their corresponding functionals $H^{(n)}_{\text{KdV}}$ and $\tilde{H}^{(n)}$ is given in Appendix A.

### Conserved charges

Since the general form of the parameters that describe the symmetries of the field equations has been explicitly found to be given by (2.39), by virtue of the fact that the generalized polynomials come from the functional derivatives of suitable functionals as in (2.48), the variation of the canonical generators in (2.19) reduces to

$$\delta Q\left[\eta,\tilde{\eta}\right]=-\sum_{j=0}^{\infty}\int d\phi\left[\eta_{j}\frac{\delta H^{(j)}_{\text{KdV}}}{\delta\mathcal{P}}\delta\mathcal{P}+\tilde{\eta}_{j}\left(\frac{\delta\tilde{H}^{(j)}}{\delta\mathcal{P}}\delta\mathcal{P}+\frac{\delta\tilde{H}^{(j)}}{\delta\mathcal{J}}\delta\mathcal{J}\right)\right]\,, \tag{2.49}$$

which then readily integrates as

$$Q\left[\eta,\tilde{\eta}\right]=-\sum_{j=0}^{\infty}\left(\eta_{j}H^{(j)}_{\text{KdV}}+\tilde{\eta}_{j}\tilde{H}^{(j)}\right)\,. \tag{2.50}$$

Therefore, we have explicitly found two infinite independent towers of conserved quantities, being spanned by $H^{(j)}_{\text{KdV}}$ and $\tilde{H}^{(j)}$, which by virtue of the recursion relation in (2.40), turn out to be in involution for both Poisson structures $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$, i.e.,

$$\left\{Q[\eta,\tilde{\eta}],Q[\zeta,\tilde{\zeta}]\right\}_{(2)}=\left\{Q[\eta,\tilde{\eta}],Q[\zeta,\tilde{\zeta}]\right\}_{(1)}=0\,, \tag{2.51}$$

where the subscripts for the brackets in (2.51) stand for the corresponding Poisson structures.[3]

Note that the pair of Hamiltonians that yield the same field equations in (2.35), given by (2.33) and (2.37), can then be written as

$$H^{(1)}=\tilde{H}^{(1)}+aH^{(1)}_{\text{KdV}}\quad,\quad H^{(2)}=\tilde{H}^{(2)}+aH^{(2)}_{\text{KdV}}\,. \tag{2.52}$$

In sum, as pointed out in the introduction, eq. (2.50) turns out to be an explicit realization of the infinite set of commuting conserved charges that is constructed out from precise nonlinear combinations of the generators of the $\text{BMS}_3$ algebra and their spatial derivatives. As discussed in section 2.3.3, this provides the basis to extend this integrable system to an entire hierarchy.

**Remarks on some additional symmetries**

The existence of additional symmetries, enlarging the set spanned by the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (2.39), is unveiled by relaxing our hypotheses, so that the parameters of the symmetries are now allowed to depend not just on the dynamical fields and their spatial derivatives, but also explicitly on the coordinates $t$, $\phi$. Hence, apart from the infinite set of symmetries spanned by (2.15), with $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ given by (2.39), the field equations (2.34) can also be seen to be invariant under Galilean and anisotropic scale transformations:

- *Galilean transformations.-* They are parametrized by a single constant velocity parameter $v_0$, so that the coordinates and the fields transform according to

$$\bar{\phi}=\phi-v_0t\qquad,\qquad\bar{t}=t\,, \tag{2.53}$$

[3]For an explicit proof of the involution of the conserved charges see Appendix B.

and

$$\bar{\mathcal{P}} = \mathcal{P} + \frac{v_0}{3} \qquad , \qquad \bar{\mathcal{J}} = \mathcal{J} - \frac{a}{3} v_0 \,, \tag{2.54}$$

respectively.

For simplicity, if one chooses the Poisson structure $\mathcal{D}^{(1)}$, the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ that correspond to the infinitesimal Galilean transformations are then given by

$$\begin{aligned} \varepsilon_{\mathcal{J}} &= v_0 t \mathcal{P} + \frac{\phi}{3} \,, \\ \varepsilon_{\mathcal{P}} &= v_0 t \mathcal{J} - \frac{a\phi}{3} \,, \end{aligned} \tag{2.55}$$

which manifestly fulfill their consistency conditions in (2.38). Equivalently, the parameters in (2.55) can be expressed in terms of the generalized polynomials as in (2.39), where now the coefficients $\eta_j$ and $\tilde{\eta}_j$ acquire an explicit dependence on the coordinates $t$, $\phi$, so that

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = v_0 t \tilde{K}^{(1)} + \frac{v_0 \phi}{3} \left( \tilde{K}^{(0)} - a K^{(0)} \right) \,. \tag{2.56}$$

Therefore, since $\varepsilon_{\mathcal{J}} = v_0 \frac{\delta G}{\delta \mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = v_0 \frac{\delta G}{\delta \mathcal{P}}$, one readily obtains the conserved charge that corresponds to the generator of the Galilean transformations, which reads

$$G = \int d\phi \left[ t \mathcal{J} \mathcal{P} + \frac{\phi}{3} \left( \mathcal{J} - a \mathcal{P} \right) \right] \,. \tag{2.57}$$

- *Anisotropic scaling of Lifshitz type.*- This symmetry is defined through a constant parameter $\sigma$, and it is generated by the transformations

$$\bar{t} = \sigma^3 t \quad , \quad \bar{\phi} = \sigma \phi \quad , \quad \begin{pmatrix} \bar{\mathcal{J}} \\ \bar{\mathcal{P}} \end{pmatrix} = \sigma^{-2} \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} \,, \tag{2.58}$$

which correspond to anisotropic scaling of Lifshitz type with dynamical exponent $z = 3$. In this case, in terms of the Poisson structure $\mathcal{D}^{(2)}$, the parameters that span the infinitesimal anisotropic scaling transformations in (2.58) read as

$$\begin{aligned} \varepsilon_{\mathcal{J}} &= \lambda\left(3t\mathcal{P} + \phi\right) , \\ \varepsilon_{\mathcal{P}} &= 3\lambda t\left(\mathcal{J} + a\mathcal{P}\right) , \end{aligned} \tag{2.59}$$

which satisfy the consistency conditions in (2.38). These parameters can also be written as a linear combination of the generalized polynomials as in (2.39), where the coefficients depend on $t$ and $\phi$, so that

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = 3\lambda t\left(\tilde{K}^{(1)} + aK^{(1)}\right) + \lambda\phi\tilde{K}^{(0)} . \tag{2.60}$$

Hence, the form of the generator of the anisotropic scaling transformations $D$ can be directly read from $\varepsilon_{\mathcal{J}} = \lambda\frac{\delta D}{\delta\mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = \lambda\frac{\delta D}{\delta\mathcal{P}}$, with

$$D = \int d\phi\left[3t\left(\mathcal{J}\mathcal{P} + \frac{a}{2}\mathcal{P}^2\right) + \phi\mathcal{J}\right] . \tag{2.61}$$

For later purposes, it is worth noting that both sets of conserved quantities, $H_{\rm KdV}^{(n)}$ and $\tilde{H}^{(n)}$, scale under (2.58) according to

$$\bar{H}^{(n)} = \sigma^{-(2n+1)}H^{(n)} . \tag{2.62}$$

### Equivalence with the Hirota-Satsuma coupled KdV system of type ix

Here we show that the field equations of the bi-Hamiltonian integrable system with $\mathrm{BMS}_3$ and canonical Poisson structures, described by (2.34), can be seen to be equivalent to a particular class of a generalization of the Hirota-Satsuma coupled KdV system [153]. Specifically, the equivalence can be seen as follows.

If one changes the dynamical fields and rescale time according to

$$u = \frac{1}{4c_{\mathcal{P}}}\mathcal{P} \quad , \quad v = \frac{1}{4}\left(\mathcal{J} + a\mathcal{P}\right) \quad , \quad \tau = -c_{\mathcal{P}}t\,, \tag{2.63}$$

the field equations in (2.34) read

$$\partial_\tau v = -12uv' - 12vu' + v''' + \gamma u'''\,, \tag{2.64}$$

$$\partial_\tau u = -12uu' + u'''\,, \tag{2.65}$$

with

$$\gamma \equiv c_{\mathcal{J}} + ac_{\mathcal{P}}\,. \tag{2.66}$$

The equations in (2.64), (2.65) then turn out to be precisely the ones of type ix in [154]. Thus, at the level of the field equations there are actually only two inequivalent cases. The generic one corresponds to $\gamma = 1$, since the field equation in (2.64) can always be brought to this form provided that $v$ is rescaled as $v \to \gamma v$. The remaining case is described by $\gamma = 0$, which is also known in the literature as "perturbed KdV" (see e.g. [155, 156, 157, 158, 159]).

Note that for $\gamma = 0$ ($c_{\mathcal{J}} = -ac_{\mathcal{P}}$), configurations with $v = 0$ ($\mathcal{J} = -a\mathcal{P}$) are devoid of energy, since $H = H^{(0)}$ in (2.24) vanishes. Nonetheless, they are generically endowed with both towers of conserved charges $H^{(j)}_{\text{KdV}}$ and $\tilde{H}^{(j)}$.

The structures discussed in this subsection provide all what is needed in order to generalize the integrable system to a hierarchy of them that is labeled by a nonnegative integer $n$.

### 2.3.3 The hierarchy $(n \geq 0)$

The results obtained in the previous section ensure that a hierarchy of bi-Hamiltonian integrable systems with $\text{BMS}_3$ and canonical Poisson structures, given by $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$, can be readily constructed out from choosing any of their Hamiltonians to be given by an arbitrary linear combination of the independent conserved charges $H_{\text{KdV}}^{(n)}$ and $\tilde{H}^{(n)}$ in (2.47). Hereafter we focus in the subclass of them that possesses well-defined scaling properties. In order to achieve this task, we recall that if one rescales the spacelike coordinate and the fields as in (2.58), then both sets of conserved charges scale according to (2.62). Therefore, the most general combination that possesses the suitable scaling properties that we look for is described by a Hamiltonian of the form

$$H^{(n)} = \tilde{H}^{(n)} + aH_{\text{KdV}}^{(n)}\,, \tag{2.67}$$

with $a$ constant and any fixed value of the integer $n \geq 0$.

The field equations of the corresponding hierarchy of integrable systems then read

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(n)} \\ \mu_{\mathcal{P}}^{(n)} \end{pmatrix}, \tag{2.68}$$

with

$$\mu_{\mathcal{J}}^{(n)} = \frac{\delta H^{(n)}}{\delta \mathcal{J}} \quad \text{and} \quad \mu_{\mathcal{P}}^{(n)} = \frac{\delta H^{(n)}}{\delta \mathcal{P}}\,. \tag{2.69}$$

The hierarchy defined through (2.68) is clearly bi-Hamiltonian, since by virtue of the recursion relationship in (2.40), the field equations can also be expressed as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(n)} \\ \mu_{\mathcal{P}}^{(n)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(n+1)} \\ \mu_{\mathcal{P}}^{(n+1)} \end{pmatrix}. \tag{2.70}$$

The field equations of the hierarchy can also be explicitly written in terms of the polynomials

$T^{(n)}$ defined through (2.46) and the Gelfand-Dikii polynomials $R^{(n)}$, so that they read

$$\begin{aligned}
\dot{\mathcal{J}} &= \mathcal{D}^{(\mathcal{P})}T^{(n)} + \left(\mathcal{D}^{(\mathcal{J})} + a\mathcal{D}^{(\mathcal{P})}\right)R^{(n)}\,, \\
\dot{\mathcal{P}} &= \mathcal{D}^{(\mathcal{P})}R^{(n)}\,.
\end{aligned} \tag{2.71}$$

It is worth pointing out that for any representative of the hierarchy with $n > 1$, not only the field equations in (2.71), but also the consistency condition for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries in (2.18) become severely more complicated than the simplest cases of $n = 0, 1$ (see eqs. (2.25), (2.27), and (2.34), (2.38), respectively). However, and remarkably, when $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ are assumed to depend only on the dynamical fields and their spatial derivatives, but not explicitly on $t$, $\phi$, by virtue of the theorem in [150, 151], the general solution of the consistency condition for the parameters in (2.18) for $n \geq 0$ turns out to be precisely given by the same expansion in terms of the generalized Gelfand-Dikii polynomials $K^{(j)}$ and $\tilde{K}^{(j)}$, as in (2.39) for $n = 1$. This is explicitly verified in Appendix C. Therefore, as a consequence, the corresponding canonical generators turn out to be given by the two independent sets of conserved charges given by (2.50), which are in involution, i.e., the commuting charges fulfill (2.51) for both Poisson structures. Nevertheless, depending on the choice of Poisson structure, $\mathcal{D}^{(2)}$ or $\mathcal{D}^{(1)}$, the energy of the system in (2.22), now corresponds to the Hamiltonian, $H^{(n)}$ or $H^{(n+1)}$, defined through (2.67), respectively.

Furthermore, by construction, the field equations for any representative of the hierarchy turn out to be invariant under anisotropic scaling transformations given by

$$t \to \sigma^z t \quad , \quad \phi \to \sigma\phi \quad , \quad \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} \to \sigma^{-2} \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} , \tag{2.72}$$

which is of Lifshitz type, and characterized by a dynamical exponent $z = 2n + 1$. Isotropic scaling then only holds for $n = 0$.

In terms of the BMS$_3$ Poisson structure $\mathcal{D}^{(2)}$, the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ that correspond to the infinitesimal anisotropic scaling transformation in (2.72), are then given by

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = \lambda z t \left( \tilde{K}^{(n)} + a K^{(n)} \right) + \lambda \phi \tilde{K}^{(0)} \,, \tag{2.73}$$

and fulfill the consistency conditions in (2.18). The corresponding conserved charge can then be obtained from $\varepsilon_{\mathcal{J}} = \lambda \frac{\delta D}{\delta \mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = \lambda \frac{\delta D}{\delta \mathcal{P}}$, with

$$D = z t H^{(n)} + \int d\phi \, \phi \mathcal{J} \,, \tag{2.74}$$

where $H^{(n)}$ stands for the Hamiltonian, given by eq. (2.67).

As an ending remark of this subsection, it is worth noting that the field equations of the so-called "perturbed KdV hierarchy" are precisely recovered from (2.71) for the particular case of $c_{\mathcal{J}} = a = 0$. In this case, for the entire hierarchy, according to (2.67), configurations with $\mathcal{J} = 0$ do not carry energy, which goes by hand with the fact that the Hamiltonian corresponds to the energy of a perturbation described by $\mathcal{J}$. Indeed, for this class of configurations, the entire set of conserved charges $\tilde{H}^{(j)}$ in (2.47) also vanishes. Note that the remaining ones, given by $H^{(j)}_{\text{KdV}}$ generically remain nontrivial, which can be interpreted as the conserved charges associated to an arbitrary "background configuration" described by $\mathcal{P} = \mathcal{P}(t, \phi)$ that solves the field equations of the $n$-th representative of the KdV hierarchy.

### 2.3.4 Analytic solutions

Exact analytic solutions of the KdV equation, as well as for the $k$-th representative of the KdV hierarchy, have been thoroughly studied in the literature since long ago through different methods (see e.g., [158]).

Here we show how to obtain an interesting wide class of analytic solutions of the field equations in (2.71) for an arbitrary value of the nonnegative integer $k$ that labels our hierarchy of integrable systems with $BMS_3$ Poisson structure.

As a warming up exercise, let us begin considering the simplest case, described by choosing the field $\mathcal{P}$ to be constant, so that the field equation of the KdV hierarchy in (2.71) is trivially solved for an arbitrary value of $k$. In this case, the field equation for $\mathcal{J}(t,\phi)$ in (2.71) just reduces to a dispersive linear homogeneous equation with constant coefficients, given by

$$\dot{\mathcal{J}} = \sum_{m=0}^{k} \alpha_{k,m}(-c_{\mathcal{P}})^{k-m}\mathcal{P}^m \partial_\phi^{2k-2m+1}\mathcal{J}\,, \tag{2.75}$$

with $\alpha_{k,m} \equiv \frac{(2k+1)!!}{m!(2k-2m+1)!!}$, which can be easily solved for an arbitrary member of the hierarchy. Indeed, expanding in Fourier modes according to[4]

$$\mathcal{J}\left(t,\phi\right) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\mathcal{J}_n e^{-i\left(\omega_{k,n}t+n\phi\right)}\,, \tag{2.76}$$

one finds that the corresponding dispersion relation is given by

$$\omega_{k,n} = \sum_{m=0}^{k}\alpha_{k,m}{c_{\mathcal{P}}}^{k-m}\mathcal{P}^m n^{2(k-m)+1}\,. \tag{2.77}$$

In the case of nontrivial solutions $\mathcal{P}=\mathcal{P}(t,\phi)$ of the $k$-th KdV equation, it is also possible to find generic analytic solutions for $\mathcal{J}\left(t,\phi\right)$. Remarkably, in spite of the fact that $\mathcal{J}\left(t,\phi\right)$ obeys a linear differential equation, their exact solutions are able to be nondispersive. This effect occurs because the coefficients of the linear equation for $\mathcal{J}$ in (2.71) are determined by nontrivial solutions of the $k$-th KdV equation, and it persists even in presence of a source term (with $a \neq 0$ or $c_{\mathcal{J}} \neq 0$). This is explicitly discussed in what follows.

[4] $\mathcal{J}(t,\phi)$ is real provided that the modes fulfill $(\mathcal{J}_n)^* = \mathcal{J}_{-n}$.

#### Generic analytic solutions

Let us assume that $\mathcal{P} = \mathcal{P}(t, \phi)$ corresponds to an arbitrary generic solution of the field equations of the $k$-th representative of the KdV hierarchy, described by the second line of (2.71). Note that, since the central extension $c_{\mathcal{P}}$ has not been scaled away, nontrivial solutions $\mathcal{P}(t, \phi)$ explicitly depend on $c_{\mathcal{P}}$.

In order to find the form of $\mathcal{J} = \mathcal{J}(t, \phi)$ one has to solve the remaining equation in (2.71), which turns out to be linear in $\mathcal{J}$, and possesses an inhomogeneous term that is completely specified by $\mathcal{P}$ and their spatial derivatives. Hence, the generic solution for $\mathcal{J}$ is given by the sum of the particular and the homogeneous solutions, i.e.,

$$\mathcal{J} = \mathcal{J}_h + \mathcal{J}_p \,. \tag{2.78}$$

Noteworthy, as it is shown in Appendix D, for an arbitrary value of the label $k$ of the hierarchy, a particular solution $\mathcal{J} = \mathcal{J}_p(t, \phi)$ can be analytically expressed in a very compact way, so that it reads[5]

$$\mathcal{J}_p = c_{\mathcal{J}} \frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}} \,. \tag{2.79}$$

Besides, a generic solution of the homogeneous equation can be found by virtue of the symmetries in (2.15), being spanned by the subset of parameters given by (2.39) that preserve the form of $\mathcal{P}(t, \phi)$, i.e., the ones for which $\delta\mathcal{P} = 0$. The suitable subset of symmetries we look for then becomes generated by an arbitrary combination of the generalized polynomials $K^{(j)}$, excluding $\tilde{K}^{(j)}$; and hence the parameters are given by $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ in (2.39) with $\tilde{\eta}_j = 0$. Thus, according to (2.15), the homogeneous solution is given by

$$\mathcal{J}_h = \delta_{\eta_j} \mathcal{J} = \sum_{j=0}^{\infty} \eta_j \mathcal{D}^{(\mathcal{P})} R^{(j)} \,. \tag{2.80}$$

[5] Note that this particular solution becomes trivial ($\mathcal{J}_p = 0$) in the case of "perturbed KdV" described by $c_{\mathcal{J}} = a = 0$.

Therefore, by virtue of the recursive relation of the Gelfand-Dikii polynomials in (2.43), the generic solution for $\mathcal{J}$ acquires the form

$$\mathcal{J} = \sum_{j=0}^{\infty} \eta_j \partial_\phi R^{(j+1)} + c_{\mathcal{J}} \frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}} \,. \tag{2.81}$$

As a cross-check, in Appendix D it is explicitly shown that eq. (2.81) solves the field equation for $\mathcal{J}$ in (2.71).

In sum, the generic solution for $\mathcal{J}$ in (2.81) has been generated through acting on the particular solution $\mathcal{J}_p$ with the symmetries that are spanned by the corresponding canonical generators in (2.50) with $\tilde{\eta}_j = 0$. Hence, since the generators are in involution, the full set of conserved charges for the solution characterized by the fields $\mathcal{P}$ and $\mathcal{J}_p$ must coincide with the ones for the fields $\mathcal{P}$ with $\mathcal{J}$ given by (2.81). In other words, the homogeneous part of the solution $\mathcal{J}_h$ does not contribute to the conserved charges.

The conserved charges are then described by the corresponding ones for KdV, given by $H^{(n)}_{\text{KdV}}$, together with the independent set $\tilde{H}^{(n)}$ in (2.47). Once the latter set is evaluated in the generic solution (2.81), it can be compactly written in terms of the total derivative of $H^{(n)}_{\text{KdV}}$ with respect to $c_{\mathcal{P}}$, so that it reads (see appendix D)

$$\tilde{H}^{(n)} = c_{\mathcal{J}} \frac{dH^{(n)}_{\text{KdV}}}{dc_{\mathcal{P}}} \,. \tag{2.82}$$

Therefore, for the BMS$_3$ Poisson structure $\mathcal{D}^{(2)}$, the energy of our generic solution for the $n$-th representative of the hierarchy is given by the Hamiltonian in (2.67), which reduces to

$$H^{(n)} = c_{\mathcal{J}} \frac{dH^{(n)}_{\text{KdV}}}{dc_{\mathcal{P}}} + aH^{(n)}_{\text{KdV}} \,. \tag{2.83}$$

It is worth pointing out that the conserved charges of the solution can be expressed exclusively in terms of $\mathcal{P}$ and their spatial derivatives.

Note that the generic class of solutions presented here is mapped into itself under the anisotropic scaling of Lifshitz type given in (2.72), where the arbitrary constants $\eta_j$ of the homogeneous solution (2.80) transform as $\bar{\eta}_j = \sigma^{2j-1}\eta_j$.

Additionally, as pointed in section 2.3.2, in the case of $n = 1$ the field equations are also invariant under Galilean transformations. Hence, in this case, by virtue of the Galilean boost spanned by (2.53), our solution in (2.81) acquires nontrivial zero modes once expressed in the moving frame.

In the next subsection, we explicitly describe a couple of simple and interesting particular examples of analytic solutions in the case of $n = 1$.

**Particular cases for $n = 1$**

- *Single KdV soliton on the real line.-* The integrable system described in section 2.3.2 can be extended to $\mathbb{R}^2$ provided that the angular coordinate is unwrapped ($-\infty < \phi < \infty$), so that our previous analysis still holds once the fall-off of the fields is assumed to be fast enough so as to get rid of boundary terms.

  In our conventions, the well-known single soliton solution of the KdV equation in (2.34) reads

  $$\mathcal{P} = -v \,\mathrm{sech}^2 (x) \ , \tag{2.84}$$

  where $x = \sqrt{\frac{v}{4c_{\mathcal{P}}}}(\phi - vt)$, and $v$ stands for the integration constant that parametrizes the velocity and amplitude of the soliton.

  An analytic solution for the remaining field equation in (2.34) can then be constructed out from (2.81), with $\mathcal{P}$ given by (2.84). For simplicity we consider that the integration constants in (2.81) are chosen as $\eta_j = \eta_1 \delta_{j,1}$, with $\eta_1$ arbitrary, so that the solution for $\mathcal{J}$

becomes explicitly given by

$$\mathcal{J} \;=\; \left(\eta_1 \frac{v^{3/2}}{\sqrt{c_{\mathcal{P}}}} - \frac{vx(ac_{\mathcal{P}} + c_{\mathcal{J}})}{c_{\mathcal{P}}}\right) \tanh(x) \operatorname{sech}^2(x) + av \operatorname{sech}^2(x) \; . \tag{2.85}$$

Therefore, although $\mathcal{J}$ obeys a linear differential equation, the solution in (2.85) clearly maintains its shape as it evolves in time. The profile of $\mathcal{J}$ in (2.85) is depicted in Figure 2.1.

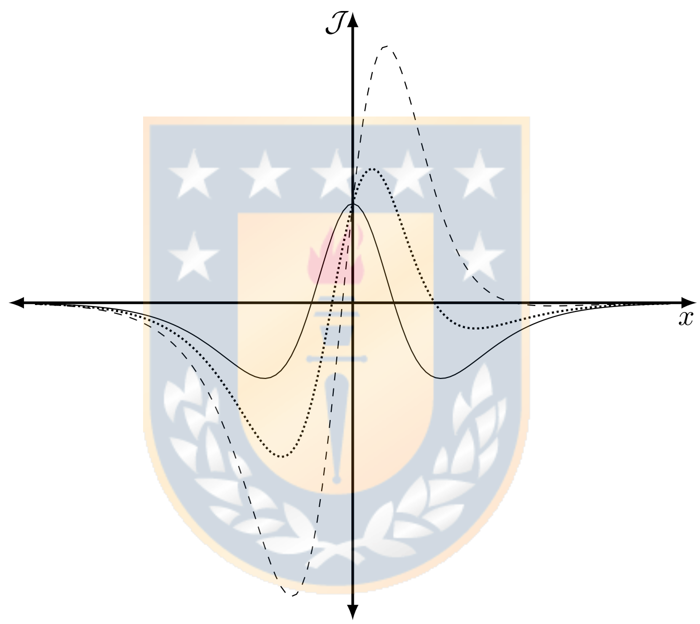


Figure 2.1: The form of $\mathcal{J}$ in (2.85) is plotted for generic fixed values of $c_{\mathcal{P}}$, $c_{\mathcal{J}}$ , $a$, $v$ and for different values of $\eta_1$. Note that when the integration constant $\eta_1$ vanishes, $\mathcal{J}$ becomes an even function of $x$ (solid line).

- *Cnoidal wave on* $S^1$.- In the case of periodic boundary conditions ($-\pi \leq \phi < \pi$) an analytic solution for the KdV equation in (2.34) of solitonic type is known as a "cnoidal wave", since it is described in terms the Jacobi elliptic cosine (cn) (see e.g. [158]). The solution is given by

$$\mathcal{P} = 4c_{\mathcal{P}} \left[A - \alpha \mathrm{cn}^2(y, m)\right] \; , \tag{2.86}$$

with $y=\sqrt{\frac{\alpha}{m}}(\phi-c_{\mathcal{P}}vt)$, and the velocity parameter is related to the remaining integration constants as

$$v=4\alpha\left(2-\frac{1}{m}\right)-12A\,. \tag{2.87}$$

The wavelength of the solution is given by $2\sqrt{\frac{m}{\alpha}}K(m)$, where $K$ stands for a complete elliptic integral of the first kind. Accordingly, the elliptic parameter $m$ can take values within the range $0<m<1$, satisfying $2\sqrt{\frac{m}{\alpha}}K(m)=\frac{2\pi}{n}$ with $n\in\mathbb{N}$.

As in the previous example, we then construct the analytic solution for $\mathcal{J}$ from the generic one in (2.81), with $\mathcal{P}$ given by the cnoidal wave in (2.86). For the sake of simplicity, we again choose the integration constants to be given by $\eta_j=\eta_1\delta_{j,1}$, and hence the searched for analytic solution becomes

$$\mathcal{J}=4c_{\mathcal{J}}\left[A-\alpha\mathrm{cn}^2\left(y,m\right)\right]+\left[8\alpha c_{\mathcal{P}}\sqrt{\frac{\alpha}{m}}(\eta_1-(ac_{\mathcal{P}}+c_{\mathcal{J}})vt)\right]\mathrm{cn}\left(y,m\right)\mathrm{sn}\left(y,m\right)\mathrm{dn}\left(y,m\right)\,, \tag{2.88}$$

where sn and dn stand for the elliptic sine and the delta amplitude, respectively. Note that the profile of $\mathcal{J}$ preserves its form as it evolves in time only in the case of $\gamma=ac_{\mathcal{P}}+c_{\mathcal{J}}=0$ (perturbed KdV), otherwise the amplitude grows linearly with time. Nonetheless, the energy in (2.83) as well as the remaining conserved charges given by $H^{(n)}_{\mathrm{KdV}}$, and $\tilde{H}^{(n)}$ in (2.47), turn out to be finite regardless the value of $\gamma$. The profiles of $\mathcal{P}$ and $\mathcal{J}$ are sketched in Figure 2.2.

As an ending remark of this section, it is worth pointing out that in the special case of $c_{\mathcal{J}}=a=0$, the field equation for $\mathcal{J}$ becomes devoid of a source. The particular solutions, given by (2.85) for the real line, and by (2.88) in the case of $S^1$, in this case turn out to be described by odd analytic functions that maintain its shape as they propagate with the same velocity as their corresponding KdV solitons described by $\mathcal{P}$ in (2.84) and (2.86), respectively. Note that, according to eq. (2.83), the total energy of this sort of soliton-antisoliton bound states for $\mathcal{J}$

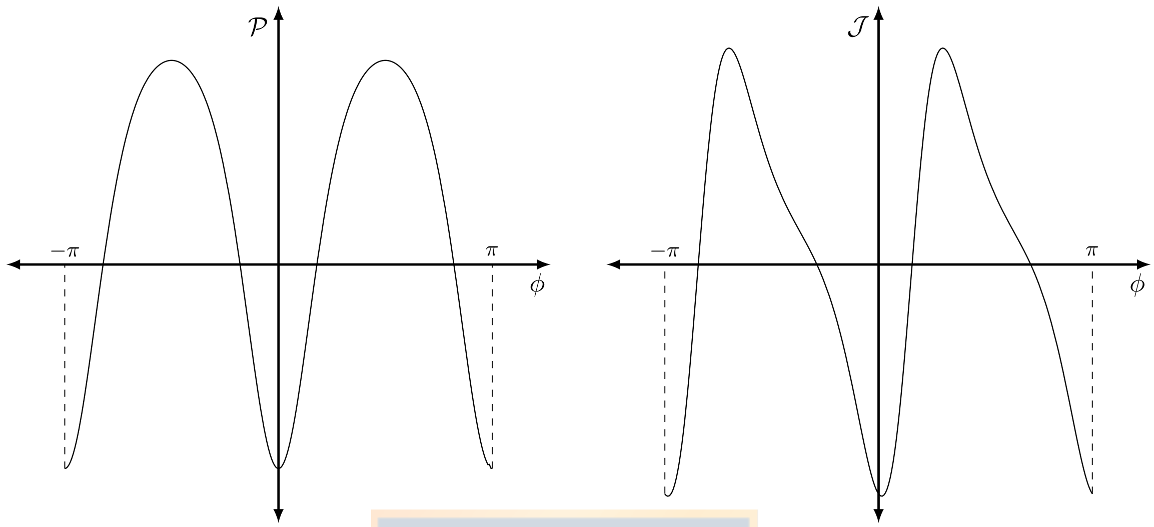


Figure 2.2: Profiles of $\mathcal{P}$ (cnoidal wave in (2.86)) and $\mathcal{J}$ in (2.88) for a fixed generic value of $A$, $\alpha$, $m$ and $t$.

vanishes, and the conserved charges $\tilde{H}^{(n)}$ in (2.47) also do. Nevertheless, the conserved charges $H^{(n)}_{\text{KdV}}$ remain being nontrivial.

## 2.4 Geometrization of the hierarchy: the dynamics of locally flat spacetimes in 3D

In this section, we show that the entire structure of the class of integrable systems with $\text{BMS}_3$ Poisson structure described above can be fully geometrized, in the sense that the dynamics turns out to be equivalently understood through the evolution of spacelike surfaces embedded in locally flat spacetimes in three dimensions.

For the sake of simplicity, here we focus in the case of $\text{BMS}_3$ Poisson structures with $c_{\mathcal{J}} = 0$. Thus, following the lines of [23], it is possible to unveil a deep link between the class of integrable systems aforementioned and General Relativity in three spacetime dimensions. Concretely, here we show that the Einstein-Hilbert action without cosmological constant in 3D can be endowed with an appropriate set of boundary conditions, so that in the reduced phase space, the Einstein equations in vacuum, which imply the vanishing of the Riemann tensor, precisely reduce to the ones of the dynamical systems with $\mathrm{BMS}_3$ Poisson structure in (2.7). As a consequence, it is possible to establish a one-to-one map between any solution of this kind of integrable systems and certain specific locally flat metric in three spacetime dimensions. Furthermore, the symmetries of the integrable systems can be seen to naturally emerge from diffeomorphisms that preserve the asymptotic form of the spacetime metric. Hence, and remarkably, the symmetries manifestly become Noetherian in our geometric framework. Therefore, the infinite set of conserved charges for the integrable system is transparently recovered from the corresponding surface integrals in the canonical approach. This can be seen as follows.

The Einstein-Hilbert action in three spacetime dimensions

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}R\,, \tag{2.89}$$

can be equivalently expressed as a Chern-Simons action for the $isl(2,\mathbb{R})$ algebra [113, 114]. Thus, up to boundary terms, the action (2.89) can be written as

$$I_{CS}[A] = \frac{k}{4\pi}\int\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\rangle\,. \tag{2.90}$$

where the Chern-Simons level is given by $k = 1/4G$, and $\langle\cdots\rangle$ stands for the invariant bilinear form defined in eq. (2.9) with $c_{\mathcal{J}} = 0$ and $c_{\mathcal{P}} = 1$. The components of the $isl(2,\mathbb{R})$-valued gauge field are then identified with the dualized spin connection and the dreibein according to

$$A = \omega^a J_a + e^a P_a\,. \tag{2.91}$$

In order to describe the asymptotic structure of the fields, as explained in [115, 116, 117] it is useful to choose the gauge so that the connection reads

$$A = b^{-1}ab + b^{-1}db\,, \tag{2.92}$$

where the radial dependence is completely captured by the group element $b = b(r)$, which as shown in [75] can be conveniently chosen as $b = e^{rP_2}$. One of the advantages of this gauge choice is that the remaining analysis can be performed in terms of the auxiliary connection

$$a = a_t dt + a_\phi d\phi\ , \tag{2.93}$$

that only depends on $t$, $\phi$. Here we propose that the asymptotic form of the auxiliary connection for the gravitational field in (2.93) is precisely given by the two-dimensional locally flat gauge field that describes the field equations of the dynamical system with BMS$_3$ Poisson structure in Section 2.2.

Their components will be described by (2.10) and (2.11), but with a slight change of conventions: $\mathcal{P} \to \frac{c_\mathcal{P}}{2}\mathcal{P}$, $\mathcal{J} \to \frac{c_\mathcal{P}}{2}\mathcal{J}$. Consequently, the components of the gauge fields that describe the boundary conditions are

$$\begin{aligned} a_\phi &= J_1 + \frac{1}{2}\left(\mathcal{J}P_0 + \mathcal{P}J_0\right)\ , \\ a_t &= \lambda\left[\mu_\mathcal{J}, \mu_\mathcal{P}\right], \end{aligned} \tag{2.94}$$

where

$$\begin{aligned} \lambda\left[\varepsilon_\mathcal{J}, \varepsilon_\mathcal{P}\right] &= \varepsilon_\mathcal{P} P_1 + \varepsilon_\mathcal{J} J_1 - \partial_\phi \varepsilon_\mathcal{P}\, P_2 - \partial_\phi \varepsilon_\mathcal{J}\, J_2 \\ &\quad + \left(\frac{1}{2}\mathcal{P}\varepsilon_\mathcal{J} - \partial_\phi^2 \varepsilon_\mathcal{J}\right) J_0 + \left(\frac{1}{2}\mathcal{J}\varepsilon_\mathcal{J} + \frac{1}{2}\mathcal{P}\varepsilon_\mathcal{P} - \partial_\phi^2 \varepsilon_\mathcal{P}\right) P_0 \end{aligned} \tag{2.95}$$

Therefore, from (2.90), the field equations imply that the connection $A$ is flat ($F = dA + A^2 =$

0), which by virtue of (2.91) amounts to deal with three-dimensional manifolds with vanishing curvature and torsion; whereas eq. (2.92) means that the field strength of the auxiliary gauge field also vanishes as in (2.13). Hence, for our boundary conditions, the Einstein equations in vacuum precisely reduce to the ones of a dynamical system with $BMS_3$ Poisson structure in (2.5).

Besides, the asymptotic symmetries, being defined as the diffeomorphisms that preserve the asymptotic form of the spacetime metric, turn out to be equivalent to the set of gauge transformations $\delta a = d\lambda + [a, \lambda]$ that maintain the asymptotic form of the above gauge field. For our boundary conditions, one then finds that the asymptotic symmetries are spanned by a Lie-algebra-valued parameter is exactly given as in eq. (2.95), including the consistency condition for the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (2.18). Furthermore, the transformation law of the dynamical fields $\mathcal{J}$, $\mathcal{P}$ precisely agrees with the transformations in eq. (2.15).

Since the asymptotic symmetries are Noetherian, the global conserved charges can be readily obtained using the canonical approach [119]. Indeed, their variation is explicitly given by surface integrals defined at the boundary of the spatial section,[6] which read

$$\delta Q[\varepsilon_{\mathcal{J}}, \varepsilon_{\mathcal{P}}] = \frac{k}{2\pi}\int d\phi \, \langle \lambda \delta a_{\phi} \rangle = \frac{k}{4\pi}\int d\phi \, \left(\varepsilon_{\mathcal{J}} \delta \mathcal{J} + \varepsilon_{\mathcal{P}} \delta \mathcal{P}\right) . \tag{2.96}$$

In particular, if the Lagrange multipliers $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ are kept fixed at the boundary according to (2.69), so that they correspond to the variation of the functional $H^{(n)}$ defined in (2.67), the Einstein equations reduce to the ones of the hierarchy of integrable systems discussed in section 2.3.3. Therefore, in this case, the variation of the surface integrals in (2.96) integrates precisely as in (2.50). Note that the angular momentum of the gravitational configuration that fulfills

[6]It is worth noting that the boundary can be located at any fixed value of the radial coordinate, and hence, not necessarily at null infinity.

these boundary conditions is then given by

$$Q[\partial_\phi] = Q[1,0] = \frac{k}{4\pi}\int d\phi \mathcal{J} \, , \tag{2.97}$$

while the variation of the energy,

$$\delta Q[\partial_t] = \delta Q[\mu_\mathcal{J}, \mu_\mathcal{P}] = \frac{k}{4\pi}\int d\phi \left(\mu_\mathcal{J}\delta\mathcal{J} + \mu_\mathcal{P}\delta\mathcal{P}\right) \, , \tag{2.98}$$

integrates by virtue of (2.6) and then reduces to

$$Q[\partial_t] = \frac{k}{4\pi}H^{(n)} = \frac{k}{4\pi}\left(H^{(n)}_{\text{KdV}} + \tilde{H}^{(n)}\right) \, . \tag{2.99}$$

In the simplest case of $n = 0$, described in section 2.3.1, we recover the set of boundary conditions proposed in [75][7] which contain the boundary conditions in [35] for a particular choice of Lagrange multipliers at the boundary. Note that in this case, the $\text{BMS}_3$ algebra is realized as the asymptotic symmetry algebra. For the remaining cases ($n \geq 1$), the new class of boundary conditions is such that the asymptotic symmetry algebra is infinite-dimensional, abelian and devoid of central charges, which is equivalent to the fact that the conserved charges of the hierarchy are in involution (see eq. (2.51)).

## 2.5 Flat space cosmology with boundary conditions associated to the integrable system

The phase space covered by the boundary conditions (2.94) is quite wide. However, in this section we will focus on the case where the state-dependent functions $\mathcal{P}, \mathcal{J}$ are constants. Within this

[7]Strictly speaking, we are dealing with the boundary conditions in [75] provided that the higher spin fields and their corresponding chemical potentials are turned-off.

sector of the phase space is the Flat Space Cosmology (FSC) [84, 85], which is described by[8]

$$\mathcal{P}^{FSC} = \hat{r}^2\,, \quad \mathcal{J}^{FSC} = 2\hat{r}r_c\,. \tag{2.100}$$

where $r_c$ correspond to the cosmological horizon of the solution and $\hat{r}$ is a parameter that plays a similar role than $r_-$ in the BTZ metric. Since the Lagrange multipliers (2.6) are functionally dependent of the dynamical fields, all the spatial derivatives of them vanish. Consequently, the leading terms of the asymptotic form of the metric writen in the Schwarzschild gauge acquires the following form,

$$ds^2 = -\mu_{\mathcal{P}}^2 \mathcal{F}\left(r\right)^2 dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + r^2\left(d\phi + \mathcal{N}^{\phi}\left(r\right)dt\right)^2\,, \tag{2.101}$$

where

$$\begin{aligned} \mathcal{F}(r)^2 &= \frac{\mathcal{J}^2}{4r^2} - \mathcal{P}\,, \\ \mathcal{N}^{\phi}(r) &= \mu_{\mathcal{J}} + \frac{\mathcal{J}}{2r^2}\mu_{\mathcal{P}}\,. \end{aligned} \tag{2.102}$$

Note that this class of configurations provides an exact solution of the field equation in (2.5) for all possible values of $n$.

### 2.5.1 Thermodynamics

In the generic case when $\mathcal{J}$ and $\mathcal{P}$ are constants. It is convenient to adopt the normalization of the Gelfand-Dikii polynomials used in (1.24). Then, taking into account that the Hamiltonians $\tilde{H}$ can be defined by those of the KdV hierarchy (2.47), the polynomials that generate the $\tilde{H}$ charges will be consistently normalized. The Lagrange multipliers $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$, acquired a much

[8]Here, we are using the metric parameters of the FSC defined in [84].

more handly form, which, by writing them in terms of $z$ instead of $n$, look like

$$\mu_{\mathcal{P}}^{(z)} = \mathcal{P}^{\frac{z-1}{2}} N + (z-1)\frac{\mathcal{J}\mathcal{P}^{\frac{z-3}{2}}}{2} N^{\phi} , \qquad \mu_{\mathcal{J}}^{(z)} = \mathcal{P}^{\frac{z-1}{2}} N^{\phi} . \tag{2.103}$$

As in Section 1.3.1, we have included the factors $N$ and $N^{\phi}$ with the purpose of keeping track of the explicit presence of chemical potentials in the metric. Thus, for $z = 1$, the Lagrange multipliers reduce to chemical potentials, $\mu_{\mathcal{P}}^{(1)} = N$ and $\mu_{\mathcal{J}}^{(1)} = N^{\phi}$.

On the other hand, demanding regularity on the Euclidean continuation of (2.101) around $r_c$, the Lagrange multipliers (in Lorentzian variables) must be set as

$$\mu_{\mathcal{P}}^{(z)} = -\frac{\pi\mathcal{J}}{\mathcal{P}^{\frac{3}{2}}} , \qquad \mu_{\mathcal{J}}^{(z)} = \frac{2\pi}{\sqrt{\mathcal{P}}} . \tag{2.104}$$

Thus, matching with (2.103), we obtain the chemical potentials for an arbitrary choice of $z$,

$$N\left[z\right] = -\frac{z\pi\mathcal{J}}{\mathcal{P}^{1+\frac{z}{2}}} , \qquad N^{\phi}\left[z\right] = 2\pi\mathcal{P}^{-\frac{z}{2}} . \tag{2.105}$$

For completeness, setting $z = 1$, we see that the temperature and angular velocity of the FSC (2.100) with standard boundary conditions are

$$T = -\frac{1}{N} = \frac{\hat{r}^2}{2\pi r_c} , \qquad \Omega = -\frac{N^{\phi}}{N} = \frac{\hat{r}}{r_c} , \tag{2.106}$$

which agrees with [84]. Note that the minus sign between the temperature and $N$ is because, in the Euclidean continuation of the solution, the cosmological horizon is located at the "south pole" of the surface $r - t_E$ (see [75] for a deeper discussion on this detail).

Therefore, the following expressions for chemical potentials hold

$$N\left[z\right] = z(2\pi)^{1-z} N[1] N^{\phi}[1]^{z-1} , \qquad N^{\phi}\left[z\right] = (2\pi)^{1-z} N^{\phi}[1]^{z} . \tag{2.107}$$

In this constant state-dependent functions case, it is possible to show that the conserved charges associated to the Lagrange multipliers (2.103), also adopt a generic compact form

$$H_{\rm KdV}^{(z)} = \frac{4\pi}{z+1}\mathcal{P}^{\frac{z+1}{2}}N\,, \qquad \tilde{H}^{(z)} = 2\pi\mathcal{J}\mathcal{P}^{\frac{z-1}{2}}N^{\phi}\,. \tag{2.108}$$

Therefore, replacing this in (2.99), we obtain the mass of the cosmological spacetime (2.100) for an arbitrary choice of boundary conditions associated to the integrable systems with $BMS_3$ Poisson structure (2.103),

$$M\left[z\right] = \frac{1}{4G}\frac{\hat{r}^{z+1}}{z+1}\,. \tag{2.109}$$

Hence, for $z = 1$, we obtain the mass of the FSC in the standard set-up [84],

$$M = \frac{\hat{r}^2}{8G}\,. \tag{2.110}$$

Finally, as in the case of the BTZ black hole, the angular momentum does not depend on the particular choice of boundary conditions (2.111). Thus, for the FSC (2.100) we obtain

$$J = \frac{r_c\hat{r}}{4G}\,. \tag{2.111}$$

Independently of the choice of boundary conditions, the FSC (2.100) must satisfy the Bekenstein-Hawking area law,

$$S = \frac{A}{4G} = \frac{\pi r_c}{2G}\,. \tag{2.112}$$

We also express this entropy in terms of the conserved charges of the integrable systems through $H_{\rm KdV}$ and $\tilde{H}$, namely

$$S = \frac{k}{2}\frac{\tilde{H}[z]}{\left(\frac{z+1}{4\pi}H_{\rm KdV}[z]\right)^{\frac{z}{z+1}}}\,. \tag{2.113}$$

Our remaining task is to explore whether this entropy could be understood through a microscopic counting by means of a some suitable modification of the Cardy formula. Since the conserved charges of the integrable systems possess the same scaling properties as the left and right energies of KdV (1.38), one might think that the anisotropic version of Cardy formula (1.44) could accomplish that task. However, If we compare (2.113) with (1.44) it is easy to see that there is no way to identify the conserved charges $H_{\text{KdV}}$ and $\tilde{H}$ with $E_{\pm}$. The problem of how is the entropy of the flat space cosmology in a non-trivial ensemble, described by (2.103), and how it can be reproduced through a holographic counting of microstates, is one of the main motivations of this thesis.

In the next chapter we obtain a formula that precisely achieves this goal. First we briefly discuss the general aspects that must be taken into account in order to obtain a Cardy-like formula. Then, we develop a simple formalism, by which we can derive some of the different versions of the Cardy formula discussed in this thesis, of course including the one that reproduces the entropy of the cosmological spacetime with boundary conditions associated to the integrable systems with $\text{BMS}_3$ Poisson structure

# Chapter 3

# Holographic counting of black hole microstates

In this section, we work on a generic 2D field theory at finite temperature, defined on a cylinder with coordinates $(t, \phi)$, where the time coordinate takes values in the real line, i.e., $-\infty < t < +\infty$, and the spatial coordinate is periodic, $\phi \sim \phi + 2\pi$. Then, in order to define the theory on the torus, we just have to perform the compactification by cutting out a finite piece from the infinite cylinder and identify the edges so that the time coordinate becomes periodic too. Nevertheless, before gluing together the edges, we could also twist the cylinder. This generates an additional non-trivial periodicity in the spatial coordinate of the torus, which, as we will see, is holographically related to the chemical potential associated to the angular momentum of black holes in $3D$ (see e.g. [162, 163]).

We will start by defining the partition function for a generic $2D$ field theory on the torus as it is done in standard statistical mechanics, i.e., as a sum over all possible configurations weighted with the Boltzmann factor $\exp(-\beta H)$. It is well-konwn that the same object can be deduced from an quantum field theory perspective with the time coordinate compactified on a circle of radius $\beta = 1/T$ [164]. The imaginary part of the (Euclidean) temporal period $\beta$ is then corresponds to the inverse of the system temperature. From now on, we deal with the grand canonical ensemble, where the chemical potentials are held fixed to arbitrary values. Then, the total energy (in Lorentzian variables) is given by $\mathcal{M} - \Omega\mathcal{J}$, where $\mathcal{M}$ and $\mathcal{J}$ corresponds to the internal energy and the momentum of the system, respectively. In consequence, the partition function is defined by

$$Z(\beta, \theta) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta\mathcal{M} - \theta\mathcal{J}}\,, \tag{3.1}$$

where $\Omega = -\theta/\beta$. Here, the trace is taken over all states in the Hilbert space of the theory. In what follows, we assume that the spectrum of the field theory possesses a gap between the ground state and the first excited states. In the following section we exploit the modular properties of the torus in order to define a notion of duality between the high and low temperatures regimes. The latter allows us to obtain a precise formula for the asymptotic growth of the number of states for fixed charges $\mathcal{M}$ and $\mathcal{J}$, which also depends on the charges of the ground state $\mathcal{M}_0$ and $\mathcal{J}_0$.

The procedure described above is a applied to different classes of $2D$ field theories. These theories are characterized by according to the type of local symmetry that each one possesses. The only features of the theories under discussion that are relevant in our analysis, in addition to their corresponding local symmetry, are those specified above, i.e., that they are defined in a torus and they possesses a gap in their spectrum.

## 3.1 Modular group of the torus

In order to properly define some useful concepts, the torus can be define by introducing arbitrary identifications on the complex plane (see e.g., [165]),

$$w \sim w + m\alpha_1 + n\alpha_2\,, \qquad m, n \in \mathbb{Z}\,, \quad w, \alpha_1, \alpha_2 \in \mathbb{C}\,. \tag{3.2}$$

As is shown in Figure 3.1, the pair $(\alpha_1, \alpha_2)$ spans a lattice, where the torus can be defined by identifying opposite edges of the smallest cell. The quantity that encodes the properties of the torus is called *modular parameter* and is defined by the rate of its complex periods,

$$\tau = \frac{\alpha_2}{\alpha_1}\,. \tag{3.3}$$

Nevertheless, different choices of the complex periods $(\alpha_1, \alpha_2)$ could span to the same lattice and therefore the same torus. In this sense, assuming that $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ define the same lattice, one should be able to write the second pair as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}\,. \tag{3.4}$$

Consistency requires that this relationship must be invertible. Thus, one can write $(\alpha_1, \alpha_2)$ in terms of $(\beta_1, \beta_2)$. In general, demanding that the inverse matrix has integer components, one obtain that $ad - bc = \pm 1$ [165]. We must also take into account that the lattice generated by $(\alpha_1, \alpha_2)$ is equivalent to that generated by $(-\alpha_1, -\alpha_2)$, and hence one has to quotient by $\mathbb{Z}_2$ action. Matrices with these properties are elements of $SL(2, \mathbb{Z})/\mathbb{Z}_2 \simeq PSL(2, \mathbb{Z})$. Thus, every equivalent pair of complex periods must be related by an $SL(2, \mathbb{Z})/\mathbb{Z}_2$ transformation.

Now, we will briefly discuss two modular transformations (transformations acting on the modular parameter $\tau$), which, in fact, constitute the complete set of generators of the modular group $SL(2, \mathbb{Z})/\mathbb{Z}_2$.

- *T-modular transformation*:

  This transformation act on the modular parameter in the following way

$$T: \quad \tau \mapsto \tau' = \tau + 1\,. \tag{3.5}$$

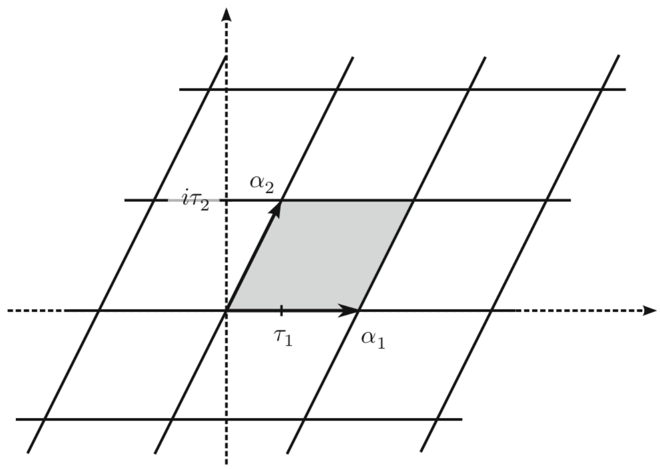


Figure 3.1: Lattice of a torus generated by $(\alpha_1, \alpha_2)$, conveniently chosen as $(1, \tau_1 + i\tau_2)$.

Under this change the lattice remains invariant and thus the torus with modular parameters $\tau$ and $\tau + 1$ are equivalent.

- *S-modular transformation*:

  This transformation act on the modular parameter as follows

$$S : \quad \tau \mapsto \tau' = -\frac{1}{\tau} \, . \tag{3.6}$$

It is worth to note that, according to the definition of the modular parameter (3.3), this operation is equivalent to swap the complex periods as

$$\begin{aligned} w \sim w + \alpha_1 \quad &\longrightarrow \quad w' \sim w' + \alpha_2' \; , \\ w \sim w + \alpha_2 \quad &\longrightarrow \quad w' \sim w' - \alpha_1' \; , \end{aligned} \tag{3.7}$$

Since the modular parameter is actually defined trough $(\alpha_1, \alpha_2)$, we will proclaim the operation in (3.7) as the fundamental realization of the S-modular invariance of the torus.

Going back to the cylinder, it is well-known that, one can obtain the geometry of the torus by a Wick rotation, so that the time coordinate is replace by, $t \to -it_E$, where "$t_E$" stands for the Euclidean time. Thus, in addition to the angular cycle, $(t_E, \phi) \sim (t_E, \phi + 2\pi)$, the identification of the temporal coordinate introduces the thermal cycle[1], $(t_E, \phi) \sim (t_E + \beta, \phi + \theta_E)$, where $\beta$ and $\theta_E$ correspond to the period of Euclidean time and the period of the angle generated by twisting the cylinder, respectively. These cycles correspond to a pair of complex periods of the form, $(\alpha_1.\alpha_2) = (2\pi, i\beta - \theta_E)$. From now on, we will work with Lorentzian variables, therefore, the angular and thermal cycles of the torus are given by

$$\text{Angular:} \qquad (t, \phi) \sim (t, \phi + 2\pi) \ , \tag{3.8}$$

$$\text{Thermal:} \qquad (t, \phi) \sim (t - i\beta, \phi - i\theta) \ , \tag{3.9}$$

Thus, the S-modular transformation (3.7) is equivalent to the following swap between the angular and thermal cycles of the torus,

$$\begin{aligned} (t, \phi) \sim (t, \phi + 2\pi) \quad &\longrightarrow \quad (t', \phi') \sim \left(t' - i\beta', \phi' - i\theta'\right) \ , \\ (t, \phi) \sim (t - i\beta, \phi - i\theta) \quad &\longrightarrow \quad (t', \phi') \sim \left(t', \phi' - 2\pi\right) \ . \end{aligned} \tag{3.10}$$

Since the S-modular invariance is an inherent feature of the torus, every $2D$ field theory defined on the cylinder (i.e., the torus in Lorentzian variables) is invariant under local transformations that preserve the relations (3.10). In what follows, we will show how different kind of transformations performs different realizations of the S-modular invariance of the torus.

[1]The reason why (3.9) is called "thermal cycle", is because in the grand canonical ensemble, $\beta$ and $\theta_E$ are interpreted as the inverse of the temperature and the chemical potential associated to spatial traslations, respectively.

## 3.2 Revisiting the asymptotic growth of the number of states

In this section we explore inequivalent realizations of the S-modular invariance of the torus (3.10). Different realizations correspond to inequivalent identifications of the periods of the cycles of the torus with the chemical potentials in the grand canonical ensemble. These realizations would play the relevant role in the high/low temperature duality of the theory, which is the key in order to derive the different kinds of generalizations of the Cardy fomula that describe the asymptotic growth of the number of states.

### 3.2.1 Standard Cardy formula

A conformal transformation corresponds to a local map that preserves the angles between any two vectors. Such definition can be expressed by a coordinate transformation that leaves the form of the metric invariant up to a conformal factor, i.e., $ds^2 \to ds'^2 = \lambda ds^2$. Then, it is easy to show that the following transformation,

$$t \to \lambda_t t' + \lambda_\phi \phi' \ , \qquad \phi \to \lambda_t \phi' + \lambda_\phi t' \ , \tag{3.11}$$

corresponds to a conformal transformation of a theory defined on the cylinder, with a conformal factor given by $\lambda_t^2 - \lambda_\phi^2$. In the present case, It is convenient to work in null coordinates, $x_\pm = t \pm \phi$, so that the above transformation can be rewritten as

$$x_\pm \to \lambda_\pm x'_\pm \ , \tag{3.12}$$

where we have defined $\lambda_\pm = \lambda_t \pm \lambda_\phi$. In terms of these variables, the swap between the angular and thermal cycles induced by the S-modular transformation (3.10) can be written as

$$
\begin{aligned}
(x_+, x_-) \sim (x_+ + 2\pi, x_- - 2\pi) \quad &\longrightarrow \quad (x'_+, x'_-) \sim (x_+ - i\beta'_+, x_- - i\beta'_-) \ , \\
(x_+, x_-) \sim (x_+ - i\beta_+, x_- - i\beta_-) \quad &\longrightarrow \quad (x'_+, x'_-) \sim (x'_+ - 2\pi, x'_- + 2\pi) \ ,
\end{aligned} \tag{3.13}
$$

where we have defined the null periods through $\beta_\pm = \beta \pm \theta$.

Requiring compatibility of the conformal transformation (3.12) with the S-modular transformation (3.13), implies that the periods of the corresponding thermal cycles are related by

$$
\beta'_\pm = \frac{4\pi^2}{\beta_\pm} \ . \tag{3.14}
$$

In terms of the modular parameter, $\tau = \alpha_2/\alpha_1 = i\beta_+/2\pi$, the above relationships reduces to the familiar S-modular transformations of the chiral and anti-chiral sectors of a CFT$_2$ [165],

$$
\tau' = -\frac{1}{\tau} \ , \qquad \bar{\tau}' = -\frac{1}{\bar{\tau}} \ . \tag{3.15}
$$

In terms of null variables, the partition function is written as

$$
Z(\beta_+, \beta_-) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta_+ E_+ - \beta_- E_-} \ , \tag{3.16}
$$

where $E_\pm = 1/2(\mathcal{M} \pm \mathcal{J})$. In this way, the system decouples into two independent subsystems, each one in the canonical ensemble where we can naturally interpret the null periods $\beta_\pm$, as the inverse of left and right temperatures (chiral and antichiral movers in the CFT$_2$ language). Note that the S-modular transformation (3.14) can be understood as a duality between the high and low temperature regimes of each subsystem. Hence, assuming that the spectrum possesses a gap, the partition function at the low temperature regime approximates as

$$Z(\beta_+, \beta_-) \approx e^{-\beta_+ E_+^0 - \beta_- E_-^0} \,, \tag{3.17}$$

where $E_\pm^0$ stand for the ground state energies of left and right movers. Thus, by virtue of (3.14), in the high temperature regime the partition function reads

$$Z(\beta_+, \beta_-) \approx e^{-\frac{4\pi^2}{\beta_+} E_+^0 - \frac{4\pi^2}{\beta_-} E_-^0} \,. \tag{3.18}$$

Therefore, at fixed energies $E_\pm \gg E_\pm^0$, the asymptotic growth of the number of states can be computed by evaluating the above partition function in the saddle-point approximation, so that the logarithm of the leading term of the asymptotic growth of the number of states is given by the following entropy

$$S = 4\pi\sqrt{-E_+^0 E_+} + 4\pi\sqrt{-E_-^0 E_-} \,. \tag{3.19}$$

This is the celebrated Cardy formula. It is worth highlighting that (3.19) has been writen in terms of the left and right ground state energies of the theory, instead of being written in terms of the central charge $c_\pm$, associated with the conformal symmetry, as in (2). Understanding the Cardy formula in terms of the ground state charges has proven to be the key in order to reproduce the Bekenstein-Hawking entropy of hairy black holes in General Relativity [166, 167]. This is also the case for other theories of three-dimensional gravity with asymptotically Lifshitz black holes [135, 168], as well as for asymptotically Warped-AdS$_3$ spacetimes [100, 169].

Remarkably, this formula exactly reproduces the entropy of the BTZ black hole, with Brown-Henneaux boundary conditions. Indeed, if we replace $E_\pm = 1/2\,(M\ell \pm J)$, where $M$ and $J$ corresponds to the mass and angular momentum of the BTZ black hole in the standard set-up, i.e., (1.40) and (1.41), respectively, and $E_\pm^0 = 1/2\left(M^0\ell \pm J^0\right)$, where $M^0 = -1/8G$ and $J^0 = 0$

corresponds to the mass and the angular momentum of the AdS$_3$ spacetime, we obtain that

$$S=\frac{\pi r_{+}}{2G}\,, \tag{3.20}$$

which is precisely the entropy of the BTZ black hole. Note that according to (2) and (3.19), the ground state energies are related to the central charges by $E^0_{\pm}=-c_{\pm}/24$ [14].

### 3.2.2 Flat analogue of the Cardy formula

The second class of theories in $2D$ that we are interested in is assume to be invariant under

$$t\rightarrow\lambda_t t'+\lambda_\phi\phi'\,,\qquad\phi\rightarrow\lambda_t\phi'\,. \tag{3.21}$$

When comparing this with (3.11), we can realize that both belong to different types of transformations, in the sense that we cannot obtain one from the other by turning off any of its parameters. Demanding compatibility of the transformations (3.21) with the S-modular transformation (3.10), the periods of the thermal cycle must be related by

$$\theta'=\frac{4\pi^2}{\theta}\,,\qquad\beta'=-\frac{4\pi^2\beta}{\theta^2}\,. \tag{3.22}$$

In this case, we work directly in the grand canonical ensemble (3.1), so the interpretation of the thermal periods $\beta$ and $\theta$ is simply the inverse of the temperature and the chemical potential associated to spatial traslations, respectively. As in the previous section, the relationships in (3.22) can also naturally be understood as a high/low temperature duality relation. Consequently, if the spectrum possesses a gap, the partition function at low temperatures approximates as

$$Z\left(\beta,\theta\right)\approx e^{-\beta\mathcal{P}_0-\theta\mathcal{J}_0}\,, \tag{3.23}$$

where $\mathcal{P}_0$ and $\mathcal{J}_0$ stands for the ground state charges. Thus, by virtue of (3.22), in the high temperature regime the partition function reads

$$Z\left(\beta,\theta\right)\approx e^{4\pi^2\frac{\beta}{\theta^2}\mathcal{P}_0-\frac{4\pi^2}{\theta}\mathcal{J}_0}\ . \tag{3.24}$$

Therefore, when the global charges are much larger than those of the ground state the asymptotic growth of the number of states can be computed by evaluating the above partition function in the saddle-point approximation. Then, the leading term of the entropy is given by

$$S=2\pi\sqrt{-\mathcal{P}_0\mathcal{P}^{-1}}\mathcal{J}+2\pi\sqrt{-\mathcal{P}\mathcal{P}_0^{-1}}\mathcal{J}_0\ . \tag{3.25}$$

This formula was derived in terms of the $\mathrm{BMS}_3$ central charges in [170]. Here, we have written it in terms of the ground state charges. Remarkably, if in (3.25) we identify $\mathcal{P}=M$ and $\mathcal{J}=J$, where $M$ and $J$ corresponds to the global charges of the FSC, (2.110) and (2.111), respectively, and if we also identifies the charges of the ground state with the mass and angular momentum of Minkowski spacetime, i.e., $M^0=-1/8G$ and $J^0=0$, we obtain that the entropy,

$$S=\frac{\pi}{4G}\frac{\mathcal{J}}{\sqrt{\mathcal{P}}}=\frac{\pi r_c}{2G}\,, \tag{3.26}$$

exactly reduces to the quarter of the area of the cosmological horizon. This result was first shown in [84, 85] where the formula was obtained for the particular case, $\mathcal{P}_0=-c_{\mathcal{P}}/2$ and $\mathcal{J}^0=0$.

## 3.3 Generalized Cardy formulae

We have already recognize the relevance that the S-modular transformation plays in the derivation of a Cardy-like formula. Here we go one step further and note that if instead of using $(\beta,\theta)$ in (3.10), we use a sort of "generalized thermal periods", defined by arbitrary fuctions of the standard canonical periods $(\beta,\theta)$. In this generalized set-up, the S-modular transformation implies that the cycles are exchanged according to

$$
\begin{aligned}
(t,\phi)\sim(t,\phi+2\pi) \quad &\longrightarrow \quad \left(t',\phi'\right)\sim\left(t'-i\beta',\phi'-i\theta'\right) \ , \\
(t,\phi)\sim\left(t-if\left(\beta,\theta\right),\phi-ig\left(\beta,\theta\right)\right) \quad &\longrightarrow \quad \left(t',\phi'\right)\sim\left(t',\phi'-2\pi\right) \ .
\end{aligned} \tag{3.27}
$$

In null coordinates, the corresponding swap reads

$$
\begin{aligned}
(x_+,x_-)\sim(x_++2\pi,x_--2\pi) \quad &\longrightarrow \quad (x'_+,x'_-)\sim(x_+-i\beta'_+,x_--i\beta'_-) \ , \\
(x_+,x_-)\sim(x_+-if_+(\beta_+),x_--if_-(\beta_-)) \quad &\longrightarrow \quad (x'_+,x'_-)\sim(x'_+-2\pi,x'_-+2\pi) \ .
\end{aligned} \tag{3.28}
$$

These generalized thermal periods, $f(\beta,\theta)$, $g(\beta,\theta)$, or equivalently $f_\pm(\beta_\pm)$, could also depend on some additional parameters that possess a well defined transformation rule under modular transformation. Note that the generalized transformation rule must be involutive, so that by successively applying the S-modular transformation twice, one has to consistently return to the original torus.

### 3.3.1 The anisotropic generalization of Cardy formula

As shown in (1.37), the boundary conditions of the KdV-type induce generalized thermal periods in the left and right inverse temperatures, $N_\pm$. In the our terms, we can rewrite these periods as

$$
f_\pm^{(z)}(\beta_\pm)=(2\pi)^{1-z}\beta_\pm^z\,. \tag{3.29}
$$

Here, we can interpret the dynamic exponent $z$, as a parameter of the generalized periods $f_\pm$, which, as we discussed above, should be subject to an appropriate transformation law associated to the S-modular invariance of the torus. As argued in [23, 168, 171], there is a good geometric reason that tells us how this transformation must be. Performing a swap between the generators of Euclidean time and space translations, it can be show that 2D Lifshitz algebras with dynamical exponents $z$ and $z^{-1}$ are in fact isomorphic [168]. Hence, the rule that the parameter has to follow under an S-modular transformation is simply

$$z \to z' = \frac{1}{z}\,. \tag{3.30}$$

Therefore, taking into account (3.30) and the aforementioned generalized thermal period, we can derive from (3.28) the anisotropic version of the conformal S-modular transformation found in [23],

$$\beta'_{\pm} = \frac{(2\pi)^{1+\frac{1}{z}}}{\beta_{\pm}^{\frac{1}{z}}}\,, \qquad z' = \frac{1}{z}\,. \tag{3.31}$$

Since in the definition of the KdV-type boundary conditions, the label $n$ is a positive integer, then $z = 2n+1$ is also an odd positive number. Thus, we can interpret (3.31) as a high/low temperature duality relation. Therefore, assuming a gap in the spectrum of the theory, one can perform the saddle-point approach in order to find the asymptotic growth of the number of states. The leading term of the entropy reads

$$S = 2\pi\,(z+1)\left[\left(\frac{|E_{+}^{0}[z^{-1}]|}{z}\right)^{z} E_{+}[z]\right]^{\frac{1}{z+1}} + 2\pi\,(z+1)\left[\left(\frac{|E_{-}^{0}[z^{-1}]|}{z}\right)^{z} E_{-}[z]\right]^{\frac{1}{z+1}}\,. \tag{3.32}$$

Note that for the case $z = 1$, this anisotropic version of the Cardy formula reduces to the standard one (3.19). The above formula also exactly agrees with the leading term of the entropy of the free boson with Lifshitz scaling [172], or equivalently, two copies of the anisotropic chiral boson [173] (up to the zero modes). Furthermore, each copy of (3.32) also agrees with the leading term of the Hardy-Ramanujan formula for the counting of partitions of an integer into $z$-th powers [171]. Remarkably, plugging into (3.32) the left and right energies of the BTZ black hole with arbitray KdV-type boundary conditions (1.38), and using $AdS_3$ as the ground state ($\mathcal{L}_{\pm}^{AdS} = -1$), the entropy (1.43) is exactly recovered. Finally, using the relationship between the global charges and the inner and outer in (1.29), we can see that entropy is consistent with the Bekenstein-Hawking area law (1).

### 3.3.2 The flat analogue of the anisotropic generalization of the Cardy formula

According to (2.107), we can see that the boundary conditions associated with the integrable systems with $BMS_3$ Poisson structure, induce generalized thermal periods on the chemical potentials $N$ and $N^{\phi}$. In our terms, we can rewrite these periods as

$$f^{(z)}(\beta,\theta)=z(2\pi)^{1-z}\beta\theta^{z-1}\,,\qquad g^{(z)}(\theta)=(2\pi)^{1-z}\theta^{z}\,. \tag{3.33}$$

As in the previous case, the parameter $z$ in these generalized periods has the same interpretation, we can inherit the transformation law (3.30) for it. Consequently, using the swap between cycles given by (3.27), we can derive the following version of the flat S-modular transformation

$$\beta'=-\frac{(2\pi)^{1+\frac{1}{z}}\beta}{z\theta^{1+\frac{1}{z}}}\,,\qquad \theta'=\frac{(2\pi)^{1+\frac{1}{z}}}{\theta^{\frac{1}{z}}}\,,\qquad z'=\frac{1}{z}\,. \tag{3.34}$$

Then, hereafter, (3.34) is regarded as a high/low temperature regime duality and we also assume a gap in the spectrum of a theory with the partition function (3.1). Note that the energy and the momentum are given by $H_{\text{KdV}}[z]$ and $\tilde{H}[z]$, respectively. Therefore, the partition function reads

$$Z(\beta,\theta)=\text{Tr}_{\mathcal{H}}\,e^{-\beta H_{\text{KdV}}[z]-\theta\tilde{H}[z]}\,. \tag{3.35}$$

Proceeding as in the previous section, the asymptotic growth of the number of states can be obtained in the saddle-point approximation, so that the leading term of the entropy gives the flat version of the anisotropic generalization of the Cardy formula,

$$S=2\pi\left(\frac{|H^{0}_{\text{KdV}}[z^{-1}]|}{zH_{\text{KdV}}[z]}\right)^{\frac{z}{z+1}}\tilde{H}[z]+2\pi\left(\frac{zH_{\text{KdV}}[z]}{|H^{0}_{\text{KdV}}[z^{-1}]|}\right)^{\frac{1}{z+1}}\tilde{H}^{0}[z^{-1}]\,. \tag{3.36}$$

It is worth pointing out that if one rewrites this formula in terms of the chemical potentials $N$ and $N^{\phi}$, instead of the charges of the ground state, it satisfies a flat version of the anisotropic Smarr relation (1.56),

$$S = (z+1)NH_{\text{KdV}} + (z+1)N^{\phi}\tilde{H}\,. \tag{3.37}$$

Note that for $z=1$ and by virtue of (2.108), the above flat version of the anisotropic Cardy formula reduces to the flat Cardy formula (3.25), with $\mathcal{P} = H_{\text{KdV}}[1]$ and $\mathcal{J} = \tilde{H}[1]$.

Finally, plugging into the flat version of the anisotropic Cardy formula (3.36) the Minkowski spacetime as the corresponding ground state, which is characterized by $\mathcal{P}^{Mink.} = -1$, $\mathcal{J}^{Mink.} = 0$, we exactly recover the entropy of the cosmology with arbitrary boundary conditions associated to the $\text{BMS}_3$ integrable systems (2.113). This result is remarkable, since it allows us to holographically understand through a microstate counting of states, the entropy of the cosmology in a non-trivial anisotropic ensemble. The anisotropic version of the flat Cardy formula (3.36) and its holographic application are undoubtedly one of the main results of this thesis.

Consistently, if we replace the global charges (2.108) into the anisotropic flat Cardy formula (3.36), provided by the state-dependent functions of the flat space cosmology (2.100) and using the Minkowski spacetime as the ground state, we can check that the FSC satisfies also the Bekenstein-Hawking area law (1).

# Conclusions

One of the main goals of this thesis was to stablish a simple set-up to understand how the Lagrange multipliers in a ADM foliation of $3D$ gravity correspond to the thermal periods of the torus and the chemical potentials in the grand canonical ensemble of the partition function of a $2D$ dual field theory.

We have then shown that boundary conditions that allow a full geometrization of a class of integrable systems (i.e., boundary conditions for $3D$ gravity, where the Lagrange multipliers are identified with the polynomials that span a hierarchy of integrable systems in one dimension less at the boundary) are correspond to a one-parameter family of generalized thermal periods of the torus, and to the chemical potentials present in the partition function of a $2D$ field theory with anisotropic scaling properties.

We have focused in two specific realizations. The first one related to the holographic computation of the entropy of the BTZ black hole, through the microscopic counting of states of a $2D$ field theory, whose partition function is given in a anisotropic ensemble. We extend the discussion to include thermal stability and shown that the phase transitions are sensitive to the generalized modular properties of the torus, associated with the generalized thermodynamic description of the chemical potentials induced by the choice of KdV-type boundary conditions. In the second example, we derived an anisotropic version of the flat analogue of Cardy formula. We have also shown that our formula successfully reproduced the entropy of the flat space cosmology with an arbitrary choice of boundary conditions for General Relativty without cosmological constant, associated to a new hierarchy of integrable systems, whose Poisson structure is given by the $\mathrm{BMS}_3$ algebra.

## Extensions of our results

As pointed out in the introduction, the $\mathrm{BMS}_3$ algebra admits interesting extensions. It is natural to expect these algebras be linked to new classes of integrable systems. In particular, an interesting nonlinear extension of $\mathrm{BMS}_3$ that includes additional generators of spin $s > 2$ was found in [74] (in full agreement with the algebra simultaneously found in [73] for $s = 3$). This kind of extensions can be regarded as "flat $W$-algebras", since they can be recovered from a suitable Inönü-Wigner contraction of two copies of certain classical $W$-algebras (for a review about $W$-algebras, see e.g. [174]). Noteworthy, preliminary results show that new hierarchies of integrable systems whose Poisson structures correspond to flat $W$-algebras indeed exist. In fact, this family of integrable systems turns out to be bi-Hamiltonian, and furthermore, following the lines of section 2.4, they can also be geometrized in terms of higher spin gravity without cosmological constant in three spacetime dimensions endowed with a suitable set of boundary conditions. Therefore, the symmetries of this novel class of integrable systems can be seen as combinations of diffeomorphisms and higher spin gauge transformations that preserve the asymptotic form of the three-dimensional configurations. As a consequence, in the three-dimensional geometric setup, the infinite set of conserved charges of the integrable systems emerge as the canonical generators that correspond to the asymptotic symmetries, being described by suitable surface integrals at the boundary, which turn out to be in involution.

Further interesting links between certain well-known classes of integrable systems and higher spin gravity on $\mathrm{AdS}_3$ have been explored in [23, 175, 176, 177, 178].

We are currently considering a new example in the geometrization of integrable systems, which would extend our results beyond General Relativity.

Topologically Massive Gravity (TMG) [179, 180], is a $3D$ gravity theory that, unlike GR, has a local degree of freedom; the graviton. This property makes TMG, in a certain sense, a better model for study gravity in $3D$. TMG with negative cosmological constant also admits the $\text{AdS}_3$ spacetime as a vacuum solution for any value of the graviton mass. However, for positive Newton constant, massive perturbations around $\text{AdS}_3$ give rise to a propagating graviton with negative energy, while if the Newton constant is negative, it is obtained that the graviton energy is positive, but the energy of the BTZ black hole turns negative. Taking this into account, $\text{AdS}_3$ does not seems to be a good ground state for the theory. Nonetheless, there is another rich choice of vacuum for the theory, given by the Warped $\text{AdS}_3$ ($\text{WAdS}_3$) spacetime whose group of isometries corresponds to $SL(2,\mathbb{R}) \times U(1)$.

Considering boundary conditions for asymptotically $\text{WAdS}_3$ spacetimes, the algebra of asymptotic symmetries for TMG with negative cosmological constant is given by the semi-direct sum of the Virasoro algebra with a Kac-Moody current $u(1)$ [181, 182, 183]. Following the same line as in Chapters 1 and 2, this infinite-dimensional algebra might define a new sort of Poisson structure of integrable systems. It is then natural to conjecture that the integrable system has a second Poisson structure, which would allow the construction of a hierarchy associated with this integrable system. Although TMG is not a strictly topological theory like GR, it still admits a Chern-Simons-type formulation [184], where the usual Hamiltonian methods for finding conserved symmetries and charges have already been studied in the literature (see e.g., [185]). Therefore, we hope that these methods can naturally adapt to our purposes.

# Appendix A

# List of conserved quantities and polynomials

Here we provide an explicit list of the first six conserved charges $H_{\text{KdV}}^{(n)}$ and $\tilde{H}^{(n)}$, and their associated polynomials $R^{(n)}$ and $T^{(n)}$.

The conserved quantities $H_{\text{KdV}}^{(n)}$ correspond to the ones of the KdV equation in (2.34), which in our conventions, generically depend on the central extension $c_{\mathcal{P}}$. Therefore, they read

$$
\begin{aligned}
H_{\text{KdV}}^{(0)} &= \int d\phi \mathcal{P}\,, \\
H_{\text{KdV}}^{(1)} &= \int d\phi \frac{\mathcal{P}^2}{2}\,, \\
H_{\text{KdV}}^{(2)} &= \int d\phi \left[\frac{1}{2}c_{\mathcal{P}}\mathcal{P}'^2 + \frac{1}{2}\mathcal{P}^3\right]\,, \\
H_{\text{KdV}}^{(3)} &= \int d\phi \left[\frac{1}{2}c_{\mathcal{P}}^2\mathcal{P}''^2 + \frac{5}{2}c_{\mathcal{P}}\mathcal{P}\mathcal{P}'^2 + \frac{5}{8}\mathcal{P}^4\right]\,, \\
H_{\text{KdV}}^{(4)} &= \int d\phi \left[\frac{1}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2 + \frac{7}{2}c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}''^2 + \frac{35}{4}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'^2 + \frac{7}{8}\mathcal{P}^5\right]\,, \\
H_{\text{KdV}}^{(5)} &= \int d\phi \Big[\frac{1}{2}c_{\mathcal{P}}^4(\mathcal{P}^{(4)})^2 + \frac{9}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2\mathcal{P} + \frac{63}{4}c_{\mathcal{P}}^2\mathcal{P}^2\mathcal{P}''^2 - 5c_{\mathcal{P}}^3\mathcal{P}''^3 \\
&\quad + \frac{105}{4}c_{\mathcal{P}}\mathcal{P}^3\mathcal{P}'^2 - \frac{35}{8}c_{\mathcal{P}}^2\mathcal{P}'^4 + \frac{21}{16}\mathcal{P}^6\Big]\,.
\end{aligned}
$$

The remaining conserved charges $\tilde{H}^{(n)}$, can then be readily obtained from (2.47), which are given by

$$
\begin{aligned}
\tilde{H}^{(0)} &= \int d\phi \mathcal{J}\,, \\
\tilde{H}^{(1)} &= \int d\phi \mathcal{J}\mathcal{P}\,, \\
\tilde{H}^{(2)} &= \int d\phi \Big[\mathcal{J}\Big(\frac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}''\Big) + \frac{1}{2}c_{\mathcal{J}}\mathcal{P}'^2\Big]\,, \\
\tilde{H}^{(3)} &= \int d\phi \Big[\mathcal{J}\Big(c_{\mathcal{P}}^2\mathcal{P}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{P}'' - \frac{5}{2}c_{\mathcal{P}}\mathcal{P}'^2 + \frac{5}{2}\mathcal{P}^3\Big) + c_{\mathcal{J}}\Big(c_{\mathcal{P}}\mathcal{P}''^2 + \frac{5}{2}\mathcal{P}\mathcal{P}'^2\Big)\Big]\,, \\
\tilde{H}^{(4)} &= \int d\phi \Big[\mathcal{J}\Big( - c_{\mathcal{P}}^3\mathcal{P}^{(6)} + 7c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P} - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'' + \frac{21}{2}c_{\mathcal{P}}^2\mathcal{P}''^2 - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}\mathcal{P}'^2 \\
&\quad +14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}' + \frac{35}{8}\mathcal{P}^4\Big) + c_{\mathcal{J}}\Big(\frac{3}{2}c_{\mathcal{P}}^2(\mathcal{P}^{(3)})^2 + 7c_{\mathcal{P}}\mathcal{P}\mathcal{P}''^2 + \frac{35}{4}\mathcal{P}^2\mathcal{P}'^2\Big)\Big]\,, \\
\tilde{H}^{(5)} &= \int d\phi \Big[\mathcal{J}\Big(c_{\mathcal{P}}^4\mathcal{P}^{(8)} - 9c_{\mathcal{P}}^3\mathcal{P}^{(6)}\mathcal{P} - \frac{69}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2 + \frac{189}{2}c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}''^2 \\
&\quad -27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{P}'' + \frac{63}{2}c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}^2 - \frac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{P}'' \\
&\quad +126c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}\mathcal{P}' + \frac{231}{2}c_{\mathcal{P}}^2\mathcal{P}'^2\mathcal{P}'' - \frac{315}{4}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'^2 + \frac{63\mathcal{P}^5}{8}\Big) \\
&\quad +c_{\mathcal{J}}\Big(2c_{\mathcal{P}}^3(\mathcal{P}^{(4)})^2 + \frac{27}{2}c_{\mathcal{P}}^2\mathcal{P}(\mathcal{P}^{(3)})^2 - 15c_{\mathcal{P}}^2\mathcal{P}''^3 + \frac{63}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}''^2 \\
&\quad -\frac{35}{4}c_{\mathcal{P}}\mathcal{P}'^4 + \frac{105}{4}\mathcal{P}^3\mathcal{P}'^2\Big)\Big]\,.
\end{aligned}
$$

Hence, according to (2.45), the Gelfand-Dikii polynomials read

$$
\begin{aligned}
R^{(0)} &= 1\,, \quad R^{(1)} = \mathcal{P}\,, \quad R^{(2)} = -c_{\mathcal{P}}\mathcal{P}'' + \frac{3}{2}\mathcal{P}^2\,, \\
R^{(3)} &= c_{\mathcal{P}}^2\mathcal{P}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{P}'' - \frac{5}{2}c_{\mathcal{P}}\mathcal{P}'^2 + \frac{5}{2}\mathcal{P}^3\,, \\
R^{(4)} &= -c_{\mathcal{P}}^3\mathcal{P}^{(6)} + 7c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P} - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'' + \frac{21}{2}c_{\mathcal{P}}^2\mathcal{P}''^2 - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}\mathcal{P}'^2 \\
&\quad +14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}' + \frac{35}{8}\mathcal{P}^4\,, \\
R^{(5)} &= c_{\mathcal{P}}^4\mathcal{P}^{(8)} - 9c_{\mathcal{P}}^3\mathcal{P}^{(6)}\mathcal{P} + \frac{63}{2}c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}^2 - \frac{69}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2 - \frac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{P}'' \\
&\quad +\frac{189}{2}c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}''^2 - \frac{315}{4}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'^2 - 27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{P}'' \\
&\quad +126c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}\mathcal{P}' + \frac{231}{2}c_{\mathcal{P}}^2\mathcal{P}'^2\mathcal{P}'' + \frac{63}{8}\mathcal{P}^5\,,
\end{aligned}
$$

while the polynomials $T^{(n)}$ can be obtained from (2.46), so that they are given by

$$
\begin{aligned}
T^{(0)} &= 0\,, \quad T^{(1)} = \mathcal{J}\,, \quad T^{(2)} = -c_{\mathcal{P}}\mathcal{J}'' - c_{\mathcal{J}}\mathcal{P}'' + 3\mathcal{J}\mathcal{P}\,, \\
T^{(3)} &= c_{\mathcal{P}}^2\mathcal{J}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{J}'' - 5c_{\mathcal{P}}\mathcal{J}\mathcal{P}'' - 5c_{\mathcal{P}}\mathcal{J}'\mathcal{P}' + \frac{15}{2}\mathcal{J}\mathcal{P}^2 \\
&\quad + c_{\mathcal{J}}\left(2c_{\mathcal{P}}\mathcal{P}^{(4)} - 5\mathcal{P}\mathcal{P}'' - \frac{5}{2}\mathcal{P}'^2\right)\,, \\
T^{(4)} &= -c_{\mathcal{P}}^3\mathcal{J}^{(6)} + 7c_{\mathcal{P}}^2\mathcal{J}^{(4)}\mathcal{P} + 14c_{\mathcal{P}}^2\mathcal{J}^{(3)}\mathcal{P}' + 21c_{\mathcal{P}}^2\mathcal{J}''\mathcal{P}'' - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{J}'' + 14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{J}' \\
&\quad + 7c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}^{(4)} - 35c_{\mathcal{P}}\mathcal{P}\mathcal{J}'\mathcal{P}' - 35c_{\mathcal{P}}\mathcal{J}\mathcal{P}\mathcal{P}'' - \frac{35}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}'^2 + \frac{35}{2}\mathcal{J}\mathcal{P}^3 - c_{\mathcal{J}}\Big(3c_{\mathcal{P}}^2\mathcal{P}^{(6)} \\
&\quad - 14c_{\mathcal{P}}\mathcal{P}\mathcal{P}^{(4)} - 21c_{\mathcal{P}}\mathcal{P}''^2 + \frac{35}{2}\mathcal{P}^2\mathcal{P}'' + \frac{35}{2}\mathcal{P}\mathcal{P}'^2 - 28c_{\mathcal{P}}\mathcal{P}^{(3)}\mathcal{P}'\Big)\,, \\
T^{(5)} &= c_{\mathcal{P}}^4\mathcal{J}^{(8)} - 9c_{\mathcal{P}}^3\mathcal{J}^{(6)}\mathcal{P} - 27c_{\mathcal{P}}^3\mathcal{J}^{(5)}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{J}^{(4)}\mathcal{P}'' + \frac{63}{2}c_{\mathcal{P}}^2\mathcal{J}^{(4)}\mathcal{P}^2 - 69c_{\mathcal{P}}^3\mathcal{J}^{(3)}\mathcal{P}^{(3)} \\
&\quad + 126c_{\mathcal{P}}^2\mathcal{J}^{(3)}\mathcal{P}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{J}'' + 189c_{\mathcal{P}}^2\mathcal{P}\mathcal{J}''\mathcal{P}'' + \frac{231}{2}c_{\mathcal{P}}^2\mathcal{J}''\mathcal{P}'^2 - \frac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{J}'' \\
&\quad - 27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{J}' + 126c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}^{(3)}\mathcal{J}' - \frac{315}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{J}'\mathcal{P}' + 231c_{\mathcal{P}}^2\mathcal{J}'\mathcal{P}'\mathcal{P}'' - 9c_{\mathcal{P}}^3\mathcal{J}\mathcal{P}^{(6)} \\
&\quad + 63c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}\mathcal{P}^{(4)} + \frac{189}{2}c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}''^2 - \frac{315}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}^2\mathcal{P}'' - \frac{315}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}\mathcal{P}'^2 + 126c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}^{(3)}\mathcal{P}' \\
&\quad + \frac{315}{8}\mathcal{J}\mathcal{P}^4 + c_{\mathcal{J}}\Big(4c_{\mathcal{P}}^3\mathcal{P}^{(8)} - 27c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}^{(6)} + 63c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}^{(4)} - \frac{207}{2}c_{\mathcal{P}}^2(\mathcal{P}^{(3)})^2 + 189c_{\mathcal{P}}\mathcal{P}\mathcal{P}''^2 \\
&\quad - 81c_{\mathcal{P}}^2\mathcal{P}^{(5)}\mathcal{P}' - 171c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}'' + 252c_{\mathcal{P}}\mathcal{P}\mathcal{P}^{(3)}\mathcal{P}' + 231c_{\mathcal{P}}\mathcal{P}'^2\mathcal{P}'' - \frac{105}{2}\mathcal{P}^3\mathcal{P}'' - \frac{315}{4}\mathcal{P}^2\mathcal{P}'^2\Big)\,.
\end{aligned}
$$

# Appendix B

# Involution of the conserved quantities

In order to prove that our set of conserved charges, $\bar{H}^{(m)} = \left(H_{\text{KdV}}^{(n)}; \tilde{H}^{(n)}\right)$, is abelian in both Poisson brackets, one can follow the lines of the proof of the same statement in the case of the pure KdV equation that can be found in standard textbooks (see e.g., [112, 150]).

Without loss of generality, let us assume that $m > n$. Thus, the Poisson bracket associated to the $\text{BMS}_3$ operator $\mathcal{D}^{(2)}$ of two conserved charges, which reads

$$\{\bar{H}^{(m)}, \bar{H}^{(n)}\}_{(2)} = \int d\phi \begin{pmatrix} \frac{\delta \bar{H}^{(m)}}{\delta \mathcal{J}} & \frac{\delta \bar{H}^{(m)}}{\delta \mathcal{P}} \end{pmatrix} \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta \bar{H}^{(n)}}{\delta \mathcal{J}} \\ \frac{\delta \bar{H}^{(n)}}{\delta \mathcal{P}} \end{pmatrix}, \tag{B.1}$$

by virtue of the recursion relation in (2.40), can be written in terms of the Poisson bracket associated to the "canonical" operator $\mathcal{D}^{(1)}$, according to

$$
\begin{aligned}
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} &= \int d\phi \begin{pmatrix} \frac{\delta \bar{H}^{(m)}}{\delta \mathcal{J}} & \frac{\delta \bar{H}^{(m)}}{\delta \mathcal{P}} \end{pmatrix} \mathcal{D}^{(1)} \begin{pmatrix} \frac{\delta \bar{H}^{(n+1)}}{\delta \mathcal{J}} \\ \frac{\delta \bar{H}^{(n+1)}}{\delta \mathcal{P}} \end{pmatrix} \\
&= \{\bar{H}^{(m)},\bar{H}^{(n+1)}\}_{(1)} \\
&= -\{\bar{H}^{(n+1)},\bar{H}^{(m)}\}_{(1)} \,.
\end{aligned}
\tag{B.2}
$$

Analogously, making use of the recursion relationship again, one finds that

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = -\{\bar{H}^{(n+1)},\bar{H}^{(m)}\}_{(1)} = \{\bar{H}^{(m-1)},\bar{H}^{(n+1)}\}_{(2)} \,. \tag{B.3}
$$

Therefore, once the procedure is applied $m-n$ times, one obtains

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = \{\bar{H}^{(n)},\bar{H}^{(m)}\}_{(2)} \,, \tag{B.4}
$$

which implies that the conserved charges are involution in both Poisson brackets, i.e.,

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = \{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(1)} = 0 \,. \tag{B.5}
$$

# Appendix C

# Solving the consistency condition of the symmetry parameters

Here we explicitly verify that eq. (2.39) solves the consistency condition for the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (2.18) for an arbitrary representative of the hierarchy, being characterized by the Hamiltonian $H^{(k)}$ given by (2.67). Since the equation is linear for the parameters, it is then enough proving that

$$\varepsilon_{\mathcal{J}}(\phi) = \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\phi)} \quad , \quad \varepsilon_{\mathcal{P}}(\phi) = \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\phi)} \, , \tag{C.1}$$

fulfills, for an arbitrary member of the set of conserved charges $\bar{H}^{(j)} = \left(H^{(j)}_{\text{KdV}}; \tilde{H}^{(j)}\right)$.

Note that the consistency condition for the parameters in (2.18) can also be written as

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} = -\int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \left( \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}(\varphi) \\ \mu_{\mathcal{P}}(\varphi) \end{pmatrix} \right)^{T} \right] \begin{pmatrix} \varepsilon_{\mathcal{J}}(\varphi) \\ \varepsilon_{\mathcal{P}}(\varphi) \end{pmatrix} , \tag{C.2}$$

which by virtue of the definition of $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ in (2.69), it reads

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} = -\int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \left( \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \right)^T \right] \begin{pmatrix} \varepsilon_{\mathcal{J}}(\varphi) \\ \varepsilon_{\mathcal{P}}(\varphi) \end{pmatrix} . \tag{C.3}$$

Therefore, once (C.1) is evaluated on (C.3), the consistency condition reduces to

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} = -\int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \left( \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \right)^T \right] \begin{pmatrix} \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} . \tag{C.4}$$

Besides, taking the time derivative of (C.1), by virtue of the field equations in (2.68) one can readily show that

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} = \int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \begin{pmatrix} \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\varphi)} & \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \right] \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} . \tag{C.5}$$

Then, integrating by parts, the latter equation reads

$$\begin{aligned} \begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} &= \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \int d\varphi \begin{pmatrix} \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\varphi)} & \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \\ &\quad - \int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \left( \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \right)^T \right] \begin{pmatrix} \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} , \end{aligned} \tag{C.6}$$

which reduces to

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}(\phi) \\ \dot{\varepsilon}_{\mathcal{P}}(\phi) \end{pmatrix} = -\int d\varphi \left[ \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \left( \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta H^{(k)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta H^{(k)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} \right)^T \right] \begin{pmatrix} \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{J}(\varphi)} \\ \frac{\delta \bar{H}^{(j)}}{\delta \mathcal{P}(\varphi)} \end{pmatrix} + \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}(\phi)} \\ \frac{\delta}{\delta \mathcal{P}(\phi)} \end{pmatrix} \{\bar{H}^{(j)}, H^{(k)}\} , \tag{C.7}$$

where we have made use of the definition of the Poisson brackets in (2.3). Therefore, since the conserved charges commute with the Hamiltonian ($\{\bar{H}^{(j)}, H^{(k)}\} = 0$), the second term at the r.h.s. of (C.7) vanishes; and consequently, the consistency condition evaluated on the parameters (C.1), given by (C.4), has been shown to be fulfilled.

# Appendix D

# Generic solution for the field equations of the hierarchy

Here we show that the generic solution in (2.81) solves the field equations (2.71) for an arbitrary value of the label of the hierarchy $k$, provided that $\mathcal{P}$ stands for an arbitrary generic solution for the field equations of the $k$-th representative of the KdV hierarchy.

In sum, we want to prove that

$$\mathcal{J} = \sum_{j=0}^{\infty} \eta_j \partial_\phi R^{(j+1)} + c_{\mathcal{J}} \frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}} \,, \tag{D.1}$$

is a solution of

$$\dot{\mathcal{J}} = \mathcal{D}^{(\mathcal{P})} T^{(k)} + \left(\mathcal{D}^{(\mathcal{J})} + a\mathcal{D}^{(\mathcal{P})}\right) R^{(k)} \,, \tag{D.2}$$

provided that $\mathcal{P}$ solves

$$\dot{\mathcal{P}} = \mathcal{D}^{(\mathcal{P})} R^{(k)} \,, \tag{D.3}$$

where the Gelfand-Dikii polynomials $R^{(k)}$, and the polynomials $T^{(k)}$ are defined through (2.45) and (2.46), with $\tilde{H}^{(k)}$ given by (2.47), i.e.,

$$R^{(k)} = \frac{\delta H^{(k)}_{\text{KdV}}[\mathcal{P}]}{\delta\mathcal{P}}\,, \tag{D.4}$$

$$T^{(k)} = c_{\mathcal{J}}\frac{\partial R^{(k)}}{\partial c_{\mathcal{P}}} + \int d\varphi \mathcal{J}(\varphi)\frac{\delta R^{(k)}(\varphi)}{\delta\mathcal{P}}\,. \tag{D.5}$$

Thus, once (D.5) is evaluated on (D.1), it reduces to

$$T^{(k)} = \sum_{j=0}^{\infty}\eta_j \int d\varphi \partial_\varphi R^{(j+1)}(\varphi)\frac{\delta R^{(k)}}{\delta\mathcal{P}(\varphi)} + c_{\mathcal{J}}\frac{dR^{(k)}}{dc_{\mathcal{P}}} + at\partial_t R^{(k)}\,, \tag{D.6}$$

where $\frac{dR^{(k)}}{dc_{\mathcal{P}}}$ stands for the total derivative of $R^{(k)}$ with respect to the central charge $c_{\mathcal{P}}$, given by

$$\frac{dR^{(k)}}{dc_{\mathcal{P}}} = \frac{\partial R^{(k)}}{\partial c_{\mathcal{P}}} + \int d\varphi \frac{\partial\mathcal{P}(\varphi)}{\partial c_{\mathcal{P}}}\frac{\delta R^{(k)}}{\delta\mathcal{P}(\varphi)}\,. \tag{D.7}$$

Besides, the time derivative of (D.1) can be written as

$$\begin{aligned}\dot{\mathcal{J}} &= \sum_{j=0}^{\infty}\eta_j\partial_\phi\left[\int d\varphi\frac{\delta R^{(j+1)}}{\delta\mathcal{P}(\varphi)}\dot{\mathcal{P}}(\varphi)\right] + c_{\mathcal{J}}\frac{\partial\dot{\mathcal{P}}}{\partial c_{\mathcal{P}}} + a\partial_t\left(t\partial_t P\right)\\ &= a\mathcal{D}^{(\mathcal{P})}R^{(k)}\\ &\quad+\partial_\phi\left[-\sum_{j=0}^{\infty}\eta_j\int d\varphi\frac{\delta\partial_\varphi R^{(j+1)}}{\delta\mathcal{P}(\varphi)}R^{(k+1)}(\varphi) + c_{\mathcal{J}}\frac{\partial R^{(k+1)}}{\partial c_{\mathcal{P}}} + at\partial_t R^{(k+1)}\right]\,,\end{aligned} \tag{D.8}$$

where we have made use of the $k$-th KdV equation in (D.3), as well as the recursion relation for the Gelfand-Dikii polynomials in (2.43). Note that by virtue of (D.4), the following identity holds

$$\int d\varphi\frac{\delta\partial_\varphi R^{(j+1)}}{\delta\mathcal{P}(\varphi)}R^{(k+1)}(\varphi) = \frac{\delta}{\delta\mathcal{P}}\int d\varphi\partial_\varphi R^{(j+1)}(\varphi)R^{(k+1)}(\varphi) - \int d\varphi\partial_\varphi R^{(j+1)}(\varphi)\frac{\delta R^{(k+1)}(\varphi)}{\delta\mathcal{P}}\,, \tag{D.9}$$

where the first term in the r.h.s. of (D.9) vanishes due to the fact that the conserved charges $H_{\text{KdV}}^{(k)}$ are in involution, i.e., $\{H_{\text{KdV}}^{(k+1)}, H_{\text{KdV}}^{(j+1)}\}_{(1)} = 0$. Hence, eq. (D.9) implies that (D.8) reduces to

$$\begin{aligned}\dot{\mathcal{J}} &= a\mathcal{D}^{(\mathcal{P})}R^{(k)} \\ &\quad +\partial_\phi\left[\sum_{j=0}^{\infty}\eta_j\int d\varphi\partial_\varphi R^{(j+1)}(\varphi)\frac{\delta R^{(k+1)}(\varphi)}{\delta\mathcal{P}} + c_{\mathcal{J}}\frac{\partial R^{(k+1)}}{\partial c_{\mathcal{P}}} + at\partial_t R^{(k+1)}\right] \\ &= a\mathcal{D}^{(\mathcal{P})}R^{(k)} + \partial_\phi T^{(k+1)}\,. \end{aligned} \tag{D.10}$$

Therefore, making use of the recursion relation for the polynomials $T^{(k)}$ in (2.44), one finally proves that eq. (D.10) reduces to the field equation in (D.2), which implies that $\mathcal{J}$ in (D.1) is indeed a solution.

Consequently, making $\eta_j = 0$ in (D.1), one concludes that $\mathcal{J}_p$ in (2.79) provides a particular solution for the field equation (D.2).

The conserved charges associated to this exact solution are then given by $H_{\text{KdV}}^{(n)}$ and $\tilde{H}^{(n)}$ defined in (2.47). Note that once $\tilde{H}^{(n)}$ is evaluated on the exact solution (D.1), the contribution due to the homogeneous part vanishes, because

$$\int d\phi\mathcal{J}_h\frac{\delta H_{\text{KdV}}^{(n)}}{\delta\mathcal{P}} = \sum_{j=0}^{\infty}\eta_j\int d\phi\partial_\phi R^{(j+1)}\frac{\delta H_{\text{KdV}}^{(n)}}{\delta\mathcal{P}} = \sum_{j=0}^{\infty}\eta_j\left\{H_{\text{KdV}}^{(n)}, H_{\text{KdV}}^{(j+1)}\right\}_{(1)} = 0\,. \tag{D.11}$$

Therefore, $\tilde{H}^{(n)}$ reduces to

$$\tilde{H}^{(n)} = c_{\mathcal{J}}\left(\frac{\partial H_{\text{KdV}}^{(n)}}{\partial c_{\mathcal{P}}} + \int d\phi\frac{\partial\mathcal{P}}{\partial c_{\mathcal{P}}}\frac{\delta H_{\text{KdV}}^{(n)}}{\delta\mathcal{P}}\right) + at\dot{H}_{\text{KdV}}^{(n)}\,, \tag{D.12}$$

where the last term in (D.12) vanishes since $H_{\text{KdV}}^{(n)}$ is conserved. Hence, (D.12) can be written in terms of the total derivative with respect to the central extension $c_{\mathcal{P}}$, according to

$$\tilde{H}^{(n)} = c_{\mathcal{J}} \frac{dH^{(n)}_{\text{KdV}}}{dc_{\mathcal{P}}} \,. \tag{D.13}$$

# Bibliography


[1] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, no. 6, 061102 (2016) doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

[2] K. Akiyama *et al.* [Event Horizon Telescope Collaboration], "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," Astrophys. J. **875**, no. 1, L1 (2019) doi:10.3847/2041-8213/ab0ec7 [arXiv:1906.11238 [astro-ph.GA]].

[3] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31**, 161 (1973). doi:10.1007/BF01645742

[4] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973). doi:10.1103/PhysRevD.7.2333

[5] S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43**, 199 (1975) Erratum: [Commun. Math. Phys. **46**, 206 (1976)]. doi:10.1007/BF02345020, 10.1007/BF01608497

[6] A. Strominger, "Five Problems in Quantum Gravity," Nucl. Phys. Proc. Suppl. **192-193**, 119 (2009) doi:10.1016/j.nuclphysbps.2009.07.049 [arXiv:0906.1313 [hep-th]].

[7] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. B **379**, 99 (1996) doi:10.1016/0370-2693(96)00345-0 [hep-th/9601029].

[8] G. 't Hooft, "Dimensional reduction in quantum gravity," Conf. Proc. C **930308**, 284 (1993) [gr-qc/9310026].

[9] L. Susskind, "The World as a hologram," J. Math. Phys. **36**, 6377 (1995) doi:10.1063/1.531249 [hep-th/9409089].

[10] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] doi:10.1023/A:1026654312961, 10.4310/ATMP.1998.v2.n2.a1 [hep-th/9711200].

[11] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[12] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) doi:10.4310/ATMP.1998.v2.n2.a2 [hep-th/9802150].

[13] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104**, 207 (1986). doi:10.1007/BF01211590

[14] A. Strominger, "Black hole entropy from near horizon microstates," JHEP **9802**, 009 (1998) doi:10.1088/1126-6708/1998/02/009 [hep-th/9712251].

[15] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69**, 1849 (1992) doi:10.1103/PhysRevLett.69.1849 [hep-th/9204099].

[16] Banados:1992wn,Banados:1992gq,Cardy:1986ie M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, "Geometry of the (2+1) black hole," Phys. Rev. D **48**, 1506 (1993) Erratum: [Phys. Rev. D **88**, 069902 (2013)] doi:10.1103/PhysRevD.48.1506, 10.1103/PhysRevD.88.069902 [gr-qc/9302012].

[17] J. L. Cardy, "Operator Content of Two-Dimensional Conformally Invariant Theories," Nucl. Phys. B **270**, 186 (1986). doi:10.1016/0550-3213(86)90552-3 50-3213(86)90552-3;

[18] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," arXiv:1703.05448 [hep-th].

[19] S. W. Hawking, "The Information Paradox for Black Holes," arXiv:1509.01147 [hep-th].

[20] A. Strominger, "Black Hole Information Revisited," arXiv:1706.07143 [hep-th].

[21] R. Sasaki and I. Yamanaka, "Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories," Adv. Stud. Pure Math. **16**, 271 (1988).

[22] T. Eguchi and S. K. Yang, "Deformations of Conformal Field Theories and Soliton Equations," Phys. Lett. B **224**, 373 (1989). doi:10.1016/0370-2693(89)91463-9

[23] A. Pérez, D. Tempo and R. Troncoso, "Boundary conditions for General Relativity on $AdS_3$ and the KdV hierarchy," JHEP **1606**, 103 (2016) doi:10.1007/JHEP06(2016)103 [arXiv:1605.04490 [hep-th]].

[24] J. de Boer and D. Engelhardt, "Remarks on thermalization in 2D CFT," Phys. Rev. D **94**, no. 12, 126019 (2016) doi:10.1103/PhysRevD.94.126019 [arXiv:1604.05327 [hep-th]].

[25] A. Dymarsky and K. Pavlenko, “Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit,” JHEP **1901**, 098 (2019) doi:10.1007/JHEP01(2019)098 [arXiv:1810.11025 [hep-th]].

[26] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, “Generalized Gibbs Ensemble and the Statistics of KdV Charges in 2D CFT,” JHEP **1903**, 075 (2019) doi:10.1007/JHEP03(2019)075 [arXiv:1810.11054 [hep-th]].

[27] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, “Thermal Correlation Functions of KdV Charges in 2D CFT,” JHEP **1902**, 044 (2019) doi:10.1007/JHEP02(2019)044 [arXiv:1810.11053 [hep-th]].

[28] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc. Roy. Soc. Lond. A **269**, 21 (1962). doi:10.1098/rspa.1962.0161

[29] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times,” Proc. Roy. Soc. Lond. A **270**, 103 (1962). doi:10.1098/rspa.1962.0206

[30] R. Sachs, “Asymptotic symmetries in gravitational theory,” Phys. Rev. **128**, 2851 (1962). doi:10.1103/PhysRev.128.2851

[31] P. Havas and J. Plebánski, “Conformal extensions of the galilei group and their relation to the schrödinger group,” *Journal of Mathematical Physics* **19** (1978), no.2, 482-488.

[32] A. Bagchi and R. Gopakumar, “Galilean Conformal Algebras and AdS/CFT,” JHEP **0907**, 037 (2009) doi:10.1088/1126-6708/2009/07/037 [arXiv:0902.1385 [hep-th]].

[33] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, “GCA in 2d,” JHEP **1008**, 004 (2010) doi:10.1007/JHEP08(2010)004 [arXiv:0912.1090 [hep-th]].

[34] A. Ashtekar, J. Bicak and B. G. Schmidt, “Asymptotic structure of symmetry reduced general relativity,” Phys. Rev. D **55**, 669 (1997) doi:10.1103/PhysRevD.55.669 [gr-qc/9608042].

[35] G. Barnich and G. Compere, “Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions,” Class. Quant. Grav. **24**, F15 (2007) doi:10.1088/0264-9381/24/5/F01, 10.1088/0264-9381/24/11/C01 [gr-qc/0610130].

[36] G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,” JHEP **1005**, 062 (2010) doi:10.1007/JHEP05(2010)062 [arXiv:1001.1541 [hep-th]].

[37] G. Barnich, L. Donnay, J. Matulich and R. Troncoso, “Asymptotic symmetries and dynamics of three-dimensional flat supergravity,” JHEP **1408**, 071 (2014) doi:10.1007/JHEP08(2014)071 [arXiv:1407.4275 [hep-th]].

[38] A. Bagchi, “Tensionless Strings and Galilean Conformal Algebra,” JHEP **1305**, 141 (2013) doi:10.1007/JHEP05(2013)141 [arXiv:1303.0291 [hep-th]].

[39] A. Bagchi, S. Chakrabortty and P. Parekh, “Tensionless Strings from Worldsheet Symmetries,” JHEP **1601**, 158 (2016) doi:10.1007/JHEP01(2016)158 [arXiv:1507.04361 [hep-th]].

[40] I. Mandal and A. Rayyan, “Super-GCA from $\mathcal{N}$ = (2,2) super-Virasoro,” Phys. Lett. B **754**, 195 (2016) doi:10.1016/j.physletb.2016.01.031 [arXiv:1601.04723 [hep-th]].

[41] E. Casali and P. Tourkine, “On the null origin of the ambitwistor string,” JHEP **1611**, 036 (2016) doi:10.1007/JHEP11(2016)036 [arXiv:1606.05636 [hep-th]].

[42] A. Bagchi, S. Chakrabortty and P. Parekh, “Tensionless Superstrings: View from the Worldsheet,” JHEP **1610**, 113 (2016) doi:10.1007/JHEP10(2016)113 [arXiv:1606.09628 [hep-th]].

[43] I. Mandal, “Addendum to ”Super-GCA from $\mathcal{N} = (2,2)$ super-Virasoro”: Super-GCA connection with tensionless strings,” Phys. Lett. B **760**, 832 (2016) doi:10.1016/j.physletb.2016.07.014 [arXiv:1607.02439 [hep-th]].

[44] E. Casali, Y. Herfray and P. Tourkine, “The complex null string, Galilean conformal algebra and scattering equations,” JHEP **1710**, 164 (2017) doi:10.1007/JHEP10(2017)164 [arXiv:1707.09900 [hep-th]].

[45] A. Bagchi, A. Banerjee, S. Chakrabortty and P. Parekh, “Inhomogeneous Tensionless Superstrings,” JHEP **1802**, 065 (2018) doi:10.1007/JHEP02(2018)065 [arXiv:1710.03482 [hep-th]].

[46] D. Francia and A. Sagnotti, “On the geometry of higher spin gauge fields,” Class. Quant. Grav. **20**, S473 (2003) [Comment. Phys. Math. Soc. Sci. Fenn. **166**, 165 (2004)] [PoS JHW **2003**, 005 (2003)] doi:10.1088/0264-9381/20/12/313, 10.22323/1.011.0005 [hep-th/0212185].

[47] D. Francia and A. Sagnotti, “Higher-spin geometry and string theory,” J. Phys. Conf. Ser. **33**, 57 (2006) doi:10.1088/1742-6596/33/1/006 [hep-th/0601199].

[48] A. Sagnotti and M. Taronna, “String Lessons for Higher-Spin Interactions,” Nucl. Phys. B **842**, 299 (2011) doi:10.1016/j.nuclphysb.2010.08.019 [arXiv:1006.5242 [hep-th]].

[49] D. Polyakov, “Interactions of Massless Higher Spin Fields From String Theory,” Phys. Rev. D **82**, 066005 (2010) doi:10.1103/PhysRevD.82.066005 [arXiv:0910.5338 [hep-th]].

[50] D. Polyakov, “Gravitational Couplings of Higher Spins from String Theory,” Int. J. Mod. Phys. A **25**, 4623 (2010) doi:10.1142/S0217751X1005041X [arXiv:1005.5512 [hep-th]].

[51] X. Bekaert, N. Boulanger and P. Sundell, “How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples,” Rev. Mod. Phys. **84**, 987 (2012) doi:10.1103/RevModPhys.84.987 [arXiv:1007.0435 [hep-th]].

[52] G. Barnich, A. Gomberoff and H. A. González, "Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field theories as the flat limit of Liouville theory," Phys. Rev. D **87**, no. 12, 124032 (2013) doi:10.1103/PhysRevD.87.124032 [arXiv:1210.0731 [hep-th]].

[53] G. Barnich and H. A. Gonzalez, "Dual dynamics of three dimensional asymptotically flat Einstein gravity at null infinity," JHEP **1305**, 016 (2013) doi:10.1007/JHEP05(2013)016 [arXiv:1303.1075 [hep-th]].

[54] C. Batlle, V. Campello and J. Gomis, "Canonical realization of ( 2+1 )-dimensional Bondi-Metzner-Sachs symmetry," Phys. Rev. D **96**, no. 2, 025004 (2017) doi:10.1103/PhysRevD.96.025004 [arXiv:1703.01833 [hep-th]].

[55] A. Bagchi, "Correspondence between Asymptotically Flat Spacetimes and Non-relativistic Conformal Field Theories," Phys. Rev. Lett. **105**, 171601 (2010) doi:10.1103/PhysRevLett.105.171601 [arXiv:1006.3354 [hep-th]].

[56] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo and R. Troncoso, "Soft Heisenberg hair on black holes in three dimensions," Phys. Rev. D **93**, no. 10, 101503 (2016) doi:10.1103/PhysRevD.93.101503 [arXiv:1603.04824 [hep-th]].

[57] D. Grumiller, A. Perez, S. Prohazka, D. Tempo and R. Troncoso, "Higher Spin Black Holes with Soft Hair," JHEP **1610**, 119 (2016) doi:10.1007/JHEP10(2016)119 [arXiv:1607.05360 [hep-th]].

[58] H. Afshar, D. Grumiller, W. Merbis, A. Perez, D. Tempo and R. Troncoso, "Soft hairy horizons in three spacetime dimensions," Phys. Rev. D **95**, no. 10, 106005 (2017) doi:10.1103/PhysRevD.95.106005 [arXiv:1611.09783 [hep-th]].

[59] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," Phys. Rev. Lett. **116**, no. 9, 091101 (2016) doi:10.1103/PhysRevLett.116.091101 [arXiv:1511.08687 [hep-th]].

[60] H. Afshar, S. Detournay, D. Grumiller and B. Oblak, "Near-Horizon Geometry and Warped Conformal Symmetry," JHEP **1603**, 187 (2016) doi:10.1007/JHEP03(2016)187 [arXiv:1512.08233 [hep-th]].

[61] L. Donnay, G. Giribet, H. A. González and M. Pino, "Extended Symmetries at the Black Hole Horizon," JHEP **1609**, 100 (2016) doi:10.1007/JHEP09(2016)100 [arXiv:1607.05703 [hep-th]].

[62] S. Deser and J. H. Kay, "Topologically Massive Supergravity," Phys. Lett. **120B**, 97 (1983). doi:10.1016/0370-2693(83)90631-7

[63] N. Marcus and J. H. Schwarz, "Three-Dimensional Supergravity Theories," Nucl. Phys. B **228**, 145 (1983). doi:10.1016/0550-3213(83)90402-9

[64] G. Barnich, L. Donnay, J. Matulich and R. Troncoso, "Super-$BMS_3$ invariant boundary theory from three-dimensional flat supergravity," JHEP **1701**, 029 (2017) doi:10.1007/JHEP01(2017)029 [arXiv:1510.08824 [hep-th]].

[65] A. Bagchi and I. Mandal, "Supersymmetric Extension of Galilean Conformal Algebras," Phys. Rev. D **80**, 086011 (2009) doi:10.1103/PhysRevD.80.086011 [arXiv:0905.0580 [hep-th]].

[66] I. Mandal, "Supersymmetric Extension of GCA in 2d," JHEP **1011**, 018 (2010) doi:10.1007/JHEP11(2010)018 [arXiv:1003.0209 [hep-th]].

[67] N. Banerjee, D. P. Jatkar, S. Mukhi and T. Neogi, "Free-field realisations of the $BMS_3$ algebra and its extensions," JHEP **1606**, 024 (2016) doi:10.1007/JHEP06(2016)024 [arXiv:1512.06240 [hep-th]].

[68] N. Banerjee, D. P. Jatkar, I. Lodato, S. Mukhi and T. Neogi, "Extended Supersymmetric $BMS_3$ algebras and Their Free Field Realisations," JHEP **1611**, 059 (2016) doi:10.1007/JHEP11(2016)059 [arXiv:1609.09210 [hep-th]].

[69] I. Lodato and W. Merbis, "Super-$BMS_3$ algebras from $\mathcal{N} = 2$ flat supergravities," JHEP **1611**, 150 (2016) doi:10.1007/JHEP11(2016)150 [arXiv:1610.07506 [hep-th]].

[70] N. Banerjee, I. Lodato and T. Neogi, "$\mathcal{N} = 4$ Supersymmetric $BMS_3$ algebras from asymptotic symmetry analysis," Phys. Rev. D **96**, no. 6, 066029 (2017) doi:10.1103/PhysRevD.96.066029 [arXiv:1706.02922 [hep-th]].

[71] R. Basu, S. Detournay and M. Riegler, "Spectral Flow in 3D Flat Spacetimes," JHEP **1712**, 134 (2017) doi:10.1007/JHEP12(2017)134 [arXiv:1706.07438 [hep-th]].

[72] O. Fuentealba, J. Matulich and R. Troncoso, "Asymptotic structure of $\mathcal{N} = 2$ supergravity in 3D: extended super-$BMS_3$ and nonlinear energy bounds," JHEP **1709**, 030 (2017) doi:10.1007/JHEP09(2017)030 [arXiv:1706.07542 [hep-th]].

[73] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, "Spin-3 Gravity in Three-Dimensional Flat Space," Phys. Rev. Lett. **111**, no. 12, 121603 (2013) doi:10.1103/PhysRevLett.111.121603 [arXiv:1307.4768 [hep-th]].

[74] H. A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, "Asymptotically flat spacetimes in three-dimensional higher spin gravity," JHEP **1309**, 016 (2013) doi:10.1007/JHEP09(2013)016 [arXiv:1307.5651 [hep-th]].

[75] J. Matulich, A. Perez, D. Tempo and R. Troncoso, "Higher spin extension of cosmological spacetimes in 3D: asymptotically flat behaviour with chemical potentials and thermodynamics," JHEP **1505**, 025 (2015) doi:10.1007/JHEP05(2015)025 [arXiv:1412.1464 [hep-th]].

[76] H. A. Gonzalez and M. Pino, "Boundary dynamics of asymptotically flat 3D gravity coupled to higher spin fields," JHEP **1405**, 127 (2014) doi:10.1007/JHEP05(2014)127 [arXiv:1403.4898 [hep-th]].

[77] O. Fuentealba, J. Matulich and R. Troncoso, "Extension of the Poincaré group with half-integer spin generators: hypergravity and beyond," JHEP **1509**, 003 (2015) doi:10.1007/JHEP09(2015)003 [arXiv:1505.06173 [hep-th]].

[78] O. Fuentealba, J. Matulich and R. Troncoso, "Asymptotically flat structure of hypergravity in three spacetime dimensions," JHEP **1510**, 009 (2015) doi:10.1007/JHEP10(2015)009 [arXiv:1508.04663 [hep-th]].

[79] R. Caroca, P. Concha, E. Rodríguez and P. Salgado-Rebolledo, "Generalizing the $\mathfrak{bms}_3$ and 2D-conformal algebras by expanding the Virasoro algebra," Eur. Phys. J. C **78**, no. 3, 262 (2018) doi:10.1140/epjc/s10052-018-5739-7 [arXiv:1707.07209 [hep-th]].

[80] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, "On the supersymmetric extension of asymptotic symmetries in three spacetime dimensions," arXiv:1908.09150 [hep-th].

[81] P. Concha, D. M. Peñafiel and E. Rodríguez, "On the Maxwell supergravity and flat limit in 2 + 1 dimensions," Phys. Lett. B **785**, 247 (2018) doi:10.1016/j.physletb.2018.08.050 [arXiv:1807.00194 [hep-th]].

[82] P. Concha, N. Merino, E. Rodríguez, P. Salgado-Rebolledo and O. Valdivia, "Semi-simple enlargement of the $\mathfrak{bms}_3$ algebra from a $\mathfrak{so}(2,2) \oplus \mathfrak{so}(2,1)$ Chern-Simons theory," JHEP **1902**, 002 (2019) doi:10.1007/JHEP02(2019)002 [arXiv:1810.12256 [hep-th]].

[83] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, "Three-dimensional Poincaré supergravity and $N$-extended supersymmetric $BMS_3$ algebra," Phys. Lett. B **792**, 93 (2019) doi:10.1016/j.physletb.2019.02.049 [arXiv:1812.05065 [hep-th]].

[84] A. Bagchi, S. Detournay, R. Fareghbal and J. Simón, "Holography of 3D Flat Cosmological Horizons," Phys. Rev. Lett. **110**, no. 14, 141302 (2013) doi:10.1103/PhysRevLett.110.141302 [arXiv:1208.4372 [hep-th]].

[85] G. Barnich, "Entropy of three-dimensional asymptotically flat cosmological solutions," JHEP **1210**, 095 (2012) doi:10.1007/JHEP10(2012)095 [arXiv:1208.4371 [hep-th]].

[86] S. Carlip, "Black Hole Entropy from Bondi-Metzner-Sachs Symmetry at the Horizon," Phys. Rev. Lett. **120**, no. 10, 101301 (2018) doi:10.1103/PhysRevLett.120.101301 [arXiv:1702.04439 [gr-qc]].

[87] S. Carlip, "Near-Horizon BMS Symmetry, Dimensional Reduction, and Black Hole Entropy," arXiv:1910.01762 [hep-th].

[88] G. Barnich, A. Gomberoff and H. A. Gonzalez, "The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes," Phys. Rev. D **86**, 024020 (2012) doi:10.1103/PhysRevD.86.024020 [arXiv:1204.3288 [gr-qc]].

[89] H. A. Gonzalez and M. Pino, "Asymptotically flat spacetimes in 3D bigravity," Phys. Rev. D **86**, 084045 (2012) doi:10.1103/PhysRevD.86.084045 [arXiv:1207.0882 [gr-qc]].

[90] A. Bagchi, S. Detournay and D. Grumiller, "Flat-Space Chiral Gravity," Phys. Rev. Lett. **109**, 151301 (2012) doi:10.1103/PhysRevLett.109.151301 [arXiv:1208.1658 [hep-th]].

[91] A. Bagchi, R. Basu, D. Grumiller and M. Riegler, "Entanglement entropy in Galilean conformal field theories and flat holography," Phys. Rev. Lett. **114**, no. 11, 111602 (2015) doi:10.1103/PhysRevLett.114.111602 [arXiv:1410.4089 [hep-th]].

[92] A. Bagchi, D. Grumiller and W. Merbis, "Stress tensor correlators in three-dimensional gravity," Phys. Rev. D **93**, no. 6, 061502 (2016) doi:10.1103/PhysRevD.93.061502 [arXiv:1507.05620 [hep-th]].

[93] G. Barnich, C. Troessaert, D. Tempo and R. Troncoso, "Asymptotically locally flat spacetimes and dynamical nonspherically-symmetric black holes in three dimensions," Phys. Rev. D **93**, no. 8, 084001 (2016) doi:10.1103/PhysRevD.93.084001 [arXiv:1512.05410 [hep-th]].

[94] C. Troessaert, D. Tempo and R. Troncoso, "Asymptotically flat black holes and gravitational waves in three-dimensional massive gravity," arXiv:1512.09046 [hep-th].

[95] A. Bagchi, M. Gary and Zodinmawia, "Bondi-Metzner-Sachs bootstrap," Phys. Rev. D **96**, no. 2, 025007 (2017) doi:10.1103/PhysRevD.96.025007 [arXiv:1612.01730 [hep-th]].

[96] A. Bagchi, M. Gary and Zodinmawia, "The nuts and bolts of the BMS Bootstrap," Class. Quant. Grav. **34**, no. 17, 174002 (2017) doi:10.1088/1361-6382/aa8003 [arXiv:1705.05890 [hep-th]].

[97] C. Erices, M. Riquelme and P. Rodríguez, "BTZ black hole with KdV-type boundary conditions: Thermodynamics revisited," arXiv:1907.13026 [hep-th].

[98] O. Fuentealba, J. Matulich, A. Pérez, M. Pino, P. Rodríguez, D. Tempo and R. Troncoso, "Integrable systems with $BMS_3$ Poisson structure and the dynamics of locally flat spacetimes," JHEP **1801**, 148 (2018) doi:10.1007/JHEP01(2018)148 [arXiv:1711.02646 [hep-th]].

[99] M. Guica, T. Hartman, W. Song and A. Strominger, "The Kerr/CFT Correspondence," Phys. Rev. D **80**, 124008 (2009) doi:10.1103/PhysRevD.80.124008 [arXiv:0809.4266 [hep-th]].

[100] S. Detournay, T. Hartman and D. M. Hofman, "Warped Conformal Field Theory," Phys. Rev. D **86**, 124018 (2012) doi:10.1103/PhysRevD.86.124018 [arXiv:1210.0539 [hep-th]].

[101] A. V. Chubukov, S. Sachdev and T. Senthil, “Quantum phase transitions in frustrated quantum antiferromagnets,” Nucl. Phys. B **426**, 601 (1994) Erratum: [Nucl. Phys. B **438**, 649 (1995)]. doi:10.1016/0550-3213(95)00016-L, 10.1016/0550-3213(94)90023-X

[102] Yang, Kun. “Ferromagnetic Transition in One-Dimensional Itinerant Electron Systems,” *Physical Review Letters* **93** (2004) 066401.

[103] J. L. Cardy, “Critical exponents of the chiral Potts model from conformal field theory,” Nucl. Phys. B **389**, 577 (1993) doi:10.1016/0550-3213(93)90353-Q [hep-th/9210002].

[104] D. M. Hofman and A. Strominger, “Chiral Scale and Conformal Invariance in 2D Quantum Field Theory,” Phys. Rev. Lett. **107**, 161601 (2011) doi:10.1103/PhysRevLett.107.161601 [arXiv:1107.2917 [hep-th]].

[105] S. Kachru, X. Liu and M. Mulligan, “Gravity duals of Lifshitz-like fixed points,” Phys. Rev. D **78**, 106005 (2008) doi:10.1103/PhysRevD.78.106005 [arXiv:0808.1725 [hep-th]].

[106] M. Taylor, “Non-relativistic holography,” arXiv:0812.0530 [hep-th].

[107] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” Class. Quant. Grav. **26**, 224002 (2009) doi:10.1088/0264-9381/26/22/224002 [arXiv:0903.3246 [hep-th]].

[108] E. D’Hoker and P. Kraus, “Holographic Metamagnetism, Quantum Criticality, and Crossover Behavior,” JHEP **1005**, 083 (2010) doi:10.1007/JHEP05(2010)083 [arXiv:1003.1302 [hep-th]].

[109] S. A. Hartnoll, “Horizons, holography and condensed matter,” arXiv:1106.4324 [hep-th].

[110] S. A. Hartnoll, D. M. Ramirez and J. E. Santos, “Emergent scale invariance of disordered horizons,” JHEP **1509**, 160 (2015) doi:10.1007/JHEP09(2015)160 [arXiv:1504.03324 [hep-th]].

[111] M. Taylor, “Lifshitz holography,” Class. Quant. Grav. **33**, no. 3, 033001 (2016) doi:10.1088/0264-9381/33/3/033001 [arXiv:1512.03554 [hep-th]].

[112] A. K. Das, “Integrable models,” World Sci. Lect. Notes Phys. **30**, 1 (1989).

[113] A. Achucarro and P. K. Townsend, “A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories,” Phys. Lett. B **180**, 89 (1986). doi:10.1016/0370-2693(86)90140-1

[114] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. Phys. B **311**, 46 (1988). doi:10.1016/0550-3213(88)90143-5

[115] O. Coussaert, M. Henneaux and P. van Driel, “The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant,” Class. Quant. Grav. **12**, 2961 (1995) doi:10.1088/0264-9381/12/12/012 [gr-qc/9506019].

[116] M. Henneaux, A. Perez, D. Tempo and R. Troncoso, "Chemical potentials in three-dimensional higher spin anti-de Sitter gravity," JHEP **1312**, 048 (2013) doi:10.1007/JHEP12(2013)048 [arXiv:1309.4362 [hep-th]].

[117] C. Bunster, M. Henneaux, A. Perez, D. Tempo and R. Troncoso, "Generalized Black Holes in Three-dimensional Spacetime," JHEP **1405**, 031 (2014) doi:10.1007/JHEP05(2014)031 [arXiv:1404.3305 [hep-th]].

[118] A. Perez, M. Riquelme, D. Tempo and R. Troncoso, "Asymptotic structure of the Einstein-Maxwell theory on $AdS_3$," JHEP **1602**, 015 (2016) doi:10.1007/JHEP02(2016)015 [arXiv:1512.01576 [hep-th]].

[119] T. Regge and C. Teitelboim, "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity," Annals Phys. **88**, 286 (1974). doi:10.1016/0003-4916(74)90404-7

[120] M. Banados, "Three-dimensional quantum geometry and black holes," AIP Conf. Proc. **484**, no. 1, 147 (1999) doi:10.1063/1.59661 [hep-th/9901148].

[121] E. Ojeda and A. Pérez, "Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies," JHEP **1908**, 079 (2019) doi:10.1007/JHEP08(2019)079 [arXiv:1906.11226 [hep-th]].

[122] H. A. González, J. Matulich, M. Pino and R. Troncoso, "Revisiting the asymptotic dynamics of General Relativity on $AdS_3$," JHEP **1812**, 115 (2018) doi:10.1007/JHEP12(2018)115 [arXiv:1809.02749 [hep-th]].

[123] E. Witten, "Multitrace operators, boundary conditions, and AdS / CFT correspondence," hep-th/0112258.

[124] D. Grumiller, A. Perez, D. Tempo and R. Troncoso, "Log corrections to entropy of three dimensional black holes with soft hair," JHEP **1708**, 107 (2017) doi:10.1007/JHEP08(2017)107 [arXiv:1705.10605 [hep-th]].

[125] M. Banados and S. Theisen, "Scale invariant hairy black holes," Phys. Rev. D **72**, 064019 (2005) doi:10.1103/PhysRevD.72.064019 [hep-th/0506025].

[126] L. Smarr, "Mass formula for Kerr black holes," Phys. Rev. Lett. **30**, 71 (1973) Erratum: [Phys. Rev. Lett. **30**, 521 (1973)]. doi:10.1103/PhysRevLett.30.521, 10.1103/PhysRevLett.30.71

[127] S. Hyun, J. Jeong, S. A. Park and S. H. Yi, "Scaling symmetry and scalar hairy Lifshitz black holes," JHEP **1510**, 105 (2015) doi:10.1007/JHEP10(2015)105 [arXiv:1507.03574 [hep-th]].

[128] B. Ahn, S. Hyun, S. A. Park and S. H. Yi, "Scaling symmetry and scalar hairy rotating $AdS_3$ black holes," Phys. Rev. D **93**, no. 2, 024041 (2016) doi:10.1103/PhysRevD.93.024041 [arXiv:1508.06484 [hep-th]].

[129] C. Erices, O. Fuentealba and M. Riquelme, "Electrically charged black hole on $AdS_3$ : Scale invariance and the Smarr formula," Phys. Rev. D **97**, no. 2, 024037 (2018) doi:10.1103/PhysRevD.97.024037 [arXiv:1710.05962 [hep-th]].

[130] M. Bravo-Gaete and M. Hassaine, "Thermodynamics of a BTZ black hole solution with an Horndeski source," Phys. Rev. D **90**, no. 2, 024008 (2014) doi:10.1103/PhysRevD.90.024008 [arXiv:1405.4935 [hep-th]].

[131] H. A. Gonzalez, M. Hassaine and C. Martinez, "Thermodynamics of charged black holes with a nonlinear electrodynamics source," Phys. Rev. D **80**, 104008 (2009) doi:10.1103/PhysRevD.80.104008 [arXiv:0909.1365 [hep-th]].

[132] B. Ahn, S. Hyun, K. K. Kim, S. A. Park and S. H. Yi, "Holography without counter terms," Phys. Rev. D **94**, no. 2, 024043 (2016) doi:10.1103/PhysRevD.94.024043 [arXiv:1512.09319 [hep-th]].

[133] S. F. Wu, X. H. Ge and Y. X. Liu, "First law of black hole mechanics in variable background fields," Gen. Rel. Grav. **49**, no. 6, 85 (2017) doi:10.1007/s10714-017-2246-3 [arXiv:1602.08661 [hep-th]].

[134] S. Carlip and C. Teitelboim, "Aspects of black hole quantum mechanics and thermodynamics in (2+1)-dimensions," Phys. Rev. D **51**, 622 (1995) doi:10.1103/PhysRevD.51.622 [gr-qc/9405070].

[135] E. Ayón-Beato, M. Bravo-Gaete, F. Correa, M. Hassaïne, M. M. Juárez-Aubry and J. Oliva, "First law and anisotropic Cardy formula for three-dimensional Lifshitz black holes," Phys. Rev. D **91**, no. 6, 064006 (2015) Addendum: [Phys. Rev. D **96**, no. 4, 049903 (2017)] doi:10.1103/PhysRevD.91.064006, 10.1103/PhysRevD.96.049903 [arXiv:1501.01244 [gr-qc]].

[136] W. G. Brenna, R. B. Mann and M. Park, "Mass and Thermodynamic Volume in Lifshitz Spacetimes," Phys. Rev. D **92**, no. 4, 044015 (2015) doi:10.1103/PhysRevD.92.044015 [arXiv:1505.06331 [hep-th]].

[137] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, "Lifshitz Black Hole in Three Dimensions," Phys. Rev. D **80**, 104029 (2009) doi:10.1103/PhysRevD.80.104029 [arXiv:0909.1347 [hep-th]].

[138] S. W. Hawking, M. J. Perry and A. Strominger, "Soft Hair on Black Holes," Phys. Rev. Lett. **116**, no. 23, 231301 (2016) doi:10.1103/PhysRevLett.116.231301 [arXiv:1601.00921 [hep-th]].

[139] S. W. Hawking, M. J. Perry and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," JHEP **1705**, 161 (2017) doi:10.1007/JHEP05(2017)161 [arXiv:1611.09175 [hep-th]].

[140] S. Haco, S. W. Hawking, M. J. Perry and A. Strominger, “Black Hole Entropy and Soft Hair,” JHEP **1812**, 098 (2018) doi:10.1007/JHEP12(2018)098 [arXiv:1810.01847 [hep-th]].

[141] O. B. Zaslavsky, “Thermodynamics of (2+1) black holes,” Class. Quant. Grav. **11**, L33 (1994). doi:10.1088/0264-9381/11/2/003

[142] J. D. Brown, J. Creighton and R. B. Mann, “Temperature, energy and heat capacity of asymptotically anti-de Sitter black holes,” Phys. Rev. D **50**, 6394 (1994) doi:10.1103/PhysRevD.50.6394 [gr-qc/9405007].

[143] B. Reznik, “Thermodynamics and evaporation of the (2+1)-dimensions black hole,” Phys. Rev. D **51**, 1728 (1995) doi:10.1103/PhysRevD.51.1728 [gr-qc/9403027].

[144] J. P. Muniain and D. D. Piriz, “Critical behavior of dimensionally continued black holes,” Phys. Rev. D **53**, 816 (1996) doi:10.1103/PhysRevD.53.816 [gr-qc/9502029].

[145] R. G. Cai, Z. J. Lu and Y. Z. Zhang, “Critical behavior in (2+1)-dimensional black holes,” Phys. Rev. D **55**, 853 (1997) doi:10.1103/PhysRevD.55.853 [gr-qc/9702032].

[146] J. Crisostomo, R. Troncoso and J. Zanelli, “Black hole scan,” Phys. Rev. D **62**, 084013 (2000) doi:10.1103/PhysRevD.62.084013 [hep-th/0003271].

[147] S. W. Hawking and D. N. Page, “Thermodynamics of Black Holes in anti-De Sitter Space,” Commun. Math. Phys. **87**, 577 (1983). doi:10.1007/BF01208266

[148] M. Dunajski, “Solitons, instantons, and twistors,” (Oxford graduate texts in mathematics. 19)

[149] V. G. Drinfeld and V. V. Sokolov, “Lie algebras and equations of Korteweg-de Vries type,” J. Sov. Math. **30**, 1975 (1984). doi:10.1007/BF02105860

[150] P. Olver, *Applications of Lie Groups to Differential Equations*. Springer New York, 2000.

[151] I. M. Gel’fand and I. Ya. Dorfman, “Hamiltonian operators and algebraic structures related to them,” Func. Anal. Appl. 13 (1979), 248-262.

[152] R. F. Penna, “$\mathrm{BMS}_3$ invariant fluid dynamics at null infinity,” Class. Quant. Grav. **35**, no. 4, 044002 (2018) doi:10.1088/1361-6382/aaa3aa [arXiv:1708.08470 [hep-th]].

[153] R. Hirota and J. Satsuma, “Soliton Solutions of a Coupled Korteweg-de Vries Equation,” Phys. Lett. A **85**, 407 (1981). doi:10.1016/0375-9601(81)90423-0

[154] S. Y. Sakovich, “Coupled KdV Equations of Hirota-Satsuma Type,” *J. Nonlinear Math. Phys.* **6** (1999) 255-262.

[155] K. M. Tamizhmani and M. Lakshmanan, “Complete Integrability Of The Korteweg-de Vries Equation Under Perturbation Around Its Solution: Lie-backlund Symmetry Approach,” J. Phys. A **16**, 3773 (1983). doi:10.1088/0305-4470/16/16/015

[156] W. X. Ma and B. Fuchssteiner, "Integrable theory of the perturbation equations," Chaos Solitons Fractals **7**, 1227 (1996) doi:10.1016/0960-0779(95)00104-2 [solv-int/9604004].

[157] A. Kalkanli Karasu, "Painleve classification of coupled KdV systems," J. Math. Phys. **38**, 3616 (1997). doi:10.1063/1.532056

[158] P. G. Drazin and R. S. Johnson, "Solitons: An Introduction," Cambridge, UK: Univ. Pr. (1989) 226 p

[159] A. Restuccia and A. Sotomayor, "Duality relation among the Hamiltonian structures of a parametric coupled Korteweg-de Vries system," Open Phys. **14**, no. 1, 95 (2016). doi:10.1515/phys-2016-0009

[160] L. Cornalba and M. S. Costa, "A New cosmological scenario in string theory," Phys. Rev. D **66**, 066001 (2002) doi:10.1103/PhysRevD.66.066001 [hep-th/0203031].

[161] A. Giacomini, R. Troncoso and S. Willison, "Three-dimensional supergravity reloaded," Class. Quant. Grav. **24**, 2845 (2007) doi:10.1088/0264-9381/24/11/005 [hep-th/0610077].

[162] G. W. Gibbons and S. W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," Phys. Rev. D **15**, 2752 (1977). doi:10.1103/PhysRevD.15.2752

[163] G. W. Gibbons and S. W. Hawking, "Euclidean quantum gravity," Singapore, Singapore: World Scientific (1993) 586 p

[164] R. Feynman and A. Hibbs, "Quantum mechanics and path integrals." International series in pure and applied physics. McGraw-Hill, 1965.

[165] R. Blumenhagen and E. Plauschinn, "Introduction to conformal field theory : with applications to String theory," Lect. Notes Phys. **779**, 1 (2009). doi:10.1007/978-3-642-00450-6

[166] F. Correa, C. Martinez and R. Troncoso, "Scalar solitons and the microscopic entropy of hairy black holes in three dimensions," JHEP **1101**, 034 (2011) doi:10.1007/JHEP01(2011)034 [arXiv:1010.1259 [hep-th]].

[167] F. Correa, C. Martinez and R. Troncoso, "Hairy Black Hole Entropy and the Role of Solitons in Three Dimensions," JHEP **1202**, 136 (2012) doi:10.1007/JHEP02(2012)136 [arXiv:1112.6198 [hep-th]].

[168] H. A. Gonzalez, D. Tempo and R. Troncoso, "Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes," JHEP **1111**, 066 (2011) doi:10.1007/JHEP11(2011)066 [arXiv:1107.3647 [hep-th]].

[169] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, "Warped AdS(3) Black Holes," JHEP **0903**, 130 (2009) doi:10.1088/1126-6708/2009/03/130 [arXiv:0807.3040 [hep-th]].

[170] A. Bagchi and R. Basu, "3D Flat Holography: Entropy and Logarithmic Corrections," JHEP **1403**, 020 (2014) doi:10.1007/JHEP03(2014)020 [arXiv:1312.5748 [hep-th]].

[171] D. Melnikov, F. Novaes, A. Pérez and R. Troncoso, "Lifshitz Scaling, Microstate Counting from Number Theory and Black Hole Entropy," JHEP **1906**, 054 (2019) doi:10.1007/JHEP06(2019)054 [arXiv:1808.04034 [hep-th]].

[172] E. Lifshitz, "On the theory of second-order phase transitions i & ii," *Zh. Eksp. Teor. Fiz* **11** (1941), no. 255, 269.

[173] O. Fuentealba, H. A. González, M. Pino and R. Troncoso, "The anisotropic chiral boson," JHEP **1911**, 123 (2019) doi:10.1007/JHEP11(2019)123 [arXiv:1909.02699 [hep-th]].

[174] P. Bouwknegt and K. Schoutens, "W symmetry in conformal field theory," Phys. Rept. **223**, 183 (1993) doi:10.1016/0370-1573(93)90111-P [hep-th/9210010].

[175] G. Compère and W. Song, "$\mathcal{W}$ symmetry and integrability of higher spin black holes," JHEP **1309**, 144 (2013) doi:10.1007/JHEP09(2013)144 [arXiv:1306.0014 [hep-th]].

[176] M. Gutperle and Y. Li, "Higher Spin Lifshitz Theory and Integrable Systems," Phys. Rev. D **91**, no. 4, 046012 (2015) doi:10.1103/PhysRevD.91.046012 [arXiv:1412.7085 [hep-th]].

[177] M. Beccaria, M. Gutperle, Y. Li and G. Macorini, "Higher spin Lifshitz theories and the Korteweg-de Vries hierarchy," Phys. Rev. D **92**, no. 8, 085005 (2015) doi:10.1103/PhysRevD.92.085005 [arXiv:1504.06555 [hep-th]].

[178] M. Gutperle and Y. Li, "Higher spin Chern–Simons theory and the super Boussinesq hierarchy," Int. J. Mod. Phys. A **33**, no. 14n15, 1850085 (2018) doi:10.1142/S0217751X18500859 [arXiv:1709.02345 [hep-th]].

[179] S. Deser, R. Jackiw and S. Templeton, "Topologically Massive Gauge Theories," Annals Phys. **140**, 372 (1982) [Annals Phys. **281**, 409 (2000)] Erratum: [Annals Phys. **185**, 406 (1988)]. doi:10.1006/aphy.2000.6013, 10.1016/0003-4916(82)90164-6

[180] S. Deser, R. Jackiw and S. Templeton, "Three-Dimensional Massive Gauge Theories," Phys. Rev. Lett. **48**, 975 (1982). doi:10.1103/PhysRevLett.48.975

[181] G. Compere and S. Detournay, "Semi-classical central charge in topologically massive gravity," Class. Quant. Grav. **26**, 012001 (2009) Erratum: [Class. Quant. Grav. **26**, 139801 (2009)] doi:10.1088/0264-9381/26/1/012001, 10.1088/0264-9381/26/13/139801 [arXiv:0808.1911 [hep-th]].

[182] G. Compere and S. Detournay, "Boundary conditions for spacelike and timelike warped $AdS_3$ spaces in topologically massive gravity," JHEP **0908**, 092 (2009) doi:10.1088/1126-6708/2009/08/092 [arXiv:0906.1243 [hep-th]].

[183] M. Henneaux, C. Martinez and R. Troncoso, "Asymptotically warped anti-de Sitter spacetimes in topologically massive gravity," Phys. Rev. D **84**, 124016 (2011) doi:10.1103/PhysRevD.84.124016 [arXiv:1108.2841 [hep-th]].

[184] W. Merbis, "Chern-Simons-like Theories of Gravity," arXiv:1411.6888 [hep-th].

[185] E. A. Bergshoeff, W. Merbis and P. K. Townsend, "On asymptotic charges in 3D gravity," arXiv:1909.11743 [hep-th].