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BLACK HOLES AND SOLITONS IN SUPERGRAVITY AND THEIR EXACT SCALAR (QUASI)-NORMAL MODES

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Tesis presentada a la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción para optar al grado académico de Magíster en Ciencias con Mención en Física

Marzo 2022
Concepción, Chile

Profesor Guía: Julio Oliva Zapata

En memoria de mi abuelo...

AGRADECIMIENTOS

Escribir esta tesis habría sido imposible sin el apoyo incondicional de mi familia, amigos y profesores. A los cuales les agradezco por todo su tiempo, cariño y consejos que me permitieron luchar contra las ecuaciones y seguir adelante para atisbar la belleza que se esconde tras ellas. En especial quiero agradecer a mi madre Carmen Gloria Catalán, mi padre Gonzalo Patricio Oyarzo, mi hermano Vicente Oyarzo y mi abuela Nelly Barriga, quienes siempre me apoyaron y me brindaron todo el cariño que necesité y más, muchas gracias!

El camino de intentar entender el mundo en donde vivimos es enrevesado, emocionante y es fácil perderse. Sin embargo, tuve la suerte de poder contar con profesores que me ayudaron a mantenerme cerca de la luz del conocimiento. En espacial quiero agradecer a mi profesor guía Dr. Julio Oliva quien siempre estuvo ahí para guiarme y aconsejarme, para así poder entender y descubrir las cosas. Gran parte de las cosas que logramos describir en esta tesis son gracias a su apoyo constante. Muchas gracias profesor! Quiero agradecer también al Dr. Fabrizio Canfora. Mucho aprendí trabajando con usted en esta tesis, además, que cuando las ecuaciones no se resolvían siempre usted estaba ahí dando aliento y nuevas ideas para triunfar en nuestra búsqueda. Agradezco a los profesores que me enseñaron muchas de las cosas que he aprendido en este camino, en espacial al Dr. Cristobal Corral, Dr. Juán Crisostomo, Dr. José Barea, Dr. Fernando Izaurieta, Dr. Joaquin Díaz de Valdés, Dr. Andrés Anabalón y Dr. Gastón Giribet, por su tiempo y dedicación, gracias.

Quiero agradecer también a mis amigos de siempre: Jorge V., Matias P., Luis M., Oscar M., Javiera Q., Valentina Q., John P., Barbara C. Byron F., Jorge L., Andree S., Claudio C., Carlos V., Mario M. y tantos otros. Ustedes siempre estuvieron ahí apoyando este camino que seguí, les agradezco por sus consejos y palabras de aliento, además sin usted seguro se me arrancaban los chivos al monte. Muchas gracias querid@s! Durante el tiempo que he estado en Concepción conocí muchos amigos, entre ellos Gonzalo S., Hernán E., Keanu M. ,José Pablo F., , Ricardo S., Leonardo S. , Scarlett R., Anibal N., Anibal O., Daniel C., Damian M., Barbara C. a los cuales les agradezco mucho por su tiempo, por las largas

discusiones sobre física y todo lo que hemos compartido.

Resumen

Esta tesis retomamos el problema de encontrar soluciones BPS en la teoría de supergravedad $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged. Reportamos una nueva solución regular en toda la variedad en el sector Abeliano de la teoría. La solución es 1/4 BPS y puede ser obtenida de la doble continuación analítica de una solución planar encontrada por Klemm en hep-th/9810090. También encontramos una solución en el sector Abeliano con simetría esférica cuya naturaleza supersimétrica fue pasada por alto en la literatura.

Estas configuraciones de agujeros negros y solitones, en el caso planar y esférico, permiten integrar de forma exacta un campo escalar de prueba, incluso con la presencia de un acomplamiento no-minimal con el escalar de Ricci. Calculamos el espectro de los modos (cuasi-) normales del campo escalar con acoplamiento no-minimal. Encontramos que las ecuaciones radiales se pueden integrar en término de funciones hipergeometricas, lo cual permite encontrar una expresión para el espectro de frecuencias de forma exacta. Los espacio tiempo considerados no son de curvatura constante asintóticamente, sin embargo adquieren un vector de Killing extra. La condiciones de borde para el caso de agujero negro es de modos entrante en el horizonte y de tipo Dirichlet en infinito. Los modos cuasi-normales no dependen del radio del agujero negro, por lo que esta familia de geometrías puede ser interpretadas como isoespectral en lo que respecta al operadores de onda acoplado no-minimalmente al escalar de Ricci. El comportamiento del campo escalar dependen de los valores de la constante de acoplamiento con el escalar de Ricci, donde encontramos configuraciones suprimidas exponencialmente en el tiempo y configuraciones inestables que crecen exponencialmente. Mostramos que las propiedades de integrabilidad del escalar de prueba son posibles en el caso de los espacios tiempo regulares supersimetrico y no-supersimetrico. La condición de borde para el caso del soliton es regular en el origen y Dirichlet de modo que la solución sea un mínimo del principio de acción. En este caso, dependiendo del valor de la constante de acoplamiento, encontramos soluciones oscilantes y soluciones con un campo escalar inestable.

También construimos soluciones en el sector no-Abelian de la teoría de supergravedad con el ansantz de meron para SU(2). Construimos soluciones de doble meron y meron cargado. Esta última se convierte en una singularidad desnuda para los valores en el espacio de los parámetros que la solución es 1/4 BPS y adquiere un vector de Killing conforme extra. También consideramos dos familias de potenciales de auto-interacción para campo escalar, de modo que estamos fuera de la teoría de supergravedad pero conservamos el acoplamiento dilatónico y el ansatz de Meron en el sector de Yang-Mills. En estas familias construimos soluciones exactas de agujeros negros que son Lifshitz topológico asintóticamente como también soluciones con propiedades asintóticas interesantes. También analizamos algunas propiedades termodinámica de estos espacios tiempos.

Keywords – Agujeros negros, Solitones, SuperGravedad, modos (cuasi)-normales, Soluciones exactas

Abstract

We revisit the problem of finding BPS solutions in $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity. We report on a new supersymmetric solution in the Abelian sector of the theory, which describes a soliton that is regular everywhere. The solution is 1/4 BPS and can be obtained from a double analytic continuation of a planar solution found by Klemm in hep-th/9810090. Also in the Abelian sector, but now for a spherically symmetric ansatz we find a new solution whose supersymmetric nature was overlooked in the previous literature.

We identify these configurations, including the planar case, as a new family of black holes and solitons that lead to the exact integration of scalar probes, even in the presence of a non-minimal coupling with the Ricci scalar which has a non-trivial profile. On these geometries, we compute the spectrum of (quasi-)normal modes for the non-minimally coupled scalar field. We find that the equation for the radial dependence can be integrated in terms of hypergeometric functions leading to an exact expression for the frequencies. The solutions do not asymptote to a constant curvature spacetime, nevertheless the asymptotic region acquires an extra conformal Killing vector. For the black hole, the scalar probe is purely ingoing at the horizon, and requiring that the solutions lead to an extremum of the action principle we impose a Dirichlet boundary condition at infinity. Surprisingly, the quasinormal modes do not depend on the radius of the black hole, therefore this family of geometries can be interpreted as isospectral in what regards to the wave operator non-minimally coupled to the Ricci scalar. We find both purely damped modes, as well as exponentially growing unstable modes depending on the values of the non-minimal coupling parameter. For the solitons we show that the same integrability property is achieved separately in a non-supersymmetric solutions as well as for the supersymmetric one. Imposing regularity at the origin and a well defined extremum for the action principle we obtain the spectra that can also lead to purely oscillatory modes as well as to unstable scalar probes, depending on the values of the non-minimal coupling.

We also construct solutions in the non-Abelian sector of the theory by considering the meron ansatz for $SU(2)$. We construct electric-meronic and double-meron solutions and show that the latter also leads to 1/4 BPS configurations that are singular and acquire an extra conformal Killing vector. We then move beyond the supergravity embedding of this theory by modifying the self-interaction of the scalar, but still within the same meron ansatz for a single gauge field, which is dilatonically coupled with the scalar. We construct exact black holes for two families of self-interactions that admit topologically Lifshitz black holes, as well as other black holes with interesting causal structures and asymptotic behavior. We analyze some thermal properties of these spacetimes.

Keywords – Black Holes, Solitons, SuperGravity, (Quasi)-normal modes, Exact solutions

Contents

Chapter 1

Introduction

It is well known that building a quantum gravity theory is needed because there are configuration where the effective theories for gravity currently available are not predictive. For example, considering General Relativity as a effective theory, the black hole solutions have a singularity inside the horizon where the geometric quantities, as the Riemann tensor and so on, blow up. Therefore, questions related to black hole formation and black hole evaporation are difficult to answer only considering the effective models.

However, thermodynamics is one of the notions that tell us something about the quantum nature of gravitational objects. The framework was developed in the seventies as a result of the no-hair theorems, which was summarized in [1], the contribution by Bekenstein who realized that black holes must have entropy and derived an heuristic formula for it [2]-[3]. The exact formula for the entropy and the physical interpretation of the temperature of a black hole were discovered by Hawking [4] where he considered quantum fields in a classical curved spacetime. These features of classical solutions give us insights about the macroscopic behaviour of a quantum system that we do not know.

One approach to construct models that could contain information about quantum gravity come from supersymmetry. These types of theories possess some symmetries generated by a global spinorial parameter in such a way that the transformation mixes fermions with bosons. The theories with this symmetry have shown to have improved quantum behaviour because of the divergence cancellation between fermions and bosons [5].

On the other hand, the standard model of particle physics showed that given a global symmetry, its local extension requires extra bosonic gauge fields which are essential to describe the interaction between particles. Thus, in the context of supersymmetry, it is natural to ask for its local version where the extra fields may describe an interaction between matter fields.

An interesting observation is that for a local supersymmetry, it is mandatory to add a Rarita-Schwinger Ψ_μ spin 3/2 fermionic field. Due to the consistency of the supersymmetry transformations, the spin 3/2 field transforms in a natural fashion with the vielbeine field $e^a_{\ \mu}$ because of the index structure of both fields, which leads to

$$\delta e^a_{\ \mu} \sim \bar{\epsilon}\gamma^a \Psi_\mu \ , \tag{1.0.1}$$

where ϵ is the local spinorial parameter of the supersymmetry transformation. Then, local supersymmetry implies naturally a coupling of matter fields to gravity [6]. These type of theories are called supergravity theory and capture some ideas of a quantum theory of gravity because of the improvement of its quantum behaviour.

Supergravity theories have been widely studied since the seventies and nowadays it is an active research field, for example in the context of localization techniques [7]-[9]. An interesting problem in this area relies on finding exact classical solutions of a supergravity theory in such a way that they allow for the existence of a Killing spinor. These configurations are called BPS and capture some aspects of the quantum theory, as was shown by Strominger and Vafa in [10] where they obtained the Bekenstein-Hawking entropy by counting BPS states.

In this thesis we will consider a particular model of supergravity theory, known as $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity or as Freedman-Schwarz model [38]. It has non-Abelian gauge fields among their matter fields. We will revisit the problem of finding BPS configurations in the Abelian and non-Abelian sectors of the theory.

The BPS configurations satisfy a first order system of equations which implies the second order system. Therefore, it is natural to consider the first order system to find new interesting configurations. For example, the Freedman-Schwarz model contains Yang-Mills fields coupled to gravity which are interesting in many gravitational scenarios, such as close to a neutron star or close to a black hole in the early cosmology [11]. In these cases the gravitational field is too strong and the curvature will influence the propagation of the matter fields, in addition to the feature that the back-reaction of the Yang-Mills fields cannot be neglected.

Solving the Einstein-Yang-Mills system has been source of great efforts in order to describe some interesting situations as the ones we mentioned above. Some numerical results are the following [12]-[15]. On the other hand, an analytic solution of the system which presents non-Abelian effects, such as the Jackiw-Rebbi-Hasenfratz-'t Hooft mechanism [62]-[63], was constructed in [60] implementeing the meron ansatz in the Yang-Mills sector. Thus, a natural question is if the meron ansatz is useful to construct new configurations in the Freedman-Schwarz model.

This thesis is structured as follows: in chapter two we will explain conventions and motivation to study self-gravitating systems and supersymmetric systems.

In chapter three we present the first part of new result in this thesis, these results were published in [97]. In section 3.1 we present new BPS and no-BPS soliton solution. Section 3.2 is devoted to finding new Abelian BPS configurations. In section 3.3 we study new solutions for the meron ansatz in the Yang-Mills sector, and due to the fact that the Freedman-Schwarz model contains two non-Abelian gauge fields we explore configurations with a meron in one gauge field and an electric Abelian gauge field in the other, which we called a charged-meron. Also we study double meron configurations which is not BPS. In section 3.4 we consider two different potentials for the dilatonic field beyond supergravity. The first potential has three parameters and is a sum of three exponentials. In this family we obtained Lifshitz topological black holes with different number of horizons. The second potential has a linear times exponential term which gives a solution with a logarithmic term in the metric function.

Chapter 4 contains the second part of novel results of this thesis and it is in process to be published in JHEP [103]. In this chapter, we consider the (quasi-)normal modes for the solitons and black holes of the Freedman-Schwarz model discussed in chapter 3. The (quasi-) normal modes are integrated analytically in term of hypergeometric functions which gives us an exact expression for the frequencies. Implementing Kummer indentities, which are fulfilled by the hypergeometric functions, we impose ingoing boundary conditions at the horizon for black holes and regularity at the origin for solitons. The geometries are not asymptotically locally flat nor asymptotically locally AdS, thus, we study the asymptotic behaviour of the scalar field in order to impose Dirichlet boundary conditions and to obtain an extremum of the action principle.

Introducción

Es bien conocido que es necesario construir una teoría de gravedad cuántica debido a que hay configuraciones donde las teorías de gravedad que actualmente se conocen no son predicativas. Por ejemplo, la teoría de Relatividad General admite soluciones de agujeros negros, los cuales presentan una singularidad al interior del horizonte donde las cantidades geométricas, como el tensor de Riemann, explotan. Por lo tanto, es difícil responder a las preguntas relacionadas con la formación y evaporación de agujeros negros considerando solo modelos efectivos.

Sin embargo, la termodinámica de agujeros negros es uno marcos conceptuales que nos podría ayudar a entender algunas características cuánticas de objetos gravitantes. Estas ideas fueron desarrolladas en los setenta como resultado de los teoremas de no-pelo [1], y de las contribuciones de Bekenstein, quien se dio cuenta que los agujeros negros deben tener entropía y derivó una formula heurística para ello [2]-[3]. Hawking consideró campos cuánticos en un espacio tiempo curvo, lo cual le permitió encontrar la fórmula exacta para la entropía de agujeros negros como también la interpretación de la temperatura de un agujero negro [4]. Estas características de las soluciones clásicas nos dicen algo sobre el comportamiento macroscópico de un sistema cuántico que aún no conocemos.

Otra forma de intentar construir modelos que pueden contener información sobre la gravedad cuántica viene de la supersimetría. Estas teorías gozan de una simetría generada por un parámetro espinoral global que mezcla fermiones con bosones. Las teorías con este tipo de simetría han mostrado tener un comportamiento cuántico mejorado debido a que hay cancelación de divergencias entre los fermiones y bosones [5].

Por otra parte, el modelo standard de física de partículas a mostrado que tomar una simetría global y convertirla en local es crucial para describir las interacciones entre partículas, debido a que esto da lugar a los bosones de gauge. Entonces en el contexto de supersimetría es natural preguntarse en convertir la supersimetría global en local y que los nuevos campos que emergen puedan describir alguna interacción a nivel cuántico.

Una observación interesante es que para hacer una supersimetría local es necesario incluir un campo fermionico de Rarita-Schiwnger Ψ_μ de spin 3/2. Debido a la consistencia de la teoría y de la estructura de índices, el campo de spin 3/2 debe transformar con el vielbeine $e^a{}_\mu$ como sigue

$$\delta e^a_\mu \sim \bar{\epsilon}\gamma^a \Psi_\mu \tag{1.0.2}$$

Donde ϵ es el parámetro espinorial local. Entonces hacer la supersimetría local lleva naturalmente a acoplar los campos de materia con la gravedad [6]. Esta teorías son llamadas de supergravedad y tienen posibilidades de capturar algunas ideas de la teoría que describe la gravedad cuántica, pues como mencionamos, la supersimetría mejora el comportamiento a nivel cuántico de la teoría en cuestión.

Las teorías de supergravedad fueron ampliamente estudiadas desde mediados de la decada del 70 y hoy en dia es una área activa de investigación en el contexto de localización y holografía [7]-[9]. Un problema interesante en este contexto es construir soluciones clásicas de la teoría de supergravedad en el sector bosónico que permitan tener espinores de Killing. Estas configuraciones son llamadas BPS y capturan algunos aspectos cuánticos de la teoría, tal como fue mostrado por Strominger y Vafa [10] donde mostraron que contando configuraciones BPS lograron reconstruir la entropía de Bekenstein-Hawking.

En esta tesis nos vamos a concentrar especialmente en un modelo de teoría de supergravedad, conocido como el modelo de freedman y schwarz, $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged supergravedad. La cual está conformada por campos de gauge no-Abelianos y campos escalares, además de la métrica. Vamos a re-visitar el problema de encontrar configuraciones BPS en esta teoría en el sector Abeliano y no-Abeliano.

Las configuraciones BPS satisfacen sistemas de ecuaciones de orden menor que las ecuaciones de campo de segundo orden. Por lo tanto, es natural considerar esta ecuaciones como una forma de resolver las de segundo orden y buscar configuraciones nuevas. Por ejemplo, el modelo de Freedman y Schwarz contiene campos de Yang-Mills que resultan interesantes en muchos escenarios gravitacionales, tales como cerca de una estrella de neutrones o cerca de un agujero negro en el universo temprano [11]. En estos casos el campo gravitacional es tan intenso que los efectos de la curvatura afectarán la propagación de los campos de materia.

Ha existido mucho esfuerzo en resolver el sistema de Einstein-Yang-Mills para construir soluciones auto-gravitantes, algunos de los resultados numéricos son [12]-[15]. Por otro lado, una solución analítica que presenta efectos no-Abelianos, tales como el mecanismo de Jackiw-Rebbi-Hasenfratz-'t Hooft [62]-[63], fue encontrada en [60] utilizando el ansatz de meron en el sector de Yang-Mills. Por lo tanto, es natural preguntar si el ansatz de meron permite encontrar configuraciones nuevas en el contexto del modelo de Freedman-Schwarz.

La presente tesis se encuentra estructurada de la siguiente forma: En el capitulo 2 se plantean las convenciones y motivaciones generales para el estudio de sistemas auto-gravitantes y sistemas supersimétricos.

En el capitulo 3 se presentan los primeros resultados de esta tesis que fue publicado en [97]. En la sección 3.1 presentamos las nuevas soluciones de solitones Abelianos BPS y no-BPS. En la sección 3.2 se muestran configuraciones Abelianas BPS que no habían sido vistas en la literatura. En la sección 3.3 estudiamos las soluciones provenientes del ansatz de meron en el sector de Yang-Mills. Dado que el modelo de Freedman-Schwarz contiene dos campos de gauge, exploramos las configuraciones con meron en un campo de gauge y un campo cargadado en el otro campo de gauge, como también un doble meron. En la sección 3.4 consideramos potenciales más generales que los presentes en la supergravedad, donde el primer potencial tiene tres parámetros y es suma de exponenciales. En esta familia construimos soluciones con 1,2 y 3 horizontes que tienen un comportamiento asintótico Lifshitz topológico. El segundo potencial tiene un término que es lineal por exponencial, donde logramos construir soluciones con una término logarítmico en la función métrica.

El capítulo 4 contiene la segunda parte de los resultados nuevo de esta tesis y se encuentra en proceso para ser publicada en JHEP [103]. En este capítulo nos concentramos en los modos (cuasi-)normales de los solitones y agujeros negros que son soluciones de la supergravedad que construimos en el capitulo anterior.

Los modos (cuasi-) normales los integramos de forma exacta en ambos casos en términos de funciones hipergeométricas, que gracias a las identidades de Kummer nos permitieron imponer las condiciones de borde causales en el agujero negro y de Dirichlet en el infinito espacial. Esto nos permitió obtener el espectro de forma cerrada para el campo escalar de prueba.

Chapter 2

Gauge theories and Gravity

Electrodynamics is the first theory of fields that humanity understood and on top of allowing us to develop technology, allows us to realize some mathematical aspects of world where we live in. One of the most exiting ideas that was developed following the symmetries of the Maxwell equations is the structure of space and time. This may be derived from the fact that the Maxwell equations are not covariant under Galileo transformations, which consider the space and time in a different footing. For example, the time difference between two events is the same on any inertial frame. Then the question is, what are the transformations that leave Maxwell equations invariant? The answer was found by Hendrik Lorentz who discovered such transformations, currently known Lorentz transformations, which leave invariant the speed of light for any inertial frame. These transformations consider the space and time in the same footing and imply that two inertial frames with different relative velocities have different notions of time.

Maxwell equations and Newton equations are incompatible because they are invariant under a different set of transformations which relate the measurements of moving observers. The way to reformulate the ideas of space and time in order to obtain a theory for massive bodies that is compatible with electrodynamics was developed by Einstein who derived the Lorentz transformations under the hypothesis of Special Relativity.

To write down some aspects of Special Relativity, let us consider the vector space $\mathbb{R}^4$ whose coordinates are x^μ and the quadratic form that is invariant under Lorentz transformation is $x^\mu x^\nu \eta_{\mu\nu}$ where $\eta_{\mu\nu} = diag\,(-+++)$, this invariance implements the constancy of the speed of light for different inertial observers. These type of spaces are called Lorentzian spacetimes and have three type of vectors v^μ: null vectors $v^\mu v^\nu \eta_{\mu\nu} = 0$, space-like vectors $v^\mu v^\nu \eta_{\mu\nu} > 0$ and time-like vectors $v^\mu v^\nu \eta_{\mu\nu} < 0$. Thus in any point, the spacetime is spitted in three regions that is useful to define causality.

Let us consider two points in the spacetime whose coordinates with respect to some inertial frame are x^μ and y^μ respectively, such that $x^t > y^t$. If these points are connected with a curve whose tangent vector is always time-like, or always null, then any inertial frame will agree with the fact that $\tilde{x}^t > \tilde{y}^t$, where the tilde denotes the coordinates of the points with respect to a second inertial frame connected with the former via a Lorentz transformation.

When the curve that connects those points is always spacelike, this statement is no longer true because for some inertial frame $\tilde{x}^t > \tilde{y}^t$ while for another set of inertial frames one has $\tilde{x}^t < \tilde{y}^t$. From this discussion we see that two events may be causally connected when they are connected with a time-like or null curve.

The idea of causality in quantum mechanics is much more subtle, because we can compute the expectation value of a free particle that propagates from $\vec{x}_0$ to $\vec{x}$ which is $P(t) = \langle \vec{x}| e^{-it\hat{H}} |\vec{x}_0\rangle$. Considering the relativistic dispersion relation $E = \sqrt{\vec{p}^2 + m^2}$ and writing the expectation value as an integral in momentum space, namely

$$
\begin{aligned}
P(t) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \langle \vec{x}| e^{-it\sqrt{\vec{p}^2+m^2}} |\vec{p}\rangle \langle \vec{p}\,|\vec{x}_0\rangle \ , \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \ .
\end{aligned}
$$

Writing the integral in spherical coordinates in momentum space and rotating the system of coordinates so that $\vec{x} - \vec{x}_0$ aligns with the p_z axis, we obtain

$$
\begin{aligned}
P(t) &= 2\pi \int_0^\infty \frac{d\rho}{(2\pi)^3} \rho^2 \int_0^\pi d\theta \sin\theta e^{-it\sqrt{\rho^2+m^2}} e^{i\rho|\vec{x}-\vec{x}_0|\cos\theta} \ , \\
&= \frac{4\pi}{|\vec{x}-\vec{x}_0|} \int_0^\infty \frac{d\rho}{(2\pi)^3} \rho \sin\left(\rho\,|\vec{x}-\vec{x}_0|\right) e^{-it\sqrt{\rho^2+m^2}} \ .
\end{aligned}
$$

Where $\rho^2 = p_x^2 + p_y^2 + p_z^2$. We are interested in points that are connected with an spacelike curve, then we can approximate the integral for $\vec{x} \gg t^2$, using saddle point approximation, then we obtain that $P(t) \sim e^{-m\sqrt{x^2-t^2}}$. Consequently, there is a non-vanishing probability of a particle traveling faster than the speed of light. This kind of troubles are cured considering a theory for fields instead of particles, where the problem is solved by microcausality, namely

$$[\hat{\phi}_H(x), \hat{\phi}_H(y)] = 0 \tag{2.0.1}$$

for x and y with a spacelike interval. This ensures that a measurement at x cannot affect a measurement at y when x and y are not causally connected. In the equation above $\hat{\phi}_H$ represents an operator in the Heisenberg picture.

2.1 Yang-Mills theories

Quantum electrodynamics (QED) is a theory for an Abelian gauge field and Dirac fields that has one of the most accurate predictions when compared with experiments. Despite of that, QED is not enough to describe all of the interactions that are realized in Nature, and then a generalization to non-Abelian gauge theories is mandatory. The renormalizability of the quantum theory restricts the possible terms present in the Lagrangian in such a way that the coupling constants must have positive mass dimension [16]. For gauge fields there are some terms that we can add to the Lagrangian such as $AA\partial A$ and A^4. In the following we will construct Maxwell electrodynamics coupled to a Dirac field in a way that admits a generalization for a general Lie group, which will give rise Yang-Mills theories.

The general idea behind the construction of gauge theories is that we start with a theory of fields that has a global symmetry of the action generated by real parameter which does not depend on the point. Then, we consider this symmetry as a local symmetry, where the parameter depends on the point. In order to do that, we have to add a new field called the gauge connection.

2.1.1 $U(1)$ Maxwell field

Let us consider a Dirac spinor ψ in $D = 4$ with the Dirac matrices γ_μ. In flat spacetime it fulfills the Clifford Algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The Lagrangian for a massive spinor is

$$\mathcal{L} = -i\bar{\psi}\left(\gamma^\mu \partial_\mu - m\right)\psi \tag{2.1.1}$$

where the Dirac conjugate is defined by $\bar{\psi} = \psi^{\dagger}\gamma^{0}$. The Lagrangian (2.1.1) is invariant under the following global transformation

$$\psi(x) \to e^{i\alpha}\psi(x) \ , \qquad \bar{\psi}(x) \to \bar{\psi}(x)\, e^{-i\alpha} \ . \tag{2.1.2}$$

The parameter α does not depend on the point. Phase transformations constitute the group of 1×1 unitary matrices called $U(1)$. The symmetry (2.1.2) means that we have the freedom to choice the global phase of the field at some time t and then the equation will determine the phase of the field at some later instant of time, thus part of the physics is encoded in phase differences and an overall phase factor is irrelevant.

Another example of theory that implements this type of symmetry is the wave function of a non-relativistic particle, where the physical quantity that we can measure is the squared modulus of the wave function which is invariant under an overall phase. The global symmetry in the latter case is important because it ensures the existence of a conserved probability current.

Imagine that we want to have the freedom of choosing the phase at any point of the spacetime. Then, the theory must be invariant under

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \ , \qquad \bar{\psi}(x) \to \bar{\psi}(x)\, e^{-i\alpha(x)} \tag{2.1.3}$$

where $\alpha(x)$ is a smooth real function. It is clear that (2.1.1) is not invariant under local transformations, because the derivative acts on the function $\alpha(x)$ and it gives an extra term in the transformed derivative of the spinor. Changing the fields by an arbitrary transformation on each spacetime point, we need to add an extra field which will carry the information of the transformation from one point to another. This quantity is the gauge field that in this case is a 1-form A_μ. As we mentioned, the problem comes from the transformation rule of the derivative of the spinor, indeed

$$\partial_\mu\psi \to (\partial_\mu\psi)' = e^{i\alpha(x)}\left(\partial_\mu\psi(x) + i\partial_\mu\alpha(x)\,\psi(x)\right) \ . \tag{2.1.4}$$

It does not transform covariantly under the local transformation (2.1.3). If we write down a derivative $D_\mu\psi$, sometimes called covariant derivative, which transforms covariantly under gauge transformation, then we will be able to construct a new

Lagrangian replacing the partial derivative by the covariant derivative. The new derivative must transform according to

$$D_\mu \psi(x) \rightarrow (D_\mu \psi(x))' = e^{i\alpha(x)} D_\mu \psi(x) \ . \tag{2.1.5}$$

The covariant derivative must contain a term that cancels the second term in (2.1.4). One way to do that is considering the covariant derivative as

$$D_\mu \psi(x) \equiv (\partial_\mu - iA_\mu(x))\, \psi \ . \tag{2.1.6}$$

We will figure out the transformation rule that A_μ fulfills in order to satisfy (2.1.5). Replacing (2.1.6) into (2.1.5) we obtain that A_μ transforms with an extra piece:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x) \ . \tag{2.1.7}$$

Then, we can construct a theory that is invariant under local transformations by adding a new field A_μ which transforms as a connection under gauge transformations. This new field is called a gauge field. The new Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= -i\left(\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi\right) \ , \\ &= -i\left(\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + \bar{\psi}\gamma^\mu A_\mu \psi\right) \ . \end{aligned}$$

The Dirac field is no longer free, there is a coupling between the gauge field A_μ and the spinor. Notice that, only imposing gauge symmetry the interaction term between the spinor and the gauge field was fixed.

The covariant derivative plays a crucial role in gauge theries, we will discuss some of its properties in the non-Abelian case. So far the gauge field does not have dynamics, it is a backgroud field that is coupled to a spinor. In principle, it is difficult to obtaining a gauge invariant term which only depends on A_μ because A_μ is a connection under gauge transformations.

However, in geometry it is constructed a covariant derivative in terms of a connection, which transports local geometric objects, like vectors, from one point to another. The curvature tensor, that transforms covariantly under general coordinate transformations, is constructed by acting with the commutator of two covariant derivatives on a vector. Doing so in the context of gauge theories we will obtain the curvature of the gauge field (also called the field strength) $F_{\mu\nu}$. The commutator acting on the spinor gives

$$\begin{aligned} [D_\mu, D_\nu]\,\psi &= D_\mu\left(D_\nu\psi\right) - D_\nu\left(D_\mu\psi\right) \ , \\ &= -i\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\psi \ . \end{aligned} \tag{2.1.8}$$

We define the field strength of the gauge field as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.1.9}$$

Due to the fact that the left hand side of (2.1.8) transforms covariantly under gauge transformations and in the right hand side ψ transforms covariantly, then we conclude that $F_{\mu\nu}$ is a covariant object under gauge transformations. Indeed it is invariant under gauge transformations, but this is only a feature present in the Abelian $U(1)$ case. Note that in a pure gauge configuration the field strength vanishes indentically because covariant derivatives commute.

Since the field strenght is invariant under gauge transformations we can use it to write a gauge invariant Lagrangian for the gauge field

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ .$$

This is the Lagrangian of the free theory for the gauge field. Its equations of motion are the Maxwell equations in vacuum. The Faraday and the Gauss law for the magnetic field are satisfy by considering $F = dA$ because $dF = 0$.

Then, the full theory that is gauge invariant and preserves the Poincare invariance is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}\left(\gamma^\mu D_\mu - m\right)\psi \ .$$

In summery, to obtain a gauge theory we have to take a Lagrangian that is invariant under a rigid transformation. Then, we gauge this symmetry by imposing that the invariance is local, which requires an extra field called the gauge connection, that allows us to construct a covariant derivative.

2.1.2 Non-Abelian gauge theories

As we explained above, in this section we will consider a set of spinors ψ_i that transform under the action of a non-Abelian Lie group G with generator X_i, $i = 1, \ldots, N_G$ which span a vector space $\mathfrak{G}$, called Lie Algebra of the group. The elements in the connected part of the group can be written as

$$U = e^{i\lambda^i X_i} \ . \tag{2.1.10}$$

Lie groups are endowed with a Lie algebra which is a direct consequence of group axioms: The product of two different elements given in the exponential form (2.1.10) must by another element in the group that can be written in the exponential from as well. One must use the Baker-Campbell-Housdorff Formula which is an infinite series of nested commutators whose first terms are

$$\exp \boldsymbol{X} \exp \boldsymbol{Y} = \exp\left(\boldsymbol{X} + \boldsymbol{Y} + \frac{1}{2}[\boldsymbol{X}, \boldsymbol{Y}] + \frac{1}{12}[\boldsymbol{X}, [\boldsymbol{X}, \boldsymbol{Y}]] - \frac{1}{12}[\boldsymbol{Y}, [\boldsymbol{X}, \boldsymbol{Y}]] - \frac{1}{24}[\boldsymbol{X}, [\boldsymbol{Y}, [\boldsymbol{X}, \boldsymbol{Y}]]] + \cdots\right) ,$$

with $\boldsymbol{X},\ \boldsymbol{Y} \in \mathfrak{G}$ and $[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{XY} - \boldsymbol{YX}$. The product of two elements is in the group when the generators fulfill an algebra

$$[X_i, X_j] = i f_{ijk} X_k \ . \tag{2.1.11}$$

The commutator is the binary operation of internal composition, namely $[\cdot, \cdot] : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, called product of the algebra. The commutator by definition satisfies the axioms of a Lie algebra: Linearity, Antisymmetry and Jacobi identity. Then, the vector space $\mathfrak{G}$ is a Lie algebra.

From the Jacobi identity, $f_{iln} f_{jkl} + f_{jln} f_{kil} + f_{kln} f_{ijl} = 0$, we can define a set of matrices in terms of the structure constants

$$(\mathbb{T}_i)_{jk} \equiv i f_{ijk} \tag{2.1.12}$$

which generate a representation of the algebra

$$[\mathbb{T}_i, \mathbb{T}_j] = i f_{ijk} \mathbb{T}_k \ . \tag{2.1.13}$$

This is called the adjoint representation and the dimension of these matrices is the number of independent generators N_G.

It is convenient to construct a scalar product that maps elements in the Lie algebra to scalars (elements in $\mathbb{R}$ or $\mathbb{C}$). This product must be invariant under the action of the group, which is important to construct an action principle that is gauge invariant. One candidate is the trace of the product of generators in some representation R of the algebra $Tr_R(t_i t_j)$. Now we will argue that it must by proportional to the Kronecker delta δ_{ij}.

Let us consider a linear transformation of the generators $X_i \to X'_i = L_{ij} X_j$,which will induce a transformation on the structure constants

$$\begin{aligned} \left[X'_i, X'_j\right] &= L_{ik} L_{jl} i f_{klm} X_m \ , \\ &= i L_{ik} L_{jl} f_{klm} L^{-1}_{\ mp} X'_p \ , \end{aligned}$$

Then, the transformed structure constant are then $f'_{ijp} = L_{ik} L_{jl} f_{klm} L^{-1}_{\ mp}$ and consequently the new matrices in the adjoint read

$$(\mathbb{T}'_i)_{jp} = L_{ik} L_{jl} \left(\mathbb{T}_k\right)_{lm} L^{-1}_{\ mp} \tag{2.1.14}$$

This means that a linear transformation on X_a induces a linear transformation on the matrices $L_{ik}\mathbb{T}_k$ and at the same time a similarity transformation $L\mathbb{T}_k L^{-1}$. In the trace the linear combination survives and the similarity transformations cancels out:

$$Tr\left(\mathbb{T}'_i \mathbb{T}'_j\right) = L_{ik} L_{jl} Tr\left(\mathbb{T}_k \mathbb{T}_l\right) \tag{2.1.15}$$

Notice that by the cyclicity of the trace the right hand side may be saw as the transformation of a symmetric matrix $M_{kl} = Tr\left(\mathbb{T}_k \mathbb{T}_l\right)$. Therefore, it is always possible to choose a basis such that $Tr\left(\mathbb{T}_k \mathbb{T}_l\right)$ is diagonal. Then,

$$Tr\left(\mathbb{T}'_k \mathbb{T}'_l\right) = \lambda^{(k)} \delta_{kl} \ . \tag{2.1.16}$$

By rescaling the generators we can normalize the eigenvalues to one (or minus one), but we cannot change the sign of the $\lambda' s$ because the left hand side is cuadratic in the rescaling. Hereafter we will consider only compact Lie algebras that have only positive $\lambda' s$, for example $SU(N)$ with special emphasis in $SU(2)$. Then, considering the $\mathbb{T}'_k$ in such a way that $\lambda^{(k)} = 1$ and dropping the primes, we have that the trace in the adjoint representation reads

$$Tr\left(\mathbb{T}_k \mathbb{T}_l\right) = \delta_{kl} \ . \tag{2.1.17}$$

In a different representation R of the Lie algebra with matrices t_i, the number in front of the Kronecker delta will be different and it is fixed because we have fixed the normalization of the structure constants by fixing the trace in the adjoint representation. The last comment on Lie Algebras is that for compact algebras the structure constant f_{ijk} are completely antisymmetric.

As we found out in the previous section, to construct a gauge theory we have to consider matter field as scalars or fermions which may be charged under the gauge group. So, let us consider a set of spinors ψ_I, where $I = 1, \ldots, N_R$, transforming in some representation R of dimension N_R of the Lie algebra of the group G,

$$\psi_I \to U_{IJ} \psi_J \ , \qquad \bar{\psi}_I \to \bar{\psi}_j U^{-1}_{IJ} \ . \tag{2.1.18}$$

We can construct a Lagrangian that is invariant under the global transformation (2.1.18):

$$\mathcal{L} = -i \bar{\psi}_I \left(\gamma^\mu \partial_\mu - m\right) \psi_I \ .$$

Motivated by the Abelian case we can gauge this symmetry, i.e. make the transformation parameter point dependent and as a consequence $U_{IJ}(x)$ depends on the spacetime point. To obtain a Lagrangian that is invariant under local transformations we have to change the partial derivative by a covariant derivative

$$\left(D_\mu\right)_{IJ} = \delta_{IJ} \partial_\mu - i g_{YM} \left(T^R_k\right)_{IJ} A^k{}_\mu \ , \tag{2.1.19}$$

where T^R_i are the matrices in the representation R. Notice that, when the matter field transforms in the adjoint representation, for example a set of scalar fields $\Phi^i \mathbb{T}_i$, then $\left(T^R_i\right)_{IJ} \to \left(\mathbb{T}_i\right)_{jk} = i f_{ijk}$ and the the covariant derivative is nothing but the commutator

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi + g_{YM} f_{ijk} A^k{}_\mu \Phi^j \mathbb{T}_i \ , \\ &= \partial_\mu \Phi - i g_{YM} \left[A_\mu, \Phi\right] \ . \end{aligned} \tag{2.1.20}$$

These types of configurations are interesting to construct solutions that have interesting topological properties, because we can construct a gauge invariant quantity by computing $Tr\left(\Phi F_{\mu\nu}\right)$, for example the 't Hooft-Polyakov monopole [17]-[18] which is similar to the Dirac monopole but without any singularity. In this sense the Higgs fields, represented by Φ in the adjoint representation, "regularizes" the Dirac monopole that arises in Yang-Mills theories.

In order to obtain an object that transforms covariantly, the derivative (2.1.19) must transform in the same way as the spinor

$$D_\mu\psi \to \left(D_\mu\psi\right)' = U\left(x\right)D_\mu\psi \ . \tag{2.1.21}$$

Again, this constraint fixes the transformation rule for the gauge field as

$$\left(T_i^R A^i{}_\mu\right)' = U\left(x\right)T_i^R A_\mu^i U^{-1}\left(x\right) + \frac{i}{g_{YM}}U\left(x\right)\partial_\mu U^{-1}\left(x\right) \ . \tag{2.1.22}$$

The object UdU^{-1} is the left invariant Maurer-Cartan 1-form which is an algebra valued object, as well as the quantity $UT_i^R A_\mu^i U^{-1}$. This is consistent with the fact that $A^{(R)} \equiv A_\mu^i T_i^R \otimes dx^\mu$ is an algebra valued object tensor product with the co-tangent space of the manifold where we are working on.

Now, we will calculate explicitly the field strength for non-Abelian gauge field in an arbitrary representation (then we will go back to the adjoint representation) using differential forms (for a detailed review on the conventions, please check the section 2.2.1).

The covariant derivative acting on the matter field is an algebra valued 1-form $D\psi = d\psi - ig_{YM}A^{(R)}\psi$, then, the commutator of covariant derivatives means acting with the covariant derivative again and due to the fact that they are differential forms the product is the wedge product:

$$\begin{aligned} D\wedge D\psi &= d\wedge D\psi - ig_{YM}A^{(R)}\wedge D\psi \ , \\ &= -ig_{YM}d\left(A^{(R)}\psi\right) - ig_{YM}A^{(R)}\wedge d\psi - g_{YM}^2 A^{(R)}\wedge A^{(R)}\psi \ , \\ &= -ig_{YM}\left(dA^{(R)} - ig_{YM}A^{(R)}\wedge A^{(R)}\right)\psi \ . \end{aligned}$$

Then, we define the field strength as the parenthesis, writing it in the adjoint representation we get

$$\begin{aligned}
F &= dA - ig_{YM}A^i\mathbb{T}_i \wedge A^j\mathbb{T}_j \ , \\
&= dA - \frac{1}{2}ig_{YM}\left(A^i_\mu A^j_\nu - A^i_\nu A^j_\mu\right)\mathbb{T}_i\mathbb{T}_j dx^\mu \wedge dx^\nu \ , \\
&= dA - \frac{1}{2}ig_{YM}A^i_\mu A^j_\nu \left[\mathbb{T}_i, \mathbb{T}_j\right] dx^\mu \wedge dx^\nu \ , \\
&= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM}f_{ijk}A^i_\mu A^j_\nu \mathbb{T}_k\right) dx^\mu \wedge dx^\nu \ .
\end{aligned}$$

Consequently we can write the field strength in components as

$$F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g_{YM}f_{ijk}A^i_\mu A^j_\nu \ , \tag{2.1.23}$$

or in terms of the commutator

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{YM}\left[A_\mu, A_\nu\right] \ . \tag{2.1.24}$$

We can show that, in spite of the fact that A_μ transforms as a connection under gauge transformations, the field strength is covariant under such gauge transformations

$$A \to UAU^{-1} + \frac{i}{g_{YM}}UdU^{-1} \qquad \text{and} \qquad F \to UFU^{-1} \ . \tag{2.1.25}$$

Due to this fact, we can write an action principle as F square, but in this case we have to add a trace because $F_{\mu\nu}F^{\mu\nu}$ is covariant under gauge transformations and the trace of it is invariant due to the cyclicity property:

$$Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \to Tr\left(UF_{\mu\nu}U^{-1}UF^{\mu\nu}U^{-1}\right) = Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{2.1.26}$$

Consequently a simple action principle for non-Abelian gauge field is

$$\mathcal{L}_{YM} = -\frac{1}{4}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \ .$$

whose equations of motion are the Yang-Mills equations

$$D_\mu F^{\mu\nu} \equiv \partial_\mu F^{\mu\nu} - ig_{YM}\left[A_\mu, F^{\mu\nu}\right] = 0 \ . \tag{2.1.27}$$

The theory is intrinsically self interacting, as we can see from the Lagrangian which has cubic a quartic terms in the gauge field.

Introducing gauge fields and then breaking gauge invariance through spontaneous symmetry breaking is the *only method* we know to describe renormalizable theory for massive spin 1 fields observed in Nature and called $W^{\pm}$ and Z^0 bosons.

In the next section we will discuss Einstein gravity in order to consider configurations where the backreaction on the spacetime due to the presence of a non-Abelian gauge field is considered.

2.2 Einstein Gravity

As we saw in the previous section, the structure of spacetime may be described by inertial frames of reference that are related to each other via Lorentz transformations. This point of view is summarized in the special theory of relativity which only assumes that the speed of light is the same for any inertial frame and that the laws of physics are the same for any inertial frame. Consequently, one derives a consistent theory for massive bodies that is invariant under the same symmetry group as the electrodynamics.

Special Relativity also implies that we cannot send information or influence a body with a signal that travels faster than the speed of light, otherwise causality would be violated. However, in real life there are not only inertial frames; gravity for example acts on bodies and change their state of motion. Newton's theory of gravity is not consistent with special relativity because one body can influence another one, instantaneously, which is disproved by special relativity.

Einstein was aware of that and instead of try to fit Newtonian gravity with special relativity, he sought a new theory of gravity motivated by two key observations:

The first one is that all bodies are influenced by gravity and all bodies fall precisely the same way in a gravitational field. This fact is called the *weak equivalence principle*, and is indeed present in Newtonian gravity under the assumption that the gravitational force on a body is proportional to its inertial mass. This principle is a key ingredient for the construction of General Relativity and it has been tested many times using different methods [19]-[21].

As a consequence of this principle the paths of free falling bodies define a preferred set of curves in spacetime just as in special relativity where the inertial frames describe straight lines in spacetime which are a preferred set of curves. Hence, we can guess that gravity has to do with the structure of spacetime itself.

The second observation is that in small regions of the spacetime any local physical experiment in a free falling laboratory is independent of the velocity of the laboratory and its location in spacetime. In other words, the small laboratory is an inertial frame for a short period of time, this principle is called strong equivalence principle. This assumption is nice because allows us to apply the ideas that we learnt in special relativity to systems that are indeed under the action of a gravitational field.

These ideas can be expressed mathematically considering the spacetime, which in special relativity is a vector space $\mathbb{R}^4$ with metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, as a curve pseudo-Riemannian manifold whose curvature is generated by the energy of the matter fields present in the spacetime as well as by the gravitational field it self since the theory is non-linear. Some mathematical aspects related to manifolds and vectors on a manifolds are summarized in the Appendix A1. This formalism also allows us to construct a theory invariant under diffeomorphisms, because obviously physics does not depends on the coordinates that we are using.

General Relativity is a theory for the spacetime manifold $\{M_d, g_{\mu\nu}\}$ that describes the gravitational field as the Riemannian curvature of the manifold. The metric tensor $g_{\mu\nu}(x)$, that depends on spacetime points, is the generalization of the flat metric $\eta_{\mu\nu}$. The metric is the dynamical field in this theory and since General Relativity does not have torsion, as was formulated originally by Einstein, all of the principal geometrical objects (the Christoffel connection and the Riemann tensor) can be computed only knowing the metric tensor. It is also important to stress that, in addition to having the metric, we have to specify the coordinates and the range of them, otherwise, the patch of the manifold is ill defined.

Let us recall some basic definitions that are useful for the physical discussion. As we said, the theory must be invariant under diffeomorphims, and the way to ensures such invariance is by using vectors and tensors that transformas under diffeomorphisms in the correct way. However, the partial derivative of a vector is not a vector anymore. We have at least three ways of fix that problem: the first one is defining a connection that transports the vector from one point to another, the second one is considering the antisymmetric part of the partial derivative acting on the 1-form and the third one is the Lie transport which requires a congruence ξ^μ on the manifold. We will discuss the second solution latter on. The former solution requires a connection which in spacetimes without torsion is the Christoffel connection

$$\Gamma^{\rho}{}_{\sigma\lambda} = \frac{1}{2} g^{\rho\nu} \left(\partial_\sigma g_{\lambda\nu} + \partial_\lambda g_{\sigma\nu} - \partial_\lambda g_{\sigma\lambda} \right). \tag{2.2.1}$$

Hence, we define the covariant derivative acting on vectors as

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma^{\nu}{}_{\mu\lambda} \xi^\lambda \tag{2.2.2}$$

which transforms covariantly under coordinate transformations. As we did in gauge theories, we can compute the commutator of covariant derivatives and obtain the curvature tensor of the manifold

$$\left[\nabla_\rho, \nabla_\sigma \right] \xi^\mu = R^{\mu}{}_{\lambda\rho\sigma} \xi^\lambda. \tag{2.2.3}$$

It measures the lack of commutativity of the covariant derivatives and is called the Riemann tensor given by the following expression

$$\frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} = \partial_{[\rho} \Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{\lambda[\rho} \Gamma^{\lambda}{}_{\sigma]\nu} . \tag{2.2.4}$$

The Riemann tensor transforms covariantly under coordinate transformations, therefore it is a good candidate to construct an action principle. Nevertheless, it depends on second derivatives on the metric tensor which could lead to equations of third order. Despite that, we will show that in some cases the equations of motion remain second order in the metric, but we have to take care of the boundary terms.

The traces of the Rieman tensor are

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \qquad \text{and} \qquad R = R_{\mu\nu} g^{\mu\nu} \tag{2.2.5}$$

that are the Ricci tensor and the Ricci scalar, respectively.

The mathematical formulation of the weak equivalence principle is as follows: a particle moving on the spacetime is described by a curve parametrized by $x^\mu(\lambda)$. If the particle is moving on a spacetime with metric $g_{\mu\nu}(x)$ and is free falling, then the curve is governed by the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{2.2.6}$$

with $\Gamma^\mu{}_{\rho\sigma}$ the Christoffel connection defined as usual. In flat spacetime $\Gamma^\mu{}_{\rho\sigma} = 0$ and the geodesics are straight lines, as we expected.

The field equations for the metric tensor were presented by Einstein in 1915 [22] and are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \ . \tag{2.2.7}$$

The left hand side of (2.2.7) is a combination of the Ricci tensor and the Ricci scalar called the Einstein tensor and is denoted by $G_{\mu\nu}$. The right hand side of (2.2.7) is the energy momentum tensor of the matter fields that are present in the spacetime. These equations are $d(d+1)/2 = 10$ equations for the metric tensor in $d = 4$ because the metric, the Ricci tensor and the energy momentum tensor are symmetric in their indices. The covariant divergence of the Einstein tensor is cero $\nabla_\mu G^{\mu\nu} = 0$ using the Bianchi identity, Consequently the equations (2.2.7) imply that the energy momentum tensor must fulfil

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{2.2.8}$$

By the consistency of the system of equations, the equations (2.2.8) must be some combination of the equation of motion of the matter field because the partial derivative contained in the covariant derivative acts on the the energy momentum tensor, that depends in general on first derivatives of the matter fields. Which give second order equations for the matter fields.

It is worth mentioning that the system of equations is, in general, a non-linear coupled system between the metric and the matter field.

The action principle that gives the equation (2.2.7) is the Einstein-Hilbert action plus the action principle of the matter fields,

$$S\left[g_{\mu\nu}\right]=\frac{1}{16\pi G}\int_{M}d^{4}x\sqrt{-g}R+\frac{1}{8\pi G}\int_{\partial M}d^{3}x\sqrt{-h}K+S_{\text{Matter}}\ . \tag{2.2.9}$$

The energy momentum tensor is defined as $T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{Matter}}}{\delta g^{\mu\nu}}$. As we mentioned above, the action principle of General Relativity has second order derivatives in the metric tensor, however, the equations of motion (2.2.7) are second order. Computing the stationary variation of (2.2.9) we note that there is a boundary term that depends on derivatives of the metric, so in general it is impossible to have a minimum of the action only imposing Dirichlet boundary conditions at the boundary ∂M of the spacetime. Nevertheless, it was shown by York [23], Gibbons and Hawing [24] that supplementing the Eintein-Hilbert action with a boundary term, that is the second term in (2.2.9), we obtain a well define action principle because the variation of the Gibbons-Hawking-York (GHY) term cancels the boundary term that comes from the variation of the Einstien-Hilbert term.

The GHY term depends on the trace of the extrinsic curvature $K=K_{\mu\nu}h^{\mu\nu}$ of the boundary surface ∂M which is a co-dimension one manifold.

In the following we will discuss some aspects and conventions of differential forms that we will use to couple fermions to gravity, which are a crucial ingredient for supersymmetry.

2.2.1 Differential forms

Let us consider a spacetime manifold $\{M_d, g_{\mu\nu}\}$ with a patch whose coordinates are $\{x^\mu\}$. As we discussed in the Appendix A1, we can construct vectors $\xi=\xi^\mu\left(x\right)\partial_\mu$ and 1-forms $A=A_\mu\left(x\right)dx^\mu$ at each point of the spacetime. When we have a notion of metric, given a vector we can compute a 1-form and viceversa, so the following discussion applies to both objects. We mentioned above that the derivative of a vector (or 1-form) is not a tensor but we can consider the antisymmetric part of it and construct a tensor again. The mathematical foundation of this statement is that under a general coordinate transformation $x^\mu=x^\mu\left(\tilde{x}^\nu\right)$ the object $\partial_\mu A_\nu$ transforms as

$$\partial_\mu A_\nu\to\tilde{\partial}_\mu\tilde{A}_\nu=\frac{\partial x^\lambda}{\partial\tilde{x}^\mu}\frac{\partial x^\rho}{\partial\tilde{x}^\nu}\partial_\lambda A_\rho+A_\sigma\frac{\partial^2 x^\sigma}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\ . \tag{2.2.10}$$

The second term in (2.2.10) is terrible because the quantity $\partial_\mu A_\nu dx^\mu \otimes dx^\nu$ transforms under coordinate transformations! In other words, if we change the coordinates the geometrical object changes, which is not consistent for objects that have an intrinsic nature, regardless of the coordinate basis used. We have fixed this problem introducing a covariant derivative with the Christoffel connection, but seeing (2.2.10) we realize that the second term is the second partial derivative of a smooth function. Then, if we compute its antisymmetric part, the last term of (2.2.10) vanishes and the derivative of a 1-form transforms in the correct fashion. So that we will consider the antisymmetric part of the vector space spanned by $\{dx^\mu \otimes dx^\nu\}$ which gives the vector space spanned by the wedge product

$$\left\{dx^\mu \wedge dx^\nu \equiv \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right)\right\} . \tag{2.2.11}$$

The vector space spanned by this basis is called the space of 2-forms and denoted by $\Omega^2(M)$. Consequently, the antisymmetric part of the partial derivative

$$dA \equiv \partial_\mu A_\nu dx^\mu \wedge dx^\nu \tag{2.2.12}$$

does not transform under coordinate transformations. In a general fashion we can define the space of $p-$forms as the vector space $\Omega^p(M)$ spanned by $\{dx^\mu \wedge \cdots \wedge dx^\nu\}$ p-times. The operator d defined in (2.2.12) is called the exterior derivative and takes a $p-$form and returns a $(p+1)-$form. Some properties that we have to keep in mind related to differential forms are the following:

$$\alpha_{[p]} \wedge \beta_{[q]} = (-1)^{pq} \beta_{[q]} \wedge \alpha_{[p]} , \tag{2.2.13}$$

$$d \wedge d\alpha_{[p]} = 0 , \tag{2.2.14}$$

$$d\left(\alpha_{[p]} \wedge \beta_{[q]}\right) = d\alpha_{[p]} \wedge \beta_{[q]} + (-1)^p \alpha_{[p]} \wedge d\beta_{[q]} . \tag{2.2.15}$$

where $\alpha_{[p]} \in \Omega^p(M)$ and $\beta_{[q]} \in \Omega^q(M)$. The first property stablishes the commutation of the differential forms. The second one means that acting two times with the exterior derivative on a $p-$form, which is smooth enough, the result vanishes. The last property is the modified Leibniz rule for differential forms.

The matter fields belong to a representations of the Lorentz group, in particular spinors are representations of spin $\frac{1}{2}$ of the Lorentz group. However, in General

Relativity, a priori, it is not clear how to obtain a Lorentz group to couple fermions in a curve spacetime. We will discuss now how to obtain a local Lorentz group in this context.

We have mentioned that the metric $g_{\mu\nu}(x)$ is a symmetric tensor that depends on the spacetime point. If we consider the metric evaluated at a point, then it is a constant symmetric matrix which always can be diagonalized. To do so, let us consider matrices $e_a{}^{\mu}(x)$, called vielbeine, in such a way that it diagonalizes the metric in each spacetime point, namely

$$e_a{}^{\mu}(x)\, e_b{}^{\nu}(x)\, g_{\mu\nu}(x) = \eta_{ab} \ . \tag{2.2.16}$$

The inverse of $e_a{}^{\mu}$ is denoted by $e^a{}_{\mu}$ that fulfills $e^a{}_{\mu} e_a{}^{\nu} = \delta^{\nu}_{\mu}$. The indices $\{a, b, \dots\}$ are called indices of a non-coordinates basis or flat indices and run in $0, 1, \dots, d-1$. Notice that $e^a{}_{\mu}$ can be thought of as a 1-form $e^a = e^a{}_{\mu} dx^{\mu}$. The equation (2.2.16) can be written as

$$g_{\mu\nu} = \eta_{ab} e^a{}_{\mu} e^b{}_{\nu} \ . \tag{2.2.17}$$

Something special emerges at this point, because transforming the vielbeine with a local Lorentz transformation $\Lambda^a{}_b(x)$ as $e^a{}_{\mu} \to e'^a{}_{\mu} = \Lambda^a{}_b e^b{}_{\mu}$, the right hand side of the equation (2.2.17) goes to $\eta_{ab} e^a{}_{\mu} e^b{}_{\nu} \to \eta_{ab} \Lambda^a{}_c \Lambda^b{}_d e'^c{}_{\mu} e'^d{}_{\nu} = \eta_{cd} e'^c{}_{\mu} e'^d{}_{\nu}$ because the flat metric η_{ab} is invariant under Lorentz transformations. Therefore, the vielbeine are defined up to a local Lorentz transformation. Notice that, the condition $\eta_{ab}\Lambda^a{}_c\Lambda^b{}_d = \eta_{cd}$ implies that $\det\Lambda = \pm 1$, but we need to exclude those transformations that have $\det\Lambda = -1$ because these change the orientation of the spacetime, hence $\Lambda^a{}_b(x) \in SO(d-1, 1)$, which means that the matrix Λ is an element of the special Lorentz group.

Given a vector ξ^{μ} or a 1-form α_{μ} in the coordinate basis we can obtain its component in the vielbeine basis as

$$\xi^a = e^a{}_{\mu}\xi^{\mu} \qquad \text{and} \qquad \alpha_a = e_a{}^{\mu}\alpha_{\mu} \ . \tag{2.2.18}$$

These objects transform naturally under local Lorentz transformations instead of the whole[1] $GL(d)$. In the previous section we discussed how to define the parallel transport of vectors that transforms under $GL(d)$. We did so by introducing the Christoffel connection and defining a covariant derivative (2.2.2) in terms of it. But now we have a different set of object that transforms under local Lorentz transformations and it is clear that the derivative of a Lorentz vector does not transform as a Lorentz tensor:

$$d\xi^a \to \Lambda^a{}_b d\xi^b + d\Lambda^a{}_b \xi^b \neq \Lambda^a{}_b d\xi^b \tag{2.2.19}$$

under local Lorentz transformations. Thus we need to introduce another connection, called the 1-form spin connection $\omega_\mu{}^a{}_b dx^\mu$, and the Lorentz covariant derivative is defined by

$$\mathcal{D}\xi^a = d\xi^a + \omega^a{}_b \xi^b \ . \tag{2.2.20}$$

Imposing that $\mathcal{D}\xi^a$ transforms as a Lorentz vector, the spin connection inherits the transformation

$$\omega^a{}_b \to \Lambda^a{}_c \omega^c{}_d \Lambda_b{}^d + \Lambda^a{}_c d\Lambda_b{}^c \ . \tag{2.2.21}$$

It transforms as a connection. Note the similarities with the transformation of the gauge field in a gauge theory with compact gauge group (2.1.25), so we can say that (2.2.21) is a gauge transformation for the group $SO(d-1,1)$. But this is a slightly different situation compared with the last section because $SO(d-1,1)$ is no compact.

So far we have two notions of parallel transport for a vector depending on whether it is a Lorentz vector or a vector under general coordinate transformation. These two notions must be the same because we are transporting the same vector and $e^a{}_\mu$ provides a map between these objects. Mathematically this means the following

$$\partial_\mu \xi^\lambda + \Gamma^\lambda{}_{\mu\nu} \xi^\nu = e_a{}^\lambda \left(\partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b \right) \ . \tag{2.2.22}$$

[1]That is another manner of saying that they are covariant under general coordinate transformations because

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} \in GL(d)$$

at a fixed point.

These equations are satisfied by any vector ξ when the condition

$$\partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda = 0 \tag{2.2.23}$$

is fulfill. We see that this condition allows us to translate the information from the connection Γ to the spin connection ω and viceversa. Notice that the connection Γ could be anything, as long as it transforms as a connection. Indeed, in the following we will see that the torsion is an object that exist only when the notion of parallel transport is different than the notion of transport given by the Christoffel connection (which is symmetric in the lower indices) and is the unique, torsionfree connection which is compatible with the metric

$$\nabla_\mu g_{\rho\sigma} = 0 \ . \tag{2.2.24}$$

It is worth mentioning that the metric compatibility condition for the Lorentz covariant derivative (i.e. $\mathcal{D}\eta_{ab} = 0$) implies that the spin connection is antisymmetric in their last indices

$$\omega_\mu{}^{ab} = -\omega_\mu{}^{ba} \tag{2.2.25}$$

Once again, computing the commutator of Lorentz covariant derivatives one can finds the curvature 2-form:

$$\mathcal{D} \wedge \mathcal{D}\xi^a = \mathcal{R}^a{}_b \xi^b \tag{2.2.26}$$

where the 2-form is given by

$$\mathcal{R}^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \tag{2.2.27}$$

In the absence of torsion, this tensor codifies the same information as the Riemann tensor, but in general they are different. In components the curvature 2-form (2.2.27) reads

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_{\nu c}{}^b - \omega_\nu{}^{ac} \omega_{\mu c}{}^b \ . \tag{2.2.28}$$

Another important object that characterizes a manifold comes from the Lorentz derivative acting on the vielbeine

$$T^a \equiv de^a + \omega^a{}_b e^b \tag{2.2.29}$$

which is called the torsion and if we write it in the coordinate basis we obtain that

$$T^a = \frac{1}{2}\left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) e^a{}_\lambda dx^\mu \wedge dx^\nu \ . \tag{2.2.30}$$

Here is the point that we wanted to stress: when we consider a manifold without torsion, the connection is symmetric in their lower indices and imposing also the metric compatibility (2.2.24), it implies that $\Gamma =$ Christoffel connection, while if the manifold has torsion the connection is different. Hereon $T^a = 0$, which implies that the right hand side of (2.2.29) vanishes:

$$\partial_{[\mu} e_{\nu]}^{\ a} + \omega_{[\mu|}{}^a{}_b e_{|\nu]}^{\ b} = 0 \ . \tag{2.2.31}$$

Usually, we have explicitly the metric tensor from which one can obtains a set of vielbeine. Then, the equation above can be used to write the torsionfree spin connection in terms of the vielbein (in the same way as the Christoffel connection is defined in terms of the metric). Defining the anholonomity coefficients

$$\Omega^c{}_{ab} \equiv e^\mu{}_a e^\nu{}_b \left(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c\right) \ , \tag{2.2.32}$$

the equation (2.2.31) can be written as the following

$$\Omega_{acb} + \omega_{cab} - \omega_{bac} = 0 \ . \tag{2.2.33}$$

The trick to solve the above equation for ω is considering the following indices permutation $(a \to b,\ b \to c,\ c \to a)$ two times, then we have three equations: (2.2.33) and

$$\Omega_{bac} + \omega_{abc} - \omega_{cba} = 0 \ , \tag{2.2.34}$$

$$\Omega_{cba} + \omega_{bca} - \omega_{acb} = 0 \ . \tag{2.2.35}$$

Now, we have to add these equation multiplying (2.2.35) by -1, then using the antisymmetry of Ω and ω in their last two indices we have the following result

$$\omega^c{}_{ab} = \frac{1}{2}\left(\Omega^c{}_{ba} - \Omega_a{}^c{}_b - \Omega_{ba}{}^c\right) \ . \tag{2.2.36}$$

The spin connection is the building block to write a theory with spinors in a curved spacetime. The covariant derivative for a fermion ψ is build up with it in the following way

$$\mathcal{D}\psi = d\psi + \frac{1}{4}\gamma^{ab}\omega_{ab}\psi \ . \tag{2.2.37}$$

We will use it in the discussion of supergravity theories where spinors are coupled to gravity and other matter fields.

2.3 Supersymmetry

Symmetry in physics is very important because it allows us to solve difficult problems and encodes deep secrets about nature. However, if we impose too much symmetry the problem may turn out to be “trivial”. For example in the Ginzburg-Landau model [25] which is a theory for a charged complex scalar field and a Maxwell field, if we impose that the scalar field is constant at infinite, then, the winding number (that is a topological invariant number) vanishes and the configuration turns out to be topologically trivial.

In particle physics the situation is different but in spirit it is the same. Fundamental particles are related to the irreducible representations of the Poincaré algebra, so we can ask about the biggest symmetry, including internal symmetries, that we can have in order to satisfy sensible assumtions. Coleman-Mandula’s theorem stated that the only possible Lie groups that can be symmetries of a relativistic theory for particles are isomorphic to the direct product of the Poincaré group and an internal symmetry group, i.e. they do not have non-trivial mixing at the algebra level.

One of the hypotesis of the theorem is that the Lie Algebra of the internal symmetry is described by commutators. In the seventies it was realized that one can avoid the theorem by taking Lie algebras based on commutators and anticommutators, called graded Lie algebras.

Thus supersymmetry (SUSY) was born, as a way of mixing internal (super)-

symmetry with Poincaré invariance. The supersymmetry generators must transform as spin 1/2 spinors under Lorentz transformations for the consistency of the theory. Consequently, under supersymmetry transformations bosons turn into fermions and fermions turn into bosons, implying that an irreducible representation of the susy algebra will correspond to a set of fields, forming the so-called *supermultiplet*. The Pauli-Lubanski vector is no longer a Casimir of the SUSY algebra, therefore different spins are allocated on the same supermultiplet. Supermultiplets always contain the same number of fermionic and bosonic degree of freedom[2] (at least it must be true on-shell) and the particles in the same multiplets must have the same mass, since $P_\mu P^\mu$ remains a Casimir of super-Poincaré.

Supersymmetry looks nice from the theoretical point of view, but the constraint on the mass for a given supermultiplet is too strong. If supersymmetry is realized in Nature, it must appear as a broken symmetry. In spite of this, there are theoretical reasons to study supersymmetry: It is the most natural extension (from theoretical human point of view) of the quantum field theory framework. It has an improved ultraviolet behaviour due to the fermionic and bosonic loops cancelation. Supersymmetric theories are in general simpler than non-SUSY ones because the symmetry constraints implies a more restrictive set of theories and they can be used as toy models that could capture features of models that describe the real word but are much more difficult. Finally supersymmetry naturally emerges from consistency requirements of string theory.

In this section we want to show explicit calculations that have common aspects with other more involved supersymmetric theories which we will use in the next chapters. These computations will give us some insight in supersymmetry in order to consider a more complicate supergravity theory such as the Freedman-Schwarz model where we will find analytic solutions in the bosonic sector of it (with fermions turned off), which may preserve some supersymmetries

This section is organized as follows: firstly we shall consider the Wess-Zumino model and we will integrate the Lagrangian in such a way that the supersymmetry transformations are a symmetry of the action. Then, we will show in certain detail the invariance of a vector multiplet under supersymmetry transformations. Finally, we will study the connection between spin 3/2 field, gravity and local

[2]The last statement may be proved rigorously [26].

supersymmetry.

2.3.1 Integrating the Wess-Zumino Lagrangian

As we mentioned above, supersymmetry theories mix fermions with bosons through the so-called supersymmetry transformation whose parameter is a spinor. In the seventies Wess and Zumino [29] suggested that the idea developed in string theory [27]-[28], where the supersymmetry was present, could be naturally extended to quantum field theory in four spacetime dimensions.

The simplest model that they constructed contains a single Majorana spinor λ, a pair of real scalar and pseudo scalar bosonic fields A and B, and a pair of real scalar and pseudo scalar F and G. Counting the degrees of freedom we note that off-shell we have $\#(\lambda) \equiv 4 = 1+1+1+1 \equiv \#(A)+\#(B)+\#(F)+\#(G)$, then they match. Nevertheless, a posteriori we note that F and G do not have propagating degrees of freedom because they are fixed by the equations of motion so the on-shell (denoted by $\approx$) counting is $\#(\lambda) \approx 2 = 1+1+0+0 \approx \#(A)+\#(B)+\#(F)+\#(G)$. It is worth empathizing that when we replace the on-shell values in the Lagrangian the commutator of the supersymmetry transformation closes on-shell, but off course the action remains invariant without using the field equations.

We want to construct the Wess-Zumino model only by imposing that the Lagrangian is invariant under supersymmetry transformations in Minkowski spacetime. Let us consider one Majorana spinor and two real scalar and pseudo-scalar A and B. We consider only for this chapter the Majorana basis for the γ matrices

$$
\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \; \gamma^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \\
\gamma^2 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; \gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \\
\gamma^5 &= i\gamma^1\gamma^2\gamma^3\gamma^0 \, .
\end{aligned}
$$

It is clear that the matrices are real and $(\gamma^0)^T = -\gamma^0$, $(\gamma^i)^T = \gamma^i$ and $(\gamma_5)^T = -\gamma_5 = -(\gamma_5)^\dagger$. γ_5 anticommutes with any gamma matrix $\{\gamma_5, \gamma^\mu\} = 0$.

The ansatz for the infinitesimal supersymmetry transformation is

$$\delta A \equiv \delta_0 A = \bar{\epsilon}\lambda \ , \tag{2.3.1}$$

$$\delta B \equiv \delta_0 B = -i\bar{\epsilon}\gamma_5\lambda \ , \tag{2.3.2}$$

$$\delta\lambda \equiv \delta_0\lambda + \delta_1\lambda \ , \tag{2.3.3}$$

here $\bar{\lambda} \equiv \lambda i\gamma^0$ and the terms in the spinor are

$$\delta_0\lambda = \partial\!\!\!/ \left(A - i\gamma_5 B\right)\epsilon \quad \text{and} \quad \delta_1\lambda = W_1(A,B)\,\epsilon - i\gamma_5 W_2(A,B)\,\epsilon \ . \tag{2.3.4}$$

$W_1(A,B)$ and $W_2(A,B)$ are arbitraty function of the scalars and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Let us consider the ansatz for the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \tag{2.3.5}$$

with

$$\mathcal{L}_0 = -\frac{1}{2}(\partial A)^2 - \frac{1}{2}(\partial B)^2 - \frac{1}{2}\bar{\lambda}\ \partial\!\!\!/\lambda \ , \tag{2.3.6}$$

$$\mathcal{L}_{int} = -V(A,B) - \frac{1}{2}U_1(A,B)\,\bar{\lambda}\lambda + \frac{i}{2}U_2(A,B)\,\bar{\lambda}\gamma_5\lambda \ . \tag{2.3.7}$$

The idea is finding the functions W_1, W_2, V, U_1 and U_2 by demanding $\delta\mathcal{L}$ =boundary term. The variation of the kinetic term of the spinor reads

$$\delta\left(\bar{\lambda}\ \partial\!\!\!/\lambda\right) = \partial_\mu\left(\delta\bar{\lambda}\gamma^\mu\lambda\right) - \partial_\mu\delta\bar{\lambda}\gamma^\mu\lambda + \bar{\lambda}\ \partial\!\!\!/\delta\lambda \tag{2.3.8}$$

The second term in the right hand side is $\partial_\mu\delta\lambda^T i\gamma^0\gamma^\mu\lambda$ and because $(\gamma^0\gamma^\mu)^T = (\gamma^\mu)^T(\gamma^0)^T = -\gamma^\mu\gamma^0 = \gamma^0\gamma^\mu$ we can interchange the spinor $\partial_\mu\delta\lambda^T$ with λ up to a minus sign as they are Grassmann numbers. Thus,

$$\delta\left(\bar{\lambda}\ \partial\!\!\!/\lambda\right) = 2\bar{\lambda}\ \partial\!\!\!/\delta\lambda + \partial \ . \tag{2.3.9}$$

Where ∂ stands for boundary terms. We will show that the free lagrangian $\mathcal{L}_0$ is invariant under the infinitesimal transformation δ_0. The variation gives

$$\begin{aligned} \delta_0\mathcal{L}_0 &= -\partial_\mu\delta_0 A\partial^\mu A - \partial_\mu\delta_0 B\partial^\mu B - \bar{\lambda}\ \partial\!\!\!/\delta_0\lambda + \partial\ , & (2.3.10)\\ &= -\bar{\epsilon}\partial_\mu\lambda\partial^\mu A + i\bar{\epsilon}\gamma_5\partial_\mu\lambda\partial^\mu B - \bar{\lambda}\gamma^\sigma\gamma^\rho\partial_\sigma\partial_\rho\left(A\epsilon + iB\gamma_5\epsilon\right) + \partial & (2.3.11)\\ &= \bar{\epsilon}\lambda\partial_\mu\partial^\mu A - i\bar{\epsilon}\gamma_5\lambda\partial_\mu\partial^\mu B - \bar{\lambda}\epsilon\partial_\mu\partial^\mu A + i\bar{\lambda}\gamma_5\epsilon\partial_\mu\partial^\mu B + \partial\ . & (2.3.12)\end{aligned}$$

The first term cancels with the third and the second one cancels with the last one because γ^0 and $\gamma^0\gamma_5$ are antisymmetric matrices. Consequently $\delta_0\mathcal{L}_0$ is a boundary term. Let us now compute the variation of the whole Lagrangian

$$\delta\mathcal{L} = \delta_0\mathcal{L}_{int} + \delta_1\mathcal{L}_{int} + \delta_1\mathcal{L}_0\ . \qquad (2.3.13)$$

Using the same tricks as before we compute the terms in the right hand side of (2.3.13):

$$\begin{aligned} \delta_0\mathcal{L}_{int} &= -\dot{V}\bar{\epsilon}\lambda + i\bar{\epsilon}\gamma_5\lambda V' + \bar{\epsilon}\ \partial\!\!\!/ A\lambda U_1 - i\bar{\epsilon}\gamma_5\ \partial\!\!\!/ B\lambda U_1 - iU_2\bar{\epsilon}\ \partial\!\!\!/ A\gamma_5\lambda & (2.3.14)\\ &\quad -U_2\bar{\lambda}\ \partial\!\!\!/ B\epsilon - \frac{1}{2}\partial_A U_1\bar{\epsilon}\lambda\bar{\lambda}\lambda + i\frac{1}{2}\partial_B U_1\left(\bar{\epsilon}\gamma_5\lambda\right)\bar{\lambda}\lambda \\ &\quad +\frac{i}{2}\partial_A U_2\left(\bar{\epsilon}\lambda\right)\bar{\lambda}\gamma_5\lambda + \frac{1}{2}\partial_B U_2\left(\bar{\epsilon}\gamma_5\lambda\right)\bar{\lambda}\gamma_5\lambda\ , \\ \delta_1\mathcal{L}_{int} &= \bar{\epsilon}\lambda\left(-U_1W_1 + U_2W_2\right) + i\bar{\epsilon}\gamma_5\lambda\left(U_1W_2 + U_2W_1\right)\ , & (2.3.15)\\ \delta_1\mathcal{L}_0 &= \bar{\epsilon}\ \partial\!\!\!/ A\lambda\dot{W}_1 + \bar{\epsilon}\ \partial\!\!\!/ B\lambda W_1' - i\bar{\epsilon}\gamma_5\ \partial\!\!\!/ A\lambda\dot{W}_2 & (2.3.16)\\ &\quad -i\bar{\epsilon}\gamma_5\ \partial\!\!\!/ B\lambda W_2' + \partial\ . & (2.3.17)\end{aligned}$$

We have denoted $(\dot{\ })\equiv\partial_A$ and $(\ )'\equiv\partial_B$. Then, the total variation reads

$$\begin{aligned} \delta\mathcal{L} &= \bar{\epsilon}\lambda\left(-\dot{V} - U_1W_1 + U_2W_2\right) + i\bar{\epsilon}\gamma_5\lambda\left(V' + U_1W_2 + U_2W_1\right) & (2.3.18)\\ &\quad +\bar{\epsilon}\ \partial\!\!\!/ A\lambda\left(\dot{W}_1 + U_1\right) - i\bar{\epsilon}\ \partial\!\!\!/ A\gamma_5\lambda\left(U_2 - \dot{W}_2\right) \\ &\quad +i\bar{\epsilon}\gamma_5\ \partial\!\!\!/ B\lambda\left(-W_2' - U_1\right) + \bar{\epsilon}\ \partial\!\!\!/ B\lambda\left(W_1' + U_2\right) \\ &\quad +\mathcal{O}\left(\lambda^2\right) + \partial\ .\end{aligned}$$

Where we have neglected cubic terms in the fermions. However the cubic terms in fermions does not add new conditions on the function, because they provide that $W_1(A,B) + iW_2(A,B)$ is an holomorphic function (see e.g. [31]), which we will derive later on. The independent equations for any A and B that we have to solve are

$$\begin{aligned} -\dot{V} - U_1 W_1 + U_2 W_2 &= 0 \ , && (2.3.19) \\ V' + U_1 W_2 + U_2 W_1 &= 0 \ , && (2.3.20) \\ \dot{W}_1 + U_1 &= 0 \ , && (2.3.21) \\ U_2 - \dot{W}_2 &= 0 \ , && (2.3.22) \\ W_2' + U_1 &= 0 \ , && (2.3.23) \\ W_1' + U_2 &= 0 \ . && (2.3.24) \end{aligned}$$

The equations coming from the substraction of (2.3.21) and (2.3.23) and (2.3.22) minus (2.3.24) are given by

$$\begin{aligned} \dot{W}_1 &= W_2' \ , && (2.3.25) \\ W_1' &= -\dot{W}_2 \ . && (2.3.26) \end{aligned}$$

Which are the Cauchy-Riemann equations for a holomorphic complex valued function $W = W_1(A,B) + iW_2(A,B)$. The equations for V are solved by

$$V = \frac{1}{2}\left(W_1^2 + W_2^2\right) \ . \qquad (2.3.27)$$

Replacing these solutions into the Lagrangian, we obtain that

$$\begin{aligned} \mathcal{L} \ = \ & -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{1}{2}\bar{\lambda} \ \partial\!\!\!/ \lambda && (2.3.28) \\ & -\frac{1}{2}\left(W_1^2 + W_2^2\right) + \frac{1}{2}\partial_A W_1 \bar{\lambda}\lambda - \frac{i}{2}\partial_B W_1 \bar{\lambda}\gamma_5 \lambda \end{aligned}$$

is invariant under susy transformations at any order in fermions, as we mentioned. It was unexpected that the interactions are controlled by a holomorphic function W. This feature makes supersymmetry so powerful in quantum field theory in four dimensions. The holomorphic function W is related to the superpotential $\mathcal{W}$, up to an irrelevant constant, as

$$\frac{d\mathcal{W}}{dz} = W \qquad (2.3.29)$$

which plays an important role in supersymmetric theories in four dimensions.

Notice that there is at least one choice of the function W such that we recover the Wess-Zumino Lagrangian [29] (for details see e.g. [30]), that is given by

$$\begin{aligned} W_1(A,B) &= \left(-A^2+B^2\right)g - Am \ , \\ W_2(A,B) &= 2ABg + Bm \ . \end{aligned}$$

The Lagrangian in this case reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\bar{\lambda}\ \partial\!\!\!/\lambda - \frac{1}{2}m\bar{\lambda}\lambda \\ &\quad -\frac{1}{2}m^2\left(A^2+B^2\right) - gmA\left(A^2+B^2\right) - \frac{1}{2}g^2\left(A^2+B^2\right)^2 \\ &\quad -g\bar{\lambda}\left(A+i\gamma_5 B\right)\lambda \ . \end{aligned} \tag{2.3.30}$$

This Lagrangian exhibits relations not only between scalar and fermions masses, but also between Yukawa interactions and scalar self couplings. This example shows that supersymmetry fixes the terms presents in the Lagrangian, up the holomorphic function W.

2.3.2 Vector multiplet invariance

In this thesis we will consider the Freedman-Schwarz supergravity model that contains non-Abelian gauge fields. We will not show the explicit invariance of the action principle of the theory under local supersymmetry transformations. Instead of that, we will consider a simpler supersymmetric model that is build up of one Abelian gauge field B_μ, a Majorana spinor λ and one auxiliary scalar field G. The Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\lambda}\ \partial\!\!\!/\lambda + \frac{1}{2}G^2 \tag{2.3.31}$$

is invariant under the following supersymmetric transformations

$$\delta B_\mu = -\bar{\epsilon}\gamma_\mu\lambda \ , \tag{2.3.32}$$

$$\delta\lambda = \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\epsilon + i\gamma_5 G\epsilon \ , \tag{2.3.33}$$

$$\delta G = i\bar{\epsilon}\gamma_5\ \partial\!\!\!/\lambda \ . \tag{2.3.34}$$

The gamma matrices with more than one index are defined by

$$\gamma^{\mu_1\dots\mu_n} = \gamma^{[\mu_1}\dots\gamma^{\mu_n]} \equiv \frac{1}{n!}\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}\gamma^{\nu_1}\dots\gamma^{\nu_n} \ . \tag{2.3.35}$$

It is possible to show that

$$\delta\bar{\lambda} = -\frac{1}{2}\bar{\epsilon}\gamma^{\mu\nu}F_{\mu\nu} + iG\bar{\epsilon}\gamma_5 \ . \tag{2.3.36}$$

Before starting, we will consider some useful identities:

$$\gamma^{\sigma\mu\nu} = \gamma^{\sigma}\gamma^{\mu\nu} + 2\gamma^{[\mu}\eta^{\nu]\sigma} \ , \tag{2.3.37}$$

$$[\gamma^{\sigma}, \gamma^{\mu\nu}] = 4\eta^{\sigma[\mu}\gamma^{\nu]} \ . \tag{2.3.38}$$

To prove (2.3.37), let us consider the definition[3] of $\gamma^{\sigma\mu\nu}$, then, using the Clifford algebra in order to put the gamma matrix with index σ on the left-hand side for each term. After rearranging terms, we will obtain the right hand side of (2.3.37). While the commutator (2.3.38) is obtained by expanding the commutator, then, using the Clifford algebra to move the gamma matrix with index σ from the left to the right in order to cancel the second term which comes from the commutator.

Let us compute the variation of the Lagrangian under the supersymmetry transformations (2.3.31)

$$\delta\mathcal{L} = -2\partial_\mu\delta B_\nu\partial^{[\mu}B^{\nu]} - \frac{1}{2}\delta\bar{\lambda}\ \not{\partial}\lambda - \frac{1}{2}\bar{\lambda}\ \not{\partial}\delta\lambda + \delta G \ , \tag{2.3.39}$$

$$= 2\bar{\epsilon}\gamma_\nu\partial_\mu\lambda\partial^{[\mu}B^{\nu]} + \frac{1}{2}\bar{\epsilon}\gamma^{\mu\nu}\gamma^{\sigma}\partial_{[\mu}B_{\nu]}\partial_\sigma\lambda - \frac{1}{2}iG\bar{\epsilon}\gamma_5\gamma^{\sigma}\partial_\sigma\lambda \tag{2.3.40}$$

$$-\frac{1}{2}\bar{\lambda}\gamma^{\sigma}\gamma^{\mu\nu}\epsilon\partial_\sigma\partial_{[\mu}B_{\nu]} - i\frac{1}{2}\bar{\lambda}\gamma^{\sigma}\gamma_5\epsilon\partial_\sigma G + Gi\bar{\epsilon}\gamma_5\gamma^{\sigma}\partial_\sigma\lambda + \partial \ .$$

Using the identity (2.3.38) we can show that

$$-\frac{1}{2}\bar{\lambda}\gamma^{\sigma}\gamma^{\mu\nu}\epsilon\partial_\sigma\partial_{[\mu}B_{\nu]} = -\bar{\lambda}\gamma^{\nu}\epsilon\partial^{\mu}\partial_{[\mu}B_{\nu]} \ , \tag{2.3.41}$$

[3]For $n = 3$ the identity (2.3.35) reads

$$\gamma^{\sigma\mu\nu} = \gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} + \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma} + \gamma^{\nu}\gamma^{\sigma}\gamma^{\mu} - \gamma^{\mu}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\sigma}\gamma^{\nu}\gamma^{\mu} - \gamma^{\nu}\gamma^{\mu}\gamma^{\sigma} \ .$$

and using (2.3.38) combined with (2.3.38) a similar result as the above is obtained

$$\frac{1}{2}\bar{\epsilon}\gamma^{\mu\nu}\gamma^{\sigma}\partial_{[\mu}B_{\nu]}\partial_{\sigma}\lambda = -\bar{\epsilon}\gamma^{\mu}\lambda\partial^{\nu}\partial_{[\mu}B_{\nu]} \ + \partial \ . \tag{2.3.42}$$

Plugging back in the variation (2.3.40) we obtain the following equation

$$\begin{aligned} \delta\mathcal{L} &= 2\bar{\epsilon}\gamma_{\nu}\partial_{\mu}\lambda\partial^{[\mu}B^{\nu]} - \bar{\epsilon}\gamma^{\mu}\lambda\partial^{\nu}\partial_{[\mu}B_{\nu]} - \frac{1}{2}iG\bar{\epsilon}\gamma_5\gamma^{\sigma}\partial_{\sigma}\lambda \\ &\quad -\bar{\lambda}\gamma^{\nu}\epsilon\partial^{\mu}\partial_{[\mu}B_{\nu]} - i\frac{1}{2}\bar{\lambda}\gamma^{\sigma}\gamma_5\epsilon\partial_{\sigma}G + Gi\bar{\epsilon}\gamma_5\gamma^{\sigma}\partial_{\sigma}\lambda + \partial \ , \\ &= \left(-\bar{\epsilon}\gamma^{\nu}\lambda - \bar{\lambda}\gamma^{\nu}\epsilon\right)\partial^{\mu}\partial_{[\mu}B_{\nu]} \\ &\quad + i\frac{1}{2}\partial_{\sigma}G\left(-\bar{\epsilon}\gamma_5\gamma^{\sigma}\lambda + \bar{\lambda}\gamma_5\gamma^{\sigma}\epsilon\right) + \partial \ . \end{aligned} \tag{2.3.43}$$

Notice that $\gamma^0\gamma_\nu$ is a symmetric matrix, then, $-\bar{\epsilon}\gamma^{\nu}\lambda = \bar{\lambda}\gamma^{\nu}\epsilon$ which cancels the first term in (2.3.43). While the matrix $\gamma^0\gamma_5\gamma^\sigma$ is antisymmetric then, $-\bar{\epsilon}\gamma_5\gamma^{\sigma}\lambda = -\bar{\lambda}\gamma_5\gamma^{\sigma}\epsilon$. Consequently, the Lagrangian is quasi-invariant under supersymmetry transformations.

2.3.3 Rarita-Schwinger field and supergravity

So far we have discussed supersymmetry whose spinorial parameter does not depend on the spacetime point. But as we saw in the Maxwell and Yang-Mills sections, promoting a global symmetry to local symmetry requires to add new fields which describe interactions between matter fields. A natural question in supersymmetry relies on who to obtain the local version of it. As we will see, the answer invokes gravity and spin 3/2 fields. In this section we will discuss aspects on the spin 3/2 field and its connection with gravity.

In supergravity the spinorial parameter of the supersymmetry transformations is a general function in spacetime $\epsilon\left(x\right)$ and the gauge field associated to it is the spin 3/2 vector-spinor $\Psi_{\mu}\left(x\right)$, called Rarita-Schwinger field. In this section we will consider the free Rarita-Schwinger which in flat spacetime enjoys the following gauge transformation:

$$\Psi_{\mu}\left(x\right) \rightarrow \Psi_{\mu}\left(x\right) + \partial_{\mu}\epsilon\left(x\right) \ . \tag{2.3.44}$$

In a general curved spacetime $\{M_4, g_{\mu\nu}\}$, the action principle for the Rarita-

Schwinger field in 4 spacetime dimensions is given by

$$S_\Psi = -\int d^4x\sqrt{-g}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho \ , \tag{2.3.45}$$

where ∇ stands for the full covariant derivative including local Lorentz covariant derivative (2.2.37), and spacetime covariant derivative:

$$\nabla_\nu\Psi_\rho = \partial_\nu\Psi_\rho + \frac{1}{4}\gamma^{ab}\omega_{ab}\Psi_\rho - \Gamma^\lambda{}_{\nu\rho}\Psi_\lambda \ . \tag{2.3.46}$$

In a torsionfree spacetime, from the point of view of the action principle the last term in (2.3.46) is irrelevant due to the presence of $\gamma^{\mu\nu\rho}$. Notice that in curved spacetime the derivative of a 1/2 spinor is a vector when we consider the Lorentz covariant derivative (2.2.37). Therefore, in curved spacetime (2.3.44) must be promoted to

$$\Psi_\mu(x) \rightarrow \Psi_\mu(x) + \mathcal{D}_\mu\epsilon(x) \ . \tag{2.3.47}$$

Let us check if $\delta\Psi_\mu = \mathcal{D}_\mu\epsilon(x)$ is a symmetry of the Rarita-Schwinger action[4]

$$\begin{aligned} \delta S_\Psi &= -\int d^4x\sqrt{-g}\left(\delta\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\delta\Psi_\rho\right) \ , \\ &= -\int d^4x\sqrt{-g}\left(\nabla_\nu\bar{\Psi}_\mu\gamma^{\rho\nu\mu}\delta\Psi_\rho + \nabla_\nu\left(\bar{\Psi}_\mu\gamma^{\rho\nu\mu}\right)\delta\Psi_\rho\right) + \partial \ . \end{aligned}$$

using the vielbeine postulate (2.2.23) which in short tell us $\nabla_\mu e^\rho{}_a = 0$ which implies that $\nabla_\nu\gamma^{\rho\nu\mu} = 0$. Then, replacing the variation in the equation above and integrating by parts, we have that

$$\begin{aligned} \delta S_\Psi &= 2\int d^4x\sqrt{-g}\nabla_\rho\nabla_\nu\bar{\Psi}_\mu\gamma^{\rho\nu\mu}\epsilon \ + \partial \ , \\ &= 2\int d^4x\sqrt{-g}\bar{\epsilon}\gamma^{\rho\nu\mu}\nabla_\rho\nabla_\nu\Psi_\mu \ + \partial \ . \end{aligned} \tag{2.3.48}$$

The commutator of covariant derivatives acting on the Rarita-Schwinger field is proportional to the Riemann tensor. Let us expand the covariant derivatives in

[4]The gamma matrices in $d = 4$ used in this section satisfy the following identity

$$\bar{\chi}\gamma_{\mu_1\ldots\mu_n}\lambda = t_n\bar{\lambda}\gamma_{\mu_1\ldots\mu_n}\chi$$

with λ and χ spinors, $t_0 = t_3 = +1$ and $t_1 = t_2 = -1$.

the integrand above

$$\begin{aligned}\gamma^{\rho\nu\mu}\nabla_\rho\nabla_\nu\Psi_\mu &= \gamma^{\rho\nu\mu}\mathcal{D}_\rho\mathcal{D}_\nu\Psi_\mu\ , \\ &= \gamma^{\rho\nu\mu}\left(\partial_{[\rho}\mathcal{D}_{\nu]}\Psi_\mu + \frac{1}{4}\gamma^{ab}\omega_{[\rho|ab}\mathcal{D}_{|\nu]}\Psi_\mu\right)\ , \\ &= \frac{1}{4}\gamma^{\rho\sigma\mu}\left(\partial_{[\rho}\omega_{\sigma]}{}^{ab}\gamma_{ab} + \frac{1}{4}\omega_{[\rho}{}^{ab}\omega_{\sigma]}{}^{cd}\gamma_{ab}\gamma_{cd}\right)\Psi_\mu\ . \end{aligned} \tag{2.3.49}$$

Only by using the Clifford algebra we can show that the gamma matrices of rank two satisfy the Lorentz algebra

$$[\gamma_{ab},\gamma_{cd}] = -2\left(\eta_{bd}\gamma_{ac} + \eta_{ac}\gamma_{bd} - \eta_{ad}\gamma_{bc} - \eta_{bc}\gamma_{ad}\right)\ . \tag{2.3.50}$$

Thus, the ω^2 term in (2.3.49) can be written as

$$\begin{aligned}\omega_{[\rho}{}^{ab}\omega_{\sigma]}{}^{cd}\gamma_{ab}\gamma_{cd} &= \frac{1}{2}\omega_\rho{}^{ab}\omega_\sigma{}^{cd}[\gamma_{ab},\gamma_{cd}]\ , \\ &= -\omega_\rho{}^{ab}\omega_\sigma{}^{cd}\left(\eta_{bd}\gamma_{ac} + \eta_{ac}\gamma_{bd} - \eta_{ad}\gamma_{bc} - \eta_{bc}\gamma_{ad}\right)\ , \\ &= 2\left(\omega_\rho{}^a{}_c\omega_\sigma{}^{cb} - \omega_\sigma{}^a{}_c\omega_\rho{}^{cb}\right)\gamma_{ab}\ . \end{aligned} \tag{2.3.51}$$

Replacing (2.3.51) in the parenthesis of (2.3.49),

$$\partial_{[\rho}\omega_{\sigma]}{}^{ab}\gamma_{ab} + \frac{1}{4}\omega_{[\rho}{}^{ab}\omega_{\sigma]}{}^{cd}\gamma_{ab}\gamma_{cd} \tag{2.3.52}$$

$$= \frac{1}{2}\left(\partial_\rho\omega_\sigma{}^{ab} - \partial_\sigma\omega_\rho{}^{ab} + \omega_\rho{}^a{}_c\omega_\sigma{}^{cb} - \omega_\sigma{}^a{}_c\omega_\rho{}^{cb}\right)\gamma_{ab}\ , \tag{2.3.53}$$

$$= \frac{1}{2}R_{\rho\sigma}{}^{ab}\gamma_{ab}\ . \tag{2.3.54}$$

We have used the Riemann tensor in components given by (2.2.28). We will use this result in the computation of the integrability condition for the Freedman-Schwarz model, at the beginning of chapter 3.

Thus, plugging back in (2.3.49) we get

$$\gamma^{\rho\nu\mu}\nabla_\rho\nabla_\nu\Psi_\mu = \frac{1}{8}\gamma^{\rho\sigma\mu}R_{\rho\sigma}{}^{ab}\gamma_{ab}\Psi_\mu \tag{2.3.55}$$

Replacing in the variation of the action principle (2.3.48), we obtain the following expression

$$\delta S_\Psi = \frac{1}{4}\int d^4x\sqrt{-g}\bar{\epsilon}\gamma^{\mu\nu\rho}\gamma_{ab}R_{\mu\nu}{}^{ab}\Psi_\rho \ + \partial \ . \tag{2.3.56}$$

Using a similar identity as (2.3.37) we can show that

$$\gamma^{\mu\nu\rho}\gamma_{ab} = \gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu\nu}{}_b\delta_a^{\rho]} + 3\gamma^{[\mu}\delta_{ba}^{\nu\rho]} \ . \tag{2.3.57}$$

The first term in the right hand side of (2.3.57) vanishes in $d = 4$. While the second term in (2.3.57) acting on the Riemann tensor gives

$$6\gamma^{[\mu\nu}{}_b\delta_a^{\rho]}R_{\rho\sigma}{}^{ab} = 2\gamma^{\mu\nu b}R_{[\mu\nu}{}^{\rho}{}_{b]} + 4\gamma^{\nu\rho b}R_{[\nu b]} = 0 \ . \tag{2.3.58}$$

The third term in (2.3.57) acting on the Riemann tensor implies the following result

$$\begin{aligned} 3\gamma^{[\mu}\delta_{ba}^{\nu\rho]}R_{\mu\nu}{}^{ab} &= 2\gamma^{\mu}\delta_{ba}^{\nu\rho}R_{\mu\nu}{}^{ab} + \gamma^{\rho}\delta_{ba}^{\mu\nu}R_{\mu\nu}{}^{ab} \ , \\ &= 4\gamma^{\mu}\left(R_{\mu}{}^{\rho} - \frac{1}{2}\delta_{\mu}^{\rho}R\right) \ . \end{aligned} \tag{2.3.59}$$

Therefore, the variation of the action principle is not vanishing, but is proportional to the Einstein tensor!

$$\delta S_\Psi = \int d^4x\sqrt{-g}\left(R_{\mu\rho} - \frac{1}{2}g_{\mu\rho}R\right)\bar{\epsilon}\gamma^{\mu}\Psi^{\rho} + \partial \ . \tag{2.3.60}$$

Some remarks: On any Ricci flat spacetime the action principle is quasi-invariant under the transformation (2.3.47). On a general curved spacetime the transformation is no longer a symmetry, but is proportional to the Einstein tensor $\delta S_\Psi \sim G_{\mu\nu}\bar{\epsilon}\gamma^{\mu}\Psi^{\nu}$. One can guess that the lack of invariance of the action may be cured by adding the Einstein-Hilbert term in the action principle, namely

$$S = \frac{1}{2\kappa^2}\int d^4x \ e \ e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \frac{1}{2\kappa^2}\int d^4x \ e \ \bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho \ . \tag{2.3.61}$$

It is indeed the case and supersymmetry transformations [6] are given by

$$\delta e^a{}_\mu = \frac{1}{2}\bar{\epsilon}\,(x)\,\gamma^a \Psi_\mu \,, \tag{2.3.62}$$

$$\delta \Psi_\mu = \mathcal{D}_\mu \epsilon\,(x) \equiv \partial_\mu \epsilon + \frac{1}{4}\gamma^{ab}\omega_{ab}\epsilon \,. \tag{2.3.63}$$

This theory corresponds to $\mathcal{N}=1,\ d=4$ supergravity theory and also is the universal part of any supergravity. The Rarita-Schwinger field is the gauge field of local supersymmetry in the same way as the Yang-Mills field are the gauge field associated to the local action of a compact group G.

In the following section we will discuss aspects on Kaluza-Klein dimensional reductions in such a way to connect with the bosonic sector of some supergravities at the end of the next section.

2.4 Kaluza-Klein dimensional reduction

In the context of Kaluza-Klein (KK) dimensional reduction, the dilatonic coupling in gravity theories with gauge fields arises in a natural fashion. Kaluza-Klein reductions were firstly proposed by Theodor Kaluza in 1921 as a generalization of General Relativity in five dimension. Its proposal was complemented by Oskar Klein in 1926 who gave an interpretation of the extra dimension. He suggested that the extra dimension must be identified with a radius of the Planck length. Nowadays, Kaluza-Klein theories, as a general idea, consists of starting from a theory in higher dimensions, formulated on a manifold M_D and through a compactification on a compact manifold or a coset space N_p obtaining a theory in lower dimensions formulated on a manifold M_d with $d \equiv D-p$. In the simplest case, it is possible to truncate out the massive modes of the fields in terms of the harmonic functions of the compact manifold. The higher dimensional manifold may be written as the product of M_d and N_p, not in the sense of a direct product, but in the topological sense.

The group manifold may have an Abelian symmetry group (i.e. $N_p = S^1 \times \cdots \times S^1$), which gives a set of Abelian gauge fields supplemented with dilatonic and axionic scalar fields without potentials. While for a group manifold with non-Abelian isometries, there are non-Abelian gauge fields and in general the scalar fields have a self-interaction potential [32].

The main goal of this section is to describe the dimensional reduction performed by Chamseddine and Volkov [41] where they started from the type IA low energy limit of string theory in $D = 10$ dimension and considered $N_p = S^3 \times S^3$ as a group manifold . They realized that the 4 dimensional theory obtained by this procedure is the $\mathcal{N} = 4$ gauged $SU(2) \times SU(2)$ supergravity theory, firstly developed by Freedman and Schwarz [38]. In order to get some intuition in the discussion let us consider the KK reduction in S^1.

2.4.1 Reduction on S^1

As we mentioned, the story of dimensional reductions start in a spacetime with manifold M_D, whose coordinates are $x^{\hat{\mu}}$, the metric tensor is denoted by $\hat{g}_{\hat{\mu}\hat{\nu}}(x, z)$. The greek letters with hat run from $0, 1, 2, \ldots, D$. For simplicity, we will consider the reduction in the Abelian group manifold $N_p = S^1$ which is the simplest one. The coordinate on the circle is $x^D = z$. In this case, one basis for the harmonic functions that take values in S^1 is $\left\{e^{in\frac{z}{L}}\ ,\ n \in \mathbb{N}\right\}$ where L is the characteristic length of the circle. Using this basis we can expand the metric tensor as

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x, z) = \sum_n \hat{g}^{(n)}_{\hat{\mu}\hat{\nu}}(x)\, e^{inz/L}\ . \tag{2.4.1}$$

Doing so, we obtain an infinite set of fields in the lower dimensional manifold M_{D-1} labeled by n. We see that, in general $M_D \neq M_{D-1} \times S^1$ because the metric M_{D-1} depends on z. We will show that those modes with $n = 0$ represent massless fields, while the fields with $n \neq 0$ have mass m_n which depends inversly on L. To stress this fact, let us consider a massless scalar field $\hat{\phi} : M_D = \mathbb{R}^{D-1}_{(-+\cdots+)} \times S^1 \to \mathbb{R}$ whose dynamics is governed by the Klein-Gordon equation. Expanding the scalar in Fourier modes we obtain that

$$\hat{\phi}(x, z) = \sum_n \phi^{(n)}(x)\, e^{inz/L}\ . \tag{2.4.2}$$

We have to replace it into the equation $\partial^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\phi} = 0$. Note that the box operator splits into the Minkowskin part and the circle part $\partial^{\hat{\mu}}\partial_{\hat{\mu}} = \partial^{\mu}\partial_{\mu} + \partial_z\partial_z$. Consequently, the equation for the modes reads

$$\sum_n e^{inz/L}\left(\partial^\mu\partial_\mu\phi^{(n)}(x)-\frac{n^2}{L^2}\phi^{(n)}(x)\right)=0\ . \qquad (2.4.3)$$

Since the functions $e^{inz/L}$ are linearly independent, all the fields $\phi^{(n)}(x)$ fulfill the Klein-Gordon equation with mass $m_n=|n|/L$. The key point in formulating theories in dimensions higher than four is that we can connect these theories with a theory in four dimensions through compact space N_p that has a characteristic length which may be small enough as to be undetectable by current experiments. This idea implies that the massive modes of the fields may have very large masses. Thus, at low energies, we do not have to take care of the massive modes that live in M_{D-1}. In the simplest case $N_p=S^1$, it is enought to consider that the metric does not depend on the compact coordinate z. This gives us three fields defined on M_{D-1}, which are related to $\hat{g}_{\mu\nu}(x)$, $\hat{g}_{\mu z}(x)$ and $\hat{g}_{zz}(x)$. These degree of freedom may be interpretated as the tensor of the manifold M_{D-1}, a 1-form A_μ and a scalar field ϕ. This intuition is right, but it is not the best parametrization to see the symmetries that the fields enjoy in the manifold M_{D-1}. In general, there is not a recipe to find the correct parametrization. However, in this simple case it is enough to consider

$$\hat{g}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}}=e^{2\alpha\phi}g_{\mu\nu}dx^\mu dx^\nu+e^{2\beta\phi}\left(dz+A\right)^2\ . \qquad (2.4.4)$$

Here α and β are constant which will take values in terms of D in order to obtain the canonical normalization of the fields in the action principle in M_{D-1}. We have not specified the theory in M_D so far, but any sensible theory that we could be interested in is invariant under diffeomorphism, because the physics does not depends on the coordinates that we will use to define the measurable quantities. Consequently, the action principle will be invariant under $x^{\hat{\mu}}\to\tilde{x}^{\hat{\mu}}=\tilde{x}^{\hat{\mu}}\left(x^{\hat{\nu}}\right)$. Let us consider an infinitesimal coordinate transformation $x^{\hat{\mu}}\to x^{\hat{\mu}}-\hat{\xi}^{\hat{\mu}}\left(x^{\hat{\nu}}\right)$, which will imply a transformation of the metric tensor in $\hat{M}_D$ which is the Lie derivative of the metric tensor $\mathcal{L}_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}}$ along the vector field $\hat{\xi}$

$$\delta_\xi\hat{g}_{\hat{\mu}\hat{\nu}}\equiv\mathcal{L}_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}}=\hat{\xi}^{\hat{\sigma}}\partial_{\hat{\sigma}}\hat{g}_{\hat{\mu}\hat{\nu}}+\partial_{\hat{\mu}}\hat{\xi}^{\hat{\sigma}}\hat{g}_{\hat{\sigma}\hat{\nu}}+\partial_{\hat{\nu}}\hat{\xi}^{\hat{\sigma}}\hat{g}_{\hat{\mu}\hat{\sigma}}\ . \qquad (2.4.5)$$

The parametrization of the metric in $\hat{M}_D$ in terms of the fields in M_{D-1} gives the transformation rules of the fields in terms of the vector $\hat{\xi}$. The parametrization (2.4.4) allows us to see the transformation rules of the fields in a simple way. For example, considering $\hat{\xi}(x,z) = \xi(x) + (\chi(x) - \lambda z)\partial_z$ and evaluating the expression above

$$\delta_{\hat{\xi}}\phi = \mathcal{L}_\xi \phi - \frac{\lambda}{\beta} \ , \tag{2.4.6a}$$

$$\delta_{\hat{\xi}} A_\mu = \mathcal{L}_\xi A_\mu + \partial_\mu \chi + \lambda A_\mu \ , \tag{2.4.6b}$$

$$\delta_{\hat{\xi}} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} + \frac{2\alpha\lambda}{\beta} g_{\mu\nu} \ . \tag{2.4.6c}$$

It is clear that in each transformation, the first term shows that the theory formulated in M_{D-1} is invariant under diffeomorphism. The second term in (2.4.6b) shows that the 1-form $A_{[1]}$ is invariant under a transformation that is generated by a local parameter, therefore it is a gauge field. The last term in each transformation from the point of view of $\hat{M}_D$ is a dilatation of the circle with parameter λ. While from the point fo view of M_{D-1} the metric and the 1-form see a dilatation and the dilaton sees a traslation. It is possible to show that under a finite dilatation, i.e. $\tilde{\phi} = \phi - \frac{\lambda}{\beta}$, $\tilde{A}_\mu = e^{\lambda} A_\mu$ and $\tilde{g}_{\mu\nu} = e^{2\alpha\lambda/\beta} g_{\mu\nu}$, the action principle transforms as $S \to e^{-\lambda} S$.

The usual example of action principle in $\hat{M}_D$ that is invariant under diffeomorphism is General Relativity

$$S\left[\hat{g}_{\hat{\mu}\hat{\nu}}\right] = \int_{\hat{M}_D} d^D x \sqrt{-\hat{g}} \hat{R} \tag{2.4.7}$$

The way to obtain the action principle in M_{D-1} is to consider the parametrization (2.4.4) and compute the Ricci scalar. This is a straightforward computation, but have to do it by hand. The easiest way to confront these kind of calculations is compute the Ricci scalar with differential forms. In order to do that we must write the metric tensor in terms of vielbeins

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{a}\hat{b}} \hat{e}_{\hat{\mu}}{}^{\hat{a}} \hat{e}_{\hat{\nu}}{}^{\hat{b}} \ . \tag{2.4.8}$$

The indices $\hat{a}, \hat{b}, \ldots$ are the flat indices in $\hat{M}_D$ which can be raised or lowered using the flat metric $\eta_{\hat{a}\hat{b}}$. While the $a, b, \ldots$ are the flat indices in M_D. In the case of (2.4.4) a sensible set of vielbein are

$$\hat{e}^a = e^{\alpha\phi} e^a \ , \qquad \hat{e}^z = e^{\beta\phi}(A + dz) \ , \tag{2.4.9}$$

and the inverse vielbeine are given by

$$\hat{e}_a = e^{-\alpha\phi}\left(e_a{}^{\mu}\partial_\mu - A_a\partial_z\right) \ , \qquad \hat{e}_z = e^{-2\beta\phi}\partial_z \ , \tag{2.4.10}$$

which fulfill $\hat{e}^{\hat{a}}{}_{\hat{\mu}}\hat{e}_{\hat{a}}{}^{\hat{\nu}} = \delta^{\hat{\nu}}_{\hat{\mu}}$. From these quantities we can compute the 1-form spin connection without torsion $\hat{\omega}_{\hat{a}\hat{b}}$ on $\hat{M}_D$ that is given in terms of the structure coefficientes $\hat{\Omega}_{\hat{a}\hat{b}\hat{c}}$ defined by (2.2.32). Using the vielbeins in (2.4.9) the structure coefficientes are

$$\begin{aligned} \hat{\Omega}_{zza} &= -\beta e^{-\alpha\phi}\partial_a\phi \ , \qquad \hat{\Omega}_{bza} = -e^{(\beta-2\alpha)\phi}F_{\mu\nu}e_a{}^{\mu}e_b{}^{\nu} \ , & (2.4.11)\\ \hat{\Omega}_{zca} &= 0 \ , \qquad\qquad\qquad \hat{\Omega}_{abc} = -2\alpha e^{-\alpha\phi}\partial_{[c}\phi\eta_{a]b} + e^{-\alpha\phi}\Omega_{abc} \ . & (2.4.12)\end{aligned}$$

where Ω_{abc} are the structure coefficientes on M_{D-1}. Then, we can compute the 1-form spin connection defined by (2.2.36) which gives

$$\begin{aligned} \hat{\omega}^{az} &= -\beta e^{-\alpha\phi}\partial^a\phi\hat{e}^z - \frac{1}{2}e^{(\beta-2\alpha)\phi}F^a{}_c\hat{e}^c \ , & (2.4.13)\\ \hat{\omega}^{ab} &= \omega^{ab} - \frac{1}{2}e^{(\beta-2\alpha)\phi}F^{ab}\hat{e}^z - \alpha e^{-\alpha\phi}\left(\partial^a\phi\hat{e}^b - \partial^b\phi\hat{e}^a\right) \ . & (2.4.14)\end{aligned}$$

The spin connection on M_{D-1} is $\omega^{ab} = \omega_c{}^{ab}e^c$ while e^c is the vielbeine 1-form on M_{D-1}. Now, we can compute the curvature 2-form defined by (2.2.27) and then computing the traces in order to get the Ricci scalar. Nevertheless, since in (2.4.7) the Ricci scalar is under an integral, we can use the Palatini identity that we proved in the Appendix A2. We need to compute the square root of the determinant of the metric in $\hat{M}_D$. To do that, we use the fact that $\sqrt{-\hat{g}} = \det \hat{e}^{\hat{a}}{}_{\hat{\mu}}$ and the determinant of a matrix may be computed as

$$\begin{aligned} D!\det\hat{e}^{\hat{a}}{}_{\hat{\mu}} &= \varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_D}\hat{e}^{\hat{a}_1}{}_{\hat{\mu}_1}\ldots\hat{e}^{\hat{a}_D}{}_{\hat{\mu}_D}\varepsilon_{\hat{a}_1\ldots\hat{a}_D} \ , & (2.4.15)\\ &= D\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_D}\hat{e}^z{}_{\hat{\mu}_1}\hat{e}^{\hat{a}_2}{}_{\hat{\mu}_2}\ldots\hat{e}^{\hat{a}_D}{}_{\hat{\mu}_D}\varepsilon_{z\hat{a}_2\ldots\hat{a}_D} \ ,\\ &= D\varepsilon^{\dot{z}\mu_2\ldots\mu_D}\hat{e}^z{}_{\dot{z}}\hat{e}^{a_2}{}_{\mu_2}\ldots\hat{e}^{a_D}{}_{\mu_D}\varepsilon_{za_2\ldots a_D}\\ &+ D\left(D-1\right)\varepsilon^{\hat{\mu}_1\dot{z}\hat{\mu}_3\ldots\hat{\mu}_D}\hat{e}^z{}_{\hat{\mu}_1}\hat{e}^{\hat{a}_2}{}_{\dot{z}}\hat{e}^{\hat{a}_3}{}_{\hat{\mu}_3}\ldots\hat{e}^{\hat{a}_D}{}_{\hat{\mu}_D}\varepsilon_{z\hat{a}_D\ldots\hat{a}_D} \ .\end{aligned}$$

where $\dot{z}$ is the coordinate index and z is the flat index. In the first term of the last line, we have taken out the hat of all indices because the Levi-Civita symbol cannot repeat indices. From (2.4.7) we see that $\hat{e}^{\hat{a}}{}_{\dot{z}} = 0$. Then, following the computation above and replacing the vielbeins and the fact that $\varepsilon^{\dot{z}\mu_2\cdots\mu_D}$ reduces to $\varepsilon^{\mu_2\cdots\mu_D}$, we get the following

$$\det \hat{e}^{\hat{a}}_{\ \hat{\mu}} = e^{\beta\phi}e^{(D-1)\alpha\phi}\frac{1}{(D-1)!}\varepsilon^{\mu_2\ldots\mu_D}e^{a_2}_{\ \mu_2}\ldots e^{a_D}_{\ \mu_D}\varepsilon_{za_D\ldots a_D} \tag{2.4.16}$$

which is nothing but

$$\sqrt{-\hat{g}} = e^{(\beta+(D-1)\alpha)\phi}\sqrt{-g} \ . \tag{2.4.17}$$

Replacing this result and the spin connection (2.4.13) in the Palatini identity A2.6 we obtain

$$\begin{aligned} S &= \int_{\hat{M}_D} d^Dx\sqrt{-\hat{g}}\hat{R} \ , \\ &= \int_{\hat{M}_D} d^Dx\sqrt{-\hat{g}}\left(\hat{\omega}^{\hat{c}\hat{a}\hat{b}}\omega_{\hat{a}\hat{b}\hat{c}} - \hat{\omega}_{\hat{c}}^{\ \hat{c}\hat{b}}\hat{\omega}_{\hat{d}\ \hat{b}}^{\ \hat{d}}\right) \ , \qquad \text{(Palaini identity)} \\ &= 2\pi L\int_{M_{D-1}} \sqrt{-g}d^{D-1}xe^{(\beta+(D-3)\alpha)\phi}\Bigg[-\frac{1}{4}e^{2(\beta-\alpha)\phi}F^{ab}F_{ab} + \omega^{cab}\omega_{abc} - \omega_c^{\ cb}\omega_{d\ b}^{\ d} \\ &\quad + \left\{-(D-4)\alpha^2 - 2\beta^2 - 2\beta\alpha(D-4) - \alpha^2(D-4)^2\right\}\partial^b\phi\partial_b\phi \\ &\quad + \left\{-2\alpha - 2\alpha(D-4) - 2\beta\right\}\omega_c^{\ cb}\partial_b\phi\Bigg] \ . \end{aligned}$$

In order to get the canonical normalization for the kinetic term of the scalar field and the Ricci scalar, we have to set $\beta = -(D-3)\alpha$ which vanishes the last term and $\alpha^{-2} = 2(D-2)(D-3)$. Then, the action principle is the Einstein-dilaton-Maxwell theory in four dimensions

$$S[g_{\mu\nu}, \phi, A] = 2\pi L\int_{M_{D-1}} d^{D-1}x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{-2(D-2)\alpha\phi}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{2.4.18}$$

One can see that the scalar field is coupled in a non-linear fashion with the Abelian gauge field. As a consequence, the gauge field appears in the equations of the scalar and viceversa. These type of couplings are called *dilatonic coupling.*

There is a linear term in the scalar in action above, which comes from the series expansion of the exponential, this fact in general implies that is subtle to turn off the scalar field. Indeed, if we compute the field equation coming from the minimization of (2.4.18) and then we set $\phi = 0$. These equations are different than the equations comming from set $\phi = 0$ in (2.4.18) and then computing the extremum of the action principle. In the former case, we obtain an extra constraint from the equation of the scalar field, namely $F_{\mu\nu}F^{\mu\nu} = 0$. Therefore, the right action principle with $\phi = 0$ must have a Lagrange multiplier which gives the constraint above.

The dilatation transformation that we discussed above transforms the action principle as $S \to e^{-\lambda}S$, which can be understood as a dilatation (or contraction) of the characteristic length L of the circle.

From this example we can see some general features of the Kaluza-Klein reduction. Such a similarities are, for example, the emergence of gauge fields related to the Lie Algebra of the compact manifold N_p, the dilatonic coupling between the scalar field, and the gauge field and the subtleties upon the truncation of the massive modes.

In the following, we will discuss some aspects of the Kaluza-Klein reduction performed by Chamsseine and Volkov in [41] which gives the $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity in $d = 4$.

2.4.2 $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity from eleven dimensional supergravity

Eleven dimensional supergravity is a unique theory in eleven dimensions with a remarkably simple field content: a vielbeine $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, one Majorana 3/2 spinor $\hat{\psi}_{\hat{\mu}}$ and one abelian 3-form $\hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}}$. The theory was discovered by Cremmer, Julia and Scherk in 1978 [33] to obtain extended $O(N)$ $(N = 1, \ldots, 8)$ supergravity theories in four dimensions, which were difficult to construct, via dimensional reduction from $11D$. This is the maximal supergravity theory in eleven dimension which does not involve higher spins in its field content (with 32 supercharges). The Lagrangian of the bosonic sector of 11D sugra is given by the following (In this section we will use the metric with signature $(+ - \cdots -)$)

$$\mathcal{L}_{11} = -\frac{1}{4}\hat{e}\hat{R} - \frac{1}{48}\hat{e}\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{C}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \frac{2}{(12)^4}\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_{11}}\hat{C}_{\hat{\mu}_1\ldots\mu_4}\hat{C}_{\hat{\mu}_1\ldots\mu_4}\hat{K}_{\mu_9\mu_{10}\mu_{11}} \qquad (2.4.19)$$

where the 4-form $\hat{C}_{[4]} = \hat{d}\hat{K}_{[3]}$ is the field strength of $K_{[3]}$.

It was shown in [34] that one can obtain $\mathcal{N} = 1$ in ten dimensions (with a half of the maximum number of supersymmetries) via Kaluza-Klein reduction in S^1

from 11D supergravity, following a similar procedure as described in the previous section. In this case the parametrization of the vielbeine, the Rarita-Schwinger and the 3-form read

$$\begin{aligned} \hat{e}^{\hat{a}}{}_{\hat{\mu}} &= \begin{pmatrix} e^{a}{}_{\mu}(x) & B^{11}{}_{\mu} \\ 0 & e^{4/3\phi(x)} \end{pmatrix} , \qquad \hat{\psi}_{\hat{\mu}} = \left(\psi_{\mu}, \psi_{\dot{1}\dot{1}}\right) , \\ \hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \left(A_{\mu\nu\rho}, A_{\mu\nu\dot{1}\dot{1}}\right) \end{aligned} \tag{2.4.20}$$

respectively. In 10D it is possible to consider the Weyl condition and the Majorana condition at the same time. Thus, each spinors coming from 11D: the Rarita-Schwinger ψ_{μ}, the spinor $\psi_{\dot{1}\dot{1}}$ and the spinorial parameter of the susy transformations $\epsilon(x)$ split into two Majorana-Weyl spinors. Then, without any truncation we would obtain $\mathcal{N} = 2$ supersymmetries in the 10D sugra [35]. Following [34] a consistent truncation that allows us to get IA sugra is

$$\frac{1}{2}(1+\gamma_{11})\psi_{\mu} \equiv \psi_{\mu} , \qquad \frac{1}{2}(1+\gamma_{11})\epsilon \equiv \epsilon , \qquad \frac{1}{2}(1-\gamma_{11})\psi_{\dot{1}\dot{1}} \equiv \psi_{\dot{1}\dot{1}} \tag{2.4.21}$$

which means that the chirality of the spinors is fixed. Also we have to truncate at bosonic level

$$B^{11}{}_{\mu} = 0 \qquad \text{and} \qquad A_{\mu\nu\rho} = 0 . \tag{2.4.22}$$

Following the same procedure as in the previous section to evaluate the action principle (2.4.19) in the ansatz imposing the constraints, we obtain the action principle in 10D with $\mathcal{N} = 1$ supersymmetry whose bosonic Lagrangian is

$$\mathcal{L}_{10}^{\mathcal{N}=1} = -\frac{1}{4}\hat{e}\hat{R} + \frac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\phi} + \frac{\hat{e}}{12}e^{-2\hat{\phi}}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} . \tag{2.4.23}$$

We write it with hat because we will reduce it again. The 3-form $\hat{H}_{[3]}$ is the field strength associated to the 2-form gauge potential coming from the 3-form in 11D. Type I is not a maximal supergravity since it only contains 16 supercharges in a single Majorana-Weyl spinor. It contains the graviton, the gravitino, the dilaton ϕ, the dilatino $\psi_{\dot{1}\dot{1}}$ and the 2-form with field strength $\hat{H}_{[3]}$.

The connection with $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged theory was found by Chamseddine and Volkov in 1998 [41]. They showed that considering the Kaluza-Klein reduction in the compact manifold $S^3 \times S^3$ of (2.4.23) including fermions, one gets the Freedman Schwarz model $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged in four dimensions. This result was also bore out in [36] where the authors considered the dimensional reduction on S^7 from 11D sugra to four dimensions and obtained the $\mathcal{N}=4$ $SO(4)$ gauged theory, and in the singular limit when the axion field is shifted by an infinite constant they recover the full Freedman-Schwarz model[5].

In what follows we will discuss how to obtain the field content of $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged including non-Abelian gauge fields in four dimensions. We use the following index convention. The manifold is splited as $M_{10}=M_4\times S^3\times S^3$. For the four dimensional spacetime indices we use $\mu,\nu,\rho,\ldots$ and $m,n,\ldots$ for the space indices in the compact manifold:

$$\{x^{\hat{\mu}}\}=\{x^{\mu}=\dot{0},\dot{1},\dot{2},\dot{3}\ ;\ z^m=\dot{1},\ldots,\dot{6}\}\ .$$

While for the flat indices we use $a,b,c\ldots$ to flat spacetime indices in M_4 and I,J,K for the flat indices in the inner manifold

$$\{\hat{a}\}=\{a=0,1,2,3\ ;\ I=1,\ldots,6\}\ .$$

It is also useful using the explicit splitting between each S^3 so that $\{I\}=\{(s=1,2)\ ;\ i=1,2,3\}$, where s indicated the "side" and i represents the 3 flat indices. Using these conventions, we write the vielbeine in 10D as

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}}=\begin{pmatrix} e^{\frac{3}{4}\phi}e_{\mu}{}^{a} & \sqrt{2}e^{\frac{1}{4}\phi}A^{I}{}_{\mu} \\ 0 & e^{-\frac{1}{4}\phi}U_{m}^{\ I}\end{pmatrix}\ ,$$

where $U_m^{\ I}$ is proportional to the left invariant Maurer-Cartan 1-form $\theta^{I}{}_{m}dz^m$ as

$$U^I\equiv U^{(s)i}=-\frac{\sqrt{2}}{e_{(s)}}\theta^{(s)i} \tag{2.4.24}$$

which fulfill

$$d\theta^{(s)i}+\frac{1}{2}\epsilon_{ijk}\theta^{(s)j}\wedge\theta^{(s)k}=0\ . \tag{2.4.25}$$

It is interesting to notice that the curvature of the spheres S^3 are related to the inverse of the gauge couplings and in the limit when the coupling constant

[5] In order to compare [41] with [36], the conventions for the Hodge dual used in [36] is given in the Appendix A3.

associated to one sphere $e_A \to 0$ or $e_B \to 0$ we obtain $S^3 \to R^3$ in the corresponding sphere. On the other hand, the dilaton is reduced as

$$\hat{\phi} = \frac{\phi}{2} \ .$$

We see that the non-Abelian gauge fields appear because the compact manifold has a group structure which is indeed the gauge group. The 3-form field strength is parametrized in terms of the non-Abelian field strength $F^I{}_{\mu\nu}$ and a pseudo-scalar $\mathbf{a}$ in the following way

$$\begin{aligned} \hat{H}_{abc} &= e^{\frac{7}{4}\phi}\epsilon_{abcd}\partial^d \mathbf{a} \ , \qquad \hat{H}_{abI} = -\frac{1}{\sqrt{2}}e^{-5\phi/4}F^I{}_{ab} \ , \\ \hat{H}_{IJK} &= \frac{1}{2\sqrt{2}}e^{3\phi/4}f_{IJK} \ , \qquad \hat{H}_{IJa} = 0 \ . \end{aligned}$$

The reduction of the ten dimensional vielbeine leads to $\det \hat{e}^{\hat{a}}{}_{\hat{\mu}} = e^{-3\phi/2}\det\left(U^I{}_m\right)\det\left(e^a{}_\mu\right)$. Then, the resulting action principle in 4 dimensions is the $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged supergravity, whose field content is given by a vielbein $e_\mu{}^a$, four Majorana spin-3/2 fields $\psi_\mu \equiv \psi^I_\mu$, as well as four Majorana spin-1/2 fields $\chi \equiv \chi^I$, where the index $I = 1, \ldots, 6$ runs in the fundamental of $SU(2)\times SU(2)$. The theory also contains a pseudoscalar axion field $\mathbf{a}(x)$, as well as a real scalar $\phi(x)$, namely the dilaton. The Yang-Mills sector consists of a vector and a pseudovector, non-abelian gauge field $A^i{}_\mu$ and $B^i{}_\mu$, respectively, with independent gauge couplings e_A and e_B, where the index $i = 1, 2, 3$ transforms in the adjoint of each corresponding $SU(2)$ copy. Following the conventions of [43], the action reads

$$\begin{aligned} \frac{\mathcal{L}}{\sqrt{-g}} = &-\frac{R}{4} + \frac{1}{2}\left[(\partial\phi)^2 + e^{4\phi}(\partial\mathbf{a})^2\right] - V(\phi) - \frac{e^{-2\phi}}{4}\left(A^{i\mu\nu}A_{i\mu\nu} + B^{i\mu\nu}B_{i\mu\nu}\right) \\ &- \frac{\mathbf{a}}{2}\left(\tilde{A}_{i\mu\nu}A^{i\mu\nu} + \tilde{B}_{i\mu\nu}B^{i\mu\nu}\right) \ , \end{aligned} \tag{2.4.26}$$

where $\sqrt{-g} = e = \det\left(e_\mu{}^a\right)$ and the self-interaction of the dilaton is unbounded from below and it is given by

$$V(\phi) = -\frac{(e_A^2 + e_B^2)}{8}e^{2\phi} \ , \tag{2.4.27}$$

As usual

$$A_{i\mu\nu} = \partial_\mu A^i{}_\nu - \partial_\nu A^i{}_\mu + e_A \epsilon_{ijk} A^j{}_\mu A^k{}_\nu \ , \tag{2.4.28}$$

$$B_{i\mu\nu} = \partial_\mu B^i{}_\nu - \partial_\nu B^i{}_\mu + e_B \epsilon_{ijk} B^j{}_\mu B^k{}_\nu \ , \tag{2.4.29}$$

$$\tilde{A}^i{}_{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon_{\mu\nu\rho\sigma} A^{i\rho\sigma} \ . \tag{2.4.30}$$

The supersymmetry transformations are generated by the local spinorial parameter $\epsilon(x) \equiv \epsilon^I(x)$, which on a purely bosonic configuration, when acting on the fermionic fields of the theory reduce to

$$\delta\bar{\chi} = \frac{i}{\sqrt{2}} \bar{\epsilon} \left(\partial_\mu \phi + i\gamma_5 e^{2\phi} \partial_\mu \mathbf{a}\right) \gamma^\mu - \frac{1}{2} e^{-\phi} \bar{\epsilon} C_{\mu\nu} \sigma^{\mu\nu} + \frac{1}{4} e^{\phi} \bar{\epsilon} \left(e_A + i\gamma_5 e_B\right) \ , \tag{2.4.31}$$

$$\delta\bar{\psi}_\rho = \bar{\epsilon} \left(\overleftarrow{D}_\rho - \frac{i}{2} e^{2\phi} \gamma_5 \partial_\rho \mathbf{a}\right) - \frac{i}{2\sqrt{2}} e^{-\phi} \bar{\epsilon} C_{\mu\nu} \gamma_\rho \sigma^{\mu\nu} + \frac{i}{4\sqrt{2}} e^{\phi} \bar{\epsilon} \left(e_A + i\gamma_5 e_B\right) \gamma_\rho \ . \tag{2.4.32}$$

The Lorentz and gauge covariant derivative is given by

$$\overleftarrow{D}_\rho = \overleftarrow{\partial}_\rho - \frac{1}{4} \omega_\rho{}^{ab} \gamma_{ab} + \frac{1}{2} e_A \alpha^i A^i{}_\rho + \frac{1}{2} e_B \beta^i B^i_\rho \ , \tag{2.4.33}$$

and the generators we will use are given by the following 4×4 matrices (see [38])

$$\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} , \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} , \qquad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} , \tag{2.4.34}$$

$$\beta^1 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} , \qquad \beta^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \beta^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} . \tag{2.4.35}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which generate the algebra $su(2) \times su(2)$, namely,

$$\alpha^i \alpha^j = -\delta^{ij} - \epsilon^{ijk} \alpha^k \ , \qquad \beta^i \beta^j = -\delta^{ij} - \epsilon^{ijk} \beta^k \ , \qquad \left[\alpha^i, \beta^j\right] = 0 \ . \tag{2.4.36}$$

$$\left[\alpha^i, \alpha^j\right] = -2\epsilon^{ijk} \alpha^k \ , \qquad \left[\beta^i, \beta^j\right] = -2\epsilon^{ijk} \beta^k \ . \tag{2.4.37}$$

Furthermore, following [38] we have defined

$$C_{\mu\nu} \equiv \alpha^i A^i{}_{\mu\nu} + i\gamma_5 \beta^i B^i{}_{\mu\nu} \ . \tag{2.4.38}$$

For the fermionic sector we consider the following conventions: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] \equiv \frac{1}{2}\gamma_{\mu\nu}$, $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$, so that $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma_a\} = 0$. When necessary, we will use the following basis

$$\gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \ \gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} , \ \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} , \ \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} . \tag{2.4.39}$$

The BPS configurations are such that they lead to non-trivial solutions for ϵ from the equation $\delta\chi = 0$ and $\delta\psi_\mu = 0$ in (2.4.31) and (2.4.32), respectively.

In order to analyze the existence of Killing spinors, it is customary to consider consistency conditions that emerge from the manipulation of equations (2.4.31) and (2.4.32). In particular, plugging the equation $\delta\chi = 0$ from (2.4.31), in equation (2.4.32) implies that BPS solutions must fulfill

$$\delta\bar{\psi}_\rho = \bar{\epsilon}\left(\overleftarrow{D}_\rho - \frac{i}{\sqrt{2}}e^{-\phi}C_{\rho\mu}\gamma^\mu + \frac{1}{2}\ \partial\!\!\!/\phi\gamma_\rho\right) - \bar{\epsilon}\frac{i}{2}e^{2\phi}\gamma_5\left(\partial_\rho \mathbf{a} - \ \partial\!\!\!/\mathbf{a}\gamma_\rho\right) = 0 \ , \tag{2.4.40}$$

An integrability conditions for the Killing spinor comes from imposing $\delta\bar{\psi}_{[\rho}\overleftarrow{D}_{\sigma]} = 0$, which after a lengthy but straightforward computation leads to

$$0 = \delta\bar{\psi}_{[\rho}\overleftarrow{D}_{\sigma]} \tag{2.4.41}$$

$$\begin{aligned}
&= -\frac{1}{8}\bar{\epsilon}R_{\sigma\rho}{}^{ab}\gamma_{ab} + \frac{1}{4}\bar{\epsilon}g_A\alpha^i A_{i\sigma\rho} + \frac{1}{4}\bar{\epsilon}g_B\beta^i B_{i\sigma\rho} + \bar{\epsilon}\frac{1}{2}e^{-2\phi}C_{[\sigma|}{}^{\nu}\gamma_\nu C_{|\rho]\mu}\gamma^\mu + \bar{\epsilon}\frac{i}{\sqrt{2}}e^{-\phi}\partial_{[\sigma}\phi C_{\rho]\mu}\gamma^\mu \\
&+ \bar{\epsilon}\frac{i}{\sqrt{2}}e^{-\phi}\left[-\alpha^i\left(\nabla_{[\sigma}A^i_{\rho]\nu} + g_A\epsilon^{lji}A^j{}_{\rho\nu}A^l{}_\sigma\right)\gamma^\nu - i\beta^i\gamma_5\left(\nabla_{[\sigma}B^i_{\rho]\nu} + g_B\epsilon^{lji}B^j{}_{\rho\nu}B^l{}_\sigma\right)\gamma^\nu\right] \\
&+ \bar{\epsilon}\frac{1}{2}\nabla_{[\sigma|}\partial_\mu\phi\gamma^\mu\gamma_{|\rho]} + \bar{\epsilon}\frac{1}{4}\partial_\mu\phi\partial^\mu\phi\gamma_{\sigma\rho} - \bar{\epsilon}\frac{1}{2}\ \partial\!\!\!/\phi\partial_{[\sigma}\phi\gamma_{\rho]} + \bar{\epsilon}\frac{i}{\sqrt{2}}e^{-\phi}\partial^\nu\phi C_{[\sigma|\nu}\gamma_{\rho]} - \bar{\epsilon}\frac{i}{\sqrt{2}}e^{-\phi}C_{\sigma\rho}\ \partial\!\!\!/\phi \\
&+ \bar{\epsilon}\frac{e^\phi}{\sqrt{2}}\gamma_5\partial_{[\sigma}\mathbf{a}C_{\rho]\mu}\gamma^\mu + \bar{\epsilon}e^{2\phi}\gamma_5\frac{i}{2}\nabla_{[\sigma|}\partial_\mu\mathbf{a}\gamma^\mu\gamma_{|\rho]} - \bar{\epsilon}\frac{1}{4}e^{4\phi}\partial_\mu\mathbf{a}\partial^\mu\mathbf{a}\gamma_{\sigma\rho} + \bar{\epsilon}\frac{1}{2}e^{4\phi}\ \partial\!\!\!/\mathbf{a}\partial_{[\sigma}\mathbf{a}\gamma_{\rho]} \\
&+ \bar{\epsilon}\frac{e^\phi}{\sqrt{2}}\gamma_5\partial^\mu\mathbf{a}C_{[\sigma|\mu}\gamma_{|\rho]} - \bar{\epsilon}\frac{e^\phi}{\sqrt{2}}\gamma_5 C_{\sigma\rho}\ \partial\!\!\!/\mathbf{a} - \bar{\epsilon}ie^{2\phi}\partial_{[\sigma}\phi\partial_{\rho]}\mathbf{a}\gamma_5 + \bar{\epsilon}\frac{ie^{2\phi}}{2}\ \partial\!\!\!/\mathbf{a}\partial_{[\sigma}\phi\gamma_{\rho]}\gamma_5 \\
&- \bar{\epsilon}\frac{e^\phi}{\sqrt{2}}\gamma_5\ \partial\!\!\!/\mathbf{a}\gamma_{[\sigma}C_{\rho]\mu}\gamma^\mu + \bar{\epsilon}\frac{i}{2}e^{2\phi}\gamma_5\partial_\mu\mathbf{a}\partial^\mu\phi\gamma_{\sigma\rho} - \bar{\epsilon}\frac{i}{2}e^{2\phi}\gamma_5\ \partial\!\!\!/\phi\partial_{[\sigma}\mathbf{a}\gamma_{\rho]}
\end{aligned} \tag{2.4.42}$$

As a matter of fact, in order to evaluate the existence of a Killing spinor in this theory, one must study both equation $\delta\bar{\chi}=0$ from (2.4.31), and (2.4.41) which respectively have the form

$$\bar{\epsilon}\Theta = 0 \quad \text{and} \quad \bar{\epsilon}\Xi_{\mu\nu} = 0 \ , \tag{2.4.43}$$

where Θ is a 16×16 matrix and $\Xi_{\mu\nu}$ are six, 16×16 matrices acting on the column arrange $\bar{\epsilon}$ of 16 components, which belongs to the tensor product of the vector space of the spinors, times the vector space of the fundamental representation of $SU(2)\times SU(2)$.

Chapter 3

New solitons and black holes solutions in Freedman Schwarz model

The problem of providing partial classifications of BPS configurations in the Freedman-Schwarz model has been considered in the literature. As mentioned above, in [39] the authors found BPS solutions which are product spacetimes of the form $AdS_2 \times \mathbb{R}^2$ where the AdS_2 factor emerges naturally since the gauge fields contribute to the dilaton effective potential providing an extremum that leads to an effective, two-dimensional, negative cosmological constant. Such configurations may preserve one-quarter or one-half of the supersymmetry. Going beyond the product space ansatz, in [40; 41] the authors constructed a $1/4$ BPS soliton, which is asymptotically locally flat and it is supported by a single gauge field (see [44] for further properties of the uplift of the soliton to ten dimensions). Some non-supersymmetric dyonic solutions where found in [45], while in [46] the author constructed planar, spherical and hyperbolic solutions of the first order BPS system, both analytically and numerically.

Restricting to the Abelian sector of both gauge fields in a double dyonic ansatz, in [43] the author constructed planar BPS black holes and identified a family of singular domain walls as supersymmetric configurations, which were previously integrated in [47; 48]. Notwithstanding this, the analysis of [43] provides no new supersymmetric configurations in the spherically symmetric case.

In the present chapter we present the first part of new results of this thesis, these results were published in [97]. The chapter is organized as follows: In Section 3.1, we will present a new supersymetric soliton. The solutions is regular, $1/4$ BPS and can be obtained from a double Wick rotation of a non-supersymetric configuration found in [43]. In Section 3.2, also in the Abelian sector of the theory we show that there are supersymmetric solutions in the spherical case. This new $1/4$ BPS solutions describe spacetimes which are singular. These spacetimes are characterized by two integration constant, and Appendix A4 is devoted to the explicit presentation of the Killing spinors. Then, in Section 3.3 we introduce the hedgehog ansatz for a meron gauge field, and construct new solutions of the $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged theory. We combine this magnetic meron ansatz with both a non-Abelian electric field or a second meronic field. Out of the new configurations, we show that only the purely magnetic, double meron leads to a supersymmetric solution. In Section 3.4 we move beyond the supergravity theory, keeping the field content fixed, but considering more general potentials $V(\phi)$ that still lead to analytic solutions with interesting thermal properties. The potentials we consider were already identified as well-behaved potentials regarding the construction of exact solutions in field theories with similar matter content (see e.g. [49]-[57]). For simplicity we will set the axion field $\mathbf{a}(x)=0$.

3.1 New BPS soliton

The following configuration for the metric, the dilaton and the gauge fields, provides a solution of $\mathcal{N}=4$ $SU(2)\times SU(2)$ gauged supergravity

$$ds^2=\rho dt^2-\frac{d\rho^2}{g(\rho)}-g(\rho)\,d\varphi^2-\rho dx^2\ , \tag{3.1.1}$$

$$g(r)=\frac{(e_A^2+e_B^2)}{2}\rho-m-\frac{2\,(Q_A^2+Q_B^2)}{\rho}\ , \tag{3.1.2}$$

$$\phi=-\frac{1}{2}\ln\rho\ , \tag{3.1.3}$$

$$A=\frac{Q_A}{\rho}d\varphi\alpha^3 \text{ and } B=\frac{Q_B}{\rho}d\varphi\beta^3\ , \tag{3.1.4}$$

where the coordinate $\rho_0 \leq \rho$ and $\varphi \in [0, \beta_\varphi]$, with

$$\rho_0 = \frac{m + \sqrt{4\left(e_A^2 + e_B^2\right)\left(Q_A^2 + Q_B^2\right) + m^2}}{\left(e_A^2 + e_B^2\right)} \ , \tag{3.1.5}$$

$$\beta_\varphi = \frac{4\pi}{g'\left(\rho_0\right)} = \frac{8\pi\rho_0^2}{\left(e_A^2 + e_B^2\right)\rho_0^2 + 4\left(Q_A^2 + Q_B^2\right)^2} \ . \tag{3.1.6}$$

Here m is an integration constants, and we have consistently removed a pure gauge, second integration constant ϕ_0 that emerges from the integration of the system. The spacetime (3.1.1) is regular everywhere and describes a charged soliton, which asymptotically approaches

$$ds^2 = \rho dt^2 - \frac{2d\rho^2}{\left(e_A^2 + e_B^2\right)\rho} - \frac{\left(e_A^2 + e_B^2\right)}{2}\rho d\varphi^2 - \rho dx^2 \ . \tag{3.1.7}$$

Notice that the asymptotic geometry acquires an extra conformal Killing vector which acts as $\rho \to \lambda\rho$. One can show that the soliton is asymptotically locally flat, since all the components of the Riemann tensor vanish as ρ goes to infinity.

Let us now move to the analysis of the supersymmetry of this solution. As mentioned above, the consistency and integrability conditions in this theory reduce to the analysis of the matrices Θ and $\Xi_{\mu\nu}$ which are of 16×16. For our soliton configuration (3.1.1)-(3.1.4), the determinant of Θ reads

$$\det\Theta = \frac{1}{2^{24}\rho^{16}}\left(4\left(e_B Q_A - e_A Q_B\right)^2 + m^2\right)^2\left(4\left(e_B Q_A + e_A Q_B\right)^2 + m^2\right)^2 \ . \tag{3.1.8}$$

Therefore the solution can be supersymmetric only if $m = 0$ and

$$e_B Q_A \pm e_A Q_B = 0 \ . \tag{3.1.9}$$

Setting $m = 0$ in $\Xi_{\mu\nu}$ leads to the following determinant

$$\det\left(\Xi_{\rho\varphi}\right) = \left(\frac{1}{2\rho}\right)^{32}\left(e_B Q_A - e_A Q_B\right)^8\left(e_B Q_A + e_A Q_B\right)^8 \ , \tag{3.1.10}$$

which actually vanishes identically given the BPS constraint coming from (3.1.9). All the remaining determinants identically vanish, even before using (3.1.9). In summary, this implies that the configuration (3.1.1)-(3.1.4) with $m = 0$ and $e_B Q_A = \mp e_A Q_B$ is supersymmetric. This solution turns out to be 1/4 BPS, and the explicit expression for the Killing spinors is presented in Appendix A. The supersymmetric solution with $m = 0$ takes a particularly simple form after performing the change of coordinates

$$\rho = \frac{2Q_A}{e_A} \cosh l \ , \tag{3.1.11}$$

with $0 \leq l < +\infty$, since it reduces to

$$ds^2_{\text{BPS-Soliton}} = \frac{Q_A}{e_A} \cosh l \left[2dt^2 - \frac{4dl^2}{e_A^2 + e_B^2} - \left(e_A^2 + e_B^2\right) \tanh^2 l d\varphi^2 - 2dx^2 \right] . \tag{3.1.12}$$

Our new solitonic solution can also be obtained from a double analytic continuation of the planar solution found in [43]. Such spacetime is characterized by an integration constant $\tilde{m}$, that maps to our m after the continuation. Nevertheless, the planar solution found in [43] with $\tilde{m} = 0$ is a naked singularity. Notwithstanding this fact, the double analytic continuation, followed by a compactification of the coordinate φ with an appropriate range, leads to a completely regular soliton spacetime. The same effect has been recently seen to work for $\mathcal{N} = 2$ gauged supergravity in $D = 4$ and $D = 5$ in [67]. The new 1/2 BPS solitons discovered in such reference are connected, via a double analytic continuation, to the plannar Reissner-Norstrom-AdS spacetime for a value of the integration constant that would lead to a naked singularity in the latter. Remarkably, the double Wick rotation leads to a smooth spacetime with unbroken supersymmetries[1] [67], as we have also reported here for $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity.

3.2 New Abelian BPS configuration

In this section we reconsider the problem of spherically symmetric, supersymmetric solutions of the Freedman-Schwarz model in the Abelian sector. In order to

[1]Double analytic continuations may also give rise to supersymmetric wormholes, when the seed spacetime is Taub-NUT-AdS in the hyperbolic foliation [68].

compare with reference [43], we consider the following ansatz for the metric[2]

$$ds^2 = f(\rho)\, dt^2 - \frac{d\rho^2}{f(\rho)} - \rho \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{3.2.1}$$

Going to the areal radial gauge is trivially achieved by introducing the coordinate r such that $\rho = r^2$. We consider the following Abelian, dyonic ansatz for the gauge fields

$$A = \left(\frac{Q_A e^{2\phi_0}}{\rho} dt - H_A \cos\theta d\varphi\right) \alpha^3 \ , \tag{3.2.2}$$

$$B = \left(\frac{Q_B e^{2\phi_0}}{\rho} dt - H_B \cos\theta d\varphi\right) \beta^3 \ , \tag{3.2.3}$$

where ϕ_0 is

$$\phi_0 = \frac{\ln\left(2\left(H_A^2 + H_B^2\right)\right)}{2} \ , \tag{3.2.4}$$

while the dilaton reads

$$\phi = \phi_0 - \frac{1}{2} \ln\rho \ . \tag{3.2.5}$$

The field equations are solved by

$$f(\rho) = \left(1 + \left(g_A^2 + g_B^2\right)\left(H_A^2 + H_B^2\right)\right)\rho - m + \frac{4\left(Q_A^2 + Q_B^2\right)\left(H_A^2 + H_B^2\right)}{\rho} \ , \tag{3.2.6}$$

with the constraint

$$H_A Q_A + H_B Q_B = 0 \ , \tag{3.2.7}$$

which comes from the field equation of the vanishing axion.

Depending on the relation between the integration constant m and the remaining charges, the spacetime (3.2.1) with the function (3.2.6) may describe a black hole, which can be extremal. The black hole asymptotically matches the metric

$$ds^2 = \left(1 + \left(g_A^2 + g_B^2\right)\left(H_A^2 + H_B^2\right)\right) r^2 dt^2 - \frac{dr^2}{\left(1 + \left(g_A^2 + g_B^2\right)\left(H_A^2 + H_B^2\right)\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{3.2.8}$$

which is asymptotically, locally flat since $R^{\mu\nu}{}_{\alpha\beta}$ goes to zero as $r \to \infty$.

[2]Notice that in [43] there is an extra γ factor in front of the sphere. Such factor can be gauged-away in the solution by an appropriate redefinition of the radial and time coordinates, including a shift in the integration constant ϕ_0 of the dilaton.

First, we are interested in revisiting the supersymmetry of this family of solutions by analyzing the consistency equations $\bar{\epsilon}\Theta = 0$ and $\bar{\epsilon}\Xi_{\mu\nu} = 0$ in (2.4.43). As in reference [43], we obtain that the 16×16 matrix Θ is block diagonal, and has the form

$$\Theta = \begin{pmatrix} \Theta_+ & 0 \\ 0 & \Theta_- \end{pmatrix} , \tag{3.2.9}$$

where $\Theta_\pm$ are 8×8 matrices, with determinants

$$\det(\Theta_\pm) = r^{-8}\left(K^0_\pm + rK^1_\pm + r^2K^2_\pm\right) . \tag{3.2.10}$$

The functions $K^{0,1,2}_\pm$ depend only on the integration constants. Requiring a nontrivial solution of $\bar{\epsilon}\Theta = 0$ for $\bar{\epsilon}$, obviously implies $\det\Theta_+ = 0$ or $\det\Theta_- = 0$, which leads to

$$(m \pm 4(H_BQ_A - H_AQ_B))^2 - 16\left(H_A^2 + H_B^2\right)^4 (Q_Ae_B \mp Q_Be_A)^2 = 0 \tag{3.2.11}$$

$$H_Ae_A \pm H_Be_B = 0 \tag{3.2.12}$$

On the other hand, the six, 16×16 matrices $\Xi_{\mu\nu}$ that can be read from (2.4.43), coming from the consistency condition of the variation of the Rarita-Schwinger field, also acquire a block-diagonal structure with non-trivial 8×8 blocks. The only components of $\Xi_{\mu\nu}$ in the consistency condition (2.4.43) with non-vanishing determinants, lead to the following expressions

$$0 = \det(\Xi_{tr}) = \frac{1}{(8r^2)^{16}}\left(-16(Q_Ae_B - Q_Be_A)^2\left(H_A^2 + H_B^2\right)^2 - 16\left(Q_A^2 + Q_B^2\right)\left(H_A^2 + H_B^2\right) + m^2\right)^4 \tag{3.2.13}$$

$$\times\left(-16(Q_Ae_B + Q_Be_A)^2\left(H_A^2 + H_B^2\right)^2 - 16\left(Q_A^2 + Q_B^2\right)\left(H_A^2 + H_B^2\right) + m^2\right)^4 \tag{3.2.14}$$

$$0 = \det(\Xi_{\theta\phi}) = \left(\frac{\sin\theta}{4}\right)^{16}(g_AH_A + g_BH_B)^8(g_AH_A - g_BH_B)^8 \tag{3.2.15}$$

For $\det\Theta_\pm = 0$ and $\det(\Xi_{tr}) = 0$, we have obtained the same equations than in reference [43]. Nevertheless, our expression for $\det(\Xi_{\theta\phi})$ differs from the one reported in [43], and in our case it can vanish for a suitable relation between the magnetic charges, which leads to a novel supersymmetric solution. Implementing all the supersymmetric constraints (3.2.11), (3.2.12), (3.2.13) and (3.2.14), the metric function $f(\rho)$ reduces to

$$f_{BPS} = \frac{((e_A^2 + e_B^2)^2 H_A^2 + e_B^2)}{e_B^2}\rho - m_{BPS} + \frac{4Q_A^2(e_A^2 + e_B^2)^2 H_A^2}{e_A^2 e_B^2 \rho} , \tag{3.2.16}$$

where

$$m_{BPS} = \frac{4H_A Q_A (e_A^2 + e_B^2)}{e_A e_B} . \tag{3.2.17}$$

Notice that the expression for m_{BPS} does not depend on the sign choice made in (3.2.11)-(3.2.12), therefore without loosing generality one can restrict to one of the two signs. One can see that $f_{BPS}(\rho)$ in (3.2.16) does not vanish, and since the spacetime (3.2.1) has a singularity at the origin $\rho = 0$, this solution represents a BPS naked singularity. It is well-known that singular spacetimes can indeed fulfill BPS conditions, as it is the case of the Reissner-Norstrom-AdS solution in $\mathcal{N} = 2$ $U(1)$ gauged supergravity, with the mass equal to the charge [58]. After a simple counting one sees that our BPS solution depends on two arbitrary integration constants.

The BPS background obtained with $Q_A = 0$, which implies $m_{BPS} = 0$, does not lead to an enhancement of the supersymmetries, and it is actually a singular spacetime, which has a divergence at the origin that is milder than the singularity at the origin of a Schwarzschild black hole, since in this case setting $H_A = H_B = m = 0$ in (3.2.6) leads to a Kretchmann scalar that diverges at the origin as r^{-4}, where r is the areal radial coordinate.

Notice that, as explained in [43] for planar black holes, the extremal configurations are $1/4$ BPS, while the background obtained by setting to zero all the integration constants, which is also the metric approached by the black holes at infinity, do acquire some extra supersymmetry leading to $1/2$ BPS solutions. As explained here, the situation for the spherically symmetric case is different and both, the metric deformed by the non-vanishing value of the charges, as well as the asymptotic metric, have the same amount of unbroken supersymmetry, namely they are $1/4$ BPS

The solutions we have identified as BPS, preserves one-quarter of the supersymmetry and one can pursue the explicit integration of the Killing spinors, which leads to the structure

$$\bar{\epsilon}_i = \Psi_1(\rho)D_i + \Psi_2(\rho)E_i \ . \tag{3.2.18}$$

The details are given in Appendix A4.

The metric (3.2.1) with the function (3.2.6) has four isometries, which close in the algebra $\mathbb{R} \times so\,(3)$. These are generated by the Killing vectors ∂_t plus the usual three Killing vectors of the round sphere spanned in spherical coordinates. We have explicitly checked that the bilinears

$$K^\mu = \bar{\epsilon}_i \gamma^\mu \epsilon_i \text{ no sum in } i \ , \tag{3.2.19}$$

do indeed lead to the isometries of the spacetime, for each of the four independent Killing spinors.

3.3 Charged merons and double meron in gauged supergravity

Hereafter we work with the following gauge for the metric

$$ds^2 = N\,(r)\, f\,(r)\, dt^2 - \frac{dr^2}{f\,(r)} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{3.3.1}$$

while the dilaton still depends only on the radial coordinate, namely $\phi = \phi\,(r)$.

3.3.1 Charged meron

Let us consider the following ansatz for the gauge fields

$$A = \xi U^{-1}_{(\alpha)} dU_{(\alpha)} \ , \ U_{(\alpha)}\,(x^\mu) \in SU\,(2) \ , \tag{3.3.2}$$

$$B = \frac{Q_B}{r^2} dt \ \beta_3 \ . \tag{3.3.3}$$

This ansatz corresponds to a superposition of an electric Abelian SU(2) gauge field and a meron configuration [59], the latter being proportional to the Maurer-Cartan left-invariant form of $su\,(2)$. It is interesting to notice that meron gauge fields also lead to black holes supported by non-Abelian gauge fields [60; 61], on which the Jackiw, Rebbi, Hasenfratz, 't Hooft mechanism of spin from isospin is present [62]-[63]. We further specialize the expressions for the meron to the hedgehog ansatz, in terms of a group valued function U given by

$$U^{\pm 1}_{(\alpha)} = \mathbf{1}\cos\Upsilon_{(\alpha)}(r) \pm \sin\Upsilon_{(\alpha)}(r)\,\hat{x}^i\alpha_i \ , \tag{3.3.4}$$

$$\hat{x}^1 = \sin\theta\cos\varphi \ , \quad \hat{x}^2 = \sin\theta\sin\varphi \ , \quad \hat{x}^3 = \cos\theta \ . \tag{3.3.5}$$

Here we are using the generator of $su(2)\times su(2)$ given (2.4.34) and (2.4.35). Setting $\Upsilon(r) = \pi/2$ we can substantially simplify the equations; the new solutions we find below, belong to such sector. The Yang-Mills equations as well as the equation for the dilaton are fulfilled when the constant ξ is fixed as

$$\xi = -\frac{1}{e_A} \ , \tag{3.3.6}$$

and the dilaton is given by

$$\phi = -\ln\left(\frac{e_A r}{\sqrt{2}}\right) \ , \tag{3.3.7}$$

therefore the gauge field A and the dilaton are devoid of integration constants. Consequently, the field strength associated to the gauge field A reads

$$A_{[2]} = \frac{1}{2}A_{\mu\nu}dx^\mu\wedge dx^\nu = \frac{1}{2e_A}\left(1-\frac{1}{2e_A}\right)\left[U^{-1}_{(\alpha)}\partial_\mu U_{(\alpha)}, U^{-1}_{(\alpha)}\partial_\nu U_{(\alpha)}\right]dx^\mu\wedge dx^\nu \ , \tag{3.3.8}$$

which implies that if the meron ansatz (3.3.2) turns out to be aligned with a single generator, its field strength would vanish. With these definitions, the gauge field A_μ has the following explicit form

$$A = -e_A^{-1}\left[(-\sin\varphi d\theta - \cos\theta\sin\theta\cos\varphi d\varphi)\,\alpha_1 + (\cos\varphi d\theta - \cos\theta\sin\theta\sin\varphi d\varphi)\,\alpha_2 + \sin^2\theta d\varphi\alpha_3\right] . \tag{3.3.9}$$

Some remarks on the nature of this ansatz are now in order. The configuration (3.3.9) is related to an Abelian one by the group element[3] $g = e^{\frac{1}{2}\theta\alpha_2}e^{\frac{1}{2}\phi\alpha_3}$, such that locally $A_{\text{monopole}} = gA_{\text{meron}}g^{-1} - \frac{2}{e_A}gdg^{-1}$. Given the fact that g does not go to the identity at infinity and it it not continuous as $r\to 0$, the two configurations A_{monopole} and A_{meron} are not gauge equivalent, since two gauge fields are gauge equivalent when they are related by a *smooth gauge transformation*

[3]We thank Andrés Anabalón for bringing this transformation to our attention.

which approaches an element of the center of the gauge group at spatial infinity. The discontinuity at the origin can be seen clearly by considering that

$$g(x=0,y=0,z=0^+) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \tag{3.3.10}$$

while

$$g(x=0,y=0,z=0^-) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} . \tag{3.3.11}$$

Such discontinuity will be a feature of any transformation relating both configurations, which confirms that they belong to different gauge equivalence classes. For a more detailed discussion, see Section V of [60]. Furthermore the transformation that relates (3.3.9) and A_{monopole} has a non-trivial winding, and therefore it is topologically non-trivial. This can be seen as follow. If the two configurations were globally gauge equivalent, it should be fulfilled that $F_{\text{monopole}} = gF_{\text{meron}}g^{-1}$. Nevertheless, the two configurations are related by

$$F_{\text{monopole}} = gF_{\text{meron}}g^{-1} + d\left(gdg^{-1}\right) - dg \wedge dg^{-1} . \tag{3.3.12}$$

For globally defined gauge transformations the last two terms in this equation cancel each other, nevertheless, this is not the case for the transformation generated by the group element $g = e^{\frac{1}{2}\theta\alpha_2}e^{\frac{1}{2}\phi\alpha_3}$, which in turn can be seen integrating equation (3.3.12) on the two dimensional surface $0 \le r \le 1$ and $0 \le \phi < 2\pi$ with $\theta = \theta_0$ and $t =$ constant. Such integration leads to

$$\int_{\text{disk}} F_{\text{monopole}} = \int_{\text{disk}} gF_{\text{meron}}g^{-1} + \int_{\partial\text{disk}=S^1} gdg^{-1} , \tag{3.3.13}$$

and the last term is non-vanishing and given by

$$w\left(gdg^{-1}\right) = \int_{\partial\text{disk}=S^1} gdg^{-1} = \pi\sin(\theta_0)\alpha_1 - \pi\cos(\theta_0)\alpha_3 . \tag{3.3.14}$$

This argument reinforces the fact that the meron configuration (3.3.9) cannot be identified as physically equivalent with the monopole configuration.

The metric of the spacetime is in the family of spherically symmetric solutions

(3.3.1) and it is explicitly given by

$$ds^2 = r^2\left(\frac{1}{2}+\frac{e_B^2}{4e_A^2}+\frac{e_A^2Q_B^2}{r^4}-\frac{\mu}{r^2}\right)dt^2-\frac{dr^2}{\left(\frac{1}{2}+\frac{e_B^2}{4e_A^2}+\frac{e_A^2Q_B^2}{r^4}-\frac{\mu}{r^2}\right)}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\ , \tag{3.3.15}$$

where μ is an integration constant. The metric (3.3.15) represents a black hole which can have an event horizon at $r=r_+$, as well as an inner Cauchy horizon located at $r=r_-$, while the integration constant μ is related to the mass of the black hole. The location of the horizons are

$$r_\pm = \sqrt{\frac{2e_A^2}{2e_A^2+e_B^2}\left(\mu\pm\sqrt{\mu^2-\left(2e_A^2+e_B^2\right)Q_B^2}\right)}\ . \tag{3.3.16}$$

Black hole solutions are present when

$$\mu^2 \geq \left(2e_A^2+e_B^2\right)Q_B^2\ , \tag{3.3.17}$$

and the spacetime becomes an extremal black hole when the bound is saturated. Asymptotically, the spacetime is locally flat and the scalar field $\phi\sim-\ln(r)$, reaches the absolute maximum of the potential (2.4.27).

The Hawking temperature in terms of r_+ reads

$$T = \frac{2e_A^2+e_B^2}{8\pi e_A^2}-\frac{e_A^2Q_B^2}{2\pi r_+^4}\ . \tag{3.3.18}$$

Notice that in the absence of the non-Abelian electric charge, namely for $Q_B=0$, the temperature reduces to a constant. The integration constant μ does not appear in the temperature when $Q_B=0$. If one pushes forward the interpretation of μ as the energy content of the spacetime, the family of black holes with $Q_B=0$ would lead to a divergence in the heat capacity $C\sim\frac{\partial\mu(T,Q)}{\partial T}$, which may be interpreted as a sign of criticality. Therefore the non-Abelian gauge field is needed in order to properly define the thermodynamics of these configurations.

Now, the analysis of the supersymmetry of this backgrounds is in order. In the presence of the meron ansatz (3.3.2), the analysis of supersymmetry is again dictated by the structure of the matrices Θ and $\Xi_{\mu\nu}$, defined in the previous sections. Actually in this case, it is enough to analyze the range of the matrix

$\Xi_{\theta\phi}$. In fact

$$\det\left(\Xi_{\theta\phi}\right) = \left(\frac{\sin\theta}{4}\right)^{16} \neq 0 \ . \tag{3.3.19}$$

This shows that the electric-meronic configuration we have constructed in this section, cannot preserve any supersymmetry.

In the next section, we move to the double meron configuration.

3.3.2 Double meron

Now, we introduce a meron ansatz in each of the $su\left(2\right)$ factors, namely

$$A = \xi_A \left[\left(-\sin\varphi d\theta - \cos\theta\sin\theta\cos\varphi d\varphi\right)\alpha_1 + \left(\cos\varphi d\theta - \cos\theta\sin\theta\sin\varphi d\varphi\right)\alpha_2 + \sin^2\theta d\varphi\alpha_3\right] \ , \tag{3.3.20}$$

$$B = \xi_B \left[\left(-\sin\varphi d\theta - \cos\theta\sin\theta\cos\varphi d\varphi\right)\beta_1 + \left(\cos\varphi d\theta - \cos\theta\sin\theta\sin\varphi d\varphi\right)\beta_2 + \sin^2\theta d\varphi\beta_3\right] \ . \tag{3.3.21}$$

Again, Yang-Mills and the dilaton equations are satisfied if

$$\xi_A = -\frac{1}{e_A} \ , \ \xi_B = -\frac{1}{e_B} \ , \tag{3.3.22}$$

and the dilaton field takes the form

$$\phi\left(r\right) = -\ln\left(\frac{e_B e_A}{\sqrt{2}\sqrt{e_A^2 + e_B^2}} r\right) \ . \tag{3.3.23}$$

Notice that as it is the case in the electrically charged meron, the gauge fields and the dilaton are completely fixed in terms of the couplings of the theory and are devoid of any integration constant. In this case the spacetime metric reduces to

$$ds^2 = r^2\left(\tilde{\Lambda}^2 - \frac{\mu}{r^2}\right)dt^2 - \frac{dr^2}{\tilde{\Lambda}^2 - \frac{\mu}{r^2}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{3.3.24}$$

where μ is an integration constant and

$$\tilde{\Lambda}^2 := \frac{e_A^4 + 3e_A^2 e_B^2 + e_B^4}{4e_A^2 e_B^2} > 0 \ . \tag{3.3.25}$$

When $\mu > 0$, this metric describes a black hole with an event horizon located at $r = r_+ = \left(\mu/\tilde{\Lambda}^2\right)^{1/2}$. It is worth emphasize that in (3.3.24) the $1/r^4$ term is absent, in contrast with (3.3.15). This implies that the electric and magnetic parts enter on different footing in these configurations. One way to understand this lack of democracy between electric and magnetic fields is the presence of the dilaton, which changes the electromagnetic duality properties of the theory.

Now, we move to the analysis of the supersymmetry of the double meron family. The non-vanishing integrability conditions for the Killing spinor, $\bar{\epsilon}\Theta = 0$ and $\bar{\epsilon}\Xi_{\mu\nu} = 0$ in (2.4.43) read

$$\det \Xi_{tr} = \frac{\mu^{16}}{r^{48}} \ ,$$
$$\det \Theta = \frac{\left(r^2 - \mu\right)^2 \left(r^2 + \mu\right)^2 \mu^4}{256 r^{32}} \ . \tag{3.3.26}$$

Therefore, the background of the black hole (3.3.24) with $\mu = 0$ preserves some supersymmetry. This background metric is actually one-quarter BPS, and the metric, which is also recovered as the asymptotic geometry of the black holes (3.3.24) reduces to

$$ds^2 = \tilde{\Lambda}^2 r^2 dt^2 - \tilde{\Lambda}^{-2} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{3.3.27}$$

This background possesses only the obvious Killing vectors as isometries, namely the time translation and the $SO\,(3)$ Killing vectors of the sphere at the r =constant surfaces of the spacelike surfaces at t =constant. Nevertheless, it is worth mentioning that this background has an extra conformal Killing vector given by

$$l = r\partial_r \ . \tag{3.3.28}$$

The temperature of the black hole (3.3.24) has the intriguing property of being independent of r_+, and it is given by

$$T = \frac{\tilde{\Lambda}^2}{2\pi} \ . \tag{3.3.29}$$

Wald's formula for the entropy yields

$$S = \frac{A}{4G} = 4\pi^2 r_+^2 \ , \tag{3.3.30}$$

since in the normalization of the Einstein term in (2.4.26) we have chosen $G =$

$(4\pi)^{-1}$. First law

$$dM = TdS \ , \tag{3.3.31}$$

provides the following value for the mass of the black hole

$$M = 2\pi\tilde{\Lambda}^2 r_+^2 \ . \tag{3.3.32}$$

Since the temperature of the black hole does not depend on its radius, we have that the heat capacity

$$C = \frac{dM}{dT} \ , \tag{3.3.33}$$

diverges, signaling the presence of a critical behavior. The free energy vanishes identically.

The expression for the temperature in (3.3.29) can be considered as evidence of the fact that the black hole we are currently analyzing is a particular case of a more general solution in the double meron sector. It would be interesting to see whether one can design a deformation of our ansatz that would allow to turn on the axion field $\mathbf{a}(x)$ in a simple enough manner as to construct new exact and possibly BPS solutions.

In the next sections, we show that there are families of potentials that go beyond the supergravity potential (2.4.27) and that allow for the construction of exact hairy black holes. As a matter of fact, we adapt the normalization and conventions in the action principle for each case, in order to simplify the presentation and analysis of the exact solutions.

3.4 Hairy black holes beyond supergravity

Since in the previous sections we have seen that the meron ansatz was fruitful in the construction of exact solutions in gauged supergravity, in what follows we will construct exact hairy black holes in the Einstein-Yang-Mills dilaton theory. Exact solutions in field theories with similar matter content have already been considered in the literature, see e.g. [49]-[57]). Here we focus on the meron sector of the theory, with a single $su(2)$ gauge field. We will introduce a suitable choice of the coefficient in the dilatonic coupling as well as a self-interaction for the dilaton, which allow for the construction of solutions in a closed form. The theories we consider in what follows can be seen as a truncation of the gauged supergravity of the previous sections, supplemented by a deformation of the theory that explicitly breaks the supersymmetry.

3.4.1 Exponential potential: topologically Lifshitz black holes

For the first family of potentials it is convenient to introduce the following suitable parametrization of the action

$$I\left[g_{\mu\nu}, A, \phi\right] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R - \frac{1}{2e^2}e^{-\frac{2}{\sqrt{z-1}}\phi}F^i_{\mu\nu}F^{i\mu\nu} - \frac{1}{2}\left(\partial\phi\right)^2 - V_1\left(\phi\right)\right) \tag{3.4.1}$$

where the potential is given by

$$V_1\left(\phi\right) = \xi e^{\sqrt{z-1}\phi} + \frac{2\left(z-1\right)}{\eta^2\left(z-2\right)}e^{\frac{\phi}{\sqrt{z-1}}} - \frac{\left(z+2\right)\left(z+1\right)}{l^2}\ , \tag{3.4.2}$$

with

$$\eta := \left(\frac{l^2}{4e^2\left(z+2\right)\left(z-1\right)}\right)^{1/4}\ . \tag{3.4.3}$$

These type of potentials are interesting by its own right and have been used in other context for Abelian and non-Abelian gauge fields with dilatonic couplings [50]-[52].

Here the field strength off the $SU(2)$ gauge field $A = A^i_\mu dx^\mu t_i$ reads

$$F = dA + A\wedge A \tag{3.4.4}$$

The range $z > 1$ will allow us to obtain a topologically Lifshitz asymptotic behavior. For $z > 2$ and $\xi \geq 0$, the potential (3.4.2) is clearly bounded from below, and it is characterized by three independent constants, ξ, l and z. The gauge coupling e also appears here. In these cases, the potential takes its minimum value (which is negative), when $\phi \to -\infty$. Notice that the potential (3.4.2) cannot be continuously connected with the supergravity potential in equation (2.4.27).

For this model, the field equations are solved by the following scalar field

$$\phi(r) = -2\sqrt{z-1}\ln\left(\frac{r}{\eta}\right) \ , \tag{3.4.5}$$

which approaches the minimum of the potential in the asymptotic region $r \to \infty$ since the metric of the spacetime in this case reads

$$ds^2 = -\frac{r^{2z}}{l^{2z-2}} g(r)\, dt^2 + \frac{dr^2}{r^2 g(r)} + r^2 d\Omega^2 \tag{3.4.6}$$

with

$$g(r) = \frac{1}{l^2} - \frac{\mu}{r^{z+2}} - \frac{1}{(z-2)\, z}\frac{1}{r^2} + \frac{\eta^{2(z-1)}\xi}{2(z-4)}\frac{1}{r^{2(z-1)}} \ . \tag{3.4.7}$$

While the gauge field as in the previous sections reads

$$A = \frac{1}{2}\left[\left(-\sin\varphi d\theta - \cos\theta\sin\theta\cos\varphi d\varphi\right)\alpha_1 + \left(\cos\varphi d\theta - \cos\theta\sin\theta\sin\varphi d\varphi\right)\alpha_2 + \sin^2\theta d\varphi\alpha_3\right] \ . \tag{3.4.8}$$

Here μ is an integration constant that will determine the energy content of the spacetime, namely its mass. It is important to stress that the spacetime configuration (3.4.6)-(3.4.7) is characterized by a single integration constant, since all the remaining variables, z, ξ, η, l, are determined by the self-interaction (3.4.2) and by the definition (3.4.3). Therefore, the black holes have a secondary hair.

Asymptotically, the spacetime defined by (3.4.6)-(3.4.7) approaches

$$ds^2_{\text{asymp}} = -\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2 dr^2}{r^2} + r^2 d\Omega^2 \ , \tag{3.4.9}$$

which is the topological extension of a Lifshitz spacetime [64]. Notice that, as $r \to \infty$ the constant r surfaces have an induced light cone structure which is non-relativistic in such limit, since for $r = r_c$ we have

$$ds^2_{\text{induced}} = -\frac{r_c^{2z}}{l^{2z}}dt^2 + r_c^2 d\Omega^2 \ , \tag{3.4.10}$$

and therefore a massless particle moving on the sphere S^2, along a trajectory $\theta = \theta(\lambda)$ and $\phi = \phi(\lambda)$, will fulfil

$$\frac{d\omega(\lambda)}{dt} = l^{2z} r_c^{2z-2} \ , \tag{3.4.11}$$

which goes to infinity as $r_c \to \infty$ when $z > 1$. Here we have defined $d\omega(\lambda) =$

$\sqrt{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2}d\lambda$. Lifshitz spacetimes are defined when the sphere in (3.4.9) is replaced by flat space. If we use Cartesian coordinates $\vec{x}$ for such flat space, the corresponding spacetime has the following anisotropic scaling symmetry $r \to \chi r$, $\vec{x} \to \chi^{-1}\vec{x}$ and $t \to \chi^{-z}t$, a symmetry that emerges for non-relativistic systems near criticality [65]. Topologically Lifshitz spacetimes break such scaling symmetry, nevertheless the non-relativistic interpretation of a potential dual theory living on the boundary of the spacetime remains, due to the previous argument [64]. A massless particle travelling radially takes a finite time to go from a point in the bulk to infinity, and the causal asymptotic behavior of (3.4.6)-(3.4.7) is that of AdS, therefore the spacetime has a timelike boundary.

The spacetime defined by (3.4.6) with (3.4.7), has a curvature singularity at $r = 0$, which may be covered by an event and a Cauchy horizon, depending on the details of the potential as well as on the value of the integration constant μ.

Since we do not have a supergravity embedding of the potential (3.4.2), which may allow us to prove the positivity of the energy, the cases with potentials bounded from below will be particularly relevant. This is ensured $z > 2$ and $\xi \geq 0$.

- For $2 < z \leq 4$, there is a minimum negative value of $\mu = \mu_e$, for which there is an extremal horizon[4]. For μ in the range $\mu_e < \mu < 0$, the topological Lifshitz black hole has an event and a Cauchy horizon. When this upper bound is saturated, the Cauchy horizon shrinks to cero, leading to a null singularity hidden by an event horizon. For $\mu > 0$, the spacetime has a single horizon, and its causal structure coincides with that of Schwarzschild-AdS.

- For $z > 4$ and $\xi = 0$, the structure of the black holes in terms of the different values of μ is exactly as the previous one. Nevertheless, for $z > 4$ and $\xi > 0$, the analysis changes. There is a minimum value of μ, that can have either sign, for which the black hole is extremal, and above which the are both an event and a Cauchy horizon.

[4] The case $z = 4$ can be integrated from scratch and it requires to impose $\xi = 0$ in the self-interaction (3.4.2). The metric function in this case reduces to (3.4.7), without the last term.

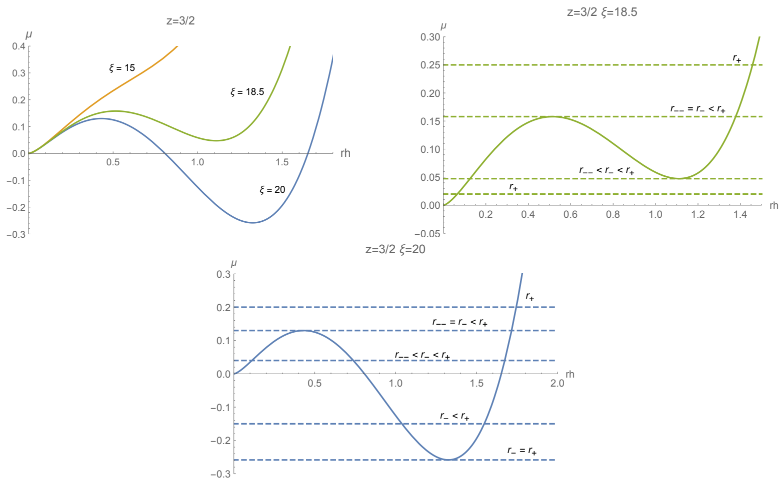

Figure 3.4.1: The panels show the structure of the curve $\mu = \mu(r_h)$ for different values of ξ for $z = 3/2$. For a given fixed value of μ the horizons are located at the intersections of the horizontal lines with the curves.

Even though the range $1 < z < 2$ leads to a potential (3.4.2) that is unbounded from below, there is a rich set of causal structures that are worth to be analyzed even when one maintains the restriction $\xi > 0$. In order to explore the parameter space, it is useful to consider the plane (r_h, μ) as well as the plane (μ, ξ), where r_h as a function of μ is obtained by solving $g\,(r_h) = 0$, with $g(r)$ given in (3.4.7).

In Figure 3.4.1, we have chosen $z = 3/2$ for the three panels, that depict the curves $\mu = \mu\,(r_h)$ for different values of the self-coupling ξ. For a given value of the integration constant μ, the horizons will be located at the crossings of the horizontal line $\mu = \mu_{\text{cte}}$ and the curves. In the first panel we see that there is a critical value of $\xi = \xi^1_{crit} \sim 17.2\sqrt{\frac{e}{l^3}}$, below which there is a single event horizon for any value of $\mu > 0$ (see e.g. the curve with $\xi = 15$). For $\xi^1_{crit} < \xi < \xi^2_{crit} \sim 18.78\sqrt{\frac{e}{l^3}}$ (see an example in the second panel), decreasing μ from infinity we have a black hole with a single horizon, which for a given value of μ develops a degenerate inner horizon, still surrounded by the external event horizon. Then, there is a range of values for μ that lead to a black hole inside a black hole structure. Such range for μ is bounded from below by a critical value that leads to an inner horizon covered by a degenerate external event horizon. Finally, for lower values of $\mu > 0$, one recovers causal structure of a unique event horizon. The second and third panels indicate the horizon structures for $\xi = 18.5$ and $\xi = 20$, respectively.

Finally, Figure 3.4.2 summarizes the causal structures that can be obtained, in the plane (μ, ξ) for $z = 3/2$.

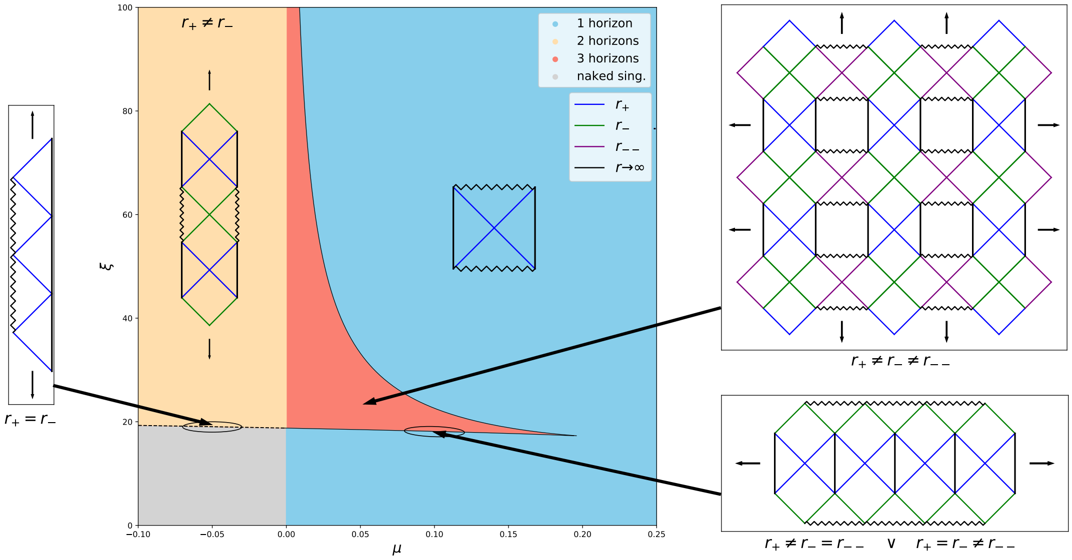

Figure 3.4.2: Causal structures that emerge for different values of the parameter ξ and the integration constant μ for $z = 3/2$.

It is also worth to discuss the thermal properties of the black holes we have considered in this section. As usual, the temperature of the event horizon can be computed requiring the regularity of the Euclidean continuation. The entropy can be computed using Wald's formula, and since all the fields are regular on the horizon, and the couplings of matter with the Einstein-Hilbert action are minimal, the entropy reduces to the Bekenstein-Hawking formula, which in the normalization of the action (3.4.1) takes its usual form

$$S = \frac{A}{4G} = \frac{\pi r_+^2}{G} \ . \tag{3.4.12}$$

Notice that the black holes that we are currently discussing are characterized by a single integration constant, that defines the energy content of the spacetime. One can obtain the mass of these black holes by the Abbott-Deser method, adapted to asymptotically, topologically Lifshitz black holes, which in this case leads to a finite result. In order to fix the absolute value of the energy using the Abbott-Deser method one requires to identify a background, nevertheless the first law will be fulfilled for any choice of such reference geometry. As discusses above, for $z > 2$ in order to have an event horizon, the value of μ is always bounded from below by an extremal case, which can be naturally chosen as a background metric (see e.g. [49]). One can show that for $z > 2$, for a given choice of the temperature, there is a unique branch of black holes, which are always thermally favoured with respect to the extremal background since the free energy of the latter is set to zero, while the free energy of the former is always negative. Figure 3.4.3 presents the corresponding plots.

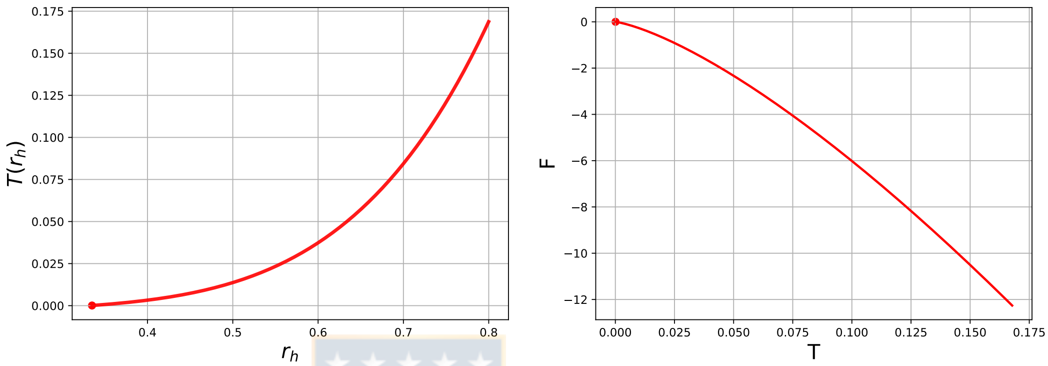

Figure 3.4.3: Temperature vs radius and Free energy vs temperature for black holes with $z = 5$, $\xi = 15\,\ell = 1$, $e = 1$.

The situation for $1 < z < 2$ is more subtle, since there is a range of ξ for which no extremal black hole exists, and therefore there is no natural choice of background in this case. In Figures 3.4.4 and 3.4.5 we have considered the thermal properties of event horizons $\xi = 18\ell$ and $\xi = 15\ell$, respectively, and fixing $z = 3/2$. The former family contains an extremal black holes while the latter does not.

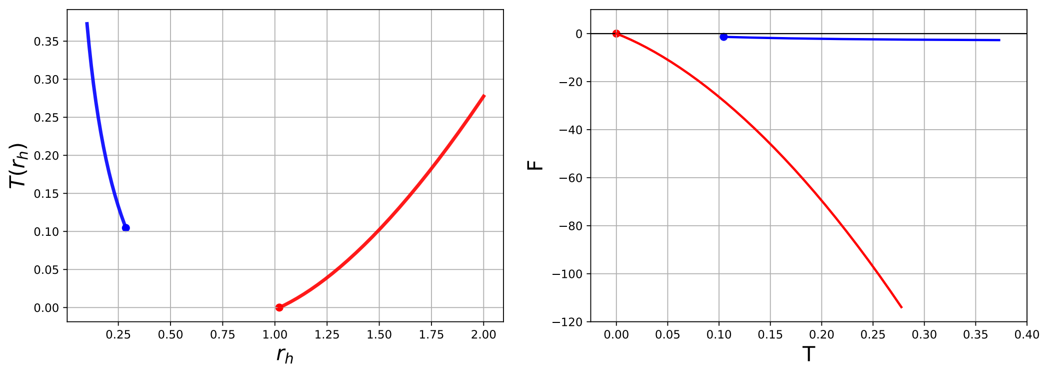

Figure 3.4.4: Temperature vs radius, and Free energy vs temperature for black holes with $z = 3/2$, $\xi = 18\,\ell = 1$, $e = 1$.

Let us discuss in some detail the case with the $z = \frac{3}{2}$ and $\xi = 18\ell$, which is particularly interesting (see Figure 3.4.4). As the mass decreases the radius of the event horizon shrinks and eventually the horizon becomes extremal at zero mass. If the mass falls below this minimum value, we still find a small black hole, which has a maximum size that is gaped with respect to radius of the extremal black hole (see also panel two of Figure 3.4.1). Large black holes, with radius above the extremal value always dominate the canonical ensemble as depicted in Figure 3.4.4. The free energy of the extremal case has been set to zero since we have used such geometry as background.

Some details of the case $z = 3/2$ and $\xi = 15$ are discussed in the caption of figure 3.4.5, a family that does not contain an extremal black holes.

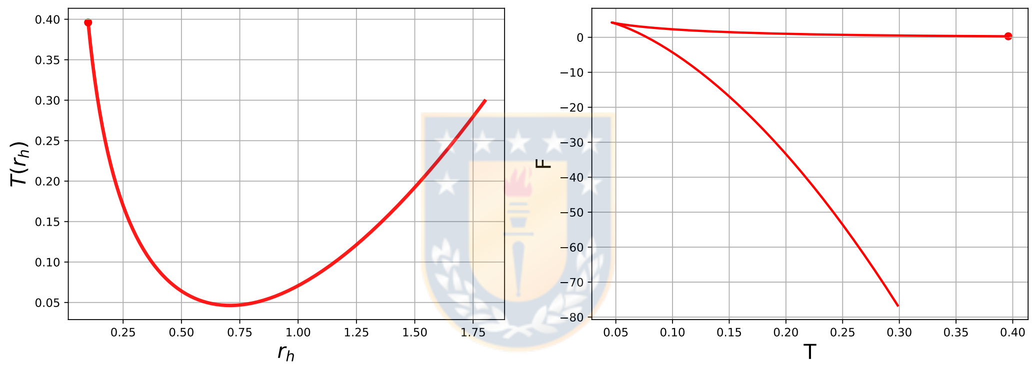

Figure 3.4.5: The figures show the temperature vs radius and Free energy vs temperature for black holes with $z = 3/2$, $\xi = 15$ and $l = 1$. There is no extremal black hole in this family. Thus, the second panel only leads to sensible information if one is comparing the free energy between the large and the small black holes. In this case the configuration with vanishing free energy is actually a naked singularity.

3.4.2 Linear times exponential potentials

The second family of potentials are given by a deformation of the scalar potential that appears in $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity (2.4.27). In this section, in order to present the exact solutions in a simple manner, we will see that it is useful to normalize the action functional as follows

$$I[g_{\mu\nu}, A, \phi] = \int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{e^{2\phi}}{2g^2}F^i_{\mu\nu}F^{i\mu\nu} - V_2(\phi)\right) \tag{3.4.13}$$

with

$$V_2(\phi) = \xi e^{-2\phi} + \zeta \phi e^{-2\phi} \ , \tag{3.4.14}$$

and

$$\zeta = \frac{(g^2\eta^2 - 1)}{\eta^4 g^2} \ , \tag{3.4.15}$$

which has to be interpreted as an equation for η in terms of the coupling g and ζ. The potential (3.4.14) is bounded from below when $\zeta > 0$ for either sign of ξ, while for $\zeta < 0$ the potential is unbounded from below. Notice that we have fixed the dilatonic coupling, which is required if we want to consider this theory as a deformation of a truncation of the mentioned supergravity theory, that is obtained by modifying the potential only. Indeed, when $\zeta = 0$, the theory (3.4.13) is recovered as a consistent truncation of $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity by setting $\xi = -\frac{g^2}{8}$ and $\eta = 1/g$. Again, the gauge field is given by the meron configuration (3.4.8) while in this case the dilaton reduces to

$$\phi(r) = \ln\left(\frac{r}{\eta}\right) \ . \tag{3.4.16}$$

For the present family of solutions, the metric of the spacetime is given by

$$ds^2 = -r^2 f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \tag{3.4.17}$$

with

$$f(r) = -\zeta \ln\left(\frac{r}{\eta}\right)\eta^2 + (\zeta - \xi)\,\eta^2 + \frac{1}{4\eta^2 g^2} - \frac{\mu}{r^2} \ . \tag{3.4.18}$$

Here μ is an integration constant. In spite of the fact that the metric has a logarithmically growing term in the areal radial coordinate, the curvature of the spacetime vanishes when $r \to \infty$, therefore the spacetime is asymptotically locally flat.

Using the Abbott-Deser method, and choosing as a background the spacetime with $\mu = 0$, leads to the following expression for the mass (cf. equation (3.4.13)):

$$M = 2\pi\mu \ . \tag{3.4.19}$$

Now we proceed to analyze the causal structures contained in this family of black holes. The equation for the zeros of the function $f(r)$ in (3.4.18) leads to the a Lambert W function[5]. For $\zeta > 0$ the largest zero of (3.4.18) represents a cosmological horizon, while for $\zeta < 0$ the largest zero of (3.4.18) gives rise to an event horizon. When ζ vanishes there could be only one horizon located at

$$r_h\big|_{\zeta=0} = \left(\frac{4\mu g^2}{g^2 - 4\xi}\right)^{\frac{1}{2}} , \tag{3.4.20}$$

which for $g^2 > 4\xi$ and $\mu > 0$ is an event horizon, while for $g^2 < 4\xi$ and $\mu < 0$ represents a cosmological horizon.

Now, let us further analyze the Lambert W function obtained for the location of the would be horizons, which will be useful in the study of the thermal properties of these black holes. Indeed, the horizons will be located at $r = r_h$ such that

$$f(r) = a + br_h^2 \ln r + cr_h^2 = 0 , \tag{3.4.21}$$

which setting $x = r_h^{-2}$ leads to

$$ax - \frac{b}{2}\ln x + c = 0 , \tag{3.4.22}$$

where

$$a = -\mu,\; b = -\zeta\eta^2,\; c = \frac{1}{4\eta^2 g^2} + (\zeta - \xi)\,\eta^2 + \zeta\eta^2 \ln \eta . \tag{3.4.23}$$

The existence, and number of solutions of (3.4.22) depends on the value of the discriminant[6]

$$\Delta = -2\frac{a}{b}e^{2\frac{c}{b}} . \tag{3.4.24}$$

We have two zeros when $-\frac{1}{e} < \Delta < 0$, which restricts the values of μ depending also on the sign of ζ (see analysis below). Within such ranges the two roots of (3.4.22) are given by

$$r_{h_2} = e^{\frac{1}{2}W_0(\Delta) - \frac{c}{b}}, \qquad r_{h_1} = e^{\frac{1}{2}W_{-1}(\Delta) - \frac{c}{b}} , \tag{3.4.25}$$

[5]The Lambert W function is defined by $W(z)\, e^{W(z)} = z$. This relation leads to a multivalued $W = W(z)$. Restricting our attention to $W(z) : \mathbb{R} \to \mathbb{R}$, there are two branches $W_0(z) : \left(-e^{-1}, \infty\right) \to (-1, +\infty)$ and $W_1 : \left(-e^{-1}, 0\right) \to (-1, -\infty)$.

[6]See reference [66] for an example of a different setup where the location of the event horizon of a black hole is given in terms of a Lambert W function. The formulae in the appendix of such reference are relevant to our analysis.

and since $W_0(x) > W_{-1}(x)$, $x \in \left(-\frac{1}{e}, 0\right)$, the root r_{h_2} is always larger than r_{h_1}. For $\zeta > 0$ in order to have two roots,μ must lie within the range $0 < \mu < \mu_{\max} = \frac{\eta^2 \zeta}{2} e^{1+\frac{2c}{b}}$, and as mentioned above $r_{h_2} = r_{++}$ and $r_{h_1} = r_+$ correspond to a cosmological and an event horizon, respectively. On the other hand, for $\zeta < 0$ and $\mu_{\min} = -\frac{\eta^2 |\zeta|}{2} e^{1+\frac{2c}{b}} < \mu < 0$, there are two horizons as well, located at $r_{h_2} = r_+$ and $r_{h_1} = r_-$, which in this case correspond to an event and a Cauchy horizon, respectively.

For $\Delta \geq 0$ and for $\Delta = -\frac{1}{e}$, there is a unique horizon, which can be an event or a cosmological horizon, depending on the values of the remaining constants. Finally for $\Delta < -e^{-1}$, there are no horizon.

Let us now provide the expression for the temperature and the entropy for a horizon located at $r = r_\star$. The former reads

$$T_\star = \frac{1}{4\pi} \sqrt{\frac{d}{dr}\left(r^2 \left(\frac{a}{r^2} + b\ln r + c\right)\right) \frac{d}{dr}\left(\left(\frac{a}{r^2} + b\ln r + c\right)\right)}\Bigg|_{r=r_\star} \tag{3.4.26}$$

$$= \frac{\sqrt{b}}{4\pi} \sqrt{b + bW_\star(\Delta) - 2ae^{-W_\star(\Delta)+2\frac{c}{b}} - 2aW_\star(\Delta)\, e^{-W_\star(\Delta)+2\frac{c}{b}}}\ , \tag{3.4.27}$$

while the entropy, is given by the Bekenstein-Hawking formula, which in the normalization of (3.4.13) leads to

$$S_\star = 4\pi^2 e^{W_\star(\Delta) - \frac{2c}{b}}\ . \tag{3.4.28}$$

Here $W_\star(\Delta)$ has to be understood as $W_{-1}(\Delta)$ or $W_0(\Delta)$ depending on whether we are dealing with a single event horizon, a cosmological and an event horizon, or an event and a Cauchy horizon.

As a final consistency check, let us verify that the first law is fulfilled, which will turn out to require using some identities satisfied by the Lambert W function. Our black holes are characterized by a single integration constant, μ, and the difference in entropy of two equilibrium configurations corresponding to the values

μ and $\mu + \delta\mu$, is given by

$$\delta S_\star = 4\pi^2 e^{W_\star(\Delta) - \frac{2c}{b}} \delta \left(W_\star (\Delta) \right) \tag{3.4.29}$$

$$= 4\pi^2 e^{W_\star(\Delta) - \frac{2c}{b}} W'_\star (\Delta) \frac{2\delta\mu}{b} e^{2\frac{c}{b}} \tag{3.4.30}$$

$$= -4\pi^2 \frac{e^{W_\star(\Delta) - \frac{2c}{b}}}{\Delta + e^{W_\star(\Delta)}} \frac{\Delta}{a} \delta\mu \ , \tag{3.4.31}$$

where in going from the before to last to the last line we have used the following identity of the derivative of the Lambert W function:

$$W'_\star (\Delta) = \left(\Delta + e^{W_\star(\Delta)} \right)^{-1} \ . \tag{3.4.32}$$

Multiplying $\delta S_\star$ times the temperature (3.4.27) leads to

$$T_{GH} \delta S_\star = -4\pi^2 \delta\mu b e^{-2\frac{c}{b}} \frac{\Delta}{a} \frac{\sqrt{b}}{4\pi} \sqrt{\frac{1 + W_\star (\Delta)}{b + \frac{W_\star(\Delta)}{\Delta} b \left(-2\frac{a}{b} e^{\frac{2c}{b}} \right)}} \tag{3.4.33}$$

$$= -\pi \delta\mu b \frac{1}{a} \left(-2\frac{a}{b} e^{2\frac{c}{b}} \right) e^{-2\frac{c}{b}} \tag{3.4.34}$$

$$= 2\pi\delta\mu \tag{3.4.35}$$

which shows that the first law is fulfilled for every potential type of horizon

$$\delta M = T_- \delta S_- \text{ and } \delta M = T_+ \delta S_+ \text{ and } \delta M = T_{++} \delta S_{++} \ . \tag{3.4.36}$$

Chapter 4

Exact scalar (Quasi-)normal modes of black hole and solitons

Quasinormal modes play a very important role both in astrophysical as well as in theoretical contexts. In the former, they dominate the ringdown dynamics of the final black hole obtained from the fusion of compact objects, and a direct measurement of the mode with the lowest damping helps obtaining the mass and angular momentum of the final object [79]. In the latter, black hole quasinormal modes, within the context of holography, allow for the computation of relaxation properties of the dual field theory living at the boundary of AdS [80],[81]. It is well-known that even for simple black holes, as for example for Schwarzschild-(A)dS the computation of quasinormal modes relies on numerical techniques. These techniques are fully reliable, notwithstanding there are particular interesting cases where the spectrum of quasinormal frequencies can be found analytically which are useful to explore the relaxation properties of perturbations outside a black hole in an exact manner as one modifies the parameters that define the background geometry. A partial list of such cases is given by [82]-[94]. We have identified a new family of black holes and solitons that allow for the exact integration of non-minimally coupled scalar probes, in the context of $SU(2) \times SU(2)$ $\mathcal{N} = 4$ gauged supergravity in four dimensions. We have identified this theory in section 2.4.2 as the dimensional reduction on $S^3 \times S^3$ from 10D supergravity [40], [41]. In this chapter we will consider the mostly plus signature for which the action principle reads

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{4} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{4\phi}\partial_\mu a\partial^\mu a + \frac{e_A^2+e_B^2}{8}e^{2\phi}\right. \qquad (4.0.1)$$
$$\left. - \frac{e^{-2\phi}}{4}\left(A^{i\mu\nu}A_{i\mu\nu} + B^{i\mu\nu}B_{i\mu\nu}\right) - \frac{\mathbf{a}}{4}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\left(A_{i\mu\nu}A^i{}_{\rho\sigma} + B_{i\mu\nu}B^i{}_{\rho\sigma}\right)\right]$$

This chapter contains the second part of new results of this thesis and it is in process to be published in JHEP [103]. We will focus on the computation of quasinormal modes of scalar probes on black holes of this theory as well as on the computation of normal frequencies of the same probe fields on the gravitational soliton constructed in the previous section, both in the supersymmetric and non-supersymmetric cases. We will deal with solutions with vanishing axion field, and since the self-interaction of the dilaton does not have a local extremum, the solutions have an asymptotic structure that has less symmetry than a maximally symmetric spacetime, although we will see the emergence of an asymptotic conformal Killing vector.

4.1 Scalar (quasi-)normal modes of black holes and solitons

The two families of black holes we will be interested in this section were constructed in [43]. The metric in both cases, namely spherical and planar, reads

$$ds^2 = -\alpha r^2\left(1-\frac{r_+^2}{r^2}\right)dt^2 + \frac{dr^2}{\alpha\left(1-\frac{r_+^2}{r^2}\right)} + r^2 d\Sigma_2^2\ , \qquad (4.1.1)$$

where Σ_2 is a two-dimensional Euclidean manifold of constant curvature $\gamma = +1, 0$.

In the spherically symmetric case, $\gamma = +1$ and $d\Sigma_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element of the round two-sphere, while the constant α, the dilaton and gauge fields read

$$\alpha = \frac{1}{2}\left(e_A^2 + e_B^2\right)\left(H_A^2 + H_B^2\right) + \frac{1}{4} \ , \tag{4.1.2}$$

$$\phi\left(r\right) = -\ln\left(\frac{r}{2\sqrt{H_A^2 + H_B^2}}\right) \ , \tag{4.1.3}$$

$$A_{[1]}^i = -H_A \cos\theta d\varphi \delta_3^i \ , \tag{4.1.4}$$

$$B_{[1]}^i = -H_B \cos\theta d\varphi \delta_3^i \ . \tag{4.1.5}$$

In the planar case, $\gamma = 0$, $d\Sigma_2^2 = dx^2 + dy^2$ the gauge fields vanish and

$$\alpha = \frac{e_A^2 + e_B^2}{8} \ , \tag{4.1.6}$$

$$\phi\left(r\right) = -\ln\left(r\right) \ . \tag{4.1.7}$$

The black holes (4.1.1) approach the background

$$ds_{\text{back}}^2 = -\alpha r^2 dt^2 + \frac{dr^2}{\alpha} + r^2 d\Sigma_2^2 \ , \tag{4.1.8}$$

with the following asymptotic behavior

$$\delta g_{tt} = \mathcal{O}\left(1\right), \delta g_{rr} = \mathcal{O}\left(r^{-2}\right) \ . \tag{4.1.9}$$

Notice that the background (4.1.8) has an extra conformal Killing vector generated by $r \to \lambda r$. The temperature of this black hole has the intriguing property of being independent of the r_+, namely a constant, and in this normalization is given by

$$T = \frac{\alpha}{2\pi} \ . \tag{4.1.10}$$

As we show below, a similar feature occurs with the quasinormal frequencies of the non-minimally coupled scalar on this geometry, which do not depend on r_+, leading to isospectral geometries in what regards to such operator. Wald's formula for the entropy yields

$$S = \frac{A}{4G} = \pi r_+^2 \text{Vol}\left(\Sigma\right) \ , \tag{4.1.11}$$

where Vol(Σ) is the volume of the Euclidean manifold Σ_2 and we have normalized the Einstein term in the action (4.0.1) such that $G=(4\pi)^{-1}$. First law

$$dM = TdS \ , \tag{4.1.12}$$

provides the following value for the mass of the black hole

$$M = \frac{\alpha r_+^2 \mathrm{Vol}\left(\Sigma\right)}{2} \ . \tag{4.1.13}$$

Here, as an avatar for the study of the stability of these black holes, we will consider a real scalar probe, coupled to the Ricci scalar in a non-minimal manner:

$$\Box\Phi - \xi R\Phi = 0 \ , \tag{4.1.14}$$

on the background (4.1.1).

Given the local isometries of the spacetime, the scalar probe admits a mode separation which is given by

$$\Phi\left(t,r,y^i\right) = \mathrm{Re}\left(\int d\omega \sum_A e^{-i\omega t} H_{\omega,A}\left(r\right) Y_A\left(y\right)\right) \ , \tag{4.1.15}$$

where y^i are the coordinates on the Euclidean manifold Σ_2 and $Y_k\left(y\right)$ are harmonic function on such manifold, which are labeled by the multi-index A. Concretely, for the spherically symmetric case the harmonic functions are standard spherical harmonics, namely $A=\{l,m\}$ and they fulfil

$$\nabla^2_{S^2} Y_{l,m} = -k^2 Y_{l,m} = -l\left(l+1\right) Y_{l,m} \ , \tag{4.1.16}$$

while for the planar case, the harmonic functions are trivially given by plane waves of the form

$$Y_A = Y_{\vec{k}} = Ce^{-i\vec{k}\cdot\vec{y}} \ , \tag{4.1.17}$$

which fulfil

$$\nabla_{R^2} Y_{\vec{k}} = -k^2 Y_{\vec{k}} = -\left(k_1^2 + k_2^2\right) Y_{\vec{k}} \ . \tag{4.1.18}$$

Hereafter, for brevity we introduce the notation $H_{\omega,A}\left(r\right) = H\left(r\right)$.

Notice that since the Ricci scalar of the spacetime has a non-trivial radial profile

$$R = \frac{2\gamma - 6\alpha}{r^2} - \frac{2\alpha r_+^2}{r^4} \ , \tag{4.1.19}$$

the non-minimal coupling term in (4.1.14) cannot be seen as an effective mass term. In spite of this fact, we will show that the equation for the radial profile of the scalar probe $H(r)$ can be solved in an exact manner in terms of hypergeometric functions.

Introducing the separation (4.1.15) on the scalar field equation (4.1.14) as a probe field on the black hole metric (4.1.1), after performing the change of variables

$$r = \frac{r_+}{(1-x)^{1/2}} \ , \tag{4.1.20}$$

which maps the region of outer communication $r \in [r_+, +\infty[$ to $x \in [0,1[$, leads to the following equation for the radial profile

$$\frac{d^2H(x)}{dx^2} + \frac{1}{x}\frac{dH(x)}{dx} + \left(\frac{\omega^2}{4\alpha^2 x^2 (1-x)^2} - \frac{k^2}{4\alpha x (1-x)^2} - \frac{(\alpha x - 4\alpha + \gamma)}{2\alpha x (1-x)^2}\xi\right) H(x) = 0 \ . \tag{4.1.21}$$

Remarkably, this equation admits a solution in terms of hypergeometric functions. After imposing ingoing boundary condition at the horizon one obtains

$$H(x) = C_1 x^{-\frac{i\omega}{2\alpha}} (1-x)^{\frac{\alpha + \sqrt{(1-6\xi)\alpha^2 + 2\xi\alpha\gamma + \alpha k^2 - \omega^2}}{2\alpha}} F(a_1, b_1, c_1, x) \ , \tag{4.1.22}$$

with

$$a_1 = \frac{1}{2} - \frac{i\omega}{2\alpha} - \frac{\sqrt{2\xi}}{2} + \frac{\sqrt{(1-6\xi)\alpha^2 + 2\xi\alpha\gamma + \alpha k^2 - \omega^2}}{2\alpha} \ , \tag{4.1.23}$$

$$b_1 = \frac{1}{2} - \frac{i\omega}{2\alpha} + \frac{\sqrt{2\xi}}{2} + \frac{\sqrt{(1-6\xi)\alpha^2 + 2\xi\alpha\gamma + \alpha k^2 - \omega^2}}{2\alpha} \ , \tag{4.1.24}$$

$$c_1 = 1 - \frac{i\omega}{\alpha} \ . \tag{4.1.25}$$

Using Kummer identities, the ingoing solution (4.1.22) can be rewritten as

$$H\left(x\right)=C_1x^{-\frac{i\omega}{2\alpha}}\left(1-x\right)^{\frac{\alpha+\sqrt{(1-6\xi)\alpha^2+2\xi\alpha\gamma+\alpha k^2-\omega^2}}{2\alpha}} \tag{4.1.26}$$

$$\times\left[\frac{\Gamma\left(c_1\right)\Gamma\left(c_1-a_1-b_1\right)}{\Gamma\left(c_1-a_1\right)\Gamma\left(c_1-b_1\right)}F\left(a_1,b_1,a_1+b_1+1-c_1,1-x\right)\right.$$
$$\left.+\left(1-x\right)^{c_1-a_1-b_1}\frac{\Gamma\left(c_1\right)\Gamma\left(a_1+b_1-c_1\right)}{\Gamma\left(a_1\right)\Gamma\left(b_1\right)}F\left(c_1-a_1,c_1-b_1,1+c_1-a_1-b_1,1-x\right)\right]\ , \tag{4.1.27}$$

which near infinity, as a function of the radial coordinate r, leads to

$$H\left(r\right)\sim_{r\to\infty}\frac{A_{bh}}{r^{\eta_+}}\left(1+O\left(\frac{1}{r}\right)\right)+\frac{B_{bh}}{r^{\eta_-}}\left(1+O\left(\frac{1}{r}\right)\right) \tag{4.1.28}$$

where

$$A_{bh}=\frac{\Gamma\left(c_1\right)\Gamma\left(c_1-a_1-b_1\right)}{\Gamma\left(c_1-a_1\right)\Gamma\left(c_1-b_1\right)}\ , \tag{4.1.29}$$

$$B_{bh}=\frac{\Gamma\left(c_1\right)\Gamma\left(a_1+b_1-c_1\right)}{\Gamma\left(a_1\right)\Gamma\left(b_1\right)}\ , \tag{4.1.30}$$

and

$$\eta_\pm=1\pm\sqrt{\left(1-6\xi\right)+\frac{\left(2\gamma\xi+k^2\right)}{\alpha}-\frac{\omega^2}{\alpha^2}}\ . \tag{4.1.31}$$

This implies that different modes will have polynomial asymptotic expansions at infinity in the radial coordinate, with an exponent that is frequency dependent. This is in contrast with the asymptotically flat case for which $R\left(r\right)\sim e^{\pm i\omega r}$, and with the asymptotically AdS case for which $\eta_\pm=\Delta_\pm$ being independent of both the angular momentum k and the frequency ω. Since in general $\omega\in\mathbb{C}$, in order to understand the possible boundary conditions at infinity, we will require the action principle to attain an extremum on the family of solutions that are ingoing at the horizon. The action principle leading to (4.1.14) reads

$$I=\int d^4x\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi-\frac{1}{2}\xi R\Phi^2\right)\ , \tag{4.1.32}$$

and its on-shell variation with respect to the scalar field leads to the boundary term

$$\delta I=-\int_M d^4x\sqrt{-g}\nabla_\mu\left(\nabla^\mu\Phi\delta\Phi\right)=-\int_{\partial M}d^3x\sqrt{-\gamma}\hat{n}_\mu\nabla^\mu\Phi\delta\Phi\ , \tag{4.1.33}$$

where γ is the determinant of the induced metric on the boundary, while $\hat{n}_\mu$ is its unit normal vector. The boundary is the union of the spatial surfaces at $t = t_i$ and $t = t_f$, with the surface $r = r_0$ with $r_0 \to \infty$. As usual, the contribution of the former vanish since we impose $\delta\Phi\left(t_i, r, y\right) = \delta\Phi\left(t_f, r, y\right) = 0$, while the latter leads to

$$- \lim_{r_0\to\infty} r^3 \partial_r H \delta H|_{r=r_0} = \lim_{r_0\to\infty} \left. \frac{\eta_+ A_{bh}^2}{r^{2\eta_+ -2}} + \frac{\eta_- B_{bh}^2}{r^{2\eta_- -2}} + \frac{A_{bh} B_{bh}\left(\eta_+ + \eta_-\right)}{r^{\eta_+ + \eta_- -2}} \right|_{r=r_0} \delta C_1 \tag{4.1.34}$$

One can check that $\eta_+ + \eta_- - 2$ vanishes, while $\mathrm{Re}\left(2\eta_- - 2\right) < 0$ and $\mathrm{Re}\left(2\eta_+ - 2\right) > 0$ on the whole complex ω-plane, therefore in order to obtain a genuine extremum of the on-shell action principle on the ingoing solution at the horizon, we need to impose $B_{bh} = 0$. From the view point of the asymptotic expansion (4.1.28), this corresponds to a Dirichlet boundary condition. Considering the expression for B_{bh} in (4.1.30) we obtain the following two equations for the spectrum

$$a_1 = \frac{1}{2} - \frac{i\omega}{2\alpha} - \frac{\sqrt{2\xi}}{2} + \frac{\sqrt{\left(1 - 6\xi\right)\alpha^2 + 2\xi\alpha\gamma + \alpha k^2 - \omega^2}}{2\alpha} = -p \text{ with } p = 0, 1, 2, \ldots \,, \tag{4.1.35}$$

$$b_1 = \frac{1}{2} - \frac{i\omega}{2\alpha} + \frac{\sqrt{2\xi}}{2} + \frac{\sqrt{\left(1 - 6\xi\right)\alpha^2 + 2\xi\alpha\gamma + \alpha k^2 - \omega^2}}{2\alpha} = -q \text{ with } q = 0, 1, 2, \ldots \,. \tag{4.1.36}$$

Equation (4.1.35) leads to the following purely imaginary spectrum

$$\omega_p = -\frac{\left(\left(2\gamma - 8\alpha\right)\xi + 2\sqrt{2\xi}\left(1 + 2p\right)\alpha - 4p\left(1 + p\right)\alpha + k^2\right)\left(1 + 2p + 2^{1/2}\xi^{1/2}\right)}{4\xi - 2(1 + 2p)^2} i \,, \tag{4.1.37}$$

which is a valid solution of (4.1.35) provided

$$\nu_p := \frac{\left(1 - 4\xi + (2p + 1)^2\right)\alpha + 2\gamma\xi + k^2 - 2\left(2p + 1\right)\alpha\sqrt{2\xi}}{4\left(1 + 2p - \sqrt{2\xi}\right)\alpha} < 0 \,. \tag{4.1.38}$$

On the other hand, equation (4.1.36) leads to the following set of frequencies

$$\omega_q = -\frac{((2\gamma - 8\alpha)\xi - 2\sqrt{2\xi}\left(1 + 2q\right)\alpha - 4q(1 + q)\alpha + k^2)(1 + 2q - \sqrt{2\xi})}{4\xi - 2(1 + 2q)^2} i \,, \tag{4.1.39}$$

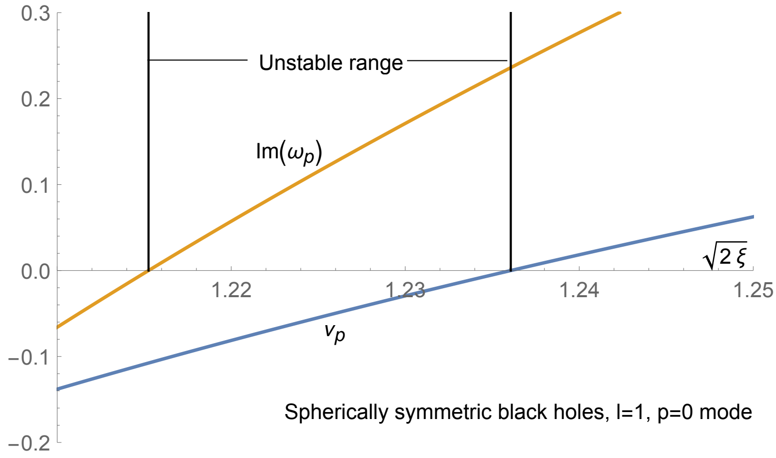

Figure 4.1.1: Frequency ω_p and ν_p on the spherically symmetric black hole $\gamma = 1$, for the non-minimally coupled scalar with non-minimal coupling ξ, on the mode with $p = 0$ and angular momentum $l = 1$. The allowed modes correspond to values of ξ such that $\nu_p < 0$. The modes with frequencies with positive imaginary part are unstable. In consequence, stability sets an upper bound on the value of the non-minimal coupling parameter.

which is instead a valid solution of (4.1.36) provided

$$\nu_q := \frac{\left(1 - 4\xi + (2p+1)^2\right)\alpha + 2\gamma\xi + k^2 + 2\,(2p+1)\,\alpha\sqrt{2\xi}}{4\left(1 + 2p + \sqrt{2\xi}\right)\alpha} < 0\ . \qquad (4.1.40)$$

It can be checked that both spectra (4.1.37) and (4.1.39), in the "s-wave" case ($k^2 = 0$) lead purely imaginary frequencies with negative imaginary part. The conditions (4.1.38) and (4.1.40) restrict the values of the non-minimal coupling parameter that lead to non-trivial spectra. An exhaustive exploration of these spectra is beyond the scope of this work, nevertheless, for the spherically symmetric black holes, with $k^2 = l\,(l+1) = 2$ we find a range of values of the non-minimal coupling ξ leading to unstable modes coming from the spectrum (4.1.37) when $p = 0$. Figure 1 depicts both $\mathrm{Im}\,(\omega_p)$ and ν_p from (4.1.37) and (4.1.38), respectively for a certain range of the non-minimal coupling, showing the presence of valid modes with $\mathrm{Im}\,(\omega_p) > 0$, therefore unstable. Notice that there is a valid mode for which $\omega = 0$, which can be interpreted as a static scalar cloud [95]. The existence of these static solutions are usually interpreted as smoking guns for the existence of a new branch of solutions in which the probe becomes fully backreacting (see e.g. [96]). Notice that in our case, the would-be static backreacting solution might be non-spherically symmetric since $l = 1$.

We can see from equations (4.1.38) and (4.1.40) that for a massless, minimally coupled scalar, namely for $\xi = 0$, it is not possible to fulfill the boundary conditions and there are no quasinormal modes of such massless scalar probe fields on the black hole background. This situation is similar to what occurs for a massless scalar probe on the asymptotically locally flat, static black holes in New Massive Gravity [93]. In the present case, a non-vanishing value of the non-minimal coupling allows for non-trivial quasinormal modes, provided (4.1.38) and (4.1.40) are fulfilled. It is very interesting to notice that such quasinormal frequencies do not depend on the black hole mass $M = M(r_+)$, and therefore all the black holes in the family (4.1.1) for different values of r_+ are isospectral in what regards the quasinormal modes of the non-minimally coupled scalars. Notice that this is the case both, in the spherically symmetric and planar cases recovered by setting $\gamma = 1$ and $\gamma = 0$, respectively.

It is also illuminating to rewrite the second order equation for the radial profile of the non-minimally coupled scalar probe in a Schroedinger-like form. This is achieved by introducing the tortoise coordinate r_* for the metric (4.1.1)

$$r_* = \frac{1}{2\alpha} \ln\left(r^2 - r_+^2\right) \to r = \sqrt{r_+^2 + e^{2\alpha r_*}} \ , \tag{4.1.41}$$

which maps $r \in]r_+, \infty[$ to the whole real line, i.e. $r_* \in]-\infty, +\infty[$. Notice that we have been able to explicitly solve r in terms of r_*, which is not possible for Schwarzschild black hole. Using this fact, we can obtain the potential of the Schroedinger-like equation explicitly in terms of r_*. Introducing

$$F(r) = \frac{H(r)}{r} \ , \tag{4.1.42}$$

leads to

$$-\frac{d^2 F}{dr_*^2} + U(r_*) F = \omega^2 F \ , \tag{4.1.43}$$

with

$$U(r_*) = \frac{\alpha\left[r_+^2 \left(2(1-4\xi)\alpha + 2\gamma\xi + k^2\right) e^{2\alpha r_*} + e^{4\alpha r_*}\left((1-6\xi)\alpha + 2\gamma\xi + k^2\right)\right]}{\left(r_+^2 + e^{2\alpha r_*}\right)^2} \ . \tag{4.1.44}$$

Notice that this potential always vanishes in the near horizon region, namely when $r_* \to -\infty$. Even more, when $\xi = 0$ as $r_* \to \infty$ the potential approaches a positive constant and has a Heviside-like shape, being a monotonically increasing function of r_*. As mentioned above, for the minimally coupled case it is impossible to find quasinormal modes, which is consistent with the basic fact that Schroedinger equation on a Heviside potential cannot have solutions that approach zero at $x \to \infty$ and that represent purely "outgoing" modes travelling towards the left as $x \to -\infty$.

In what follows we move to the problem of computing the spectrum for a non-minimally coupled scalar probe on the gravitational solitons constructed in section 3.1 in $\mathcal{N}=4$ $SU(2) \times SU(2)$ gauged supergravity, both in the supersymmetric and non-supersymmetric cases.

4.2 Spectrum of probe scalars on solitons

As we showed in section 3.1, $\mathcal{N}=4$ $SU(2) \times SU(2)$ gauged supergravity has the following soliton solution

$$ds^2 = -\rho dt^2 + g(\rho) d\varphi^2 + \frac{d\rho^2}{g(\rho)} + \rho dy^2 \ , \tag{4.2.1}$$

where

$$g(\rho) = \alpha \left(\rho - m - \frac{q^2}{\rho} \right) \ , \tag{4.2.2}$$

m and q being integration constants, α is related with the gauge couplings and φ is identified with period β_φ given by

$$\beta_\varphi = \frac{4\pi}{g'(\rho_0)} \ . \tag{4.2.3}$$

Here $g(\rho_0) = 0$, $\rho \geq \rho_0$ and the constants α, q, the gauge fields and the dilaton are given by

$$\alpha = \frac{1}{2}\left(e_A^2 + e_B^2\right) \ , \qquad q^2 = \frac{8\left(Q_A^2 + Q_B^2\right)}{e_A^2 + e_B^2} \ , \tag{4.2.4}$$

$$A^i_{[1]} = \frac{Q_A}{\rho} d\varphi \delta^i_3 \ , \qquad B^i_{[1]} = \frac{Q_B}{\rho} d\varphi \delta^i_3 \ , \tag{4.2.5}$$

$$\phi(r) = -\frac{1}{2} \ln \rho \ . \tag{4.2.6}$$

For general values of the integration constants m and q, the non-minimally coupled scalar probe does not admit a solution in a closed form. Nevertheless, for the case $q=0$ and m arbitrary, as well as for the case $m=0$ and q arbitrary, the non-minimally coupled scalar field can indeed be solved in terms of hypergeometric functions, consequently boundary conditions can be imposed in a closed manner, leading to a discrete set of frequencies. Hereafter we refer to these special cases as soliton-1 and soliton-2, which are defined by the metric (4.2.1), with $g(\rho)$ given by

$$g_{sol1}(\rho) = \alpha(\rho - m) \ , \tag{4.2.7}$$

$$g_{sol2}(\rho) = \alpha\left(\rho - \frac{q^2}{\rho}\right) \ , \tag{4.2.8}$$

respectively. The soliton-2 spacetime leads to a supersymmetric configuration that preserves 1/4 of the supersymmetry.

Defining $\varphi = \frac{\beta_\varphi}{2\pi}\phi$, the coordinate ϕ will have period 2π, and the metric (4.2.1) reduces to

$$ds^2 = -\rho dt^2 + \frac{\beta_\varphi^2}{4\pi^2} g(\rho)\, d\phi^2 + \frac{d\rho^2}{g(\rho)} + \rho dy^2 \ . \tag{4.2.9}$$

Given the isometries of this spacetime we write the following separation ansatz for a scalar probe

$$\Phi = \mathrm{Re}\left(\sum_n \int d\omega dk e^{-i\omega t + iky + in\phi} H_{\omega,k,n}(\rho)\right) \ . \tag{4.2.10}$$

The Ricci scalar of (4.2.9) has a non-trivial profile and it is given by

$$R = \frac{g(\rho)}{2\rho^2} - g''(\rho) - \frac{2g'(\rho)}{\rho} \ . \tag{4.2.11}$$

Introducing the notation $H_{\omega,k,n}(\rho) = H(\rho)$, the equation for the non-minimally coupled scalar

$$\Box\Phi - \xi R \Phi = 0 \ , \tag{4.2.12}$$

leads to the following ODE for the radial dependence

$$2\rho^2 g^2 \beta_\varphi^2 H'' + 2g\rho\beta_\varphi^2 (g\rho)' H' + \left(g\beta_\varphi^2\left(2g''\rho^2 + 4g'\rho - g\right)\xi - 2\rho\left(4\pi^2 n^2 \rho + g\beta_\varphi^2\left(k^2 - \omega^2\right)\right)\right) H = 0 \ . \tag{4.2.13}$$

Here the prime denotes derivative with respect to ρ. In what follows we analyze this equation for both soliton-1 and soliton-2 spacetimes, separately.

4.2.1 Non-supersymmetric soliton

For the family of solitons defined by the function soliton-1 in (4.2.7), we have $\rho_0 = m$, and $g'(\rho_0) = \alpha$, therefore $\beta_\varphi = \frac{4\pi}{\alpha}$. Introducing the coordinate x such that

$$\rho = \frac{\rho_0}{1-x} \ , \tag{4.2.14}$$

which maps $\rho \in [\rho_0, \infty[$ to $x \in [0,1[$, leads to an equation for the radial profile that can be integrated in terms of hypergeometric functions. Imposing regularity at the origin $\rho = \rho_0$ ($x = 0$) leads to the following solution

$$H(\rho(x)) = C_1 x^{\frac{|n|}{2}} (1-x)^{\frac{1}{2}\left(1-\sqrt{(1-6\xi)+n^2+\frac{4(k^2-\omega^2)}{\alpha}}\right)} F(\alpha_1, \beta_1, \gamma_1, x) \ , \tag{4.2.15}$$

with

$$\alpha_1 = \frac{1}{2}\left(1+|n|+\sqrt{2\xi}\right) - \frac{1}{2}\sqrt{(1-6\xi+n^2)+\frac{4(k^2-\omega^2)}{\alpha}} \ , \tag{4.2.16}$$

$$\beta_1 = \frac{1}{2}\left(1+|n|-\sqrt{2\xi}\right) - \frac{1}{2}\sqrt{(1-6\xi+n^2)+\frac{4(k^2-\omega^2)}{\alpha}} \ , \tag{4.2.17}$$

$$\gamma_1 = 1+|n| \ .$$

As in the previous section, using Kummer identities allows to rewrite (4.2.15) as

$$\begin{aligned} H(\rho(x)) = {} & C_1 x^{\frac{|n|}{2}} (1-x)^{\frac{1}{2}\left(1-\sqrt{(1-6\xi)+n^2+\frac{4(k^2-\omega^2)}{\alpha}}\right)} \times \\ & \left[\frac{\Gamma(\gamma_1)\Gamma(\gamma_1-\alpha_1-\beta_1)}{\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_1-\beta_1)} F(\alpha_1, \beta_1, \alpha_1+\beta_1+1-\gamma_1, 1-x) \right. \qquad (4.2.18) \\ & \left. + (1-x)^{\gamma_1-\alpha_1-\beta_1} \frac{\Gamma(\gamma_1)\Gamma(\alpha_1+\beta_1-\gamma_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} F(\gamma_1-\alpha_1, \gamma_1-\beta_1, 1+\gamma_1-\alpha_1-\beta_1, 1-x)\right] \end{aligned} \tag{4.2.19}$$

which leads to the following two leading terms on each branch of the asymptotic behavior as $x \to 1$

$$H(x) \underset{x\to 1}{\sim} A_1 (1-x)^{\delta-} + B_1 (1-x)^{\delta+} \ , \tag{4.2.20}$$

with

$$\delta_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{(1-6\xi) + n^2 + \frac{4\left(k^2-\omega^2\right)}{\alpha}}\right) , \tag{4.2.21}$$

and

$$A_1 = \frac{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_1-\alpha_1-\beta_1\right)}{\Gamma\left(\gamma_1-\alpha_1\right)\Gamma\left(\gamma_1-\beta_1\right)} , \tag{4.2.22}$$

$$B_1 = \frac{\Gamma\left(\gamma_1\right)\Gamma\left(\alpha_1+\beta_1-\gamma_1\right)}{\Gamma\left(\alpha_1\right)\Gamma\left(\beta_1\right)} . \tag{4.2.23}$$

Since the exponents in the asymptotic behavior (4.2.20) are ω dependent, we must be careful when imposing the boundary conditions. Again, the boundary term coming from the on-shell variation of the action principle (4.1.32)-(4.1.33) leads to a single contribution at infinity coming from the surface $x = x_0 \to 1$. In terms of the coordinate x, the non-supersymmetric soliton spacetime reads

$$ds^2 = -\frac{\rho_0}{1-x}dt^2 + \frac{4\rho_0 x}{1-x}d\phi^2 + \frac{\rho_0}{\alpha x\left(1-x\right)^3}dx^2 + \frac{\rho_0}{1-x}dy^2 , \tag{4.2.24}$$

while the boundary term of the on-shell variation of the action reads

$$\begin{aligned}&\lim_{r\to\infty}\int d^3x\sqrt{-\gamma}\hat{n}_\mu\nabla^\mu\Phi\delta\Phi \\ &\sim \lim_{x\to 1}\left(\delta_-^2 A_1^2\left(1-x\right)^{2\delta_- -1} + \delta_+^2 B_1^2\left(1-x\right)^{2\delta_+ -1} + A_1 B_1\left(\delta_+ + \delta_-\right)\left(1-x\right)^{\delta_+ + \delta_- -1}\right)\delta C_1\end{aligned} \tag{4.2.25}$$

It can be checked that $\mathrm{Re}\left(2\delta_- - 1\right) < 0$ on the whole complex ω-plane, while $\mathrm{Re}\left(2\delta_+ - 1\right) > 0$ and $\delta_+ + \delta_- - 1 = 0$. Therefore, in order to make the boundary term to vanish when evaluated on-shell on the branch that is regular at the origin, we must impose

$$A_1 = \frac{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_1-\alpha_1-\beta_1\right)}{\Gamma\left(\gamma_1-\alpha_1\right)\Gamma\left(\gamma_1-\beta_1\right)} = 0 . \tag{4.2.26}$$

Notice that this is actually a Dirichlet boundary condition as can be seen from

(4.2.20). The spectrum is therefore obtained from

$$\gamma_1 - \alpha_1 = \frac{\sqrt{(1-6\xi+n^2)\alpha + 4\left(k^2-\omega^2\right)} + \sqrt{\alpha}\left(1+|n|-\sqrt{2\xi}\right)}{2\sqrt{\alpha}} = -p \ , \tag{4.2.27}$$

$$\gamma_1 - \beta_1 = \frac{\sqrt{(1-6\xi+n^2)\alpha + 4\left(k^2-\omega^2\right)} + \sqrt{\alpha}\left(1+|n|+\sqrt{2\xi}\right)}{2\sqrt{\alpha}} = -q \ , \tag{4.2.28}$$

with q and p in $\{0, 1, 2, ...\}$. One can also check that the second quantization condition (4.2.28) cannot be fulfilled, nevertheless the quantization condition (4.2.27) leads to the spectrum

$$\omega_p = \pm\frac{1}{2}\sqrt{2\alpha\left(|n|+2p+1\right)\sqrt{2\xi} + 4k^2 - 2\left(|n|\left(1+2p\right)+4\xi+2p\left(1+p\right)\right)\alpha} \ , \tag{4.2.29}$$

which is a valid solution of (4.2.27) provided

$$\nu_{ns} := |n| + 1 - \sqrt{2\xi} + 2p < 0 \ . \tag{4.2.30}$$

As in the case of the black hole, for the non-supersymmetric soliton requiring regularity at the origin and Dirichlet boundary condition at infinity leads to an eigenvalue problem with a void spectrum when $\xi = 0$. Nevertheless, the presence of the non-minimal coupling leads to non-trivial probe modes.

The spectrum of the scalar on the non-supersymmetric soliton can be of diverse nature. Depending on the values of the parameters, it could be void, purely oscillatory namely with real frequencies (4.2.29) or unstable. The different behavior can be seen as separated by thresholds in the value of the non-minimal coupling ξ. Figure 2 shows two possible spectra.

4.2.2 Supersymmetric soliton

The 1/4 supersymmetric soliton is given by the metric (4.2.9) with the function $g\left(\rho\right)$ given by

$$g_{sol2}\left(\rho\right) = \alpha\left(\rho - \frac{q^2}{\rho}\right) \ . \tag{4.2.31}$$

In this case the smooth origin of the spacetime is located at $\rho = \rho_0 = q$ and the equation for the radial profile of the non-minimally coupled scalar probe has the

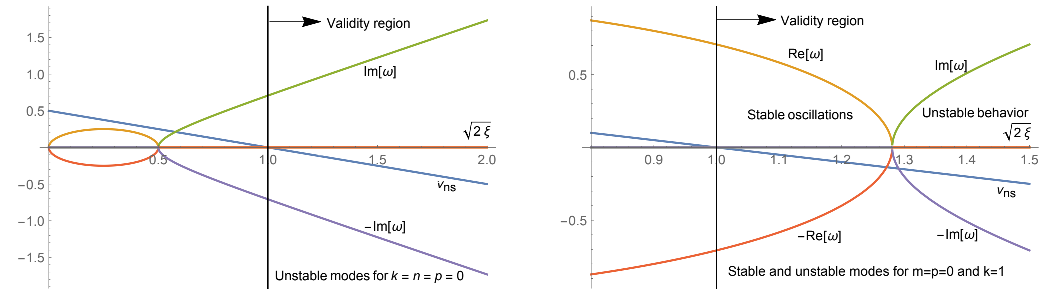

Figure 4.2.1: The panels show the spectra of stable and unstable modes of the non-minimally coupled scalar probe on the background of the non-supersymmetric soliton. Valid solutions for the quantization equation leading to the frequencies require $\nu_p < 0$, therefore in both panels, to the left of the vertical black line, there are no allowed modes given our boundary conditions.

following solution which is regular at the origin

$$H\left(x\right)=C_1x^{\frac{|n|}{2}}\left(1-x\right)^{\frac{1}{4}-\frac{1}{4}\sqrt{(1-6\xi)+4n^2+\frac{(k^2-4\omega^2)}{\alpha}}}F\left(\alpha_2,\beta_2,\gamma_2,x\right)\ , \tag{4.2.32}$$

where in this case the coordinate x is conveniently chosen as

$$\rho=\frac{\rho_0}{\left(1-x\right)^{1/2}}\ . \tag{4.2.33}$$

Here, the parameters of the hypergeometric function in (4.2.32) are given by

$$\alpha_2=-\frac{1}{4}\sqrt{1-6\xi+4n^2+\frac{4\left(k^2-\omega^2\right)}{\alpha}}+\frac{1}{2}\left(|n|+1+\frac{1}{2}\sqrt{1+2\xi}\right)\ , \tag{4.2.34}$$

$$\beta_2=-\frac{1}{4}\sqrt{1-6\xi+4n^2+\frac{4\left(k^2-\omega^2\right)}{\alpha}}+\frac{1}{2}\left(|n|+1-\frac{1}{2}\sqrt{1+2\xi}\right)\ , \tag{4.2.35}$$

$$\gamma_2=1+|n|\ . \tag{4.2.36}$$

Using Kummer identity in (4.2.32) leads to the following leading terms of the two branches of asymptotic behavior

$$H\left(x\right)\sim_{x\to1}A_2\left(1-x\right)^{\lambda_-}+B_2\left(1-x\right)^{\lambda_+}\ , \tag{4.2.37}$$

with

$$\lambda_\pm=\frac{1}{4}\pm\frac{1}{4}\sqrt{(1-6\xi)+4n^2+\frac{(k^2-4\omega^2)}{\alpha}}\ , \tag{4.2.38}$$

and

$$A_2 = \frac{\Gamma(\gamma_2)\Gamma(\gamma_2-\alpha_2-\beta_2)}{\Gamma(\gamma_2-\alpha_2)\Gamma(\gamma_2-\beta_2)} ,$$
$$B_2 = \frac{\Gamma(\gamma_2)\Gamma(\alpha_2+\beta_2-\gamma_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} . \tag{4.2.39}$$

As in the previous section, when the variation of the action is evaluated on the solution that is regular at the origin, one obtains a boundary term that vanishes iff

$$A_2 = \frac{\Gamma(\gamma_2)\Gamma(\gamma_2-\alpha_2-\beta_2)}{\Gamma(\gamma_2-\alpha_2)\Gamma(\gamma_2-\beta_2)} = 0 . \tag{4.2.40}$$

In consequence, this implies the following two quantization conditions for the spectrum

$$\gamma_2-\alpha_2 = \frac{1}{2}+\frac{|n|}{2}+\frac{1}{4}\sqrt{1-6\xi+4n^2+\frac{4(k^2-\omega^2)}{\alpha}}-\frac{1}{4}\sqrt{1+2\xi} = -p , \tag{4.2.41}$$

$$\gamma_2-\beta_2 = \frac{1}{2}+\frac{|n|}{2}+\frac{1}{4}\sqrt{1-6\xi+4n^2+\frac{4(k^2-\omega^2)}{\alpha}}+\frac{1}{4}\sqrt{1+2\xi} = -q , \tag{4.2.42}$$

with p and q elements of $\{0,1,2,3,...\}$. It can be shown that the second condition cannot be fulfilled, while the former leads to the spectrum

$$\omega = \pm\sqrt{k^2+(1+|n|+2p)\,\alpha\sqrt{1+2\xi}-\alpha\left(2\xi+(1+2p)^2+2|n|\,(1+2p)\right)} , \tag{4.2.43}$$

which are genuine solutions of (4.2.41) provided

$$\nu_{susy} = 2+|n|+4p-\sqrt{1+2\xi} < 0 \tag{4.2.44}$$

Depending on the ranges of the parameters one can obtain the same qualitative spectra as in the non-supersymmetric soliton, namely there is a range of values for the non-minimal coupling for which the spectrum is void, while in the complementary range one can have both stable and unstable modes. Stable oscillatory behavior can be achieved provided one restricts the values of the non-minimal coupling.

Chapter 5

Conclusion

In this thesis, we have found a new family of solutions of the Freedman-Schwarz supergravity model. Some of these spacetimes have a geometry simple enough to obtain the spectrum of a scalar probe non-minimally coupled. Also, we went beyond supergravity and found two families of potentials for the scalar field in such a way that the equations are solved by topologically Lifshitz black holes with interesting causal structures and spacetimes with a logarithmic branch in the metric function.

In sections 3.1 and 3.2 we started by revisiting the problem of BPS solutions in the $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity. In the Abelian sector of gauge fields of the theory, we have found a new completely regular, soliton spacetime, which preserves one-quarter of the supersymmetry. The soliton is charged, and asymptotically admits an extra conformal Killing vector. This spacetime can also be obtained from the double analytic continuation of a plannar solution found in [43]. The BPS conditions lead to a naked singularity in [43], nevertheless due to the double Wick rotation, the conditions for unbroken supersymmetry in our case lead to a regular spacetime. Also in the Abelian sector of the theory, but now assuming spherical symmetry on the metric, we have also found a new BPS configuration which preserves one-quarter of the supersymmetries, and that describes a naked singularity. Notice that singular spacetimes are known to appear as BPS solutions in gauged supergravities, as it is the case of the Reissner-Nordstrom solution in AdS with $Q = M$ [58].

The supergravity theory considered in this work has a potential for the dilaton without a local extremum, which leads to asymptotically locally flat spacetimes, instead of locally AdS as it is the case in other gauged supergravities. Then, we have moved to the construction of new, non-Abelian solutions, by considering the meron ansatz [59]. Since the supergravity theory contains two $su(2)$ gauge fields, we have constructed electric-meronic solutions as well as double-meron solutions with a spacetime that is spherically symmetric. The latter leads to one-quarter BPS configurations where the spacetime is again singular. It is interesting to remark that the non-BPS black holes in the double-meron sector, have a temperature that is independent of the mass of the spacetime, which can be seen as a signal of criticality. It would be interesting to allow non-trivial profiles for the axion, which has been set to zero along our work, and to explore the construction of more general exact solutions in this supergravity model.

A thorough exploration of self-interactions that allow for the explicit construction of black holes has proven to be a worth task in different contexts, since for example in the series of works [69]-[73] such analysis turned out to lead to a one-parameter deformation of the four, single scalar truncations of the maximal supergravity in four dimensions [74], as well as to the potentials of the two cases that admit an ω-deformation [75] in the single dilaton consistent sectors identified in [76]. With this in mind, in Section 3.4 we have moved beyond supergravity, but keeping the metric, one gauge field and the dilaton as field content of the theory. Within the meron ansatz for the gauge field, we have found that there are at least two families of self-interactions for the scalar field which allow to find exact analytic black holes with interesting properties. The first family of self-interactions leads to topologically Lifshitz black holes [64] with a variety of causal structures, containing for example a black hole inside a black hole. The second family of potentials can be seen as a two-parameter deformation of the dilaton potential of the Freedman-Schwarz model. The spacetime metric contains a logarithmically growing term, in spite of which the asymptotic region is locally flat. In this case there can be a single event horizon, a single cosmological horizon, an event horizon surrounded by a cosmological one, and finally an event horizon hiding a Cauchy horizon; the latter configuration can achieve extremality.

The spacetimes where we have integrated the scalar probe are solutions of the Freedman-Schwarz model and approach a background at infinity which is not maximally symmetric, but possesses and extra conformal Killing vector, which is due to the fact that the dilatonic potential of the theory does not have local extrema. The solitonic geometries presented in section 3.1 are smooth at the origin and can preserve $1/4$ of the supersymmetries. At the origin and in the near horizon region, the boundary conditions are clear and are given by regularity and purely ingoing modes, respectively. Due to the non-trivial geometry at infinity, the behavior of the scalar probe in the asymptotic region is given by powers of the radial coordinate which depend on the frequencies. In order to select a consistent boundary condition at infinity we impose that the on-shell variation of the action functionals must vanish. This leads to a Dirichlet boundary condition and allows to write the spectra in a closed form. For the massless scalar probe it is impossible to fulfil these boundary conditions. For the black holes, this is consistent with the fact that the effective Schroedinger-like potential controlling the radial dependence of the scalar probe in terms of the tortoise coordinate, has a Heaviside function shape. Including a non-minimal coupling allows for a non-trivial spectrum which surprisingly, in the case of the black hole, does not depend on the value of the mass of the spacetime. Therefore all these geometries are isospectral in what regards to the non-minimally coupled wave operator. Given the integrability properties of this potential it will be interesting to compare our results with the recently reported potentials coming from a geometric approach to spectral theory in connection with $SU(2)$ Seiberg-Witten theory with fundamental hypermultiplets (see Section 2 of [99]). Such potentials are also given in terms of ratios of linear combinations of exponentials, and the technique elaborates on the previous work [100] (see also the recent [101] and [102]).

Stability of the modes is achieved for a certain range of non-minimal couplings, above which one finds modes that are exponentially growing in time, and that are in consequence unstable. The stable and unstable regimes are separated by solutions to the boundary eigenvalue problem which are time independent. These solutions have the same properties as the scalar clouds found in [95] which from the point of view of the fully backreacting theory are branching spacetimes to a new family of solutions (see e.g. [96]).

We have been able to solve in a closed form the non-minimally coupled scalar probe on a family of black holes and solitons of the Freedman-Schwarz model, even in the case of $1/4$-BPS geometries. Such scalar probe goes beyond the field content of the theory, and it would be interesting to see whether some of the exact results we have obtained here, are also present in the context of gravitational perturbation theory considering only the fields that lead to the supersymetric model even if one has to rely on numerical or perturbative methods.

Conclusión

En esta tesis, encontramos nuevas soluciones al modelo de supergravedad de Freedman-Schwarz. Algunos de estos espacios tiempos tienen geometrías suficientemente simples para obtener el espectro de un escalar de prueba no-minimalmente acoplado. Además, consideramos teorías mas generales que la supergravedad de Freedman-Schwarz donde encontramos dos familias de potencias para el campo escalar. Las ecuaciones de campo son resultas por agujeros negros Lifshitz topológicos asintóticamente con estructuras causales interesante y espacios tiempos con términos logarítmicos en la función métrica.

En las secciones 3.1 y 3.2 revisamos el problema de encontrar soluciones BPS a la supergravedad $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged. En el sector Abeliano de la teoría, encontramos soluciones regulares nuevas que preservan $1/4$ de la supersimetría. Estos solitones estan cargado, y asintóticamente adquieren un nuevo vector de Killing conforme. Este espacio tiempo puede ser obtenido de la doble continuación analítica de una solución planar de agujero negro encontrada en [43]. Sin embargo, la condición BPS lleva a una singularidad desnuda en [43] pero, gracias a doble rotación de Wick la condición BPS lleva a un espacio que es regular. Exigiendo simetría esférica y aún en el sector Abeliano de la teoría, encontramos configuraciones BPS nuevas que preservan $1/4$ de la supersimetría y describen una singularidad desnuda. Los espacios singulares BPS aparecen en teorías de supergravedad, tal como el caso de las soluciones de Reissner-Nordstrom en AdS con $Q = M$ [58].

La teoría de supergravedad que consideramos en este trabajo tienen un potencial para el campo escalar que no tiene un mínimo local. Esto lleva a soluciones planas asintóticamente en vez de AdS asintóticamente como es el caso de otras teorías de supergravedad gauged. Después construimos soluciones no-Abelianas nuevas utilizando el ansatz de meron [59]. Debido a que la supergravedad contiene dos campos de gauge no-Abelianos, construimos soluciones de meron cargado y de doble meron en espacios tiempo con simetría esférica. La solución de doble meron es 1/4 BPS y el espacio tiempo es singular. Una propiedad curiosa de la solución de doble meron es que la temperatura no depende de la masa, lo que se puede entender como un signo de criticalidad. Sería interesante desformar estas soluciones y permitir un perfil no trivial para el axion, que fue apagado en este trabajo.

Explorar de forma exhaustiva los potenciales auto-interactuantes que permiten soluciones de agujero negro es una tarea que ha mostrado ser importante en varios contextos. Por ejemplo, este tipo de análisis fueron hechos en la serie de trabajos [69]-[73], que dio lugar a una deformación uniparamétrica de las cuatro supergravedades maximalmente simétricas truncadas con un solo campo escalar [74], como también dio lugar a los dos potenciales que admiten una ω-desformación [75] en el sector con un dilaton que es consistente y que fue identificado en [76]. Con estas ideas en mente, en la sección 3.4 consideramos potenciales más generales que los de la supergravedad en cuestión, pero dejamos como contenido de campos la métrica, un campo de gauge y el dilaton. Siguiendo con el ansantz de meron en el sector de Yang-Mills, encontramos que hay al menos dos familias de auto-interacciones para el campo escalar que admiten soluciones exactas de agujeros negros con propiedades interesantes. La primera familia de auto-interacciones admite soluciones de tipo agujero negro con comportamiento asintótico Lifshitz topológico [64] con amplia variedad de estructuras causales, como por ejemplo un agujero negro dentro de un agujero negro. La segunda familia de soluciones puede ser pensada como una deformación biparamétrica del potencial presente en el modelo de Freedman-Schwarz. Los espacios tiempo que resuelven las ecuaciones de campo en esta familia tienen un término logarítmico, y a pesar de eso, son planos asintóticamente. En estos casos puede haber un horizonte de eventos, un horizonte cosmológico, un horizonte de eventos rodeado por un horizonte cosmológico, y finalmente un horizonte de eventos que rodea a un horizonte de Cauchy. Éste último puede alcanzar la extremalidad.

Lo espacios tiempo de donde logramos integral el campo escalar de prueba son solución al modelo de Freedman-Schwarz y se aproximan a un espacio que no es máximamente simétrico asintóticamente, pero posee un vector de killing conforme extra, el cual es consecuencia de que el potencial del dilaton no tiene un extremo local. La geometrías del solitón fue presentada en la sección 3.1 son regulares en el origen y preservan 1/4 de la supersimetría. Las condiciones de borde en el origen y en la región cercana al horizonte son de regulares y causales respectivamente. Debido a la geometría no trivial en la región asintótica, el campo escalar de prueba es dado en términos de potencias, que dependen de la frecuencia, de la coordenada radial. Las condiciones de borde fueron impuestas de modo que la variación de la acción evaluada en la solución sea cero. Esto dio lugar a condiciones de borde tipo Dirichlet, que nos permitió escribir el espectro en forma cerrada. Es importante enfatizar que los campos escalares de prueba sin masa no pueden satisfacer estas condiciones de borde. En el caso de agujeros negros, esto es consistente con el hecho que el potencial efectivo tipo Schroedinger que controla la dependencia radial del escalar en términos de la coordenada tortuga, tiene una forma de función de Heaviside. Cuando se incluye un acomplamiento no-minimal, es posible obtener el espectro de forma analítica. En el caso del agujero negro, es sorprendente que el espectro no depende del valor de la masa del espacio tiempo. Por lo que estas geometrías son isoespectrales en lo que se refiere al operador de onda acoplado no-minimalmente al escalar de Ricci. Las propiedades de integrabilidad de este potencial nos permiten contrastarlo con los resultados reportados recientemente, que vienen de un enfoque geométrico de teoría espectral en conexión con la teoría de Seiberg-Witten $SU(2)$ con hipermultiplete fundamental (para detalles ver la sección 2 de [99]). Estos potenciales son dados en términos de cocientes de combinaciones lineales de exponenciales, y usando técnicas elaboradas en el artículo previo [100] (vea los artículos recientes [101] y [102]).

La estabilidad de los modos se logra para cierto rango del parámetro que regular el acoplamiento no-minimal, sobre cierto valor se encuentran modos inestables que crecen exponencialmente en el tiempo. Los regímenes estables e inestables están separados por soluciones que son independientes del tiempo. Estas soluciones tienes propiedades similar a una nube escalar encontradas en [95], que desde el punto de vista de la teoría donde la gravedad es dinámica, estos son espacios tiempos que se ramifican a nuevas familias de soluciones (vea e.g. [96]).

Hemos logrado resolver de forma cerrada las ecuaciones para un campo escalar no-minimalmente acoplado en una familia de soluciones de agujero negros y de solitones al modelo Freedman-Schwarz, incluso en los casos supersimétricos. Este tipo de escalar de prueba está fuera del contenido de campos de la teoría, y es interesante estudiar los resultados presentados en esta tesis siguen presentes en el contexto perturbativo de la teoría de supergravedad.

A1 Manifolds and vectors

The first concept that we want to discuss is the concept of a pseudo-Riemannian manifold. The main property of a manifold is that it is made of points for which we can give a name using at least one set of coordinates. The coordinates are a set of real numbers $x^1(p), x^2(p), \ldots, x^d(p)$ associated to each point $p \in M$. The adjective "Riemannian" briefly speaking means that the manifold is also equipped with a metric $g_{\mu\nu}$ that allows us to calculate distances between two events or points and "pseudo" means that the signature of the metric is Lorentzian.

Mathematically a $d-$dimension manifold M_d is a set together with a collection of subsets$\{\mathcal{O}_j\}$ of M_d, such that:

(i) each point $p \in M$ lies in at least in one subset $\mathcal{O}_j$.

(ii) For each j there is a one-to-one map $\phi_j : \mathcal{O}_j \to U_j$, where U_j is a open subset of $\mathbb{R}^d$.

iii) If two sets $\mathcal{O}_j$ and $\mathcal{O}_k$ overlap then we can construct a map between $U_j \to U_k$ that is a map between two subsets of $\mathbb{R}^d$.

In general, we will not able to obtain a single chart that covers the whole manifold, but a set of $\{\mathcal{O}_j\}$ that perform such task always exists. We want to stress that the range of coordinates (i.e. the set U_j) must be specified, otherwise the patch is ill defined.

The second concept that we will discuss is the concept of vectors in a manifold. In Special Relativity the concept of vector is straightforward because we formulated it in $\mathbb{R}^4$ that is a vector space, but General Relativity is formulated in a curved manifold then, at first sight it is not clear how to define vectors on it. The idea of vector is closely related to tanget vector of a curve. In a manifold we define a curve as a map $C(t) : [0,1] \to \mathcal{O}_i \subset M_d$.

Then, we define a smooth real function from a subset of the manifold $\mathcal{O}_i$ to the real line $f : \mathcal{O}_i \subset M_d \to \mathbb{R}$, the composite function $f(C(t)) \equiv g(t)$ maps $[0,1]$ to $\mathbb{R}$, thus, we can calculate its derivative with respect to the parameter t at $t = 0$ that reads as follows

$$\left.\frac{dg}{dt}\right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left.\frac{dx^\mu}{dt}\right|_{t=0} . \tag{A1.1}$$

Notice that if we specify the explicit parametrization of the curve $x^\mu(t)$ one can compute the quantity $c^\mu = \frac{dx^\mu}{dt}\Big|_{t=0} \in \mathbb{R}$ then we have

$$\frac{dg}{dt} = c^\mu \frac{\partial f}{\partial x^\mu} \ . \tag{A1.2}$$

The equation (A1.2) can be understood as the action of a differential operator on the space of smooth functions

$$\boldsymbol{t} \equiv c^\mu \frac{\partial}{\partial x^\mu} \tag{A1.3}$$

The set of operators defined by (A1.3) satisfy the axioms of vector space. This discussion justifies that the tangent space $T_p(M_d)$ at $p \in M_d$ is defined as the vector space of first order differential operators acting on smooth functions. The basis of this space is $\{\partial_\mu\}$, in the coordinates x^μ that we are using. It is worth to emphasize that the transformations law for the component of a vector under a diffeomorphism $x^\mu = x^\mu(\tilde{x}^\nu)$ is as usual $c^\mu \to \tilde{c}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} c^\mu$.

From here it is clear that there exist a dual vector space called the cotangent vector space $T_p^*(M_d)$ whose basis is $\{dx^\mu\}$. We can compute the tensor product between these spaces and construct tensors with more legs, for example the object

$$T = T_{\mu_1 \ldots \mu_n}{}^{\nu_1 \ldots \nu_m} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_n} \otimes \partial_{\nu_1} \otimes \cdots \otimes \partial_{\nu_m} \tag{A1.4}$$

belongs to $T_p^*(M_d) \otimes \cdots \otimes T_p^*(M_d) \otimes T_p(M_d) \otimes \cdots \otimes T_p(M_d)$.

A2 Palatini identity

In Kaluza-Klein dimensional reduction we have to compute the Einstein-Hilbert action in the ansatz for the vielbeine. Thus, we compute the spin connection 1-form and then we compute curvature 2-form given by (2.2.27) which can be written as

$$\mathcal{R}^{ab}_{[2]} = \frac{1}{2} \left(\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^a{}_c \omega_\nu{}^{cb} - \omega_\nu{}^a{}_c \omega_\mu{}^{cb} \right) dx^\mu \wedge dx^\nu \ . \tag{A2.1}$$

Plugging in the Einstein-Hilbert action in forms language we get

$$\int R\sqrt{-g}d^dx = \frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \left(d\omega^{ab}+\omega^a{}_d\wedge\omega^{db}\right)\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}}, \tag{A2.2}$$

$$= \frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \left(d\omega^{ab}\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}}\right. \tag{A2.3}$$

$$\left. + \ \omega^a{}_d\wedge\omega^{db}\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}}\right) \ .$$

Integrating by parts the first term, neglecting the boundary term and using the torsionless constraint $de^a+\omega^a{}_b e^b=0$, we obtain that

$$\begin{aligned}
\int R\sqrt{-g}d^dx &= \frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega^{ab}\wedge\left(de^{c_1}\wedge\cdots\wedge e^{c_{d-2}}-e^{c_1}\wedge de^{c_2}\cdots\wedge e^{c_{d-2}}+\cdots\right)\\
&\quad+\frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega^a{}_d\wedge\omega^{db}\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}} \ ,\\
&= \frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int (d-2)\,\omega^{ab}\wedge de^{c_1}\wedge\cdots\wedge e^{c_{d-2}}\\
&\quad+\frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega^a{}_d\wedge\omega^{db}\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}} \ ,\\
&= -\frac{1}{(d-3)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega^{ab}\wedge\omega^{c_1}{}_e\wedge e^e\wedge\cdots\wedge e^{c_{d-2}}\\
&\quad+\frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega^a{}_d\wedge\omega^{db}\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}} \ .
\end{aligned}$$

Writting the component of the spin connection explicitlly and using the relation between the coordinate basis and the vielbein, we get that

$$\begin{aligned}
\int R\sqrt{-g}d^dx &= -\frac{1}{(d-3)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega_d{}^{ab}\omega_f{}^{c_1}{}_e e^d\wedge e^f\wedge e^e\wedge\cdots\wedge e^{c_{d-2}} \\
&\quad +\frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega_d{}^a{}_e\omega_f{}^{eb}e^d\wedge e^f\wedge e^{c_1}\wedge\cdots\wedge e^{c_{d-2}}\ , \\
&= -\frac{1}{(d-3)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega_d{}^{ab}\omega_f{}^{c_1}{}_e\epsilon^{dfec_2\ldots c_{d-2}}\sqrt{-g}d^dx \\
&\quad +\frac{1}{(d-2)!}\epsilon_{abc_1\ldots c_{d-2}}\int \omega_d{}^a{}_e\omega_f{}^{eb}\epsilon^{dfc_1\ldots c_{d-2}}\sqrt{-g}d^dx\ , \\
&= -\frac{1}{(d-3)!}\int\sqrt{-g}d^dx\,\omega_d{}^{ab}\omega_f{}^{c_1}{}_e\delta^{dfec_2\ldots c_{d-2}}_{abc_1c_2\ldots c_{d-2}} \\
&\quad +\frac{1}{(d-2)!}\int\sqrt{-g}d^dx\,\omega_d{}^a{}_e\omega_f{}^{eb}\delta^{dfc_1\ldots c_{d-2}}_{abc_1\ldots c_{d-2}}\ , \\
&= -\frac{1}{(d-3)!}\int\sqrt{-g}d^dx\,\omega_d{}^{ab}\omega_f{}^{c_1}{}_e\,(d-3)!\delta^{dfe}_{abc_1} \\
&\quad +\frac{1}{(d-2)!}\int\sqrt{-g}d^dx\,\omega_d{}^a{}_e\omega_f{}^{eb}\,(d-2)!\delta^{df}_{ab}\ , \\
&= \int\sqrt{-g}d^dx\left(-\omega_d{}^{ab}\omega_f{}^{c_1}{}_e\delta^{dfe}_{abc_1}+\omega_d{}^a{}_e\omega_f{}^{eb}\delta^{df}_{ab}\right)\ .
\end{aligned} \tag{A2.4}$$

expanding the Kronecker delta as

$$\delta^{dfe}_{abc_1} = \delta^d_a\delta^f_b\delta^e_{c_1}+\delta^d_b\delta^f_{c_1}\delta^e_a+\delta^d_{c_1}\delta^f_a\delta^e_b-\delta^d_b\delta^f_a\delta^e_{c_1}-\delta^d_a\delta^f_{c_1}\delta^e_b-\delta^d_{c_1}\delta^f_b\delta^e_a\ , \tag{A2.5}$$

we obtain that

$$\int R\sqrt{-g}d^dx = \int\sqrt{-g}d^dx\left[-\omega_b{}^{ba}\omega_c{}^c{}_a+\omega_c{}^{ab}\omega_{ab}{}^c\right]\ . \tag{A2.6}$$

This is the so-called Palatini identity. It is useful because computing the spin connection, we can compute the Einstein-Hilbert action without spending time computing derivatives of the spin connection.

A3 Another convention for the Hodge dual

In this appendix we want to write down the conventions used in [36] where the authors wrote the following action principle

$$\begin{aligned}\mathcal{L}_4^{FS} &= R *_4 1 - \frac{1}{2} *_4 d\phi \wedge d\phi - \frac{1}{2}e^{2\phi} *_4 d\chi \wedge d\chi + 2\left(g^2 + \tilde{g}^2\right) e^{\phi} *_4 1 \\ &\quad -\frac{1}{2}e^{-\phi} *_4 F^i_{(2)} \wedge F^i_{(2)} - \frac{1}{2}e^{-\phi} *_4 \tilde{F}^i_{(2)} \wedge \tilde{F}^i_{(2)} - \frac{1}{2}\chi F^i_{(2)} \wedge F^i_{(2)} - \frac{1}{2}\chi \tilde{F}^i_{(2)} \wedge \tilde{F}^i_{(2)}\end{aligned} \tag{A3.1}$$

which is equivalent to (2.4.26) up to a field re-definitions and coupling constants re-definitions. The convention used in [36] for the Hodge dual in a manifold $\{M_d, g_{\mu\nu}\}$ reads

$$*_d\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) = \frac{1}{p!}\epsilon_{\nu_1 \ldots \nu_d}{}^{\mu_1 \ldots \mu_d} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_d} \ , \tag{A3.2}$$

then

$$*_d\alpha_{[p]} = \frac{1}{p!\,(d-p)!}\epsilon_{\nu_1 \ldots \nu_{d-p}\mu_1 \ldots \mu_p}\alpha^{\mu_1 \ldots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{d-p}} \ .$$

Where we used the following

$$\begin{aligned}\epsilon_{\mu_1 \ldots \mu_d} &= \sqrt{-g}\,\varepsilon_{\mu_1 \ldots \mu_d} \ , \qquad \epsilon^{\mu_1 \ldots \mu_d} = \frac{1}{\sqrt{-g}}\varepsilon^{\mu_1 \ldots \mu_d} \\ \text{and} \qquad \varepsilon_{01 \ldots (d-1)} &= 1 \ , \qquad \varepsilon^{01 \ldots (d-1)} = -1 \ .\end{aligned} \tag{A3.3}$$

Therefore,

$$\epsilon_{\mu \ldots \nu}\epsilon^{\rho \ldots \sigma} = -\delta^{\rho \ldots \sigma}_{\mu \ldots \nu} \ .$$

The wedge product $*A \wedge B$, for p-forms A and B, in this convention is given by

$$
\begin{aligned}
& *_d A \wedge B && \text{(A3.4)}\\
= \;& \frac{1}{p!\,(d-p)!}\epsilon_{\nu_1\ldots\nu_{d-p}\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}\frac{1}{p!}B_{\rho_1\ldots\rho_p}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{d-p}}\wedge dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_p}\ ,\\
= \;& \frac{1}{(p!)^2\,(d-p)!}\epsilon_{\nu_1\ldots\nu_{d-p}\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}B_{\rho_1\ldots\rho_p}\frac{1}{d!}\delta^{\nu_1\ldots\nu_{d-p}\rho_1\ldots\rho_p}_{\alpha_1\ \ \ldots\ \ \alpha_d}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_d}\ ,\\
= \;& \frac{1}{(p!)^2\,(d-p)!}\delta^{\nu_1\ldots\nu_{d-p}\rho_1\ldots\rho_p}_{\nu_1\ldots\nu_{d-p}\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}B_{\rho_1\ldots\rho_p}\sqrt{-g}d^dx\ ,\\
= \;& \frac{1}{(p!)^2\,(d-p)!}\,(d-p)!\,\delta^{\rho_1\ldots\rho_p}_{\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}B_{\rho_1\ldots\rho_p}\sqrt{-g}d^dx\ ,\\
= \;& \frac{1}{p!}A^{\mu_1\ldots\mu_p}B_{\mu_1\ldots\mu_p}\sqrt{-g}d^dx\ ,\\
\equiv \;& \frac{1}{p!}A\cdot B *_d 1\ .
\end{aligned}
$$

This is consistent with the notation in (A3.1).

A4 Explicit expressions for the Killing spinors of Sections 3.1 and 3.2 and 3.3

Killing spinors for the Soliton: The Killing spinors for the soliton presented in Section II are given by

$$
\bar\epsilon_1 = (\cosh l)^{-1/4}\begin{pmatrix} -\sqrt{e_A^2+e_B^2}\sqrt{\cosh l-1}\\ \sqrt{\cosh l+1}\\ 0\\ \frac{ie_A}{e_B}\sqrt{\cosh l+1}\\ i\frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l-1}\\ -i\sqrt{\cosh l+1}\\ 0\\ \frac{e_A}{e_B}\sqrt{\cosh l+1}\\ 0\\ 0_{8\times1}\end{pmatrix}\ ,\ \bar\epsilon_2 = (\cosh l)^{-1/4}\begin{pmatrix} i\sqrt{e_A^2+e_B^2}\sqrt{\cosh l+1}\\ i\sqrt{\cosh l-1}\\ 0\\ -\frac{e_A}{e_B}\sqrt{\cosh l-1}\\ -\frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l+1}\\ -\sqrt{\cosh l-1}\\ 0\\ -i\frac{e_A}{e_B}\sqrt{\cosh l-1}\\ 0\\ 0_{8\times1}\end{pmatrix}
$$

$$\bar{\epsilon}_3 = (\cosh l)^{-1/4} \begin{pmatrix} i\frac{e_A}{e_B}\sqrt{\cosh l+1} \\ 0 \\ \sqrt{\cosh l+1} \\ \frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l-1} \\ -\frac{e_A}{e_B}\sqrt{\cosh l+1} \\ 0 \\ i\sqrt{\cosh l+1} \\ i\frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l-1} \\ 0 \\ 0_{8\times 1} \end{pmatrix} , \ \bar{\epsilon}_4 = (\cosh l)^{-1/4} \begin{pmatrix} -\frac{e_A}{e_B}\sqrt{\cosh l-1} \\ 0 \\ i\sqrt{\cosh l-1} \\ -i\frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l+1} \\ i\frac{e_A}{e_B}\sqrt{\cosh l-1} \\ 0 \\ \sqrt{\cosh l-1} \\ -\frac{\sqrt{e_A^2+e_B^2}}{e_B}\sqrt{\cosh l+1} \\ 0 \\ 0_{8\times 1} \end{pmatrix}$$

which shows that the soliton spacetime preserves $1/4$ of the supersymmetries of the theory.

Killing spinors for the spherically symmetric, supersymmetric solution in the Abelian sector: For the Abelian BPS solution discussed in section III, the four Killing spinors that preserve the supersymmetry transformations have half of the components vanishing. It is not possible to factorize the dependence on the radial coordinate ρ. Nevertheless, the explicit form of the Killing spinors can be given in a compact manner as follows:

$$\bar{\epsilon}_1 = \Psi_1(\rho) \begin{pmatrix} -e_A c_\varphi s_\theta \\ -ie_A s_\varphi s_\theta \\ \frac{2\Lambda}{\sin\theta} s_\varphi c_\theta^2 s_\theta + ie_B c_\varphi s_\theta \\ i\Lambda c_\varphi c_\theta - e_B s_\varphi s_\theta \\ \Lambda c_\varphi c_\theta + \frac{2ie_B}{\sin\theta} s_\varphi s_\theta^2 c_\theta \\ -\frac{2i\Lambda}{\sin\theta} s_\varphi c_\theta^2 s_\theta + e_B c_\varphi s_\theta \\ -\frac{2e_A}{\sin\theta} s_\varphi c_\theta s_\theta^2 \\ ie_A c_\varphi s_\theta \\ 0_{8\times 1} \end{pmatrix}^T + \Psi_2(\rho) \begin{pmatrix} -e_A c_\varphi c_\theta \\ ie_A s_\varphi c_\theta \\ -\frac{2\Lambda}{\sin\theta} s_\varphi s_\theta^2 c_\theta + ie_B c_\varphi c_\theta \\ i\Lambda c_\varphi s_\theta + e_B s_\varphi c_\theta \\ -\Lambda c_\varphi s_\theta + \frac{i2e_B}{\sin\theta} s_\varphi s_\theta c_\theta^2 \\ -\frac{i2\Lambda}{\sin\theta} s_\varphi s_\theta^2 c_\theta - e_B c_\varphi c_\theta \\ -\frac{2e_A}{\sin\theta} s_\varphi c_\theta^2 s_\theta \\ -ie_A c_\varphi c_\theta \\ 0_{8\times 1} \end{pmatrix}^T$$

$$\bar{\epsilon}_2 = \Psi_1(\rho)\begin{pmatrix} -e_A c_\theta c_\varphi \\ i e_A s_\varphi c_\theta \\ \frac{2\Lambda}{\sin\theta} s_\varphi s_\theta^2 c_\theta + i e_B c_\varphi c_\theta \\ e_B s_\varphi c_\theta - i\Lambda c_\varphi s_\theta \\ -\Lambda c_\varphi s_\theta - \frac{2ie_B}{\sin\theta} s_\varphi c_\theta^2 s_\theta \\ -\frac{i2\Lambda}{\sin\theta} s_\varphi s_\theta^2 c_\theta + e_B c_\varphi c_\theta \\ \frac{2e_A}{\sin\theta} s_\varphi s_\theta c_\theta^2 \\ i e_A c_\varphi c_\theta \\ 0_{8\times 1} \end{pmatrix}^T + \Psi_2(\rho)\begin{pmatrix} e_A c_\varphi s_\theta \\ i e_A s_\theta s_\varphi \\ \frac{2\Lambda}{\sin\theta} s_\varphi c_\theta^2 s_\theta - i e_B s_\theta c_\varphi \\ i c_\varphi c_\theta \Lambda + e_B s_\theta s_\varphi \\ -c_\varphi c_\theta \Lambda + \frac{2ie_B}{\sin\theta} s_\varphi s_\theta^2 c_\theta \\ \frac{2i\Lambda}{\sin\theta} s_\varphi c_\theta^2 s_\theta + e_B c_\varphi s_\theta \\ -\frac{2e_A}{\sin\theta} s_\varphi s_\theta^2 c_\theta \\ i e_A c_\varphi s_\theta \\ 0_{8\times 1} \end{pmatrix}^T$$

$$\bar{\epsilon}_3 = \Psi_1(r)\begin{pmatrix} e_A s_\varphi s_\theta \\ -e_A i c_\varphi s_\theta \\ \frac{2\Lambda}{\sin\theta} c_\varphi c_\theta^2 s_\theta - i e_B s_\varphi s_\theta \\ -e_B c_\varphi s_\theta - i\Lambda s_\varphi c_\theta \\ -\Lambda s_\varphi c_\theta + \frac{i2e_B}{\sin\theta} c_\varphi s_\theta^2 c_\theta \\ -\frac{i2\Lambda}{\sin\theta} c_\varphi c_\theta^2 s_\theta - e_B s_\varphi s_\theta \\ -\frac{2e_A}{\sin\theta} c_\varphi c_\theta s_\theta^2 \\ -i e_A s_\varphi s_\theta \\ 0_{8\times 1} \end{pmatrix}^T + \Psi_2(r)\begin{pmatrix} e_A s_\varphi c_\theta \\ i e_A c_\varphi c_\theta \\ -\frac{2\Lambda}{\sin\theta} c_\varphi s_\theta^2 c_\theta - i e_B s_\varphi c_\theta \\ e_B c_\varphi c_\theta - i\Lambda s_\varphi s_\theta \\ \Lambda s_\varphi s_\theta + \frac{i2e_B}{\sin\theta} c_\varphi c_\theta^2 s_\theta \\ -\frac{i2\Lambda}{\sin\theta} c_\varphi s_\theta^2 c_\theta + e_B s_\varphi c_\theta \\ -\frac{2e_A}{\sin\theta} c_\varphi c_\theta^2 s_\theta \\ i e_A s_\varphi c_\theta \\ 0_{8\times 1} \end{pmatrix}^T$$

$$\bar{\epsilon}_4 = \Psi_1(\rho)\begin{pmatrix} -e_A s_\varphi c_\theta \\ -i e_A c_\varphi c_\theta \\ -\frac{2\Lambda}{\sin\theta} c_\varphi s_\theta^2 c_\theta + i e_B c_\theta s_\varphi \\ -i\Lambda s_\theta s_\varphi - c_\varphi e_B c_\theta \\ -\Lambda s_\varphi s_\theta + \frac{i2e_B}{\sin\theta} c_\varphi c_\theta^2 s_\theta \\ \frac{i2\Lambda}{\sin\theta} c_\varphi s_\theta^2 c_\theta + e_B s_\varphi c_\theta \\ -\frac{2e_A}{\sin\theta} c_\varphi c_\theta^2 s_\theta \\ i e_A s_\varphi c_\theta \\ 0_{8\times 1} \end{pmatrix}^T + \Psi_2(\rho)\begin{pmatrix} e_A s_\varphi s_\theta \\ -i e_A c_\varphi s_\theta \\ -\frac{2\Lambda}{\sin\theta} c_\varphi c_\theta^2 s_\theta - i e_B s_\varphi s_\theta \\ i\Lambda s_\varphi c_\theta - e_B c_\varphi s_\theta \\ -\Lambda s_\varphi c_\theta - \frac{i2e_B}{\sin\theta} c_\varphi s_\theta^2 c_\theta \\ -\frac{i2\Lambda}{\sin\theta} c_\varphi c_\theta^2 s_\theta + e_B s_\varphi s_\theta \\ \frac{2e_A}{\sin\theta} c_\varphi s_\theta^2 c_\theta \\ i e_A s_\theta s_\varphi \\ 0_{8\times 1} \end{pmatrix}^T$$

where

$$\begin{aligned} \Psi_1(\rho) &= \frac{\sqrt{(\rho e_A + i2Q_A)\Lambda^2 H_A - i e_A e_B \rho}}{\rho^{1/4}}\,, \\ \Psi_2(\rho) &= \frac{e_A e_B \sqrt{f_{BPS}(\rho)\,\rho}}{\rho^{1/4}\sqrt{(\rho e_A + i2Q_A)\Lambda^2 H_A - i e_A e_B \rho}}\,, \end{aligned}$$

and $f_{BPS}(\rho)$ is defined by (3.2.16).

Killing spinors for the double-meron: In section IV we have shown that the double-meron solution with $\mu = 0$ admits a set of four Killing spinors that satisfy the equations (2.4.31) and (2.4.32). Now we will provide the explicit form of these spinors, which take the form

$$\bar{\epsilon}_i = \sqrt{r}\left(A \otimes B_i + \eta \otimes C_i\right)^T \tag{A4.1}$$

where T means transpose and the vectors A and η are common for the four spinors and are given by

$$A = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\phi \\ \cos\theta \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice that the radial dependence is factorized on the killing spinor and the depends on the angles are in the vector and spinors A, B_i and C_i which are given by $\left(\tilde{\Lambda} \to \tilde{\Lambda} 2 e_A e_B\right)$

$$B_1 = \begin{pmatrix} c_\varphi s_\theta \\ i s_\varphi s_\theta \\ \frac{2e_B\tilde{\Lambda}}{\Lambda} s_\varphi s_\theta + i\frac{e_B}{\Lambda} s_\varphi c_\theta - i\frac{e_B}{e_A} c_\varphi s_\theta \\ -i\frac{2e_B\tilde{\Lambda}}{\Lambda}\left(c_\varphi s_\theta\right) - \frac{e_B}{\Lambda} c_\varphi c_\theta + \frac{e_B}{e_A} s_\varphi s_\theta \end{pmatrix}, C_1 = \begin{pmatrix} i\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} s_\varphi c_\theta + \frac{e_B^2}{\Lambda^2} s_\varphi s_\theta + \frac{\Lambda}{e_A} c_\varphi c_\theta \\ -\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} c_\varphi c_\theta - i\frac{e_B^2}{\Lambda^2} c_\varphi s_\theta - i\frac{\Lambda}{e_A} s_\varphi c_\theta \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(i s_\varphi s_\theta - 2\tilde{\Lambda} s_\varphi c_\theta\right) \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(c_\varphi s_\theta - 2i\tilde{\Lambda} c_\varphi c_\theta\right) \end{pmatrix}$$

$$B_2 = \begin{pmatrix} c_\varphi c_\theta \\ -i s_\varphi c_\theta \\ -\frac{2e_B\tilde{\Lambda}}{\Lambda} s_\varphi c_\theta + i\frac{e_B}{\Lambda} s_\varphi s_\theta - i\frac{e_B}{e_A} c_\varphi c_\theta \\ -i\frac{2e_B\tilde{\Lambda}}{\Lambda} c_\varphi c_\theta + \frac{e_B}{\Lambda} c_\varphi s_\theta - \frac{e_B}{e_A} s_\varphi c_\theta \end{pmatrix}, C_2 = \begin{pmatrix} i\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} s_\varphi s_\theta - \frac{e_B^2}{\Lambda^2} s_\varphi c_\theta - \frac{\Lambda}{e_A} c_\varphi s_\theta \\ \frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} c_\varphi s_\theta - i\frac{e_B^2}{\Lambda^2} c_\varphi c_\theta - i\frac{\Lambda}{e_A} s_\varphi s_\theta \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(-i s_\varphi c_\theta - 2\tilde{\Lambda} s_\varphi s_\theta\right) \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(c_\varphi c_\theta + 2i\tilde{\Lambda} c_\varphi s_\theta\right) \end{pmatrix}$$

$$B_3 = \begin{pmatrix} i s_\varphi s_\theta \\ c_\varphi s_\theta \\ -i\frac{2e_B\tilde{\Lambda}}{\Lambda} c_\varphi s_\theta + \frac{e_B}{\Lambda} c_\varphi c_\theta + \frac{e_B}{e_A} s_\varphi s_\theta \\ \frac{2e_B\tilde{\Lambda}}{\Lambda} s_\varphi s_\theta - i\frac{e_B}{\Lambda} s_\varphi c_\theta - i\frac{e_B}{e_A} c_\varphi s_\theta \end{pmatrix}, C_3 = \begin{pmatrix} \frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} c_\varphi c_\theta - i\frac{e_B^2}{\Lambda^2} c_\varphi s_\theta + i\frac{\Lambda}{e_A} s_\varphi c_\theta \\ -i\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2} s_\varphi c_\theta + \frac{e_B^2}{\Lambda^2} s_\varphi s_\theta - \frac{\Lambda}{e_A} c_\varphi c_\theta \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(c_\varphi s_\theta + 2i\tilde{\Lambda} c_\varphi c_\theta\right) \\ \frac{4e_A e_B}{\Lambda^2\sin^2\theta} s_\theta^2 c_\theta^2\left(i s_\varphi s_\theta + 2\tilde{\Lambda} s_\varphi c_\theta\right) \end{pmatrix}$$

$$B_4 = \begin{pmatrix} -is_\varphi c_\theta \\ c_\varphi c_\theta \\ -i\frac{2e_B\tilde{\Lambda}}{\Lambda}c_\varphi c_\theta - \frac{e_B}{\Lambda}c_\varphi s_\theta - \frac{e_B}{e_A}s_\varphi c_\theta \\ -\frac{2e_B\tilde{\Lambda}}{\Lambda}s_\varphi c_\theta - i\frac{e_B}{\Lambda}s_\varphi s_\theta - i\frac{e_B}{e_A}c_\varphi c_\theta \end{pmatrix}, \quad C_4 = \begin{pmatrix} -\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2}c_\varphi s_\theta - i\frac{e_B^2}{\Lambda^2}c_\varphi c_\theta + i\frac{\Lambda}{e_A}s_\varphi s_\theta \\ -i\frac{2e_B^2\tilde{\Lambda}}{\Lambda^2}s_\varphi s_\theta - \frac{e_B^2}{\Lambda^2}s_\varphi c_\theta + \frac{\Lambda}{e_A}c_\varphi s_\theta \\ \frac{4e_Ae_B}{\Lambda^2\sin^2\theta}s_\theta^2c_\theta^2\left(c_\varphi c_\theta - 2i\tilde{\Lambda}c_\varphi s_\theta\right) \\ \frac{4e_Ae_B}{\Lambda^2\sin^2\theta}s_\theta^2c_\theta^2\left(-is_\varphi c_\theta + 2\tilde{\Lambda}s_\varphi s_\theta\right) \end{pmatrix}$$

Where $c_\varphi = \cos\frac{\varphi}{2}$, $s_\varphi = \sin\frac{\varphi}{2}$, $c_\theta = \cos\frac{\theta}{2}$, $s_\theta = \sin\frac{\theta}{2}$.

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