Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
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On the homogeneity of topological spaces

Sobre la homogeneidad de espacios topológicos

Tesis para optar al grado de Doctor en Matemática

SEBASTIÁN ANDRÉS BARRÍA BURGOS
Marzo 2024
Concepción, Chile

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A esas personas que decidieron no seguir con nosotros

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Resumen

En esta tesis estudiamos la preservación de homogeneidad (y no homogeneidad) de contraejemplos universales no metrizables bajo productos e hiperespacios, con el fin de responder las siguientes preguntas: ¿Es la ω-ésima potencia del plano de Niemytzki homogénea? [Fitzpatrick Jr. and Zhou (1990), Problem 5] y ¿Es el hiperespacio de los cerrados no vacíos de la doble flecha homogéneo? [Arkhangel'skiĭ (1987), Problem II.1].

Para abordar la primera pregunta, investigamos subespacios de la ω-ésima potencia del plano de Niemytzki y la respondemos parcialmente demostrando la homogeneidad del producto entre el plano de Niemytzki y la ω-ésima potencia de un abierto básico. Como consecuencia, concluimos que el producto de la ω-ésima potencia del plano de Niemytzki con la ω-ésima potencia de un abierto básico es también homogéneo.

Para responder a la segunda pregunta, analizamos hiperespacios de la doble flecha y ofrecemos una respuesta parcial probando que los espacios de uniones de a lo más una cantidad finita de intervalos cerrados, así como todos los productos simétricos excepto el primero, no son homogéneos. Como contraparte, demostramos que el segundo producto simétrico de la recta de Sorgenfrey es homogéneo. Además, logramos dar una imagen completa de cómo lucen los autohomeomorfismos de potencias finitas de la doble flecha. Mostramos que cualquier autohomeomorfismo de una potencia finita de la doble flecha es localmente (fuera de un conjunto nunca denso) un producto de encajes monótonos que van desde un intervalo abierto-cerrado de la doble flecha a esta, seguido de una permutación de las coordenadas.

Keywords – espacio homogéneo, hiperespacio, producto simétrico, plano de Niemytzki, recta de Sorgenfrey, flecha doble, sucesión convergente no trivial

Abstract

In this thesis we study the preservation of homogeneity (and non-homogeneity) of nonmetrizable universal counterexamples under products and hyperspaces. The main objective is to answer the following questions: Is ω^{th} power of the Niemytzki plane homogeneous? [Fitzpatrick Jr. and Zhou (1990), Problem 5], Is the hyperspace of nonempty closed subsets of the double arrow homogeneous? [Arkhangel'skiĭ (1987), Problem II.1].

To answer the first question, we study subspaces of the ω^{th} power of the Niemytzki plane and we answer it partially by showing that the product of the ω^{th} power of the Niemytzki plane and the ω^{th} power a basic open is homogeneous. As a consequence, the product of the ω^{th} power of the Niemytzki plane and the ω^{th} power of a basic open is homogeneous.

To answer the second question, we study hyperspaces of the double arrow and we answer it partially by showing that the spaces of all unions of at most a finite number of closed intervals and all symmetric products, except the first one, are nonhomogeneous. As a counterpart, we prove that the second symmetric product of the Sorgenfrey line is homogeneous. Moreover, we give a complete picture on how the autohomeomorphisms of finite powers of the double arrow looks like, by showing that any autohomeomorphism of a finite power of the double arrow is locally (outside of a nowhere dense set) a product of monotone embeddings from a clopen interval of the double arrow to the double arrow, followed by a permutation of the coordinates.

Keywords – homogeneous space, hyperspace, symmetric product, Niemytzki plane, Sorgenfrey line, double arrow, nontrivial convergent sequence

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Introducción

Un espacio es homogéneo si para cada par de puntos de este existe un autohomeomorfismo que lleva un punto en el otro. Intuitivamente, esto significa que la estructura topológica en cada punto es la misma, es decir, que no depende del punto. Algunos ejemplos de espacios homogéneos son grupos topológicos y variedades conexas.

A pesar de que la homogeneidad es un concepto bastante natural, aún no es bien comprendido. Muchas veces la homogeneidad de un espacio no es fácil de verificar, sobre todo en ausencia de metrizabilidad (ver Arhangel'skii and van Mill (2014)). Dicho lo anterior, en el contexto de "contraejemplos universales" no metrizables (ver Steen and Seebach (1995)), queremos dar un poco de luz al asunto estudiando la preservación de homogeneidad (o no homogeneidad) bajo las operaciones topológicas de producto e hiperespacio.

Es fácil ver que la homogeneidad se comporta bien bajo productos. Más aún, el producto de espacios no homogéneos puede ser homogéneo. Por ejemplo, el cubo de Hilbert $[0,1]^\omega$ [Keller (1931)], el producto numerable de variedades conexas metrizables con frontera [Fort (1962),Yang (1992)] y el producto numerable de espacios primero contables 0-dimensionales [Dow and Pearl (1997)]. Observamos que no siempre la homogeneidad puede ser inducida; por ejemplo, ninguna potencia de $\{0\} \cup [1,2]$ es homogénea. Así, la homogeneidad de un producto numerable de espacios se vuelve bastante compleja.

En [Fitzpatrick Jr. and Zhou (1990), Problem 5], y más recientemente en [Hrušák and van Mill (2018), Problem 5)], los autores hacen la siguiente pregunta:

Pregunta 1. *¿Es la ω-ésima potencia del plano de Niemytzki homogénea?*

En esta tesis respondemos parcialmente esta pregunta probando que:

Teorema 1 (Teorema 2.2.9)**.** *El producto del plano de Niemytzki con la ω-ésima potencia de un abierto básico del plano de Niemyztki es homogéneo. En particular,* *el producto de la ω-ésima potencia del plano de Niemytzki con la ω-ésima potencia de un abierto básico del plano de Niemyztki es homogéneo.*

Dado un espacio X, el hiperespacio $\mathrm{Exp}(X)$ es el conjunto de todos los subconjuntos cerrados no vacíos de X con la topología de Vietoris. Esta topología generaliza la métrica de Hausdorff en el caso de espacios metrizables compactos (ver Michael (1951) e Illanes and Nadler (1999)). Dado un entero positivo m, el producto simétrico $\mathcal{F}_m(X)$ es el subespacio de $\mathrm{Exp}(X)$ de todos los subconjuntos de X con a lo más m elementos. Dado un espacio linealmente ordenado X, consideramos $\mathcal{C}_m(X)$ como el subespacio de $\mathrm{Exp}(X)$ de todas las uniones de a lo más m intervalos cerrados.

Varios resultados clásicos sobre homogeneidad involucran el estudio de hiperespacios. En los años 70, R. Schori y J. West (Schori and West (1975)) mostraron que $\mathrm{Exp}([0,1])$ es homeomorfo al cubo de Hilbert. En particular, $\mathrm{Exp}(X)$ puede ser homogéneo sin que X lo sea. Por otro lado, si $\kappa > \aleph_1$, entonces $\mathrm{Exp}\left(2^{\kappa}\right)$ no es homogéneo (Ščepin (1976)). Por consiguiente, se puede ver que la homogeneidad de un hiperespacio es bastante sutil.

Sea $\mathbb{A}$ la flecha doble de Alexandroff-Uryshon. En [Arkhangel'skiĭ (1987), Problem II..1], A.V. Arhangel'skiĭ pregunta lo siguiente,

Pregunta 2. *¿Es el hiperespacio* $\mathrm{Exp}(\mathbb{A})$ *homogéneo?*

Para comprender mejor este hiperespacio, estudiamos la homogeneidad de algunos de sus subespacios más conocidos, tales como los productos simétricos y el espacio de sucesiones convergentes no triviales. Este último fue introducido en García-Ferreira and Ortiz-Castillo (2015) para espacios métricos y estudiado de forma más general en Maya et al. (2018). Hoy en día tiene gran interés entre topólogos.

Respondemos parcialmente la pregunta 2 mostrando que.

Teorema 2 (Teorema 4.3.5)**.** *El producto simétrico* $\mathcal{F}_m(\mathbb{A})$ *no es homogéneo para todo* $m \geq 2$.

Teorema 3 (Teorema 4.4.5)**.** $\mathcal{C}_m(\mathbb{A})$ *no es homogéneo para todo entero positivo* m.

A pesar de la fuerte conexión entre $\mathbb{A}$ y la recta de Sorgenfrey $\mathbb{S}$, como contraparte demostramos que

Teorema 4 (Teorema 3.2.3)**.** *El producto simétrico $\mathcal{F}_2(\mathbb{S})$ es homeomorfo a $\mathbb{S}^2$. En particular, es homogéneo.*

Durante el desarrollo del Teorema 2, estudiamos el grupo de autohomeomorfismos de ${}^m\mathbb{A}$ y obtuvimos el siguiente resultado que nos da una imagen completa de la estructura de tales morfismos.

Teorema 5 (Teorema 4.2.7)**.** *Si $h : {}^m\mathbb{A} \to {}^m\mathbb{A}$ un homeomorfismo, entonces existe una sucesión de cajas abiertas-cerradas básicas disjuntas $U_n := \prod_{j\in m} I_n^j (n \in \omega)$ tales que $\bigcup_{n\in\omega} U_n$ es denso en ${}^m\mathbb{A}$ y $h \restriction U_n = \sigma \circ (h^0 \times \cdots \times h^{m-1})$, donde cada $h^j : I_n^j \to \mathbb{A}$ es un homeomorfismo estrictamente monótono sobreyectivo a un intervalo abierto-cerrado, y σ es una permutación de ${}^m\mathbb{A}$.*

A continuación, se presenta la estructura del trabajo. En el Capítulo 1 proporcionamos definiciones y resultados básicos sobre homogeneidad e hiperespacios necesarios para comprender el resto de los contenidos. Además, probamos un teorema de metrización (Proposición 1.3.1).

En el Capítulo 2 estudiamos la homogeneidad de algunos subespacios de la ω-ésima potencia del plano de Niemytzki. En la sección 2.1 hablamos sobre cierto tipo de n-celdas (Definición 2.1.1) y recordamos un fuerte teorema de los orígenes de la topología de dimensión infinita (Teorema 2.1.2). En la sección 2.2, analizando los abiertos básicos del plano de Niemytzki (Lema 2.2.2) y ocupando el teorema mencionado, demostramos que la ω-ésima potencia de los dichos básicos es homogénea (Corolario 2.2.4). Luego, por extensión probamos el Teorema 1.

En el Capítulo 3 analizamos la homogeneidad de tres subespacios de $\operatorname{Exp}(\mathbb{S})$. En la sección 3.1 mostramos que el espacio de sucesiones convergentes no triviales de $\mathbb{S}$ es homogéneo (Proposición 3.1.2). En la sección 3.2, haciendo uso de una partición muy ingeniosa (Proposición 3.2.1) probamos el Teorema 4. En la sección 3.3 demostramos que el espacio de los intervalos cerrados no vacíos de $\mathbb{S}$ es homogéneo (Proposición 3.3.2).

En el Capítulo 4 investigamos la homogeneidad de varios subespacios de $\operatorname{Exp}(\mathbb{A})$. En la sección 4.1 probamos que el espacio de sucesiones convergentes no triviales de $\mathbb{A}$ es homogéneo (Proposición 4.1.2). En la sección 4.2 demostramos el Teorema 5 haciendo uso de funciones monótonas definidas sobre $\mathbb{A}$. En la sección 4.3 mostramos el Teorema 2. En la sección 4.4, dado un espacio linealmente ordenado X damos una caracterización para $\mathcal{C}_m(X)$ (Proposición 4.4.2) y probamos el Teorema 3.

Emplearemos el libro de R. Engelking Engelking (1989) como referencia para topología. Para homogeneidad utilizaremos el artículo de A.V. Arhangel'skiǐ y J. van Mill Arhangel'skii and van Mill (2014). Para hiperespacios ocuparemos el artículo de E. Michael Michael (1951) y el libro de A. Illanes con S.B. Nadler Jr. Illanes and Nadler (1999).

Introduction

A space is *homogeneous* if for any two of its points there exists an autohomeomorphism that carries one point into the other. Intuitively, this means that all points have the same topological properties, that is, the topological structure does not depend on the point. Some examples are topological groups and connected manifolds.

Even though homogeneity is a very natural concept, it is not well understood yet. Very often the homogeneity of a space is not easy to verify, especially in the absence of metrizability (see Arhangel'skii and van Mill (2014)). Having said the above, in the context of "universal counterexamples" (see Steen and Seebach (1995)), we want to shed some light on the issue by studying the preservation of homogeneity (or non-homogeneity) under the topological operations of product and hyperspace.

It is easy to see that homogeneity behaves well under products. Moreover, the infinite product of non homogeneous spaces can be homogeneous. For example; the Hilbert cube $[0,1]^\omega$ [Keller (1931)], the countable infinite product of connected metrizable manifolds with boundary [Fort (1962); Yang (1992)] and any product of countably many 0-dimensional first countable spaces [Dow and Pearl (1997)]. On the other hand, we observe that homogeneity not always can be induced; for example, no power of $\{0\} \cup [1,2]$ is homogeneous. Thus, homogeneity of a product can be quite complex.

In [Fitzpatrick Jr. and Zhou (1990), Problem 5], and more recently in [Hrušák and van Mill (2018), Problem 5], the authors asks the following.

Question 1. *Is the ω^{th} power of the Niemytzki plane homogeneous?*

In this thesis we partially solves this question proving that.

Theorem 1. *The product of the Niemytzki plane and the ω^{th} power of a basic neighborhood of the Niemytzki plane is homogeneous. In particular, the product of*

ω^{th} power of the Niemytzki plane and the ω^{th} power of a basic neighborhood of the Niemytzki plane is homogeneous.

Given a space X, the hyperspace $\mathrm{Exp}(X)$ is the set of all nonempty closed subsets of X with the Vietoris topology. This topology generalizes the Hausdorff metric for compact metrizable spaces (see Michael (1951) and Illanes and Nadler (1999)). Given a positive integer m, the symmetric product $\mathcal{F}_m(X)$ is the subspace of $\mathrm{Exp}\,(X)$ consisting of all nonempty subsets of X with at most m elements. For a linearly ordered space X, we consider $\mathcal{C}_m(X)$ as the subspace of $\mathrm{Exp}(X)$ of all unions of at most m closed intervals.

Several classic results on homogeneity involve the study of the hyperspaces. In the 1970's, it was shown by R. Schori and J. West Schori and West (1975) that $\mathrm{Exp}([0,1])$ is homeomorphic to the Hilbert cube. In particular, it is possible that the hyperspace $\mathrm{Exp}(X)$ is homogeneous while X is not. On the other hand, if $\kappa > \aleph_1$, then $\mathrm{Exp}\,(2^{\kappa})$ is not homogeneous (see Ščepin (1976)). Thus, the question of homogeneity of the hyperspace turns out to be quite subtle.

Let $\mathbb{A}$ be the Alexandroff-Uryshon double arrow space. In [Arkhangel'skiǐ (1987), Problem II.1], A.V. Arhangel'skiǐ asked the following question.

Question 2. *Is the hyperspace* $\mathrm{Exp}\,(\mathbb{A})$ *homogeneous?*

To better understand this space, we study the homogeneity of some of the most well-known subspaces of it, such as symmetric products and the space of nontrivial convergent sequences. The later space was introduced in García-Ferreira and Ortiz-Castillo (2015) for metric spaces and studied in a more general setting in Maya et al. (2018). Today has a great interest among topologists.

We partially answer Question 2, by showing that.

Theorem 2 (Theorem 4.3.5)**.** *The symmetric product $\mathcal{F}_m(\mathbb{A})$ is not homogeneous for any $m \geq 2$.*

Theorem 3 (Theorem 4.4.5)**.** *$\mathcal{C}_m(X)$ is not homogeneous for any positive integer m.*

Despite the strong connection between $\mathbb{A}$ and the Sorgenfrey line $\mathbb{S}$, as a counterpart we prove that.

Theorem 4 (Theorem 3.2.3)**.** *The symmetric product $\mathcal{F}_2(\mathbb{S})$ is homeomorphic to $\mathbb{S}^2$. In particular, it is homogeneous.*

In the course of proving Theorem 2, we study the group of autohomeomorphisms of ${}^m\mathbb{A}$ and obtain the following Theorem which gives us a complete picture on the structure of such autohomeomorphisms.

Theorem 5 (Theorem 4.2.7)**.** *Let $h : {}^m\mathbb{A} \to {}^m\mathbb{A}$ be a homeomorphism. Then there is a pairwise disjoint sequence of basic clopen boxes $U_n := \prod_{j\in m} I_n^j (n \in \omega)$ such that $\bigcup_{n\in\omega} U_n$ is dense in ${}^m\mathbb{A}$ and $h \restriction U_n = \sigma \circ (h^0 \times \cdots \times h^{m-1})$, where each $h^j : I_n^j \to \mathbb{A}$ is an strictly monotone homeomorphism onto a clopen interval, and σ is a permutation of ${}^m\mathbb{A}$.*

This thesis is organized as follows. In Chapter 1 we will give definitions and basic results about homogeneity and hyperspaces needed to understand the other chapters. Also, we prove a metrization theorem (Proposition 1.3.1).

In Chapter 2 we study the homogeneity of some subspaces of the ω^{th} power of the Niemytzki plane. In section 2.1 we talk about a special type of n-cells (Definition 2.1.1) and recall a strong theorem from the origins of the infinite-dimensional topology (Theorem 2.1.2). In section 2.2, analyzing the basic open sets of the Niemytzki plane (Lemma 2.2.2) and using the theorem above, we show that the ω^{th} power of such basic sets is homogeneous (Corollary 2.2.4). Finally, by extension we prove Theorem 1.

In Chapter 3 we study the homogeneity of three subspaces of $\mathrm{Exp}(\mathbb{S})$. In section 3.1 we show that the space of nontrivial convergent sequences of $\mathbb{S}$ is homogeneous (Proposition 3.1.2). In section 3.2, via a very ingenious partition (Proposition 3.2.1) we prove Theorem 4. In section 3.3 we show that the space of nonempty closed intervals of $\mathbb{S}$ is homogeneous (Proposition 3.3.2).

In Chapter 4 we study the homogeneity of several subspaces of $\mathrm{Exp}(\mathbb{A})$. In section 4.1 we show that the space of nontrivial convergent sequences of $\mathbb{A}$ is homogeneous (Proposition 4.1.2). In section 4.2 we prove Theorem 5 via monotone functions defined on $\mathbb{A}$. In section 4.3 we prove Theorem 2. In section 4.4, for a compact linearly ordered space X, we give a characterization for $\mathcal{C}_m(X)$ (Proposition 4.4.2) and we show Theorem 3.

We will use the book of R. Engelking Engelking (1989) as a basic reference on topology. For homogeneity we will use the article of A.V. Arhangel'skiǐ and J. van Mill Arhangel'skii and van Mill (2014). For hyperspaces we will use the article of E. Michael Michael (1951) and the book of A. Illanes with S.B. Nadler Jr. Illanes and Nadler (1999).

Chapter 1

Homogeneity and hyperspaces

By a space we mean a topological space. In the first two sections of this chapter we will give definitions and basic results on homogeneity and hyperspaces. In section 1.1 we prove that finite powers of the Niemytzki plane are nonhomogeneous and we give basic properties of the Sorgenfrey line and the double arrow. In section 1.2 we recall definitions and properties related to the Vietoris topology and a characterization for symmetric products. In the last section we give a metrization theorem that was obtained in our efforts to prove Theorem 4.3.5 and generalizes a classical result on compact spaces.

1.1 Homogeneity

Definition 1.1.1. *A space X is homogeneous if for every $x, y \in X$ there exists $h \in \mathrm{Aut}(X)$ such that $h(x) = y$, where $\mathrm{Aut}(X)$ denotes the group of autohomeomorphisms of X.*

Example 1.1.2. *The real numbers, the rational numbers and the Hilbert cube are homogeneous spaces.*

The homogeneity of the Hilbert cube was first proved in Keller (1931). More recent proofs can be found in [van Mill (2001), Theorem 1.6.6] and [Sakai (2020), Theorem 2.1.2].

The proof of the following proposition is straightforward.

Proposition 1.1.3. *Let $\{X_s\}_{s\in S}$ be a family of spaces. If X_s is homogeneous for all $s \in S$, then $\prod_{s\in S} X_s$ is homogeneous. In particular, all powers of a homogeneous space are homogeneous.*

We will now introduce one of the main objects of study in this thesis. Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, $\mathbb{L} = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and $\mathbb{H}^+ = \mathbb{H} \setminus \mathbb{L}$. For every point $x \in \mathbb{L}$, let $\mathcal{B}(x)$ the family of all sets of the form $D \cup \{x\}$ with $D \subset \mathbb{H}^+$ an open disc tangent to $\mathbb{L}$ at x. For every $x \in \mathbb{H}^+$, let $\mathcal{B}(x)$ be the family of all open discs in $\mathbb{H}^+$ centered at x. The set $\mathbb{H}$ with the topology generated by the neighborhood system $\{\mathcal{B}(x)\}_{x\in\mathbb{H}}$ is called *Niemytzki plane* and will be denoted by $\mathbb{H}_N$. This space was defined (and attributed to Niemytzki) by Alexandroff and Hopf in Alexandroff and Hopf (1935). We note that $\mathbb{H}_N$ is first-countable.

Proposition 1.1.4. *For all $m \in \mathbb{Z}^+$, $\mathbb{H}_N^m$ is not homogeneous.*

Proof. Let $m \in \mathbb{Z}^+$ and $x \in \mathbb{H}_N^m$ such that all its coordinates are equal to $(0, 0)$. We claim that x does not have an open neighborhood with compact closure and we proceed by contradiction. Let $B \subset \mathbb{H}_N^m$ a basic neighborhood of x. In this way, there are open discs D_i tangent to $(0, 0)$ such that $B = \prod_{i=1}^{m}(D_i \cup \{(0, 0)\})$. Assume that $\overline{B}$ is compact. In particular, the boundary of B, ∂B, is also compact. Since $\mathbb{H}_N^m$ is first-countable, by [(Engelking, 1989), Theorem 3.10.31] ∂B is sequentially compact. Choose a sequence $(x_n) = (x_{n,1}, \ldots, x_{n,m})(n \in \omega)$ such that for each i, $(x_{n,i})(n \in \omega)$ is a sequence in $\partial(D_i \cup \{(0, 0)\})$ $(x_{n,i} \neq (0, 0))$ that converges to $(0, 0)$ in the Euclidean topology.

Claim 1.1.5. *$(x_n)(n \in \omega)$ do not have convergent subsequences.*

Proof. If x_n have a convergent subsequence, then the only candidate for limit is x. If for each i we choose an open disc C_i tangent to $(0, 0)$ that is strictly contained in D_i, then $\prod_{i=1}^{m}(C_i \cup \{(0, 0)\})$ is an open neighborhood of x that leaves out all the elements of the sequence (x_n). $\square$

Since (x_n) is a sequence in $\prod_{i=1}^{m} \partial(D_i \cup \{(0, 0)\}) \subset \partial B$, there is a convergent subsequence of (x_n), which contradicts the claim. $\square$

The *Sorgenfrey line* $\mathbb{S}$ is the set of the real numbers with the topology generated by the base $\mathcal{B} = \{[a, b[: a, b \in \mathbb{R}, a < b\}$. This space appeared in Alexandroff (1929), but only after Sorgenfrey's paper Sorgenfrey (1947) did it become one of the "universal counterexamples" in general topology.

We recall that a space is 0-*dimensional* if it is T_1 and has a base consisting of clopen sets.

Proposition 1.1.6. *$\mathbb{S}$ is homogeneous, 0-dimensional and homeomorphic to every element of $\mathcal{B}$ with the subspace topology.*

Proof. We have that $\mathbb{S}$ is 0-dimensional, since every element in $\mathcal{B}$ is clopen.

For the homogeneity, let $a, b \in \mathbb{S}$. The map $f : \mathbb{S} \to \mathbb{S}$ defined by $f(x) = x+b-a$ is a homeomorphism with $f(a) = b$.

Since all the elements in $\mathcal{B}$ are homeomorphic, we will only prove that $\mathbb{S} \cong [0, 1[$. Let $[0, \to [\, = \{x \in \mathbb{R} : x \geq 0\}$. For each $n \in \omega$ we have that $[\frac{1}{n+2}, \frac{1}{n+1}[\, \cong [n, n+1[$. In addition, $[n, n+\frac{1}{2}[\, \cong [n, n+1[$ and $[n+\frac{1}{2}, n+1[\, \cong [-n-1, n[$. In this way, $[n, n+1[\, = [n, n+\frac{1}{2}[\cup [n+\frac{1}{2}, n+1[\, \cong [-n-1, n[\cup [n, n+1[$. Since every element in $\mathcal{B}$ is clopen, $[0, 1[\, = \bigcup_{n\in\omega}[\frac{1}{n+2}, \frac{1}{n+1}[\, \cong \bigcup_{n\in\omega}([-n-1, n[\cup [n, n+1[) = \mathbb{S}$. □

Remark 1.1.7. *Since the Sorgenfrey line is homeomorphic to $[0, 1[$ with the subspace topology, we will asume that the $\mathbb{S} = [0, 1[$.*

Let $(X, <)$ be a linear order. For $a, b \in X$ we define the basic intervals

$$] \leftarrow, a[\, = \{x \in X : x < a\}$$

$$]a, \to [\, = \{x \in X : x > a\}$$

$$]a, b[\, = \{x \in X : a < x < b\}$$

The family of all this sets is a base for a topology called the *order topology*. We say that $(X, <)$ is a *linearly ordered space* if we consider it with the order topology.

Let $\mathbb{A}_0 = \,]0, 1] \times \{0\}$, $\mathbb{A}_1 = [0, 1[\times \{1\}$ and $\mathbb{A} = \mathbb{A}_0 \cup \mathbb{A}_1$. Define the lexicographical ordering $\langle a, r\rangle \prec \langle b, s\rangle$ if $a < b$ or $a = b$ and $r < s$. The set $\mathbb{A}$ with the order topology is the *Alexandroff-Urysohn double arrow space* (*double arrow* for short). This space was defined in (Alexandroff, 1929). We observe that $\mathcal{D} := \{[\langle a, 1\rangle, \langle b, 0\rangle] : 0 \leq a < b \leq 1]\}$ is a base for $\mathbb{A}$.

Remark 1.1.8. *Since the map $f : [\langle a, 1\rangle, \langle b, 0\rangle] \to \mathbb{A}$ defined by $f(\langle x, i\rangle) = \langle \frac{x-a}{b-a}, i\rangle (i \in \{0, 1\})$ is a homeomorphism, we have that every element in $\mathcal{D}$ is homeomorphic to $\mathbb{A}$.*

Proposition 1.1.9. *$\mathbb{A}$ is homogeneous, 0-dimensional, first countable and compact.*

Proof. Since every element of $\mathcal{D}$ is clopen, we have that $\mathbb{A}$ is 0-dimensional.

To prove homogeneity, let $w, z \in \mathbb{A}$. Assume that $w \prec z$. We consider several cases.

Case 1. If $w, z \in A_0$, then $w = \langle a, 0\rangle$ and $z = \langle b, 0\rangle$ for $a, b \in \,]0, 1]$ with $a < b$. The function $f : [\langle 0, 1\rangle, \langle a, 0\rangle] \to [\langle a, 1\rangle, \langle b, 0\rangle]$ defined by $f(\langle x, i\rangle) = \langle b - \frac{b-a}{b}(b - \frac{bx}{a}), i\rangle$ is a homeomorphism. By the previous remark, there is a homeomorphism g that sends $\mathbb{A} \setminus [\langle 0, 1\rangle, \langle a, 0\rangle]$ onto $\mathbb{A} \setminus [\langle a, 1\rangle, \langle b, 0\rangle]$. Thus, $h := f \cup g \in \mathrm{Aut}(\mathbb{A})$ and $h(w) = z$.

Case 2. For $w, z \in A_1$ the procedure is analogous to the previous case.

Case 3. If $w \in A_0$ and $z \in A_1$, then there are $a, b \in \,]0, 1[$ with $a \leq b$ such that $w = \langle a, 0\rangle$ and $z = \langle b, 1\rangle$. The function $f : [\langle 0, 1\rangle, \langle a, 0\rangle] \to [\langle b, 1\rangle, \langle 1, 0\rangle]$ defined by $f(\langle x, 0\rangle) = \langle 1 - \frac{x(1-b)}{a}, 1\rangle$ and $f(\langle x, 1\rangle) = \langle 1 - \frac{x(1-b)}{a}, 0\rangle$ is a homeomorphism. By the previous remark, there is a homeomorphism g that sends $\mathbb{A} \setminus [\langle 0, 1\rangle, \langle a, 0\rangle]$ onto $\mathbb{A} \setminus [\langle b, 1\rangle, \langle 1, 0\rangle]$. Thus, $h := f \cup g \in \mathrm{Aut}(\mathbb{A})$ and $h(w) = z$.

Case 4. For $w \in A_1$ and $z \in A_0$ the procedure is similar to the previous case except when $w = \langle 0, 1\rangle$ or $z = \langle 1, 0\rangle$. If $w = \langle 0, 1\rangle$ and $z = \langle a, 0\rangle$ for some $a \in \,]0, 1]$, then the function $f : [\langle 0, 1\rangle, \langle a, 0\rangle] \to [\langle 0, 1\rangle, \langle a, 0\rangle]$ defined by $f(\langle x, 0\rangle) = \langle a - x, 1\rangle$ and $f(\langle x, 1\rangle) = \langle a - x, 0\rangle$ is a homeomorphism. If $a = 1$, then we are done. If $a < 1$, then by the previous proposition there is a homeomorphism g that sends $\mathbb{A} \setminus [\langle 0, 1\rangle, \langle a, 0\rangle]$ onto itself. Thus, $h := f \cup g \in \mathrm{Aut}(\mathbb{A})$ and $h(w) = z$. Similarly for $w = \langle a, 1\rangle$ and $z = \langle 1, 0\rangle$ with $a \in [0, 1[$.

We conclude that $\mathbb{A}$ is homogeneous. The other properties can be found in Engelking (1989). $\square$

Remark 1.1.10. *We note that $\mathbb{A}_0$ and $\mathbb{A}_1$ have the Sorgenfrey line topology as subspaces of $\mathbb{A}$.*

1.2 Hyperspaces

Given a space X, we denote by $\mathrm{Exp}\,(X)$ the set of all non-empty closed subsets of X. For a non-empty open set V of X, let $[V] = \{F \in \mathrm{Exp}\,(X) : F \subset V\}$ and $\langle V\rangle = \{F \in \mathrm{Exp}\,(X) : F \cap V \neq \emptyset\}$.

Definition 1.2.1 (Michael (1951))**.** *The collection of all sets $[V]$ and $\langle V\rangle$ is a subbase for a topology on $\mathrm{Exp}\,(X)$ called the Vietoris topology.*

From now on, $\mathrm{Exp}(X)$ will be considered with the Vietoris topology.

Remark 1.2.2. *The collection of all sets of the form*

$$\left\{F \in \operatorname{Exp}(X) : F \subset \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i\right\}$$

with $U_1, \ldots, U_n$ non-empty open sets of X; is a base for a topology. Originally, the Vietoris topology was defined in Vietoris (1922) as the topology with such a base.

Since $\langle \cup_i V_i \rangle = \cup_i \langle V_i \rangle$ for any collection of non-empty open sets V_i; if X is generated by a base, then the Vietoris topology on $\operatorname{Exp}(X)$ is generated by the subbase of all sets of the form $[V]$ and $\langle W \rangle$ with V open sets and W basic sets. It is known that if X is compact, then $\operatorname{Exp}(X)$ is also compact [(Vietoris, 1922)].

Definition 1.2.3. *Given a space X, a hyperspace of X is any subspace of* $\operatorname{Exp}(X)$.

All subsets of $\operatorname{Exp}(X)$ will be considered hyperspaces.

Definition 1.2.4. *For $m \in \mathbb{Z}^+$, the m^{th} symmetric product of X, $\mathcal{F}_m(X)$, is the collection of all subsets of X with cardinality at most m.*

In this way, all symmetric products are hyperspaces. We recall the following result by E. Michael.

Proposition 1.2.5 (Michael (1951))**.** *If X is a T_1 space, then $X \cong \mathcal{F}_1(X)$.*

Given a linearly ordered space $(X, <)$, we denote by

$$\Delta_m(X) = \{x \in X^m : \forall i \in \{1, \ldots, m-1\}(\pi_i(x) \leq \pi_{i+1}(x))\}$$

Let $\rho : \Delta_m(X) \to \mathcal{F}_m(X)$ be the map given by $\rho(x) = \{\pi_1(x), \ldots, \pi_m(x)\}$ and let $\sim$ the equivalence relation on $\Delta_m(X)$ defined by $x \sim y$ if and only if $\rho(x) = \rho(y)$. Let $q : \Delta_m(X) \to \Delta_m(X)/\sim$ be the quotient map. We will sometimes write $[x]$ instead of $q(x)$ to represent the equivalence class. We consider $\Delta_m(X)/\sim$ as a topological space with the quotient topology.

The following classical fact gives us a more geometric representation of $\mathcal{F}_m(X)$.

Proposition 1.2.6 (Ganea (1954))**.** *If X is a linearly ordered space, then the map $\tilde{\rho} : \Delta_m(X)/\sim \to \mathcal{F}_m(X)$ given by $\tilde{\rho}([x]) = \rho(x)$ is a homeomorphism.*

Definition 1.2.7 (Maya et al. (2018))**.** *Let X be a Hausdorff space. A set $S \subset X$ will be called a nontrivial convergent sequence in X if S is countably infinite and* *there is $x \in S$ such that $S \setminus V$ is finite for any open neighborhood V of x. The point x is called the limit of S and we will say that S converges to x. The set of all nontrivial convergent sequences in X will be denoted $\mathcal{S}_c(X)$.*

Remark 1.2.8. *In the usual sense, a convergent sequence in X is a function $f : \omega \to X$ for which there exists $x \in X$ such that for each open neighborhood V of x, there is $m \in \omega$ with $f(n) \in U$ for all $n \geq m$. If $f''(\omega)$ is infinite, then $(\{x\} \cup f''(\omega)) \in \mathcal{S}_c(X)$.*

The hyperspaces $\mathcal{F}_m(X)$ and $\mathcal{S}_c(X)$ will be relevant in chapters 3 and 4.

1.3 A metrization theorem

Proposition 1.3.1. *Let X be a compact Hausdorff space. If there exists a G_δ-set $C \subset X^2$ homeomorphic to X such that for every $x \in X$ there are $a \in C$ and a unique $b \in C$ with $x = \pi_1(a) = \pi_2(b)$, then X is metrizable.*

Proof. For each $n \in \omega$, let G_n be an open subset of X^2 such that $C = \bigcap_{n\in\omega} G_n$ and $G_{n+1} \subset G_n$. Since X^2 is normal and C is closed, we can define a sequence of open sets U_n as follows. Let $U_0 = G_0$ and for $n > 0$ let U_n such that $C \subset U_n \subset \overline{U_n} \subset U_{n-1} \cap G_n$. It follows that $C = \bigcap_{n\in\omega} \overline{U_n}$. Let $x \in X$ and $(s, x) \in C$. For $n \in \omega$, let $U_n[s] := \{y \in X : (s, y) \in U_n\}$. Hence, $\overline{U_n[s]} \subset U_{n-1}[s]$ for any $n > 0$, since

$$\begin{aligned}\overline{U_n[s]} = \overline{\pi_2''(U_n \cap (\{s\} \times X))} &= \pi_2''(\overline{U_n \cap (\{s\} \times X)}) \subset \pi_2''(\overline{U_n} \cap \overline{\{s\} \times X}) \\ &\subset \pi_2''(U_{n-1} \cap (\{s\} \times X)) = U_{n-1}[s]\end{aligned}$$

where the second equality follows from [Engelking (1989), Corollary 3.1.11].

Claim 1.3.2. *For any open neighborhood V of x, there exists $n \in \omega$ such that*

$$x \in U_n[s] \subset V$$

Proof. We proceed by contradiction. Let V be an open neighborhood of x with $U_n[s] \not\subset V$ for any $n \in \omega$. We choose $x_n \in U_n[s] \setminus V$. It follows that $\bigcap_{n\in\omega} \overline{U_n[s]} = \{x\}$, since if $z \in \bigcap_{n\in\omega} \overline{U_n[s]} \subset \bigcap_{n\in\omega} U_n[s]$, then $(s, z) \in \bigcap_{n\in\omega} U_n \subset C$. Since X is compact, the pseudocharacter and the character of x are equal [Engelking (1989), Exercise 3.1.F.(a) (proved in (Alexandroff, 1924))]. Since the sets $U_n[s]$

are open, the pseudocharacter of x is countable. Hence, X is first countable. By [(Engelking, 1989), Theorem 3.10.31] X is sequentially compact. In this way, there exists a convergent subsequence of $(x_n)(n \in \omega)$, let us say with limit L. Since each set $\overline{U_m[s]}$ is closed and contains infinitely many elements of such subsequence, we have that L belongs to each one of them. Thus, $L \in \bigcap_{m\in\omega} \overline{U_m[s]} = \{x\}$. By convergence, there are infinite elements of the subsequence in V, which is a contradiction. $\square$

For each $(s,t) \in C$ and $n \in \omega$ we can choose open neighborhoods $B_{s,n}$ and $B_{t,n}$ such that $B_{s,n} \times B_{t,n} \subset U_n$. Since X is compact, there exists a finite subcover $\mathcal{B}_n$ of $\{B_{t,n} : t \in X\}$. We claim that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a countable base for X. Let $x \in X$ and V an open neighborhood of x. By the claim, there is $n \in \omega$ such that $x \in U_n[s] \subset V$. We choose $B_{z,n}$ from $\mathcal{B}_n$ with $x \in B_{z,n}$. Therefore, $x \in B_{z,n} \subset U_n[s] \subset V$.

By the Urysohn's metrization theorem, X is metrizable. $\square$

As a consequence, we obtain the following classical fact.

Corollary 1.3.3 (Šneĭder (1945)). *Let X a compact Hausdorff space. If the diagonal of X is a G_δ-set, then X is metrizable.*

Chapter 2

Subspaces of countable products of Niemytzki planes

In section 2.1 we give the definition and some properties of near n-cells. In particular, Anderson has found when the countable infinite product of near n-cells is homeomorphic to $\mathbb{R}^\omega$. In section 2.2, via a geometric construction we show that basic neighborhoods of points in $\mathbb{H}$ with the Niemytzki topology are homeomorphic to basic neighborhoods with the Euclidean topology. With this results, we prove the ω^{th} power of basic neighborhoods of the origen are homeomorphic to $\mathbb{R}^\omega$. By extension, we prove that the product of the Niemytzki plane with the ω^{th} power of a basic neighborhood of it is homogeneous and as a consequence, we show that the product of the ω^{th} power of the Niemytzki plane with the ω^{th} power of a basic neighborhood is homogeneous too.

2.1 About n-cells

We recall that for $n \in \mathbb{Z}^+$, a *closed n-cell* is a product of n closed intervals of the real line.

Definition 2.1.1 (Anderson (1967)). *Let $Y \subset \mathbb{R}^n$. We have that Y is a near n-cell if Y is a subset of a closed n-cell V and the interior of V is contained in Y. Additionally, if $V \setminus Y$ is a G_δ of the boundary of V, then Y is called a G_δ near n-cell. A proper near n-cell is a proper subset of a near n-cell.*

We now recall the following result by R.D. Anderson.

Theorem 2.1.2 (Anderson (1967), Theorem 9.3)**.** *A countable infinite product of near n-cells is homeomorphic to $\mathbb{R}^\omega$ if and only if all its factors are G_δ near n-cells and infinite many of them are proper.*

By the previous theorem is natural to ask if, in general, an infinite product of near n-cells is homogeneous. The answer is negative.

Example 2.1.3. *Let D be the interior of a closed 2-cell and b a point of its boundary. We have that $(D \cup \{b\}) \times [0,1]^\omega$ is not homogeneous, since points with coordinate b do not have open neighborhoods with compact closure.*

However, we have the following result.

Proposition 2.1.4. *The ω^{th} power of a product of G_δ near n-cells is homogeneous.*

Proof. Let $\{A_s\}_{s\in S}$ be a family of G_δ near n-cells. Let P be the set of all indexes $s \in S$ with A_s a proper G_δ near n-cell. Thus,

$$\begin{aligned}\left(\prod_{s\in S} A_s\right)^\omega = \left(\prod_{s\in P} A_s\right)^\omega \times \left(\prod_{s\in S\setminus P} A_s\right)^\omega &= \prod_{s\in P} A_s^\omega \times \prod_{s\in S\setminus P} A_s^\omega \\ &\cong \prod_{s\in P} \mathbb{R}^\omega \times \prod_{s\in S\setminus P} [0,1]^\omega\end{aligned}$$

Since $\mathbb{R}$ and $[0,1]^\omega$ are homogeneous and product of homogeneous spaces is homogeneous, we have that $\left(\prod_{s\in S} A_s\right)^\omega$ is homogeneous. $\square$

2.2 Homogeneity of some subspaces of the ω^{th} power of the Niemytzki plane

Proposition 2.2.1. *If $a, b \in \mathbb{H}^+$ (or $\mathbb{L}$), then there is $f \in \mathrm{Aut}(\mathbb{H}_N)$ such that $f(a) = b$.*

Proof. Let π_1 and π_2 be the projections onto the x-axis and y-axis, respectively.

Let $a, b \in \mathbb{L}$. Assume that $\pi_1(a) \leq \pi_1(b)$. In this way, the map $f : \mathbb{H}_N \to \mathbb{H}_N$ defined by $f(x) = (\pi_1(x) + \pi_1(b) - \pi_1(a), \pi_2(x))$ is as required. For $\pi_1(a) > \pi_1(b)$ the argument is similar.

Let $a, b \in \mathbb{H}^+$. We have two main subcases. If $\pi_2(a) = \pi_2(b)$, then without loss of generality assume that $\pi_1(a) < \pi_1(b)$. Thus, the function $f : \mathbb{H}_N \to \mathbb{H}_N$ defined by $f(x) = (\pi_1(x) + \pi_1(b) - \pi_1(a), \pi_2(x))$ is a homeomorphism and $f(a) = b$. If $\pi_1(a) = \pi_1(b)$, then assume that $\pi_2(a) < \pi_2(b)$. The function $f : \mathbb{H}_N \to \mathbb{H}_N$ defined by $f(x) = (\pi_1(x), \pi_2(b)\pi_2(x)/\pi_2(a))$ is a homeomorphism with $f(a) = b$. Finally, let us analyze the general case. We can assume that $\pi_1(a) < \pi_1(b)$ and $\pi_2(a) < \pi_2(b)$. By the previous subcases, we can send a to b via the homeomorphism that sends a to $(\pi_1(a), \pi_2(b))$ composition the homeomorphism that sends $(\pi_1(a), \pi_2(b))$ to b. □

We denote $\mathbb{H}$ with the Euclidean topology by $\mathbb{H}_E$. For any $A \subset \mathbb{H}$, A_N and A_E denote A with the Niemytzki and Euclidean topologies, respectively. Let $\mathbb{B} = \mathbb{H}^+ \cup \{(0,0)\}$.

Lemma 2.2.2. *$\mathbb{B}_N$ is homeomorphic to $\mathbb{B}_E$.*

Proof. Let $f : \mathbb{B}_N \to \mathbb{B}_E$ be the function defined by

$$f(x,y) = \begin{cases} \left(\dfrac{x\sqrt{x^2+y^2}}{y}, \sqrt{x^2+y^2}\right) & \text{if } x \in \mathbb{R} \text{ and } y > 0 \\ (0,0) & \text{if } x = y = 0 \end{cases}$$

We will prove that f is a homeomorphism. First of all, f is bijective with inverse

$$f^{-1}(s,t) = \begin{cases} \left(\dfrac{st}{\sqrt{s^2+t^2}}, \dfrac{t^2}{\sqrt{s^2+t^2}}\right) & \text{if } s \in \mathbb{R} \text{ and } t > 0 \\ (0,0) & \text{if } s = t = 0 \end{cases}$$

Since the Niemytzki topology coincides with the Euclidean topology on $\mathbb{H}^+$ and $f \restriction \mathbb{H}^+ \in \mathrm{Aut}(\mathbb{H}_E^+)$, we have that a set is open in $\mathbb{H}_N^+$ if and only if its image under f is open in $\mathbb{H}_E^+$.

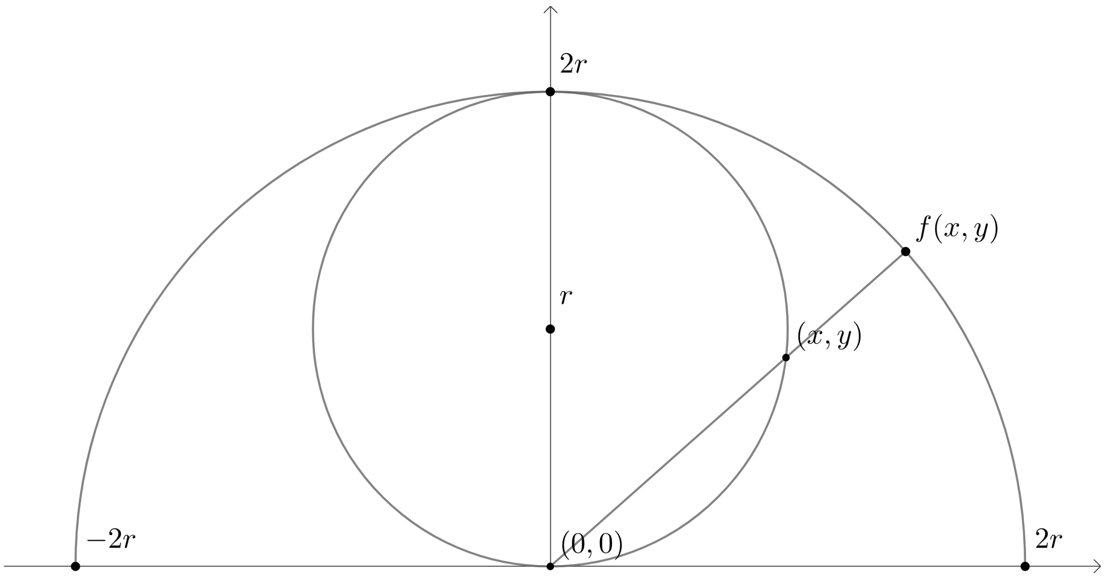
Let U be a basic neighborhood of $(0,0)$ in $\mathbb{B}_N$. There is $r > 0$ such that $U = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-r)^2 < r^2\} \cup \{(0,0)\}$. Define $V = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4r^2 \wedge y > 0\} \cup \{(0,0)\}$.

Claim 2.2.3. *$f''(U) = V$.*

Proof. If $(x,y) \in U \setminus \{(0,0)\}$, then $x^2 + (y-r)^2 < r^2$, that is to say $x^2 + y^2 < 2ry$. In this way,

$$\frac{x^2(x^2+y^2)}{y^2} + x^2 + y^2 = \frac{(x^2+y^2)^2}{y^2} < \frac{(2ry)^2}{y^2} = 4r^2$$

Figure 2.2.1: Construction of f



which means that $f(x,y) \in V$.

If $(s,t) \in V \setminus \{(0,0)\}$, then $s^2+t^2 < 4r^2$, or equivalently, $1 < \dfrac{2r}{\sqrt{s^2+t^2}}$. Thus, $(s,t) = f\left(\dfrac{st}{\sqrt{s^2+t^2}}, \dfrac{t^2}{\sqrt{s^2+t^2}}\right)$ and

$$\left(\frac{st}{\sqrt{s^2+t^2}}\right)^2 + \left(\frac{t^2}{\sqrt{s^2+t^2}} - r\right)^2 = t^2\left(1 - \frac{2r}{\sqrt{s^2+t^2}}\right) + r^2 < r^2$$

implies that

$$\left(\frac{st}{\sqrt{s^2+t^2}}, \frac{t^2}{\sqrt{s^2+t^2}}\right) \in U$$

□

In a similar way, if V is a basic neighborhood of $(0,0)$ in $\mathbb{B}_E$, then $f^{-1}(V) = U$ for a basic neighborhood U of $(0,0)$ in $\mathbb{B}_N$. Therefore, we can send basic neighborhoods of $(0,0)$ with the Niemytzki topology onto basic neighborhoods of $(0,0)$ with the Euclidean topology and viceversa.

We conclude that a set is open in $\mathbb{B}_N$ if and only if its image under f is open in $\mathbb{B}_E$. □

Corollary 2.2.4. *The spaces $\mathbb{B}_N^\omega$ and $\mathbb{R}^\omega$ are homeomorphic. In particular, $\mathbb{B}_N^\omega$ is homogeneous.*

Proof. Since $\mathbb{B}_E$ is homeomorphic to the G_δ near 2-cell $(]-1,1[\times]0,1[) \cup \{0,0\}$, by Theorem 2.1.2 $\mathbb{B}_E^\omega$ is homeomorphic to $\mathbb{R}^\omega$. By Lemma 2.2.2, $\mathbb{B}_N^\omega$ and $\mathbb{R}^\omega$ are homeomorphic. □

In our efforts to show that $\mathbb{B}_N^\omega$ is homogeneous using the technique in the proof of [Lemma 5 ,Yang (1992)] we obtain the following result.

Corollary 2.2.5. *There is no $h \in \mathrm{Aut}(\mathbb{B}_N^2)$ that carries a point of $\mathbb{B}_N^2 \setminus (\mathbb{H}_N^+)^2$ to $((0,0),(0,0))$.*

Proof. By Lemma 2.2.2, we can consider $\mathbb{B}_E$ instead of $\mathbb{B}_N$. We suppose that such h exists. Without loss of generality, let $a \in \mathbb{H}_E^+$ such that $h(a,(0,0)) = ((0,0),(0,0))$. Since the points in $(\mathbb{H}_E^+)^2$ have compact neighborhoods and the points in $\mathbb{B}_E^2 \setminus (\mathbb{H}_E^+)^2$ do not, we have that $h''((\mathbb{H}_E^+)^2) = (\mathbb{H}_E^+)^2$ and $h''(\mathbb{B}_E^2 \setminus (\mathbb{H}_E^+)^2) = \mathbb{B}_E^2 \setminus (\mathbb{H}_E^+)^2$. But this is a contradiction, since $(\mathbb{B}_E^2 \setminus (\mathbb{H}_E^+)^2) \setminus \{(a,(0,0))\}$ is connected and $(\mathbb{B}_E^2 \setminus (\mathbb{H}_E^+)^2) \setminus \{((0,0),(0,0))\}$ is not. □

Question 2.2.6. *Is there an $h \in \mathrm{Aut}(\mathbb{B}_N^3)$ that carries a point of $\mathbb{B}_N^3 \setminus (\mathbb{H}_N^+)^3$ to $((0,0),(0,0),(0,0))$?*

Definition 2.2.7 (Ford (1954), Definition 4.1)**.** *A space X will be called strongly locally homogeneous, SLH for short, if for every $p \in X$ and every open subset U of X such that $p \in U$, there is an open set V with $p \in V \subset U$ such that if $q \in V$ there is $h \in \mathrm{Aut}(X)$ with $h(p) = q$ and $h(x) = x$ for all $x \in X \setminus V$.*

We recall the following result by L.R. Ford Jr.

Proposition 2.2.8 (Ford (1954), Theorem 4.3)**.** $\mathbb{R}^\omega$ *is SLH.*

Theorem 2.2.9. $\mathbb{H}_N \times \mathbb{B}_N^\omega$ *is homogeneous.*

Proof. Let $\pi_{\mathbb{H}_N}$ and $\pi_{\mathbb{B}_N^\omega}$ be the projections onto $\mathbb{H}_N$ and $\mathbb{B}_N^\omega$ respectively. Let $a,b \in \mathbb{H}_N \times \mathbb{B}_N^\omega$. We have three cases.

Case 1: $\pi_{\mathbb{H}_N}(a), \pi_{\mathbb{H}_N}(b) \in \mathbb{L}$. There is $f_1 \in \mathrm{Aut}(\mathbb{H}_N)$ such that $f_1(\pi_{\mathbb{H}_N}(a)) = \pi_{\mathbb{H}_N}(b)$ by Proposition 2.2.1. Since $\mathbb{B}_N^\omega$ is homogeneous by Corollary 2.2.4, there is $f_2 \in \mathrm{Aut}(\mathbb{B}_N^\omega)$ such that $f_2(\pi_{\mathbb{B}_N^\omega}(a)) = \pi_{\mathbb{B}_N^\omega}(b)$. In this way, $f_1 \times f_2 \in \mathrm{Aut}(\mathbb{H}_N \times \mathbb{B}_N^\omega)$ and $(f_1 \times f_2)(a) = b$.

Case 2: $\pi_{\mathbb{H}_N}(a), \pi_{\mathbb{H}_N}(b) \in \mathbb{H}^+$. By an analogous procedure as for the previous case, there is $h \in \mathrm{Aut}(\mathbb{H}_N \times \mathbb{B}_N^\omega)$ such that $h(a) = b$.

Case 3: $\pi_{\mathbb{H}_N}(a) \in \mathbb{H}^+$ and $\pi_{\mathbb{H}_N}(b) \in \mathbb{L}$. By Proposition 2.2.1, there is $g_1 \in \mathrm{Aut}(\mathbb{H}_N)$ such that $g_1(0,0) = \pi_{\mathbb{H}_N}(b)$. Let us define $c = \big((0,0), \pi_{\mathbb{B}_N^\omega}(b)\big)$ and $U = (D \cup \{(0,0)\})_N \times \mathbb{B}_N^\omega$ with D an open disc tangent to $\mathbb{L}$ at $(0,0)$. Since $\mathbb{B}_N \times \mathbb{B}_N^\omega$ is SLH by Corollary 2.2.4 and Proposition 2.2.8, there is an open neighborhood $V \subset U$ of c such that for $d \in (\mathbb{H}_N^+ \times \mathbb{B}_N^\omega) \cap V$ there is $f \in \mathrm{Aut}(\mathbb{B}_N \times \mathbb{B}_N^\omega)$ with $f(d) = c$ and f is the identity on $(\mathbb{B}_N \times \mathbb{B}_N^\omega) \setminus V$. For $x \in \mathbb{L} \setminus \{(0,0)\}$, we choose a basic neighborhood A_x disjoint from $\pi''_{\mathbb{H}_N}(V)$. Let

$$g = f \cup \bigcup_{x \in \mathbb{L} \setminus \{(0,0)\}} Id_{A_x \times \mathbb{B}_N^\omega}$$

We have that $g \in \mathrm{Aut}(\mathbb{H}_N \times \mathbb{B}_N^\omega)$.

By the second case, there is $h \in \mathrm{Aut}(\mathbb{H}_N \times \mathbb{B}_N^\omega)$ such that $h(a) = d$. We conclude that $(g_1 \times Id) \circ g \circ h \in \mathrm{Aut}(\mathbb{H}_N \times \mathbb{B}_N^\omega)$ and $(g_1 \times Id) \circ g \circ h(a) = b$. □

Corollary 2.2.10. *$\mathbb{H}_N^\omega \times \mathbb{B}_N^\omega$ is homogeneous.*

Proof. Since $\mathbb{H}_N \times \mathbb{B}_N^\omega$ is homogeneous by the previous proposition, $(\mathbb{H}_N \times \mathbb{B}_N^\omega)^\omega \cong \mathbb{H}_N^\omega \times \mathbb{B}_N^\omega$ is homogeneous too. □

Question 2.2.11. *Is $\mathbb{H}_N^\omega$ homogeneous?*

Chapter 3

Hyperspaces of the Sorgenfrey line

In section 3.1 we prove that the space of nontrivial convergent sequences of $\mathbb{S}$ is homogeneous in a very natural way. In section 3.2, with a very technical partition as one of the main tools, we show that the second symmetric product of $\mathbb{S}$ is homogeneous. In section 3.3 we prove that the space of non-empty closed intervals of $\mathbb{S}$ is homogeneous via a geometric characterization.

3.1 Homogeneity of the space of nontrivial convergent sequences

Proposition 3.1.1. *If $S, T \in \mathcal{S}_c(\mathbb{S})$, then there exists a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $h''(S) = T$.*

Proof. Let $S, T \in \mathcal{S}_c(\mathbb{S})$. First, we will prove that if $S = \{x\} \cup \{x_n : n \in \mathbb{Z}^+\}$ and $P := \{0\} \cup \{1/2^n : n \in \mathbb{Z}^+\} \in \mathcal{S}_c(\mathbb{S})$, then there exists a homeomorphism $h_1 : \mathbb{S} \to \mathbb{S}$ such that $h_1''(S) = P$. Since $\mathbb{S}$ is homogeneous, there is a homeomorphism $f : \mathbb{S} \to \mathbb{S}$ with $f(x) = 0$. We have that the sequence $f(x_n)$ converges to $f(x) = 0$, so we can define inductively $z_1 = \max\{f(x_n) : n \in \mathbb{Z}^+\}$ and $z_m = \max\{f(x_n) : n \in \mathbb{Z}^+\} \setminus \{z_1, \ldots, z_{m-1}\}$ for $m \geq 2$. By convergence, we can choose a clopen neighborhood V_1 of 0 such that $f(x_n) \in V_1$ for every n with $f(x_n) \neq z_1$ and $z_1 \notin V_1$. Because $\mathbb{S} \setminus V_1$ and $[1/2, 1[$ are homeomorphic to $\mathbb{S}$ and $\mathbb{S}$ is homogeneous, there exists a homeomorphism $g_1 : \mathbb{S} \setminus V_1 \to [1/2, 1[$ such that $g_1(z_1) = 1/2$. As before, we can choose a clopen neighborhood V_2 of 0 such that $f(x_n) \in V_2$ for every n with $f(x_n) \neq z_1, z_2$ and $z_1, z_2 \notin V_2$. There exists a homeomorphism $g_2 : V_1 \setminus V_2 \to [1/2^2, 1/2[$ with $g_2(z_2) = 1/2^2$. Recursively, we can choose a clopen neighborhood V_m of 0 such that $f(x_n) \in V_m$ for every n with $f(x_n) \neq z_1, \ldots, z_m$ and $z_1, \ldots, z_m \notin V_m$. There exists a homeomorphism $g_m : V_{m-1} \setminus V_m \to [1/2^m, 1/2^{m-1}[$ with $g_m(z_m) = 1/2^m$.

We define the homeomorphism $g = \bigcup g_m :]0,1[\to]0,1[$. Hence, we have the homeomorphism $\overline{g} : \mathbb{S} \to \mathbb{S}$ with $\overline{g}(x) = g(x)$ if $x \neq 0$ and $\overline{g}(0) = 0$. In this way, $h_1 := \overline{g} \circ f$ is the desired homeomorphism.

Finally, by the previous argument there is a homeomorphism $h_2 : \mathbb{S} \to \mathbb{S}$ such that $h_2''(P) = T$. Therefore, the homeomorphism $h := h_2 \circ h_1$ is as required. $\square$

Proposition 3.1.2. *$\mathcal{S}_c(\mathbb{S})$ is homogeneous.*

Proof. Given $S, T \in \mathcal{S}_c(\mathbb{S})$, consider $h \in \mathrm{Aut}(\mathbb{S})$ as in the previous proposition so that $h''(S) = T$. Let us define $\overline{h} : \mathcal{S}_c(\mathbb{S}) \to \mathcal{S}_c(\mathbb{S})$ such that $\overline{h}(X) = h''(X)$. If $X \in \mathcal{S}_c(\mathbb{S})$, then $h^{-1}(X) \in \mathcal{S}_c(\mathbb{S})$, so $\overline{h}(h^{-1}(X)) = X$ and $\overline{h}$ is onto. If $X, Y \in \mathcal{S}_c(\mathbb{S})$ and $\overline{h}(X) = \overline{h}(Y)$, then $h''(X) = h''(Y)$, so $X = Y$ by the injectivity of h. Hence, $\overline{h}$ is bijective and $\overline{h}(S) = T$.

We will prove that $\overline{h}$ is continuous. Let B a basic set of $\mathcal{S}_c(\mathbb{S})$. We have two cases. If $B = \mathcal{S}_c(\mathbb{S}) \cap [V]$ with V an open set of $\mathbb{S}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{S}) \cap \overline{h}^{-1}([V]) = \mathcal{S}_c(\mathbb{S}) \cap [h^{-1}(V)]$. If $B = \mathcal{S}_c(\mathbb{S}) \cap \langle V \rangle$ with V a basic set of $\mathbb{S}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{S}) \cap \overline{h}^{-1}(\langle V \rangle) = \mathcal{S}_c(\mathbb{S}) \cap \langle h^{-1}(V) \rangle$. Therefore, $\overline{h}$ is continuous.

To end, we will prove that $\overline{h}$ is an open map. Let B a basic set of $\mathcal{S}_c(\mathbb{S})$. If $B = \mathcal{S}_c(\mathbb{S}) \cap [V]$ with V an open set of $\mathbb{S}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{S}) \cap \overline{h}''([V]) = \mathcal{S}_c(\mathbb{S}) \cap [h''(V)]$. If $B = \mathcal{S}_c(\mathbb{S}) \cap \langle V \rangle$ with V a basic set of $\mathbb{S}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{S}) \cap \overline{h}''(\langle V \rangle) = \mathcal{S}_c(\mathbb{S}) \cap \langle h''(V) \rangle$. $\square$

3.2 Homogeneity of the second symmetric product

In this section we prove Theorem 3.2.3. It is worth mentioning that our proofs are based on work of Bennett, Burke and Lutzer Bennett et al. (2012) and we will also borrow some of its notation.

A *Sorgenfrey rectangle* is a set of the form $[a,b[\times[c,d[$ where $a,b,c,d \in [0,1[$; $a < b$ and $c < d$. By the *Euclidean closure* of such a rectangle we mean its closure in the euclidean topology of $[0,1[^2$. Let $\Delta_2 := \{(x,y) \in \mathbb{S}^2 : x \leq y\}$ and let Δ be the diagonal of $\mathbb{S}$. For each $k \in \omega$, let L_k be the straight line joining the points $(0, \frac{1}{k+1})$ and $(1,1)$.

Proposition 3.2.1 (Bennett et al. (2012), Proposition 2.1)**.** *There is a countable collection $\mathcal{T}$ of pairwise disjoint Sorgenfrey rectangles such that:*

(1) $\bigcup \mathcal{T} = \Delta_2 \setminus \Delta$*;*

(2) for each $T \in \mathcal{T}$*, the Euclidean closure of* T *is disjoint from* Δ*;*

(3) for each $x \in [0,1[$ *the set* $\{T \in \mathcal{T} : T \cap (\{x\} \times]x,1[) \neq \emptyset\}$ *is infinite and can be indexed as* $\{T_m : m \in \mathbb{Z}^+\}$ *in such a way that for all* k*, points of* T_k *lie above points of* T_{k+1}*.*

(4) for each $T \in \mathcal{T}$*, there is* $k \in \omega$ *such that* T *is between* L_k *and* L_{k+2}*.*

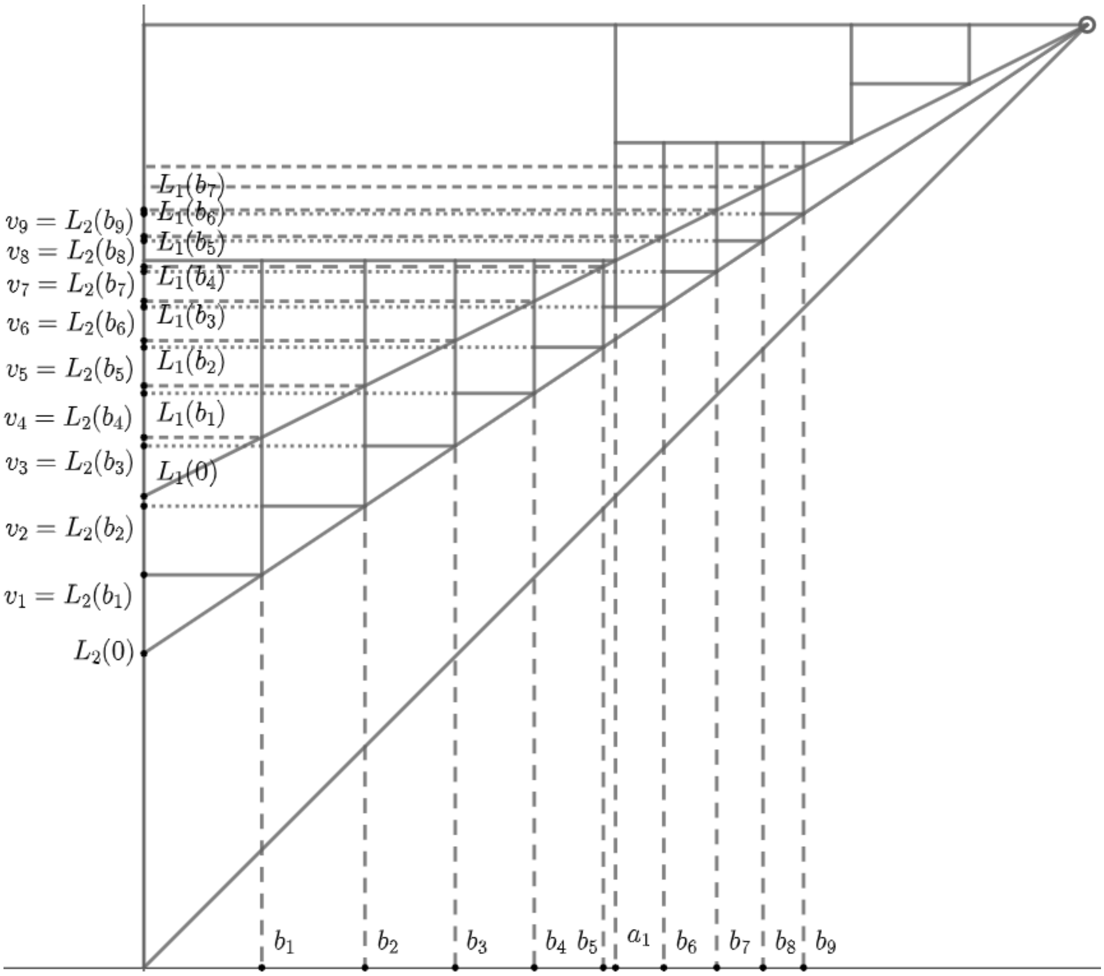
Proof. For $k \in \mathbb{Z}^+$, we claim that there is a step function $S_k : [0,1[\to]0,1[$ such that:

- the graph of S_k lies strictly between the graphs of L_k and L_{k-1};
- the jump points of S_k occur at rational numbers and those jump points are an increasing sequence that converges to 1;
- for each $x \in [0,1[$, $S_k(x)$ is rational;
- the horizontal segments of the graphs of S_k contains their left endpoints, but not their right endpoints.

For $x \in [0,1[$ and $k \in \omega$, let $L_k(x)$ be the second coordinate of the point in $L_k \cap (\{x\} \times [0,1[)$. We will show how to construct S_2 betweeen L_2 and L_1. The other constructions are analogous. Let v_1 the average between $L_2(0)$ and $L_1(0)$. Find $b_1 \in [0,1[$ with $L_2(b_1) = v_1$ and let v_2 be the average of $v_1 = L_2(b_1)$ and $L_1(b_1)$. Find b_2 with $L_2(b_2) = v_2$ and let v_3 be the average of $v_2 = L_2(b_2)$ and $L_1(b_2)$. In general, given $b_1, \dots, b_n$ and $v_1, \dots, v_n$; let v_{n+1} be the average of $v_n = L_2(b_n)$ and $L_1(b_n)$ and find b_{n+1} such that $L_2(b_{n+1}) = v_{n+1}$. This recursion gives rational numbers b_n and v_n, which will be the jump points of S_1 and the set of values of S_2 respectively. For $x \in [0, b_1[$ we define $S_2(x) = v_1$. For $n \geq 2$ and $x \in [b_{n-1}, b_n[$ we define $S_2(x) = v_n$. Notice that because the graph of S_2 lies between L_2 and L_1, while the graph of S_1 is constructed between L_1 and L_0, we have $S_2(x) < S_1(x)$ for all $x \in [0,1[$.

We will use the graphs of the functions S_k and their jump points to describe the edges of the Sorgenfrey rectangles of the collection $\mathcal{T}$. Using S_1 the rectangles

Figure 3.2.1: Partition $\mathcal{T}$ of $\Delta_2 \setminus \Delta$



at the top are described. List the jump points of S_1 as $a_0 := 0 < a_1 < a_2 < \dots$ and choose the Sorgenfrey rectangles $[0, a_1[\times[S_1(0), 1[$ and $[a_j, a_{j+1}[\times[S_1(a_j), 1[$ for $j \in \mathbb{Z}^+$.

The next tier of rectangles is defined using S_1 and S_2. List the jump points of S_2 as $b_0 := 0 < b_1 < b_2 < \dots$. If for $j \in \mathbb{Z}^+$ there is no jump point of S_1 in $[b_j, b_{j+1}[$, then choose the Sorgenfrey rectangle $[b_j, b_{j+1}[\times [S_2(b_j), S_1(b_j)[$. If there are jump points of S_1 in $[b_j, b_{j+1}[$, then list them as $b_j < c_{j_1} < \cdots < c_{j_i} < b_{j+1}$ and choose the Sorgenfrey rectangles $[b_j, c_{j_1}[\times[S_2(b_j), S_1(b_j)[$, $[c_{j_1}, c_{j_2}[\times[S_2(c_{j_1}), S_1(c_{j_1})[$,..., $[c_{j_i}, b_{j+1}[\times[S_2(c_{j_i}), S_1(c_{j_i})[$. This process is repeated for each pair S_k, S_{k+1} of consecutive step functions. The resulting collection of Sorgenfrey rectangles is as required. $\square$

Lemma 3.2.2 ((Bennett et al., 2012), Lemma 2.2)**.** *Let* $a, b, c, d \in [0, 1[$ *with*

$a < b$ and $c < d$. The function

$$h_{abcd}(x) = c + \frac{d-c}{b-a}(x-a)$$

is an order-isomorphism from $[a,b[$ onto $[c,d[$, and the inverse of h_{abcd} is h_{cdab}.

For the main Theorem of the section we will give two proofs. In the first one, we define a homeomorphism between Δ_2 and $\mathbb{S}^2$ that carries Δ onto $\mathbb{S} \times \{0\}$ and it is inspired in the construction of the homeomorphism in the proof of Proposition 3.1 in (Bennett et al., 2012). In the second one, we send Δ onto itself, which results into a more intuitive and shorter proof.

Theorem 3.2.3 (Barría and Martínez-Ranero (2023), Theorem 1.4)**.** *We have that Δ_2 and $\mathbb{S}^2$ are homeomorphic.*

Proof 1. We will define an homeomorphism $\varphi : \Delta_2 \to \mathbb{S}^2$. First, we will describe some special notation, then define the function φ in steps D-1 to D-4. After that we will prove that φ is 1-1 and onto, then define the inverse of φ in steps I-1 through I-4. Finally, we will prove that φ and its inverse φ^{-1} are continuous in steps C-1 through C-4 and IC-1 through IC-4 respectively.

Special notation. For $k \in \mathbb{Z}^+$ let $D(k) := [1/2^k, 1/2^{k-1}[^2$. The sets $D(k)$ will be called the *basic diamonds* of $[0,1[^2$ and the sets $\Delta_2 \cap D(k)$ will be called *basic triangles* of Δ_2. We will denote π_2 for the second projection.

Define two step functions σ and τ, as follows, both having domain $]0,1[$ and range $]0,1]$. For each $x \in]0,1[$ there is a unique $k \in \mathbb{Z}^+$ such that $1/2^k \leq x < 1/2^{k-1}$, so we define $\sigma(x) = 1/2^k$ and $\tau(x) = 1/2^{k-1}$. The horizontal pieces of the graphs of σ and τ are, respectively, the bottom and top of the basic diamonds $D(k)$. It will be important to note that each horizontal segment of the graph of σ contains a left endpoint, but not a right one, and the same is true for τ.

For each $x \in [0,1[$ let $B(x) := \{x\} \times [0, \tau(x)[$, and for $n \in \mathbb{Z}^+$, subdivide $B(x)$ into disjoint subsegments $B(x,n) := \{x\} \times [\tau(x)/2^n, \tau(x)/2^{n-1}[$.

For each $x \in \mathbb{S}$ with $x \neq 0$, there is a unique diamond $D(k)$ containing (x,x). Note that (x,x) might be the southwest corner point of $D(k)$, but it cannot be the northeast corner point of $D(k)$. Define $VL(x) = \{x\} \times]x, 1/2^{k-1}[$.

For each $x \in]0,1[$ we subdivide $VL(x)$ as follows. Find the unique k such that $(x,x) \in D(k)$. By Proposition 3.2.1, we list all members of the collection $\mathcal{T}$ that intersect $VL(x)$ as $T_1, T_2, \ldots$, where for all j, each point of T_j lies above each point of T_{j+1}. For each j we can write $T_j = [a_j, b_j[\times[c_j, d_j[$, and then we have $\cdots < d_3 = c_2 < d_2 = c_1 < 1/2^{k-1} \leq d_1$. Define $VL(x,n) = VL(x) \cap T_n = \{x\} \times [c_n, d_n[$ for each $n \geq 2$, and $VL(x,1) = \{x\} \times [c_1, 1/2^{k-1}[$.

Definition of $\varphi(x,y)$. The definition of $\varphi(x,y)$ has four parts, called D-1 through D-4, depending on the location of $(x,y) \in \Delta_2$. We proceed by cases.

D-1. Let $\varphi(x,x) = (x,0)$ for each $x \in \mathbb{S}$.

D-2. Let $\varphi(0,y) = (0,y)$ for each $y \in \mathbb{S}$.

D-3. If $(x,y) \in \Delta_2$ is such that $\tau(x) \leq y$, then define $\varphi(x,y) = (x,y)$ (φ is the identity map above the basic triangles $\Delta_2 \cap D(k)$).

D-4. If $(x,y) \in \Delta_2 \cap D(k)$ for some $k \in \mathbb{Z}^+$, then (x,y) is in the vertical line $VL(x)$ and therefore in a unique subsegment $VL(x,n)$ which has the form $VL(x,n) = \{x\} \times [p,q[$. The vertical segment $B(x) = \{x\} \times [0, \tau(x)[$ contains the subsegment $B(x,n)$ which has the form $\{x\} \times [r,s[$. By Lemma 3.2.2, we have an order isomorphism $h_{pqrs} : [p,q[\to [r,s[$. Define $\varphi(x,y) = (x, h_{pqrs}(y))$.

We have a function $\varphi : \Delta_2 \to \mathbb{S}^2$.

The function φ is 1-1 and onto. Let $(x,y), (v,w) \in \Delta_2$ with $(x,y) \neq (v,w)$. If $x \neq v$, then $\varphi(x,y) \in \{x\} \times [0,1[$ and $\varphi(v,w) \in \{v\} \times [0,1[$. Since this sets are disjoint, we have that $\varphi(x,y) \neq \varphi(v,w)$. If $x = v$ and $y \neq w$, then we have four cases.

Case 1. If (x,y) and (v,w) are not in a set $D(k)$, then $\varphi(x,y) = (x,y) \neq (v,w) = \varphi(v,w)$.

Case 2. If for some $k \in \mathbb{Z}^+$ we have that $(v,w) \in D(k)$ and $(x,y) \notin D(k)$, then $\varphi(x,y) = (x,y) \notin B(x)$. Since $\varphi(v,w) \in B(x)$, $\varphi(x,y) \neq \varphi(v,w)$.

Case 3. If for some $k \in \mathbb{Z}^+$ both points (x,y) and (v,w) are in $D(k)$ with $(x,y) \in VL(x,n)$ and $(v,w) \in VL(x,m)$ for $n \neq m$, then $\varphi(x,y) \in B(x,n)$ and $\varphi(v,w) \in B(x,m)$. Since $B(x,n) \cap B(x,m) = \emptyset$, we have that $\varphi(x,y) \neq \varphi(v,w)$.

Case 4. If for $k,n \in \mathbb{Z}^+$ both points (x,y) and (v,w) are in $D(k) \cap VL(x,n)$, we write $VL(x,n) = \{x\} \times [p,q[$ and $B(x,n) = \{x\} \times [r,s[$, then by Lemma 3.2.2 there is a homeomorphism $h_{pqrs} : [p,q[\to [r,s[$ such that $\varphi(x,y) = (x, h_{pqrs}(y))$ and $\varphi(v,w) = (x, h_{pqrs}(w))$. Since $h_{pqrs}(y) \neq h_{pqrs}(w)$, we have that $\varphi(x,y) \neq \varphi(v,w)$.

We conclude that φ is 1-1.

Let $(x,y) \in \mathbb{S}^2$. If (x,y) is above the basic diamonds $D(k)$, then $\varphi(x,y) = (x,y)$. If $y = 0$, then $\varphi(x,x) = (x,y)$. If $(x,y) \in \,]0,1[^2$ and $\tau(x) > y$, then there exists $n \in \mathbb{Z}^+$ such that $(x,y) \in B(x,n)$. Let $B(x,n) := \{x\} \times [r,s[$ and $VL(x,n) := \{x\} \times [p,q[$. By Lemma 3.2.2, there exists a homeomorphism

$h_{pqrs} : [p, q[\to [r, s[$. In this way, $\varphi(x, h_{rspq}(y)) = (x, y)$. Therefore, φ is onto.

Definition of $\varphi^{-1}(x, y)$. The definition of $\varphi^{-1}(x, y)$ has four parts, called I-1 through I-4, depending on the location of $(x, y) \in \mathbb{S}^2$.

I-1. Define $\varphi^{-1}(x, 0) = (x, x)$ for each $x \in \mathbb{S}$.

I-2. Let $\varphi^{-1}(0, y) = (0, y)$ for each $y \in \mathbb{S}$.

I-3. If $(x, y) \in \mathbb{S}^2$ is such that $\tau(x) \leq y$, then define $\varphi^{-1}(x, y) = (x, y)$.

I-4. If $\tau(x) > y$, then define $\varphi^{-1}(x, y) = (x, h_{rspq}(y))$ with h as in the last case of the proof of the surjectivity.

Continuity of φ. To prove that φ is continuous at (x, y) requires different arguments for points in different parts of Δ_2. We consider four separate cases that we call C-1 through C-4.

C-1. First, suppose $(x, y) \in \Delta_2 \setminus \{(0, 0)\}$ with $x > 0$ and $\tau(x) \leq y < 1$, or $x = 0$ and $y > 0$. Since the set of all points of this type is an open subset of Δ_2 and $\varphi(x, y) = (x, y)$, we have φ is continuous at each such point.

C-2. Second, let $(x, y) = (0, 0)$ and $(x_n, y_n)(n \in \omega)$ be a sequence that converges to $(0, 0)$. We may assume that $(x_n, y_n) \neq (0, 0)$ for each n. Separately, consider two subsequences, namely, those with $\tau(x_n) \leq y_n$ and those with $\tau(x_n) > y_n$. If there are infinitely many points of the first type, then their images converges to $(0, 0)$ because φ is the identity for such points. We observe that every point of the second type has their image below the graph of τ. Therefore, if the subsequence of such points is infinite, then their images converges to $(0, 0)$. We conclude that $\varphi(x_n, y_n)$ converges to $(0, 0)$.

C-3. Third, consider a point $(x, y) = (x, x)$ on the diagonal with $x \neq 0$. There is a unique basic triangle $\Delta_2 \cap D(k)$ that contains (x, x). Let $\epsilon > 0$ and $V = [x, x + \epsilon[\times [0, \epsilon[$ be a neighborhood of $\varphi(x, x) = (x, 0)$. We may assume that $\epsilon < \sigma(x)$ and $\sigma(x') = \sigma(x)$ for each $x' \in [x, x + \epsilon[$ (this is posible because no horizontal segment of the graph of σ contains its right endpoint). We will find a $\delta > 0$ such that if $U := \Delta_2 \cap [x, x + \delta[^2$, then $\varphi''(U) \subset V$.

Since $\sigma(x) = \sigma(x')$ for $x \leq x' < x + \epsilon$, we know that $\tau(x) = \tau(x')$, so $B(x) = \{x\} \times [0, \tau(x)[$ and $B(x') = \{x'\} \times [0, \tau(x')[$ have the same set of second coordinates. In this way, $\pi_2''(B(x)) = \pi_2''(B(x'))$ and $\pi_2''(B(x, k)) = \pi_2''(B(x', k))$ for all $x' \in [x, x + \epsilon[$. Since $\lim_{j \to \infty} \sigma(x)/2^j = 0$ and $B(x, j + 2) \subset \{x\} \times [0, \sigma(x)/2^j[$, we may choose N such that $\bigcup\{B(x, j) : j \geq N\} \subset \{x\} \times [0, \epsilon[\subset V$. For each

$x' \in [x, x+\epsilon[$ we have $\bigcup\{B(x', j) : j \geq N\} \subset \{x'\} \times [0, \epsilon[\subset V$ and therefore,

$$\bigcup\{B(x', j) : x \leq x' < x+\epsilon, j \geq N\} \subset [x, x+\epsilon[\times [0, \epsilon[= V \tag{3.2.1}$$

Consider the vertical line $VL(x) = \{x\} \times]x, 1/2^{k-1}[$, which lies inside the triangle $\Delta_2 \cap D(k)$. By Proposition 3.2.1, we can list all members of $\mathcal{T}$ that intersect $VL(x)$ as $T_1, T_2, \dots$ where each point of T_j lies above of each point of T_{j+1}. The Sorgenfrey rectangles T_j has the form $T_j = [a_j, b_j[\times [c_j, d_j[$ and we have that $x < \cdots < d_3 = c_2 < d_2 = c_1 < d_1$. Also, since $\lim_{j\to\infty} c_j = x$ we can choose $M \geq N$ such that $c_j < x + \epsilon$ when $j \geq M$.

Recall that $VL(x, j) = VL(x) \cap T_j = \{x\} \times [c_j, d_j[$. Since no set $T_1, \dots, T_M$ contains its right edge, there exists $\eta > 0$ such that $[x, x+\eta[\times [c_i, d_i[\subset T_i$ for $1 \leq i \leq M$. Consider any $x' \in [x, x+\eta[$ and let T'_n be the listing in decreasing order of all members of $\mathcal{T}$ that intersect $VL(x')$ as in Proposition 3.2.1. Since the collection $\mathcal{T}$ is pairwise disjoint and $VL(x') \cap T_1 \neq \emptyset$, we conclude that $T'_1 = T_1$. Similarly, $T'_i = T_i$ for $1 \leq i \leq M$. Thus, $VL(x', j) \subset \{x'\} \times]x', c_M[\subset [x, x+\eta[\times [x, c_M[$ when $j \geq M$.

Let $\delta = \min\{\epsilon, \eta, c_M - x\}$. We note that if $x' \in [x, x+\delta[$, then $x \leq x' \leq x+\eta$, so that the subset $\{x'\} \times]x', c_M[$ of $VL(x')$ is such that

$$\{x'\} \times]x', c_M[\subset \bigcup\{VL(x', j) : j \geq M\} \tag{3.2.2}$$

To end this case, let $(x_1, y_1) \in \Delta_2 \cap [x, x+\delta[^2$. We have that $(x_1, x_1) \in \Delta$ and $(x_1, y_1) \in VL(x_1)$. Since $x \leq x_1 < x+\delta \leq x+\eta$ and $x \leq x_1 < y_1 < x+\delta \leq c_M$, equation 3.2.2 gives

$$(x_1, y_1) \in \{x_1\} \times]x_1, c_M[\subset \bigcup\{VL(x_1, j) : j \geq M\} \tag{3.2.3}$$

Since there is a unique k such that $(x_1, y_1) \in VL(x_1, k)$, we have that $\varphi(x_1, y_1) \in B(x_1, k)$. Because $k \geq M \geq N$, equation 3.2.1 gives $\varphi(x_1, y_1) \in B(x_1, k) \subset V$.

C-4. Fourth and finally, consider any point $(x, y) \in \Delta_2 \cap D(k) \setminus \Delta$. Since $(x, y) \in VL(x)$, we can list all members of $\mathcal{T}$ that intersect $VL(x)$ as $T_1, T_2, \dots$ where the points of T_j lie above the points of T_{j+1}. Let $T_j = [a_j, b_j[\times [c_j, d_j[$. These sets subdivide $VL(x)$ into vertical segments $VL(x, j) := VL(x) \cap T_j = \{x\} \times [c_j, d_j[$. Choose the unique N with $(x, y) \in VL(x, N)$. For notational convenience we write

$VL(x,N) = \{x\} \times [c,d[$. Consider the set $B(x)$ which is divided into subsegments $B(x,k)$, and write $B(x,N) = \{x\} \times [e,f[$. Thus, $\varphi(x,y) = (x, h_{cdef}(y))$ where $h_{cdef} : [c,d[\to [e,f[$ is the order-isomorphism from Lemma 3.2.2.

For $p =: h_{cdef}(y)$ and $\epsilon > 0$ consider any neighborhood V of $\varphi(x,y) = (x,p)$ of the form $V = [x, x+\epsilon[\times [p, p+\epsilon[$. We may assume that $p + \epsilon < \tau(x)$ and that if $x \leq x' < x + \epsilon$, then $\tau(x') = \tau(x)$. In this way, $\pi_2''(B(x,j)) = \pi_2''(B(x',j))$ for all j and $x' \in [x, x+\epsilon[$. In particular, we have that $B(x',N) = \{x'\} \times [e,f[$ for $x \leq x' < x + \epsilon$.

We will find a neighborhood $U = [x, x+\delta[\times [y, y+\delta[$ such that $\varphi''(U) \subset V$. First, since h_{cdef} is continuous at $y \in [c,d[$ and is order-preserving, there exists $\delta_1 > 0$ such that if $y \leq y' < y + \delta_1 < d$, then $p = h_{cdef}(y) \leq h_{cdef}(y') < h_{cdef}(y) + \epsilon = p + \epsilon$. Second, consider the sets $T_1, \ldots, T_N$ where $T_j = [a_j, b_j[\times [c_j, d_j[$. Because none of this sets contains its right edge, there exists an $\eta > 0$ such that $[x, x+\eta[\times [c_j, d_j[\subset T_j$ for $1 \leq j \leq N$. We may assume that $\eta < \epsilon$. In this way, for any $x' \in [x, x+\eta[$ we have that $VL(x',j) = \{x'\} \times [c_j, d_j[$ where $1 \leq j \leq N$.

Since we are writing $[c_N, d_N[= [c,d[$, for $x \leq x' < x + \eta$ we have $VL(x',N) = \{x'\} \times [c,d[$ and for any $(x',y') \in VL(x',N)$ we have $\varphi(x',y') = (x', h_{cdef}(y'))$.

Let $\delta = \min\{\delta_1, \eta\}$ and consider $U = [x, x+\delta[\times [y, y+\delta[$. Therefore, if $(x_1, y_1) \in U$, then $\varphi(x_1, y_1) \in [x, x+\epsilon[\times [p, p+\epsilon[= V$, as required to prove continuity of φ at any point $(x,y) \in \Delta_2 \cap D(k) \setminus \Delta$.

Continuity of φ^{-1}. To prove that φ^{-1} is continuous at (x,y) requires different arguments for points in different parts of $\mathbb{S}^2$. We consider four separate cases that we call IC-1 through IC-4.

IC-1. First, suppose $(x,y) \in \mathbb{S}^2 \setminus \{(0,0)\}$ with $x > 0$ and $\tau(x) \leq y < 1$, or $x = 0$ and $y > 0$. Since the set of all points of this type is an open subset of $\mathbb{S}^2$ and $\varphi^{-1}(x,y) = (x,y)$, we have that φ^{-1} is continuous at each such point.

IC-2. Second, let $(x,y) = (0,0)$ and $(x_n, y_n)(n \in \omega)$ be a sequence in $\mathbb{S}^2$ that converges to $(0,0)$. We may assume that $(x_n, y_n) \neq (0,0)$ for each n. Separately, consider two subsequences, namely, those with $\tau(x_n) \leq y_n$ and those with $\tau(x_n) > y_n$. If there are infinitely many points of the first type, then their images under φ^{-1} converges to $(0,0)$ because φ^{-1} is the identity for such points. We observe that every point of the second type has their image in Δ_2 below the graph of τ. Therefore, if the subsequence of such points is infinite, then their images converges to $(0,0)$. We have that $\varphi^{-1}(x_n, y_n)$ converges to $(0,0)$.

IC-3. Third, consider $(x,y) \in \mathbb{S}^2$ with $x > 0$ and $0 < y < \tau(x)$. Since

$(x,y) \in B(x)$, there is a unique n with $(x,y) \in B(x,n)$. Let $B(x,n) = \{x\} \times [r,s[$. There is a basic triangle $\Delta_2 \cap D(k)$ such that $\varphi^{-1}(x,y) \in VL(x,n) \subset \Delta_2 \cap D(k)$. As usual, we can list all members of $\mathcal{T}$ that intersect $VL(x)$ as $T_1, T_2, \ldots$ in such a way that points of T_j lie above the points of T_{j+1}. We write $T_j = [t_j, u_j[\times [v_j, w_j[$, so that $VL(x,n) = \{x\} \times [v_n, w_n[$. For notational convenience we let $v = v_n$ and $w = w_n$. In this way, $\varphi^{-1}(x,y) = (x, h_{rsvw}(y))$ where h_{rsvw} is the order-isomorphism given by Lemma 3.2.2. For $\epsilon > 0$, consider the neighborhood $V = [x, x+\epsilon[\times [h_{rsvw}(y), h_{rsvw}(y) + \epsilon[$ of $\varphi^{-1}(x,y)$.

We will find $\delta > 0$ so that if $U = [x, x+\delta[\times [y, y+\delta[$, then $\varphi^{-1}(U) \subset V$. Our first step is to find a $\delta_1 > 0$ so that $y + \delta_1 < \tau(x)$ and for each $x' \in [x, x+\delta_1[$ we have $\tau(x') = \tau(x)$. In this way, $\pi_2''(B(x',j)) = \pi_2''(B(x,j))$ for all j and $x' \in [x, x+\delta_1[$. In particular, for $x' \in [x, x+\delta_1[$ we have $B(x',n) = \{x'\} \times [r,s[$.

Since the function $h_{rsvw} : [r,s[\to [v,w[$ is continuous and order-preserving, there is a $\delta_2 > 0$ such that if $y \leq y' < y + \delta_2$, then $h_{rsvw}(y) \leq h_{rsvw}(y') < h_{rsvw}(y) + \epsilon$.

Consider $\eta > 0$ such that $[x, x+\eta[\times [v_i, w_i[\subset T_i$ for $1 \leq i \leq n$. Consider $x' \in [x, x+\eta[$ and list the members of $\mathcal{T}$ that intersect $VL(x')$ as $T_1', T_2', \ldots$ so that the points of T_j' lie above the points of T_{j+1}' for all $j \in \mathbb{Z}^+$. Because $\mathcal{T}$ is pairwise disjoint and $VL(x') \cap T_i \neq \emptyset$, necessarily $T_i' = T_i$ for $1 \leq i \leq n$. Thus, $VL(x',n) = \{x'\} \times [v_n, w_n[= \{x'\} \times [v,w[$.

Let $\delta = \min\{\delta_1, \delta_2, \eta, \epsilon\}$ and $(x',y') \in U := [x, x+\delta[\times [y, y+\delta[$. We have that $\tau(x') = \tau(x)$ and $\pi_2''(B(x',n)) = \pi_2''(B(x,n)) = [r,s[$. Therefore, $\varphi^{-1}(x',y') \in [x, x+\epsilon[\times [h_{rsvw}(y), h_{rsvw}(y) + \epsilon[= V$.

IC-4. Finally, consider $(x,0) \in \mathbb{S}^2$ with $x > 0$. We know that $\varphi^{-1}(x,0) = (x,x)$ belongs to some basic triangle $\Delta_2 \cap D(k)$. For $\epsilon > 0$, consider the basic neighborhood $V = \Delta_2 \cap [x, x+\epsilon[^2 \subset \Delta_2 \cap D(k)$ of (x,x). We will find a neighborhood U of $(x,0)$ such that if $(x',y') \in U$ and $(x',y') \in B(x',j)$ for some j, then $\varphi^{-1}(x',y') \in V$.

Let $T_1, T_2, \ldots$ be all the members of $\mathcal{T}$ that intersect $VL(x) \subset \Delta_2 \cap D(k)$, where the points of T_j lie above the points of T_{j+1}. Write $T_j = [a_j, b_j[\times [c_j, d_j[$. Because the sequence d_n converges to x, there is some N with $x < d_N < x + \epsilon$. On the other hand, there is some $\eta > 0$ such that $[x, x+\eta[\times [c_j, d_j[\subset T_j$ for $1 \leq j \leq N$. We may assume that $\eta < \epsilon$. If we consider $x \leq x' < x + \eta$, then

$\pi_2''(VL(x',j)) = \pi_2''(VL(x,j)) = [c_j, d_j[$ for $1 \leq j \leq N$. In this way,

$$\text{If } x \leq x' < x+\eta \text{ and } j \geq N, \text{ then } VL(x',j) \subset \{x'\}\times]x', d_N] \subset \Delta_2 \cap [x, x+\epsilon[^2 = V \tag{3.2.4}$$

The segment $B(x) = \{x\} \times [0, \tau(x)[$ is partitioned by the sets $B(x,j)$. Let t_N be the top point of $B(x,N)$ and $\delta := \min\{t_N, \eta, \epsilon\}$. Define $U = [x, x+\delta[\times [0, \delta[$ and suppose $(x', y') \in U$. Since $(x', y') \in B(x', j)$ for some j, we have that $y' \in \pi_2''(B(x',j)) = \pi_2''(B(x,j))$, so $y' < \delta \leq t_N$ gives $j \geq N$. Therefore, $\varphi^{-1}(x', y') \in VL(x', j)$ and $VL(x', j) \subset V$ by 3.2.4. □

Proof 2. Let T be an element of the partition $\mathcal{T}$. If $T = [a, b[\times [c, d[$, then define $T^U = [a, b[\times [\frac{c+d}{2}, d[$, $T^L = [a, b[\times [c, \frac{c+d}{2}[$ and $T^S = [c, d[\times [a, b[$. Finally, let $\mathcal{T}^S = \{T^S : T \in \mathcal{T}\}$. [1]

We shall define a homeomorphism $h : \Delta_2 \to \mathbb{S}^2$. First, we will define the function h. After that we will prove that h is a bijection. Finally, we will prove that h and its inverse h^{-1} are continuous. For each $T = [a, b[\times [c, d[\in \mathcal{T}$, we consider the following homeomomorphisms

$$h_{T^U, T^S} : T^U \to T^S \text{ and } h_{T^L, T} : T^L \to T$$

given by $h_{T^U,T^S}(x, y) = (2y - d, x)$ and $h_{T^L,T}(x, y) = (x, 2y - c)$, respectively. Let

$$h := Id_\Delta \cup \bigcup_{T \in \mathcal{T}} \left(h_{T^L,T} \cup h_{T^U,T^S}\right),$$

where Id_Δ represents the identity function restricted to the diagonal. Since $\Delta_2 = \Delta \sqcup \bigsqcup_{T\in\mathcal{T}} (T^U \sqcup T^L)$ and $\mathbb{S}^2 = \Delta \sqcup \bigsqcup_{T\in\mathcal{T}} (T \sqcup T^S)$, it follows that $h : \Delta_2 \to \mathbb{S}^2$ is a bijection. Notice that $h \restriction (\Delta_2 \setminus \Delta)$ and $h^{-1} \restriction (\mathbb{S}^2 \setminus \Delta)$ are continuous, as h_{T^U,T^S} and $h_{T^L,T}$ are homeomorphism between clopen subspaces.

We will now show that $h \restriction \Delta$ is continuous. Let $(x, x) \in \Delta$ and $(x_n, y_n)(n \in \omega)$ be a sequence that converges to (x, x). We may assume, without loss of generality, that all elements of the sequence belong to the open neighborhood $[x, 1[\times [x, 1[$ of (x, x). Thus, we have that $x \leq x_n \leq y_n$ for any $n \in \omega$. Separately, consider three subsequences, namely, those on the diagonal, those in the sets of the form T^U and those in the sets of the form T^L. Let A_0, A_1, A_2 denote the indexes of

[1]Note that T^U is the upper half of T, T^L is the lower half of T and T^S is the reflection of T across the diagonal Δ.

the corresponding subsequences. If there are infinitely many points of the first type, then their images under h converges to (x, x) since h is the identity at such points.

If there are infinitely many points of the second type, then their images under h have the form $(2y_n - d_n, x_n)$, where $T_n = [a_n, b_n[\times [c_n, d_n[$ is the element of $\mathcal{T}$ so that T_n^U contains (x_n, y_n). Notice that $x < c_n$, as the elements of $\mathcal{T}$ are strictly above the diagonal and $(x_n, y_n) \in [x, 1[\times [x, 1[$. Since $(x_n, y_n) \in T_n^U$, we have that $\frac{c_n+d_n}{2} \leq y_n < d_n$. It follows

$$2y_n - d_n < (y_n + d_n) - d_n = y_n \quad \text{and} \quad x < c_n = 2\left(\frac{c_n + d_n}{2}\right) - d_n \leq 2y_n - d_n$$

Thus, $x < 2y_n - d_n < y_n$ which implies $\lim\limits_{n \in A_1} h(x_n, y_n) = \lim\limits_{n \in A_1} (2y_n - d_n, x_n) = (x, x)$.

If there are infinitely many points of the third type, then their images under h have the form $(x_n, 2y_n - c_n)$, where $T_n = [a_n, b_n[\times [c_n, d_n[$ is the element of $\mathcal{T}$ so that T_n^L contains (x_n, y_n). Notice that $x < c_n$, as the elements of $\mathcal{T}$ are strictly above the diagonal and $(x_n, y_n) \in [x, 1[\times [x, 1[$. Since $(x_n, y_n) \in T_n^L$, we have that $c_n \leq y_n < \frac{c_n+d_n}{2}$. It follows

$$x < c_n \leq y_n + (y_n - c_n) = 2y_n - c_n \quad \text{and} \quad 2y_n - c_n < 2\left(\frac{c_n + d_n}{2}\right) - c_n = d_n$$

Hence, $x < 2y_n - c_n < d_n$. Therefore, it is sufficient to prove that:

Claim 3.2.4. *If* $\lim\limits_{n \in A_2} (x_n, y_n) = (x, x)$, *then* $\lim\limits_{n \in A_2} (x_n, d_n) = (x, x)$.

Proof. Let $V = [x, x + \epsilon[^2 \cap \Delta_2$ be a given open neighborhood of (x, x). Fix k such that $\frac{1}{k} < \frac{\epsilon}{2}$. Since (x_n, y_n) converges to (x, x), we can find N so that $(x_n, y_n) \in [x, x + \frac{\epsilon}{2}[^2$ and it is below the line L_{k+2} for all $n \geq N$. We are left to show that $(x_n, d_n) \in V$ for all $n \geq N$. Let $n \geq N$ be given. Since (x_n, y_n) is below the line L_{k+2} and every rectangle $T \in \mathcal{T}$ is between two lines L_ℓ and $L_{\ell+2}$ for some ℓ, it follows that $T_n = [a_n, b_n[\times [c_n, d_n[$ is below L_k. Hence, $d_n - c_n < \frac{1}{k} < \frac{\epsilon}{2}$ and $x < y_n < x + \frac{\epsilon}{2}$. Therefore, $(x_n, d_n) \in V$ as required. $\square$

We are left to show that the inverse mapping h^{-1} is continuous on the diagonal. Let $(x, x) \in \Delta$ be given and let $(x_n, y_n)(n \in \omega)$ be a sequence that converges to (x, x). We may assume, without loss of generality, that all elements of the sequence belong to the open neighborhood $[x, 1[\times [x, 1[$ of (x, x). Thus, we have that $x \leq x_n, x \leq y_n$ for any $n \in \omega$. Separately, consider three subsequences, namely, those on the diagonal, those elements above the diagonal and those in the sets below the diagonal. Let A_0, A_1, A_2 denote the indexes of the corresponding subsequences. If there are infinitely many points of the first type, then their images under h^{-1} converges to (x, x) since h^{-1} is the identity at such points.

If there are infinitely many points of the second type, then their images under h^{-1} have the form $(x_n, \frac{c_n+y_n}{2})$ where $T_n = [a_n, b_n[\times [c_n, d_n[$ are the elements of $\mathcal{T}$ that contains (x_n, y_n). Notice that $x < c_n$, as the elements of $\mathcal{T}$ are strictly above the diagonal and $(x_n, y_n) \in [x, 1[\times [x, 1[$. Since $(x_n, y_n) \in T_n$, we have that $c_n \leq y_n < d_n$. It follows

$$x < c_n = \frac{c_n + c_n}{2} \leq \frac{c_n + y_n}{2} \leq \frac{y_n + y_n}{2} = y_n$$

Thus, $x < \frac{c_n+y_n}{2} < y_n$ which implies $\lim\limits_{n\in A_1} h(x_n, y_n) = \lim\limits_{n\in A_1} \left(x_n, \frac{c_n+y_n}{2}\right) = (x, x)$.

If there are infinitely many points of the third type, then their images under h^{-1} have the form $(y_n, \frac{x_n+d_n}{2})$ where $T_n^S = [c_n, d_n[\times [a_n, b_n[$ are the elements of $\mathcal{T}^S$ such that T_n^S contains (x_n, y_n). Notice that $x < c_n$ since all elements of $\mathcal{T}^S$ are strictly below the diagonal and $(x_n, y_n) \in [x, 1[\times [x, 1[$. Since $x < \frac{x_n+d_n}{2} < d_n$, it is sufficient to prove that:

Claim 3.2.5. *If* $\lim\limits_{n\in A_2}(x_n, y_n) = (x, x)$*, then* $\lim\limits_{n\in A_2}(y_n, d_n) = (x, x)$*.*

Proof. Let $V = [x, x + \epsilon[^2$ be a given open neighborhood of (x, x). Let $\tilde{L}_k$ denote the line from $(\frac{1}{k}, 0)$ to $(1, 1)$ for $k \geq 1$. Fix k such that $\frac{1}{k} < \frac{\epsilon}{2}$. Since (x_n, y_n) converges to (x, x), we can find N so that $(x_n, y_n) \in [x, x + \frac{\epsilon}{2}[^2$ and it is above the line $\tilde{L}_{k+2}$ for all $n \geq N$. We are left to show that $(y_n, d_n) \in V$ for all $n \geq N$. Let $n \geq N$ be given. Since (x_n, y_n) is above the line $\tilde{L}_{k+2}$ and every rectangle $T \in \mathcal{T}^S$ is between two lines $\tilde{L}_\ell$ and $\tilde{L}_{\ell+2}$ for some ℓ, it follows that $T_n^S = [c_n, d_n[\times [a_n, b_n[$ is above $\tilde{L}_k$. Hence, $d_n - c_n < \frac{1}{k} < \frac{\epsilon}{2}$ and $x < x_n < x + \frac{\epsilon}{2}$. Therefore, $(y_n, d_n) \in V$ as required. □

This concludes the proof of the Theorem. □

Proposition 3.2.6 (Ganea (1954))**.** *The map* $\rho : \Delta_2 \to \mathcal{F}_2(\mathbb{S})$ *given by* $\rho(x) = \{\pi_1(x), \pi_2(x)\}$ *is a homeomorphism.*

Corollary 3.2.7. $\mathcal{F}_2(\mathbb{S})$ *is homogeneous.*

Proof. By proposition 3.2.6 and theorem 3.2.3, $\mathcal{F}_2(\mathbb{S})$ is homeomorphic to $\mathbb{S}^2$. As $\mathbb{S}^2$ is homogeneous, $\mathcal{F}_2(\mathbb{S})$ is also homogeneous. □

By Proposition 1.2.5 we have that $\mathbb{S}$ is homeomorphic to $\mathcal{F}_1(\mathbb{S})$. Since $\mathbb{S}$ is homogeneous, it is natural to ask the following.

Question 3.2.8. *Is $\mathcal{F}_n(\mathbb{S})$ homogeneous for all $n \in \mathbb{Z}^+$?*

3.3 Homogeneity of the space of non-empty closed intervals

Let $\mathcal{C}_n(\mathbb{S}) \subset \mathrm{Exp}(\mathbb{S})$ be the hyperspace of all unions of at most n non-empty closed intervals of $\mathbb{S}$.

Proposition 3.3.1. *The function $\rho : \Delta_2 \to \mathcal{C}_1(\mathbb{S})$ defined by $\rho(a,b) = [a,b]$ is a homeomorphism.*

Proof. It is easy to see that ρ is a bijection. For the continuity, we will prove that the preimages under ρ of $[V]$ and $\langle W \rangle$, with V an open set and $W = [c,d[$ a basic open set, are open.

Let V an open set of $\mathbb{S}$. There exists basic intervals V_j such that $V = \bigcup_j V_j$. Let $(a,b) \in \rho^{-1}([V]) = \{(x,y) \in \Delta_2 : [x,y] \subset \bigcup_j V_j\}$. We define $B = \bigcup\{V_j : [a,b] \cap V_j \neq \emptyset\}$. We have that B is an interval and open set that contains $[a,b]$. Let $(x,y) \in \Delta_2 \cap B^2$. Since $x, y \in B$, we have that $[x,y] \subset B \subset \bigcup_j V_j = V$. Therefore, $(a,b) \in \Delta_2 \cap B^2 \subset \rho^{-1}([V])$ and $\rho^{-1}([V])$ is open.

Let $W = [c,d[$ a basic interval of $\mathbb{S}$ and $(a,b) \in \rho^{-1}(\langle W \rangle)$. By definition, $[a,b] \cap W \neq \emptyset$. We have two cases.

Case 1. If $c \leq b < d$, let us consider $(x,y) \in \Delta_2 \cap (\mathbb{S} \times W)$. Thus, $[x,y] \cap W \neq \emptyset$. In this way, $(a,b) \in \Delta_2 \cap (\mathbb{S} \times W) \subset \rho^{-1}(\langle W \rangle)$.

Case 2. If $b \geq d$, necessarily $a < d$. Let $(x,y) \in \Delta_2 \cap (]\leftarrow, d[\times [d, \rightarrow[)$. By definition, $[x,y] \cap W \neq \emptyset$. Therefore, $(a,b) \in \Delta_2 \cap (]\leftarrow, d[\times [d, \rightarrow[) \subset \rho^{-1}(\langle W \rangle)$. We conclude that $\rho^{-1}(\langle W \rangle)$ is open.

To show that ρ^{-1} is continuous, we will prove that ρ is an open map. Without loss of generality, let $V = \Delta_2 \cap (C \times \mathbb{S})$ an open set of Δ_2, with C a basic interval of $\mathbb{S}$ and $[a,b] \in \rho''(V)$. Thus, $(a,b) \in V$, that is to say $a \leq b$ and $a \in C$. Let $B = [a, \rightarrow[$ and consider $[x,y] \in \langle C \rangle \cap [B]$. Since $[x,y] \cap C \neq \emptyset$ and $[x,y] \subset B$, we have that $x \in C$, so $(x,y) \in V$. In this way, $[a,b] \in \langle C \rangle \cap [B] \subset \rho''(V)$. Therefore, $\rho''(V)$ is an open set. □

Corollary 3.3.2. *$\mathcal{C}_1(\mathbb{S})$ is homogeneous.*

Proof. By the previous proposition, $\mathcal{C}_1(\mathbb{S})$ is homeomorphic to Δ_2. By theorem 3.2.3 the results holds. $\square$

Question 3.3.3. *Is the hyperspace $\mathcal{C}_2(\mathbb{S})$ homogeneous?*

Chapter 4

Hyperspaces of the double arrow

In section 4.1 we prove that the space of nontrivial convergent sequences of $\mathbb{A}$ is homogeneous in a very similar way as for $\mathbb{S}$. In section 4.2 we prove Theorem 4.2.7. In section 4.3 we prove Theorem 4.3.5. Finally, in section 4.4 we give a geometric characterization for spaces of unions of at most m closed intervals of a compact linearly ordered space and we prove that in the case of the double arrow this spaces are non-homogeneous in a similar way as for symmetric products.

4.1 Homogeneity of the space of nontrivial convergent sequences

Proposition 4.1.1. *If $S, T \in \mathcal{S}_c(\mathbb{A})$, then there exists a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ such that $h''(S) = T$.*

Proof. Let $S, T \in \mathcal{S}_c(\mathbb{A})$. First, we will prove that if $S = \{x\} \cup \{x_n : n \in \mathbb{Z}^+\}$ and $P = \{\langle 0, 1\rangle\} \cup \{\langle 1/2^n, 1\rangle : n \in \mathbb{Z}^+\}$, then there is a homeomorphism $h_1 : \mathbb{A} \to \mathbb{A}$ such that $h_1''(S) = P$. Since $\mathbb{A}$ is homogeneous, there is a homeomorphism $f : \mathbb{A} \to \mathbb{A}$ with $f(x) = \langle 0, 1\rangle$. We have that the sequence $f(x_n)$ converges to $f(x) = \langle 0, 1\rangle$, so we can define inductively $z_1 = \max\{f(x_n) : n \in \mathbb{Z}^+\}$ and $z_m = \max\{f(x_n) : n \in \mathbb{Z}^+\} \setminus \{z_1, \ldots, z_{m-1}\}$ for $m \geq 2$. By convergence, we can choose a clopen neighborhood V_1 of $\langle 0, 1\rangle$ such that $f(x_n) \in V_1$ for every n with $f(x_n) \neq z_1$ and $z_1 \notin V_1$. Because $\mathbb{A} \setminus V_1$ and $[\langle 1/2, 1\rangle, \langle 1, 0\rangle]$ are homeomorphic to $\mathbb{A}$ and $\mathbb{A}$ is homogeneous, there exists a homeomorphism $g_1 : \mathbb{A} \setminus V_1 \to [\langle 1/2, 1\rangle, \langle 1, 0\rangle]$ such that $g_1(z_1) = \langle 1/2, 1\rangle$. As before, we can choose a clopen neighborhood V_2 of $\langle 0, 1\rangle$ such that $f(x_n) \in V_2$ for every n with $f(x_n) \neq z_1, z_2$ and

$z_1, z_2 \notin V_2$. There exists a homeomorphism $g_2 : V_1 \setminus V_2 \to [(\langle 1/2^2, 1\rangle, \langle 1/2, 0\rangle]$ with $g_2(z_2) = \langle 1/2^2, 1\rangle$. Recursively, we can choose a clopen neighborhood V_m of $\langle 0, 1\rangle$ such that $f(x_n) \in V_m$ for every n with $f(x_n) \neq z_1, \ldots, z_m$ and $z_1, \ldots, z_m \notin V_m$. There exists a homeomorphism $g_m : V_{m-1} \setminus V_m \to [\langle 1/2^m, 1\rangle, \langle 1/2^{m-1}, 0\rangle]$ with $g_m(z_m) = \langle 1/2^m, 1\rangle$.

We define the homeomorphism $g = \bigcup g_m :]\langle 0, 1\rangle, \langle 1, 0\rangle] \to]\langle 0, 1\rangle, \langle 1, 0\rangle]$. Hence, we have the homeomorphism $\overline{g} : \mathbb{A} \to \mathbb{A}$ with $\overline{g}(x) = g(x)$ if $x \neq \langle 0, 1\rangle$ and $\overline{g}(\langle 0, 1\rangle) = \langle 0, 1\rangle$. In this way, $h_1 := \overline{g} \circ f$ is the desired homeomorphism.

Finally, by the previous argument there is a homeomorphism $h_2 : \mathbb{A} \to \mathbb{A}$ such that $h_2''(P) = T$. Therefore, the homeomorphism $h := h_2 \circ h_1$ is as required. □

Proposition 4.1.2. *$\mathcal{S}_c(\mathbb{A})$ is homogeneous.*

Proof. Let $S, T \in \mathcal{S}_c(\mathbb{A})$ and $h \in \mathrm{Aut}(\mathbb{A})$ as in the previous proposition. Let $\overline{h} : \mathcal{S}_c(\mathbb{A}) \to \mathcal{S}_c(\mathbb{A})$ such that $\overline{h}(X) = h''(X)$. If $X \in \mathcal{S}_c(\mathbb{A})$, then $h^{-1}(X) \in \mathcal{S}_c(\mathbb{A})$, so $\overline{h}(h^{-1}(X)) = X$ and $\overline{h}$ is onto. If $X, Y \in \mathcal{S}_c(\mathbb{A})$ and $\overline{h}(X) = \overline{h}(Y)$, then $h''(X) = h''(Y)$, so $X = Y$ by the injectivity of h. Thus, $\overline{h}$ is bijective and $\overline{h}(S) = T$.

We will prove that $\overline{h}$ is continuous. Let B a basic set of $\mathcal{S}_c(\mathbb{A})$. We have two cases. If $B = \mathcal{S}_c(\mathbb{A}) \cap [V]$ with V an open set of $\mathbb{A}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}^{-1}([V]) = \mathcal{S}_c(\mathbb{A}) \cap [h^{-1}(V)]$. If $B = \mathcal{S}_c(\mathbb{A}) \cap \langle V\rangle$ with V a basic set of $\mathbb{A}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}^{-1}(\langle V\rangle) = \mathcal{S}_c(\mathbb{A}) \cap \langle h^{-1}(V)\rangle$. Therefore, $\overline{h}$ is continuous.

To end, we will prove that $\overline{h}$ is an open map. Let B a basic set of $\mathcal{S}_c(\mathbb{A})$. If $B = \mathcal{S}_c(\mathbb{A}) \cap [V]$ with V an open set of $\mathbb{A}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}''([V]) = \mathcal{S}_c(\mathbb{A}) \cap [h''(V)]$. If $B = \mathcal{S}_c(\mathbb{A}) \cap \langle V\rangle$ with V a basic set of $\mathbb{A}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}''(\langle V\rangle) = \mathcal{S}_c(\mathbb{A}) \cap \langle h''(V)\rangle$. □

4.2 Autohomeomorphisms of the finite powers of the double arrow

It will be convenient to introduce some notation. Let $\pi : \mathbb{A} \to [0, 1]$ be the projection onto the first factor $\pi(\langle x, r\rangle) = x$. We will think of an element of the finite power $x \in {}^m\mathbb{A}$ as function $x : m \to \mathbb{A}$. For any $a \in \mathbb{A}$ we will denote by $\overline{a}$ the constant sequence a of finite length m, where the value of m should be understood by context. Let $\pi_i : {}^m\mathbb{A} \to \mathbb{A}$ be the projection onto the i-coordinate, and for any function $h : {}^m\mathbb{A} \to {}^m\mathbb{A}$, let $h_i = \pi_i \circ h$ denote its i-th coordinate function. Recall that a partial function $f : \mathbb{A} \to \mathbb{A}$ is monotone if it is either non-decreasing or non-increasing, and f is strictly monotone if it is either strictly increasing or strictly decreasing.

We now recall the following result by R. Hernández-Gutiérrez.

Proposition 4.2.1 (Hernández-Gutiérrez (2013), Proposition 3.1)**.** *Let $h : \mathbb{A} \to \mathbb{A}$ be a continuous function, then there exists a pairwise disjoint sequence $J_n (n \in \omega)$ of clopen intervals such that $\bigcup_{n\in\omega} J_n$ is dense in $\mathbb{A}$ and $h \restriction J_n$ is monotone for any $n \in \omega$.*

The following proposition tell us how continuous monotone functions look like locally. It will be a key factor in the proof of the main theorems of the chapter.

Proposition 4.2.2 (Barría and Martínez-Ranero (2023), Proposition 2.2)**.** *Let $h : \mathbb{A} \to \mathbb{A}$ be a monotone continuous function. Then there is a clopen interval J so that either $h \restriction J$ is constant or $h \restriction J$ is strictly monotone.*

Proof. We may assume that h is a non-decreasing function since the argument is similar in the other case. On one hand, if there is a clopen interval J so that $h \restriction J$ is an injection, then there is nothing to prove. On the other hand, if there are $x, y \in \mathbb{A}$ so that $\pi(x) \neq \pi(y)$ and $h(x) = h(y)$, then there is a clopen interval J such that $h \restriction J$ is constant. If neither of the above alternatives hold, then:

- For any nonempty open set U there is $a \in [0, 1]$ such that $\langle a, 0\rangle, \langle a, 1\rangle \in U$ and $h(\langle a, 0\rangle) = h(\langle a, 1\rangle)$;
- For any $x, y \in \mathbb{A}$, if $\pi(x) < \pi(y)$ then $h(x) < h(y)$.

Pick $\langle a, 0\rangle, \langle a, 1\rangle \in \mathbb{A}$ such that $h(\langle a, 0\rangle) = h(\langle a, 1\rangle)$. We may assume that $h(\langle a, 0\rangle) = \langle b, 0\rangle$ since the other case is analogous. Choose a strictly decreasing sequence $a_n (n \in \omega)$ of real numbers converging to a. Hence, the sequence $\langle a_n, 0\rangle$ converges to $\langle a, 1\rangle$. Since $a < a_n$, it follows that $h(\langle a, 1\rangle) < h(\langle a_n, 0\rangle)$. However, this implies that $h(\langle a_n, 0\rangle)$ does not belong to the open neighbourhood $[\langle 0, 1\rangle, \langle b, 1\rangle[$ of $\langle b, 0\rangle$ for any $n \in \omega$, which contradicts the continuity of h. This finishes the proof of the Proposition. □

Definition 4.2.3. *Let $h : \mathbb{A} \to {}^m\mathbb{A}$ and $j_0 \in m$ be given. We say that a clopen interval J is j_0-good for h if $h_{j_0} \restriction J$ is strictly monotone and $h_j \restriction J$ is constant for any $j \in m \setminus \{j_0\}$. We say that J is good for h if it is j_0-good for some $j_0 \in m$.*

Remark 4.2.4. *Informally, J is j_0-good for h if h sends J into a line parallel to the j_0-th coordinate axis.*

The following lemma gives an indication as to why this definition will play a role. It will be used in the verification of theorem 4.2.7.

Lemma 4.2.5 (Barría and Martínez-Ranero (2023), Lemma 2.5)**.** *Let $h : \mathbb{A} \to {}^m\mathbb{A}$ be an embedding such that $h''(\mathbb{A})$ is a G_δ set in ${}^m\mathbb{A}$. Then there exists a pairwise disjoint sequence $J_n (n \in \omega)$ of clopen intervals such that $\bigcup_{n\in\omega} J_n$ is dense in $\mathbb{A}$ and for each n, there is $j \in m$ such that J_n is j-good for h.*

Proof. Let U denote the union of all clopen intervals which are good for h. Since $\mathbb{A}$ is separable, it suffices to show that U is dense. In order to get a contradiction, suppose that there is a nonempty clopen interval J disjoint from U. By going to a clopen sub-interval of J if necessary, and applying, Proposition 4.2.1 and Proposition 4.2.2, m times, we may assume that $h_j \restriction J$ is either constant or strictly monotone, for any $j \in m$. Since h is an embedding there exists $j_0 \in m$, such that h_{j_0} is non-constant (equivalently, strictly monotone). As J is not good for h, it follows that there is a $j_1 \neq j_0$ such that h_{j_1} is also strictly monotone.

Claim 4.2.6. *The subspace $X = h''(J]$ is not a G_δ set in ${}^m\mathbb{A}$.*

Proof. Let $X \subseteq \bigcap_{n\in\omega} U_n$, where each U_n is an open set. Since X is compact, we may assume that $U_n = \bigcup_{i\in k_n} \prod_{j\in m} I^j_{n,i}$, where $I^j_{n,i}$ are clopen intervals and $k_n \in \omega$. Let A be equal to

$$\{\pi(x) : \exists n \in \omega \exists i < k_n \ (x \in \{\min(I^{j_0}_{n,i}), \max(I^{j_0}_{n,i})\})\}$$

Pick $x_0 \in J$ such that $\pi(h_{j_0}(x_0))$ does not belong to A, this is possible as A is countable and $h_{j_0} \restriction J$ is an injection. We claim that both points

$$h(x_0) \restriction (m \setminus \{j_0\}) \cup \{(j_0, \langle \pi(h_{j_0}(x_0)), 0\rangle)\} \text{ and}$$

$$h(x_0) \restriction (m \setminus \{j_0\}) \cup \{(j_0, \langle \pi(h_{j_0}(x_0)), 1\rangle)\}[1]$$

belong to $\bigcap_{n\in\omega} U_n$. However, only one of them belongs to X as $h_{j_1} \restriction J$ is injective, which implies that X is not a G_δ set. In order to prove this, fix $N \in \omega$. As

[1]Since $h(x_0) \in {}^m\mathbb{A}$, we consider $h(x_0)$ as a subset of $m \times \mathbb{A}$.

$h(x_0) \in X \subseteq \bigcap_{n\in\omega} U_n$, we can find $i \in k_N$ so that $h(x_0) \in \prod_{j\in m} I^j_{N,i}$. Notice that, $\langle \pi(h_{j_0}(x_0)), 0\rangle$ and $\langle \pi(h_{j_0}(x_0)), 1\rangle$, both belong to $I^{j_0}_{N,i}$, as neither of them are the maximum nor the minimum of the interval. Thus, both $h(x_0) \restriction (m \setminus \{j_0\}) \cup \{(j_0, \langle \pi(h_{j_0}(x_0)), 0\rangle)\}$ and $h(x_0) \restriction (m \setminus \{j_0\}) \cup \{(j_0, \langle \pi(h_{j_0}(x_0)), 1\rangle)\}$ belong to U_N as required. This finishes the proof of the Claim. □

Observe that since $h''(\mathbb{A})$ is a G_δ set in ${}^m\mathbb{A}$ and J is clopen in $\mathbb{A}$, then it follows that X is also a G_δ set in ${}^m\mathbb{A}$, which contradicts the previous Claim. □

We are now ready to prove the main theorem of the section.

Theorem 4.2.7 (Barría and Martínez-Ranero (2023), Theorem 1.5). *Let $h : {}^m\mathbb{A} \to {}^m\mathbb{A}$ be a homeomorphism. Then there is a pairwise disjoint sequence of basic clopen boxes $U_n := \prod_{j\in m} I^j_n (n \in \omega)$ such that $\bigcup_{n\in\omega} U_n$ is dense in ${}^m\mathbb{A}$ and $h \restriction U_n = \sigma \circ (h^0 \times \cdots \times h^{m-1})$, where each $h^j : I^j_n \to \mathbb{A}$ is an strictly monotone homeomorphism onto a clopen interval, and σ is a permutation of ${}^m\mathbb{A}$.*

Proof. Since ${}^m\mathbb{A}$ is separable, and every clopen box is homeomorphic to ${}^m\mathbb{A}$ via a product of strictly increasing homeomorphisms, then it suffices to show that there is a clopen box, so that h restricted to it, is as desired. For each $i \in m$ and $a \in {}^{m\setminus\{i\}}\mathbb{A}$ define the line $E_{a,i} = \{x \in {}^m\mathbb{A} : x \restriction (m \setminus \{i\}) = a\}$, and define an embedding $e_{a,i} : \mathbb{A} \to E_{a,i}$ by $e_{a,i}(p) = a \cup \{(i, p)\}$. For each $i \in m$, let $\pi_{m\setminus\{i\}} : {}^m\mathbb{A} \to {}^{m\setminus\{i\}}\mathbb{A}$ be the projection defined by $\pi_{m\setminus\{i\}}(x) = x \restriction (m \setminus \{i\})$. Notice that $h''(E_{a,i})$ is a G_δ set in ${}^m\mathbb{A}$, since $E_{a,i}$ is a G_δ set in ${}^m\mathbb{A}$, and h is a homeomorphism. We will recursively construct, for $j \in m$, clopen boxes $V_j := \prod_{i\in m} J^j_i \subset {}^m\mathbb{A}$ and functions $\sigma_j : \{0, \ldots, j\} \to m$ such that

i. $V_{j+1} \subset V_j$ for $j \in m - 1$.

ii. For each $i \leq j$ and $a \in \pi''_{m\setminus\{i\}}(V_j)$, J^j_i is $\sigma_j(i)$-good for $h \circ e_{a,i}$. [2]

iii. $\sigma_{j+1} \restriction j = \sigma_j$ and σ_j is injective for $j \in m - 1$.

Suppose we have constructed V_j, σ_j for $j < k \leq m$. By applying Lemma 4.2.5 to the map $h \circ e_{a,k} \restriction J^{k-1}_i$, we can find for each $a \in A := \pi''_{m\setminus\{k\}}(V_{k-1})$, rationals numbers $q_a, r_a \in \mathbb{Q}$ and an integer $j_{a,k}$ such that $[\langle q_a, 1\rangle, \langle r_a, 0\rangle]) \subset J^{k-1}_i$ is

[2]Roughly speaking, in case $m = 2$, we construct a rectangle V_0 where all vertical lines in V_0 are mapped into horizontal lines say, and a subrectangle V_1 where all vertical lines are mapped into horizontal lines and also all horizontal lines are mapped into vertical lines.

$j_{a,k}$-good for $h \circ e_{a,k}$. Since A is a Baire space, there exists j_0, q, r such that $A_{j_0,q,r} := \{a \in A : q_a = q, r_a = r, j_{a,0} = j_0\}$ is dense in some clopen box $V := J_0 \times \cdots \times J_{k-1} \times J_{k+1} \times \cdots \times J_{m-1} \subseteq A$.

Claim 4.2.8. *The interval $[\langle q, 1\rangle, \langle r, 0\rangle]$ is j_0-good, for $h \circ e_{x,k}$ for any $x \in V$.*

Proof. Let

$$A^{<}_{j_0,q,r} := \{a \in A_{j_0,q,r} : h_{j_0} \circ e_{a,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle] \text{ is strictly increasing}\}$$

and

$$A^{>}_{j_0,q,r} := \{a \in A_{j_0,q,r} : h_{j_0} \circ e_{a,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle] \text{ is strictly decreasing}\}.$$

We may assume, without loss of generality, that the set $A^{<}_{j_0,q,r}$ is dense in V. Fix $x \in V$. We shall first show that $h_j \circ e_{x,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle]$ is constant for any $j \in m \setminus \{j_0\}$. In order to do so, pick $j \in m \setminus \{j_0\}, x \in V$ and $s < t \in [\langle q, 1\rangle, \langle r, 0\rangle]$. Choose a sequence x_n of elements of $A^{<}_{j_0,q,r}$ converging to x, this is possible as $A^{<}_{j_0,q,r}$ is dense in V. Notice that $e_{x_n,k}(s)$ and $e_{x_n,k}(t)$ converges to $e_{x,k}(s)$ and $e_{x,k}(t)$, respectively. By assumption, we have that $h_j(e_{x,k}(s)) = \lim_{n\to\infty} h_j(e_{x_n,k}(s)) = \lim_{n\to\infty} h_j(e_{x_n,k}(t)) = h_j(e_{x,k}(t))$. We are now left to show that $h_{j_0} \circ e_{x,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle]$ is strictly monotone. Observe that $h_{j_0} \circ e_{x,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle]$ is injective as $h \circ e_{x,k}$ is injective and all the other coordinate functions are constant. Notice that, by assumption, $h_{j_0}(e_{x_n,k}(s)) < h_{j_0}(e_{x_n,k}(t))$ for any $n \in \omega$. Hence, it follows that $h_{j_0}(e_{x,k}(s)) = \lim_{n\to\infty} h_{j_0}(e_{x_n,k}(s)) \leq \lim_{n\to\infty} h_{j_0}(e_{x_n,k}(t)) = h_{j_0}(e_{x,k}(t))$. Since $s < t$ were arbitrary, it follows that $h_{j_0} \circ e_{x,k} \restriction [\langle q, 1\rangle, \langle r, 0\rangle]$ is strictly monotone as required. □

We now define

$$V_k = J_0 \times \cdots \times J_{k-1} \times [\langle q, 1\rangle, \langle r, 0\rangle] \times J_{k+1} \times \cdots \times J_{m-1} \text{ and } \sigma_k = \sigma_{k-1} \cup \{(k, j_0)\}.$$

It follows from the previous claim that properties i. and ii. hold and also clearly σ_k extends σ_{k-1}. Hence, we are only left to show that σ_k is injective. Aiming towards a contradiction, assume that $\sigma_k(i) = \ell_0 = \sigma_k(j)$ for some $i \neq j$.

In order to simplify the notation, define $x^\ell = x \restriction (m \setminus \{\ell\})$ for any $x \in {}^m\mathbb{A}$ and any $\ell \in m$. Pick $a \in V_k$ and let $P = \{x \in V_k : x \restriction (m \setminus \{i, j\}) = a \restriction (m \setminus \{i, j\})\}$.

We shall prove that $h''(P) \subseteq E_{h(a)^{\ell_0},\ell_0}$. In order to do so, fix $b \in P$. Notice that

$$a \restriction (m \setminus \{i,j\}) \cup \{(j,a(j)),(i,b(i))\} \in E_{a^i,i} \cap E_{b^j,j}.$$

Thus, it follows that $h''(E_{a^i,i} \cap P) \cap h''(E_{b^j,j} \cap P) \neq \emptyset$. By definition of σ_k we have that:

- $h''(E_{a^i,i} \cap P) \subseteq E_{h(a)^{\sigma_k(i)},\sigma_k(i)} = E_{h(a)^{\ell_0},\ell_0}$,
- $h''(E_{b^j,j} \cap P) \subseteq E_{h(b)^{\sigma_k(j)},\sigma_k(j)} = E_{h(b)^{\ell_0},\ell_0}$.

Hence, $h(b) \in E_{h(b)^{\ell_0},\ell_0} = E_{h(a)^{\ell_0},\ell_0}$. It follows that, $h \restriction P : P \to E_{h(a)^{\ell_0},\ell_0}$ is an embedding. However, P is homeomorphic to ${}^2\mathbb{A}$ and $E_{h(a)^{\ell_0},\ell_0}$ is homeomorphic to $\mathbb{A}$, which is impossible since $\mathbb{A}$ is hereditarily normal [Engelking (1989), Problem 2.7.5.(c)] and ${}^2\mathbb{A}$ contains the non-normal subspace ${}^2\mathbb{S}$.

Finally, let $\sigma = \sigma_{m-1}$ and let $h^j = h_j \circ e_{a^j,j} \restriction I_j^{m-1}$ for some fixed $a \in V_{m-1}$ and $j \in m$. It follows from our construction that $h \restriction V_{m-1} = \sigma \circ (h^0 \times \cdots \times h^m)$ as required. $\square$

Definition 4.2.9. *A space X is countable dense homogeneous if it is Hausdorff, separable and given two countable dense subsets D and E of X there is a homeomorphism $h : X \to X$ such that $h''(D) = E$.*

Examples 4.2.10. *The euclidean spaces, the Cantor set and the Hilbert cube are countable dense homogeneous Klee (1957).*

The previous theorem gives us a posteriori explanation of why the space ${}^m\mathbb{A}$ is not countable dense homogeneous for any $m \geq 2$. This was first proved in Arhangel'skii and van Mill (2013) for $m = 1$ and for $m \geq 2$ in Hernández-Gutiérrez (2013).

Corollary 4.2.11 (Hernández-Gutiérrez (2013), Corollary 2.5)**.** *The space ${}^m\mathbb{A}$ is not countable dense homogeneous for any $m \in \mathbb{Z}^+$.*

Proof. Let $Q = \mathbb{Q} \cap]0,1[$, $D_0 = {}^m(Q \times \{0\})$ and $D_1 = {}^m(Q \times \{0,1\})$. It is easy to see that no autohomeomorphism of ${}^m\mathbb{A}$ can map D_1 onto D_0. $\square$

4.3 Non-homogeneity of symmetric products

In this section, for $m \in \mathbb{Z}^+ \setminus \{1\}$ we denote $\Delta_m = \{x \in^m \mathbb{A} : \forall i \in m-1(x(i) \leq x(i+1))\}$.

We recall proposition 1.2.6 with $X = \mathbb{A}$ that gives us a more geometric representation of $\mathcal{F}_m(\mathbb{A})$.

Proposition 4.3.1. *The map $\tilde{\rho} : \Delta_m/\sim \to \mathcal{F}_m(\mathbb{A})$ given by $\tilde{\rho}([x]) = \rho(x)$ is a homeomorphism.*

Proposition 4.3.2 (Barría and Martínez-Ranero (2023), Proposition 3.2)**.** *Every clopen subset of ${}^m\mathbb{A}$ is homeomorphic to ${}^m\mathbb{A}$.*

Proof. Since ${}^m\mathbb{A}$ is compact, then any clopen subset is a finite union of clopen boxes. Thus, it is sufficient to show that any clopen subset is equal to a disjoint union of clopen boxes. In order to do this, we shall prove the following Claim.

Claim 4.3.3. *Let $I^i(i \in N)$ be a finite sequence of clopen intervals of $\mathbb{A}$. Then there exists a finite sequence of pairwise disjoint clopen intervals $J^j(j \in M)$ such that:*

1. $\bigcup\limits_{i \in N} I^i = \bigsqcup\limits_{j \in M} J^j$;
2. *For any $i \in N$ and $j \in M$ either $J^j \subset I^i$ or $J^j \cap I^i = \emptyset$.*

Proof. Let $A = \{\pi(x) : \exists i \in N(x \in \{\min(I^i), \max(I^i)\})\}$, and let $\{a_0, \ldots, a_\ell\}$ be the increasing enumeration of A. Consider the following pairwise disjoint sequence of clopen intervals

$$J^0 := [\langle a_0, 1\rangle, \langle a_1, 0\rangle], \ldots, J^j := [\langle a_j, 1\rangle, \langle a_{j+1}, 0\rangle], \ldots, J^{\ell-1} := [\langle a_{\ell-1}, 1\rangle, \langle a_\ell, 0\rangle].$$

Set $F = \{j \in \ell : J^j \cap \bigcup\limits_{i \in N} I^i \neq \emptyset\}$. We claim that the sequence $J^j(j \in F)$ is as required. First of all, notice that $\bigcup\limits_{i \in N} I^i \subseteq \bigcup\limits_{j \in F} J^j$ as $\bigcup\limits_{i \in N} I^i \subseteq [\langle a_0, 1\rangle, \langle a_\ell, 0\rangle] = \bigcup_{j \in \ell} J^j$. We are left to show the sequence satisfies clause (2). Fix $j_0 \in F$, and let $i_0 \in N$ be given. If $J^{j_0} \cap I^{i_0} = \emptyset$, then there is nothing to show. So we may assume that $J^{j_0} \cap I^{i_0} \neq \emptyset$. Observe that if $J^{j_0} \not\subset I^{i_0}$, then either $\min(I^{i_0}) < \min(J^{j_0}) < \max(I^{i_0})$ or $\min(I^{i_0}) < \max(J^{j_0}) < \max(I^{i_0})$, which contradicts the definition of J^{j_0}. [3] Therefore, the sequence $J^j(j \in F)$ is as required. □

Let V be a clopen subset of ${}^m\mathbb{A}$. We can express V as a finite union of clopen boxes, say,

$$V = \bigcup_{k\in N} \prod_{k\in m} I_k^i$$

where I_k^i are clopen intervals for any $i \in N, k \in m$.

By the previous Claim, we can find a sequence $J_k^j(j \in M_k)$ witnessing clauses (1) and (2) for the sequence $I_k^i(i \in N)$, for any $k \in m$ respectively. For each function $\sigma \in \prod\limits_{k\in m} M_k$, let $C_\sigma = \prod\limits_{k\in m} J_k^{\sigma(k)}$, and let $F := \{\sigma \in \prod_{k\in m} M_k : C_\sigma \cap V \neq \emptyset\}$. We claim that $V = \bigsqcup\limits_{\sigma\in F} C_\sigma$. It follows from (1) that $V \subseteq \bigsqcup\limits_{\sigma\in F} C_\sigma$. Notice that if $C_\sigma \cap \prod\limits_{k\in m} I_k^i \neq \emptyset$, then $J_{\sigma(k)}^j \subset I_k^i$. Thus, $C_\sigma \subseteq \prod\limits_{k\in m} I_k^i$ which implies $\bigsqcup\limits_{\sigma\in F} C_\sigma \subseteq V$. This finishes the proof of the Proposition. □

Lemma 4.3.4 (Barría and Martínez-Ranero (2023), Lemma 3.4)**.** *If $\mathcal{F}_m(\mathbb{A})$ is homogeneous, then it is homeomorphic to ${}^m\mathbb{A}$.*

Proof. Suppose $\mathcal{F}_m(\mathbb{A})$ is homogeneous, then there is an autohomeomorphism $h : \Delta_m/{\sim} \to \Delta_m/{\sim}$ such that $h([\overline{\langle 0,1\rangle}]) = [x]$, where x is some fixed point such that $x(0) < x(1) < \cdots < x(m-1)$. On one hand, notice that, if $J_0 < \cdots < J_{m-1}$ is a sequence of pairwise disjoint clopen intervals with $x(i) \in J_i$ for $i \in m$, then $q \restriction \prod\limits_{i\in m} J_i : \prod\limits_{i\in m} J_i \to \Delta_m/{\sim}$ is an embedding. On the other hand, observe that for any $0 < \epsilon < 1$ the clopen cube ${}^m[\langle 0,1\rangle, \langle \epsilon, 0\rangle]$ is a saturated neighborhood of $\overline{\langle 0,1\rangle}$ such that $q''({}^m[\langle 0,1\rangle, \langle \epsilon, 0\rangle])$ is homeomorphic to $\Delta_m/{\sim}$. Since h is continuous, there is an $\epsilon > 0$ such that $h''({}^m[\langle 0,1\rangle, \langle \epsilon, 0\rangle]/\sim) \subseteq \prod\limits_{i\in m} J_i$. Thus, we have that

$${}^m\mathbb{A} \cong \prod_{i\in m} J_i \cong h''({}^m[\langle 0,1\rangle, \langle \epsilon, 0\rangle])/{\sim}) \cong {}^m[\langle 0,1\rangle, \langle \epsilon, 0\rangle]/{\sim} \cong \Delta_m/{\sim}$$

where the second homeomorphism follows from Proposition 4.3.2. □

We are now ready to prove the main result of the section.

Theorem 4.3.5 (Barría and Martínez-Ranero (2023), Theorem 1.3)**.** *The symmetric product $\mathcal{F}_m(\mathbb{A})$ is not homogeneous for any $m \geq 2$.*

[3]For any $j \in M$, the interval J^j contains exactly two extreme points of the intervals I^i's.

Proof. Aiming towards a contradiction, assume that there is a homeomorphism $h : \Delta_m/\sim \to {}^m\mathbb{A}$, and let $\Gamma = \{[\overline{x}] \in \Delta_m/\sim : x \in \mathbb{A}\}$. Recall that the diagonal $\{(x,x) \in {}^2\mathbb{A} : x \in \mathbb{A}\}$ is not a G_δ subspace as $\mathbb{A}$ is a non-metrizable compact space. It follows from this that Γ is not a G_δ in $\Delta_m/\sim$ as otherwise this would imply that $\pi''_{\{0,1\}}(q^{-1}(\Gamma)) = \{(x,x) \in {}^2\mathbb{A} : x \in \mathbb{A}\}$ would also be one, where $\pi_{\{0,1\}} : {}^m\mathbb{A} \to {}^2\mathbb{A}$ denotes the projection onto the first 2 coordinates. Notice that, since $\mathbb{A} \times {}^{m-1}\{\langle 0,1\rangle\} = \bigcap_{n\in\omega} \mathbb{A} \times {}^{m-1}\{[\langle 0,1\rangle, \langle \frac{1}{n}, 0\rangle]\} \subseteq {}^m\mathbb{A}$ and $\mathbb{A}$ is a perfect space, then every closed subset of $\mathbb{A} \times {}^{m-1}\{\langle 0,1\rangle\}$ is a G_δ set in ${}^m\mathbb{A}$. Analogously, every closed subset of any line parallel to one of coordinates axis, is also a G_δ set in ${}^m\mathbb{A}$. We now consider the embedding $\alpha : \mathbb{A} \to {}^m\mathbb{A}$ given by $\alpha(x) = h([\overline{x}])$. By applying Proposition 4.2.2 m-times, we can find a clopen interval J such that $\alpha_j := \pi_j \circ \alpha \restriction J$ is monotone for every $j \in m$. Since $h''(\Gamma)$ is not a G_δ in ${}^m\mathbb{A}$, it follows, by our previous observations, that there exists $j_0 \neq j_1 \in m$ such that α_{j_0} and α_{j_1} are strictly monotone restricted to J. We will assume that both $\alpha_{j_0} \restriction J, \alpha_{j_1} \restriction J$ are strictly increasing, as the other cases are analogous.

Claim 4.3.6. *There is a countable subset $C \subseteq \pi''(J)$ such that*

$$\pi(\alpha_{j_0}(\langle a,0\rangle)) = \pi(\alpha_{j_0}(\langle a,1\rangle))$$

and

$$\pi(\alpha_{j_1}(\langle a,0\rangle)) = \pi(\alpha_{j_1}(\langle a,1\rangle))$$

for any $a \in \pi''(J) \setminus C$. In other words, $\alpha_{j_k}(\langle a,1\rangle)$ is the immediate successor of $\alpha_{j_k}(\langle a,0\rangle)$ for $k \in 2$.

Proof. Let $C_k = \{a \in \pi''(J) : \pi(\alpha_{j_k}(\langle a,0\rangle)) < \pi(\alpha_{j_k}(\langle a,1\rangle))\}$ for $k \in 2$. For each $a \in C_k$, pick a rational r_a such that $\pi(\alpha_{j_k}(\langle a,0\rangle)) < r_a < \pi(\alpha_{j_k}(\langle a,1\rangle))$. Observe that since α_{j_k} is strictly increasing, the map $f : C_k \to \mathbb{Q}$ given by $f(a) = r_a$, is one-to-one. Thus, $C = C_0 \cup C_1$ is countable as desired. □

For each $a \in A := \pi''(J) \setminus C$, let $P_a^- = \alpha(\langle a,0\rangle), Q_a^+ = \alpha(\langle a,1\rangle)$ and let

$$P_a^+ = \alpha(\langle a,0\rangle) \restriction_{(m\setminus\{j_0\})} \cup (j_0, \langle \pi(\alpha_{j_0}(\langle a,0\rangle)), 1\rangle)$$

and

$$Q_a^- = \alpha(\langle a,1\rangle) \restriction_{(m\setminus\{j_1\})} \cup (j_1, \langle \pi(\alpha_{j_1}(\langle a,0\rangle)), 0\rangle).$$

Pick an element $[x_a]$ belonging to $h^{-1}(\{P_a^+, Q_a^-\})\setminus\tilde{\rho}^{-1}(\{\langle a,0\rangle,\langle a,1\rangle\})$. Observe that, by our choice of x_a, there is a $\ell_a \in m$ so that $\pi(x_a(\ell_a)) \neq a$. Let

$$A^{P,<} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) < a\},$$

$$A^{P,>} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) > a\},$$

$$A^{Q,<} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) < a\}$$

and

$$A^{Q,>} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) > a\}.$$

We may assume, without loss of generality, that $A^{P,<}$ is uncountable as the other cases are similar. By successively refining $A^{P,<}$, we can find an uncountable subset $B \subseteq A^{P,<}$, a natural number ℓ and a rational number $r \in \mathbb{Q}$ such that $\ell_a = \ell$ and $\pi(x_a(\ell)) < r < a$ for any $a \in B$.

Consider the disjoint clopen sets

$$U := \bigcup_{j\in m} \pi_j^{-1}([\langle 0,1\rangle,\langle r,0\rangle]) \text{ and } V := \bigcap_{j\in m} \pi_j^{-1}([\langle r,1\rangle,\langle 1,0\rangle]).$$

Since U, V are saturated, then we have that $\tilde{U} := q''(U)$ and $\tilde{V} := q''(V)$ are clopen and disjoint. Notice that $X := \{[x_a] : a \in B\} \subset \tilde{U}$ and $Y := \{[\overline{\langle a,0\rangle}] : a \in B\} \subset \tilde{V}$. Also observe that as B is infinite (uncountable) and $\Delta_m/\sim$ is compact, then the accumulation points X' and Y' of X and Y, respectively, are both nonempty. It follows that $X' \cap Y' = \emptyset$.

Claim 4.3.7. *The sets $h''(X) = \{P_a^+ : a \in B\}$ and $h''(Y) := \{P_a^- : a \in B\}$ have the same accumulations points.*

Proof. We shall prove that the accumulation points of $h''(X)$ are contained in the accumulation points of $h''(Y)$ as the other case is analogous. Let P be an accumulation point of $h''(X)$ and let $W := \prod_{j\in m} J_j$ be a clopen neighborhood of P where each J_j is a clopen interval. Since P is an accumulation point, then there is an infinite subset $B' \subseteq B$ such that $\{P_a^+ : a \in B'\} \subseteq W$. By construction $P_a^-(j) = P_a^+(j)$ for any $j \in m \setminus \{j_0\}$ and $a \in B$. In particular, $P_a^-(j) \in J_j$ for any $j \in m \setminus \{j_0\}$ and $a \in B'$. Observe that $P_a^+(j_0) \neq P_b^+(j_0)$ for any $a \neq b \in B$ as $\alpha_{j_0} \restriction J$ is strictly monotone. Thus, there is an infinite subset $B'' \subseteq B'$ such that $\pi(P_a^+(j_0)) \notin \{\pi(\min(J_{j_0})), \pi(\max(J_{j_0}))\}$ for all $a \in B''$. It follows that, $\{P_a^- : a \in B''\} \subseteq W$ and hence, P is an accumulation point of $h''(Y)$ as required. $\square$

Since h is a homeomorphism, then X and Y have the same accumulation points which is a contradiction. This finishes the proof of the Theorem. $\square$

It would be interesting to see if the above theorem can be extended to the hyperspace of all non-empty finite subsets $\mathcal{F}(\mathbb{A})$.

Question 4.3.8. *Is the hyperspace $\mathcal{F}(\mathbb{A})$ homogeneous?*

4.4 Non-homogeneity of the space of unions of at most m closed intervals

Let $(X,<)$ be a linearly ordered space. For $m \in \mathbb{Z}^+$, we denote $\mathcal{C}_m(X) \subset \mathrm{Exp}(X)$ as the subspace of all unions of at most m non-empty closed intervals in X and $\Delta_{2m} = \{x \in {}^{2m}X : \forall i \in 2m-1(x(i) \leq x(i+1))\}$. Let $\varrho : \Delta_{2m}(X) \longrightarrow \mathcal{C}_m(X)$ be the map defined by $\varrho(x) = \bigcup_{i\in m}[x(2i), x(2i+1)]$ and let $\approx$ the equivalence relation on $\Delta_{2m}(X)$ defined by $x \approx y$ if and only if $\varrho(x) = \varrho(y)$. Let $p : \Delta_{2m}(X) \to \Delta_{2m}(X)/{\approx}$ be the quotient map. We will sometimes write $[x]$ instead of $p(x)$ to represent the equivalence class. We consider $\Delta_{2m}(X)/{\approx}$ as a topological space with the quotient topology.

Proposition 4.4.1. *If $(X,<)$ is a linearly ordered space, then ϱ is continuous.*

Proof. We will prove that the preimages under ϱ of $[V]$ and $\langle W\rangle$, with V an open set and W a basic interval, are open.

Let V an open set of X. There exists basic intervals V_j such that $V = \bigcup_{j\in J} V_j$. Let $x \in \varrho^{-1}([V]) = \{y \in \Delta_{2m}(X) : \bigcup_{i\in m}[y(2i), y(2i+1)] \subset \bigcup_{j\in J} V_j\}$. For each $i \in m$ we define $W_i = \bigcup\{V_j : [x(2i), x(2i+1)] \cap V_j \neq \emptyset\}$. We have that W_i is an open interval that contains $[x(2i), x(2i+1)]$. Let $y \in \Delta_{2m}(X) \cap \prod_{i\in m} W_i^2$. For all i, $y(2i)$ and $y(2i+1)$ are in W_i, so $\bigcup_{i\in m}[y(2i), y(2i+1)] \subset \bigcup_{i\in m} W_i \subset \bigcup_{j\in J} V_j = V$. Therefore, $x \in \Delta_{2m}(X) \cap \prod_{i\in m} W_i^2 \subset \varrho^{-1}([V])$ and $\varrho^{-1}([V])$ is open.

Let W be a basic open interval of X and let $x \in \varrho^{-1}(\langle W\rangle)$ be given. By definition, there exists j such that $[x(2j), x(2j+1)] \cap W \neq \emptyset$. If $W =]\leftarrow, a[$, then we define $B = \prod_{i\in 2m} B_i$ with $B_i = X$ if $i \neq 2j$ and $B_{2j} = W$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$, that is to say $\bigcup_{i\in m}[y(2i), y(2i+1)] \cap W \neq \emptyset$. In this way, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W \rangle)$. The proof for $W =]a, \rightarrow[$ is similar. When $W =]a, b[$ we have two cases.

Case 1. $a < x(2j+1) < b$. Define $B = \prod_{i \in 2m} B_i$ with $B_i = X$ if $i \neq 2j+1$ and $B_{2j+1} = W$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$. We have that $\bigcup_{i \in m}[y(2i), y(2i+1)] \cap W \neq \emptyset$. In this way, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W \rangle)$.

Case 2. $x(2j+1) \geq b$. Necessarily $x(2j) < b$. Define $B = \prod_{i \in 2m} B_i$ with $B_i = X$ if $i \in 2m \setminus \{2j, 2j+1\}$, $B_{2j} =]\leftarrow, b[$ and $B_{2j+1} =]a, \rightarrow[$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$. Therefore, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W \rangle)$.

We conclude that $\varrho^{-1}(\langle W \rangle)$ is open. $\square$

Analogously to Proposition 1.2.6, the following result gives us a more geometric representation of $\mathcal{C}_{2m}(X)$.

Corollary 4.4.2. *If $(X, <)$ a compact linearly ordered space, then the map $\tilde{\varrho} : \Delta_{2m}(X)/\approx \,\rightarrow \mathcal{C}_m(X)$ given by $\tilde{\varrho}([x]) = \varrho(x)$ is a homeomorphism.*

Proof. Since ϱ is continuous, we have that $\tilde{\varrho}$ is a continuous bijection. Let $x \in {}^{2m}X \setminus \Delta_{2m}(X)$. There are $i, j \in 2m$ such that $i < j$ and $x(i) > x(j)$. Since X is Hausdorff, there exists two disjoint basic intervals V and W with $W < V$ such that $x(i) \in V$ and $x(j) \in W$. Let $A = \prod_{k \in 2m} X_k$ an open neighborhood of x with $X_k = X$ for all $k \in 2m \setminus \{i, j\}$, $X_i = V$ and $X_j = W$. We have that $x \in A \subset {}^{2m}X \setminus \Delta_{2m}(X)$, so $\Delta_{2m}(X)$ is closed in ${}^{2m}X$. Since ${}^{2m}X$ is compact, so is $\Delta_{2m}(X)$. Therefore, $\Delta_{2m}(X)/\approx$ is compact and $\tilde{\varrho}$ is a homeomorphism. $\square$

Remark 4.4.3. *We note that $\Delta_2(X) = \Delta_2(X)/\approx$. By the previous Corollary and Proposition 1.2.6, we have that $\mathcal{F}_2(X) \cong \mathcal{C}_1(X)$.*

Lemma 4.4.4. *If $\mathcal{C}_m(\mathbb{A})$ is homogeneous, then it is homeomorphic to ${}^{2m}\mathbb{A}$.*

Proof. Suppose $\mathcal{C}_m(\mathbb{A})$ is homogeneous, then there is an autohomeomorphism $h : \Delta_{2m}/\approx \,\rightarrow \Delta_{2m}/\approx$ such that $h([\overline{\langle 0, 1 \rangle}]) = [x]$, where x is some fixed point such that $\pi(x(0)) < \pi(x(1)) < \cdots < \pi(x(2m-1))$. On one hand, notice that, if $J_0 < \cdots < J_{2m-1}$ is a sequence of pairwise disjoint clopen intervals with $x(i) \in J_i$ for $i \in 2m$ and $\max(\pi(J_i)) < \min(\pi(J_{i+1}))$ for $i \in 2m-1$, then $p \restriction \prod\limits_{i \in 2m} J_i : \prod\limits_{i \in 2m} J_i \rightarrow \Delta_{2m}/\approx$ is an embedding. On the other hand, observe that for any $0 < \epsilon < 1$ the clopen cube ${}^{2m}[\langle 0, 1 \rangle, \langle \epsilon, 0 \rangle]$ is a saturated neighborhood of $\overline{\langle 0, 1 \rangle}$ such that $p''({}^{2m}[\langle 0, 1 \rangle, \langle \epsilon, 0 \rangle])$ is homeomorphic to $\Delta_{2m}/\approx$. Since h is continuous, there is an $\epsilon > 0$ such that $h''({}^{2m}[\langle 0,1\rangle, \langle \epsilon, 0\rangle]/\approx) \subseteq \prod_{i\in 2m} J_i$. Thus, we have that

$$ {}^{2m}\mathbb{A} \cong \prod_{i\in 2m} J_i \cong h''({}^{2m}[\langle 0,1\rangle, \langle \epsilon, 0\rangle]/\approx) \cong {}^{2m}[\langle 0,1\rangle, \langle \epsilon, 0\rangle]/\approx \,\cong \Delta_{2m}/\approx $$

where the second homeomorphism follows from Proposition 4.3.2. □

Theorem 4.4.5. *$\mathcal{C}_m(\mathbb{A})$ is not homogeneous for any $m \in \mathbb{Z}^+$.*

Proof. We proceed by contradiction. Suppose that there is a homeomorphism $h : \Delta_{2m}/\approx \to {}^{2m}\mathbb{A}$, and let $\Gamma = \{[\overline{x}] \in \Delta_{2m}/\approx : x \in \mathbb{A}\}$. Recall that the diagonal $\{(x,x) \in {}^{2}\mathbb{A} : x \in \mathbb{A}\}$ is not a G_δ subspace as $\mathbb{A}$ is a non-metrizable compact space. It follows from this that Γ is not a G_δ in $\Delta_{2m}/\approx$ as otherwise this would imply that $\pi''_{\{0,1\}}(p^{-1}(\Gamma)) = \pi''_{\{0,1\}}(\{\overline{x} \in \Delta_{2m} : x \in \mathbb{A}\}) = \{(x,x) \in {}^{2}\mathbb{A} : x \in \mathbb{A}\}$ would also be one, where $\pi_{\{0,1\}} : {}^{2m}\mathbb{A} \to {}^{2}\mathbb{A}$ denotes the projection onto the first 2 coordinates. Notice that, since $\mathbb{A} \times {}^{2m-1}\{\langle 0,1\rangle\} = \bigcap_{n\in\omega} \mathbb{A} \times {}^{2m-1}\{[\langle 0,1\rangle, \langle \frac{1}{n}, 0\rangle]\} \subseteq {}^{2m}\mathbb{A}$ and $\mathbb{A}$ is a perfect space, then every closed subset of $\mathbb{A} \times {}^{2m-1}\{\langle 0,1\rangle\}$ is a G_δ set in ${}^{2m}\mathbb{A}$. Analogously, every closed subset of any line parallel to one of coordinates axis, is also a G_δ set in ${}^{2m}\mathbb{A}$. We now consider the embedding $\alpha : \mathbb{A} \to {}^{2m}\mathbb{A}$ given by $\alpha(x) = h([\overline{x}])$. By applying Proposition 4.2.2 $2m$-times, we can find a clopen interval J such that $\alpha_j := \pi_j \circ \alpha \restriction J$ is monotone for every $j \in 2m$. Since $h''(\Gamma)$ is not a G_δ in ${}^{2m}\mathbb{A}$, it follows, by our previous observations, that there exists $j_0 \neq j_1 \in 2m$ such that α_{j_0} and α_{j_1} are strictly monotone restricted to J. We will assume that both $\alpha_{j_0} \restriction J, \alpha_{j_1} \restriction J$ are strictly increasing, as the other cases are analogous.

The proof of the following result is analogous to the proof of Claim 4.3.6.

Claim 4.4.6. *There is a countable subset $C \subseteq \pi''(J)$ such that*

$$ \pi(\alpha_{j_0}(\langle a,0\rangle)) = \pi(\alpha_{j_0}(\langle a,1\rangle)) $$

and

$$ \pi(\alpha_{j_1}(\langle a,0\rangle)) = \pi(\alpha_{j_1}(\langle a,1\rangle)) $$

for any $a \in \pi''(J) \setminus C$. In other words, $\alpha_{j_k}(\langle a,1\rangle)$ is the immediate successor of $\alpha_{j_k}(\langle a,0\rangle)$ for $k \in 2$.

For each $a \in A := \pi''(J) \setminus C$, let $P_a^- = \alpha(\langle a, 0\rangle)$, $Q_a^+ = \alpha(\langle a, 1\rangle)$ and let

$$P_a^+ = \alpha(\langle a, 0\rangle) \restriction_{(2m\setminus\{j_0\})} \cup (j_0, \langle \pi(\alpha_{j_0}(\langle a, 0\rangle)), 1\rangle)$$

and

$$Q_a^- = \alpha(\langle a, 1\rangle) \restriction_{(2m\setminus\{j_1\})} \cup (j_1, \langle \pi(\alpha_{j_1}(\langle a, 0\rangle)), 0\rangle).$$

Pick an element $[x_a]$ belonging to $h^{-1}(\{P_a^+, Q_a^-\}) \setminus \tilde{\varrho}^{-1}([\langle a, 0\rangle, \langle a, 1\rangle])$. Observe that, by our choice of x_a, there is a $\ell_a \in 2m$ so that $\pi(x_a(\ell_a)) \neq a$. Let

$$A^{P,<} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) < a\},$$

$$A^{P,>} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) > a\},$$

$$A^{Q,<} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) < a\}$$

and

$$A^{Q,>} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) > a\}.$$

We may assume, without loss of generality, that $A^{P,<}$ is uncountable as the other cases are similar. By successively refining $A^{P,<}$, we can find an uncountable subset $B \subseteq A^{P,<}$, a natural number ℓ and a rational number $r \in \mathbb{Q}$ such that $\ell_a = \ell$ and $\pi(x_a(\ell)) < r < a$ for any $a \in B$.

Consider the clopen sets

$$U := \bigcup_{j\in 2m} \pi_j^{-1}([\langle 0, 1\rangle, \langle r, 0\rangle]) \text{ and } V := \bigcap_{j\in 2m} \pi_j^{-1}([\langle r, 1\rangle, \langle 1, 0\rangle]).$$

Claim 4.4.7. *The sets U and V are saturated.*

Proof. Let $x \in p^{-1}(p''(U))$. There is $y \in U$ such that $\bigcup_{i\in m}[x(2i), x(2i+1)] = \bigcup_{i\in m}[y(2i), y(2i+1)]$. Since there is $j \in 2m$ and $k \in m$ with $\langle 0, 1\rangle \le y(j) \le \langle r, 0\rangle$ and $y(j) \in [x(2k), x(2k+1)]$, then $\langle 0, 1\rangle \le x(2k) \le \langle r, 0\rangle$. Thus, $x \in U$.

Let $x \in p^{-1}(p''(V))$. There is $y \in V$ such that $\bigcup_{i\in m}[x(2i), x(2i+1)] = \bigcup_{i\in m}[y(2i), y(2i+1)]$. Since $y(j) \in [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$, we have that $\bigcup_{i\in m}[x(2i), x(2i+1)] \subset [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$. It follows that $x(j) \in [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$, that is to say, $x \in V$. $\square$

We have that $\tilde{U} := p''(U)$ and $\tilde{V} := p''(V)$ form a clopen partition of $\Delta_{2m}/\approx$. Notice that $X := \{[x_a] : a \in B\} \subset \tilde{U}$ and $Y := \{[\overline{\langle a, 0\rangle}] : a \in B\} \subset \tilde{V}$. Since B

is infinite (uncountable) and $\Delta_{2m}/\approx$ is compact, then the set of accumulation points X' and Y' of X and Y, respectively, are both non-empty. It follows that $X' \cap Y' = \emptyset$.

Claim 4.4.8. *The sets $h''(X) = \{P_a^+ : a \in B\}$ and $h''(Y) := \{P_a^- : a \in B\}$ have the same accumulations points.*

Proof. We shall prove that the accumulation points of $h''(X)$ are contained in the accumulation points of $h''(Y)$ as the other case is analogous. Let P be an accumulation point of $h''(X)$ and let $W := \prod_{j \in 2m} J_j$ be a clopen neighborhood of P where each J_j is a clopen interval. Since P is an accumulation point, then there is an infinite subset $B' \subseteq B$ such that $\{P_a^+ : a \in B'\} \subseteq W$. By construction $P_a^-(j) = P_a^+(j)$ for any $j \in 2m \setminus \{j_0\}$ and $a \in B$. In particular, $P_a^-(j) \in J_j$ for any $j \in 2m \setminus \{j_0\}$ and $a \in B'$. Observe that $P_a^+(j_0) \neq P_b^+(j_0)$ for any $a \neq b \in B$ as $\alpha_{j_0} \restriction J$ is strictly monotone. Thus, there is an infinite subset $B'' \subseteq B'$ such that $\pi(P_a^+(j_0)) \notin \{\pi(\min(J_{j_0})), \pi(\max(J_{j_0}))\}$ for all $a \in B''$. It follows that, $\{P_a^- : a \in B''\} \subseteq W$ and hence, P is an accumulation point of $h''(Y)$ as required. □

Since h is a homeomorphism, then X and Y have the same accumulation points which is a contradiction. This finishes the proof of the Theorem. □

Since $\mathcal{F}_m(\mathbb{A}) \subset \mathcal{C}_m(\mathbb{A})$ for $m \geq 2$, now we are a little more closer to answer Question 2.

It would be interesting to see if the above theorem can be extended to the hyperspace of all finite unions of non-empty closed intervals $\mathcal{C}(\mathbb{A})$.

Question 4.4.9. *Is the hyperspace $\mathcal{C}(\mathbb{A})$ homogeneous?*

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Appendix A

Symbols used

$\mathbb{Z}^+$: the positive integers

ω : $\mathbb{Z}^+ \cup \{0\}$

$\mathbb{Q}$: the rational numbers

$\mathbb{R}$: the real numbers

$f \restriction A$: f restricted to the set A

$f''(A)$: image of the set A, $\{f(x) : x \in A\}$

$f^{-1}(A)$: preimage of the set A, $\{x : f(x) \in A\}$

$\cong$: homeomorphic

$\mathrm{Aut}(X)$: homeomorphisms from X to X

∂A: topological boundary of A

π_i: projection on the i^{th} coordinate