Universidad de Concepción

Facultad de Ciencias Físicas y Matemáticas

Programa Doctorado en Matemática

Cox rings of K3 surfaces of Picard number three and four

Anillos de Cox de superficies K3 de número de Picard tres y cuatro.

Tesis para optar al grado de Doctora en Matemática

CLAUDIA INÉS CORREA DEISLER

CONCEPCIÓN - CHILE

octubre - 2020

Profesor Guía: Michela Artebani

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Resumen

En esta tesis estudiamos anillos de Cox de superficies K3 Mori dream, es decir superficies proyectivas suaves X con $H^1(X, \mathcal{O}_X) = \{0\}$ y con clase canónica trivial cuyo anillo de Cox es finitamente generado. Hacia el 2009, comienza la investigación sobre los anillos de Cox de las superficies K3 con el trabajo de Artebani, Hausen y Laface [AHL10] y McKernan [McK10], donde los autores probaron independientemente que el anillo de Cox de una superficie K3 es finitamente generado si y sólo si su cono efectivo es poliedral, o equivalente si su grupo de automorfismos es finito. Las superficies K3 proyectivas con número de Picard ≥ 3 y con grupo de automorfismos finito han sido clasificadas, y se sabe que hay un número finito de familias con dicha propiedad.

El objetivo principal de esta tesis es desarrollar técnicas y herramientas computacionales para calcular anillos de Cox de superficies K3 Mori dream, es decir encontrar generadores y relaciones para el anillo de Cox.

Un primer resultado en esta dirección se basa en sucesiones exactas de tipo Koszul, el cual nos permite probar un teorema general para los anillos de Cox de superficies K3 (no necesariamente Mori dream), es decir demostramos que los grados de un conjunto minimal de generadores del anillo de Cox $R(X)$ son o bien clases de (-2)-curvas, o bien clases de divisores nef los cuales son suma de a lo más tres elementos de la base

de Hilbert del cono nef (permitiendo repeticiones), o bien clases de divisores de la forma $2(F + F')$ donde F, F' son curvas elípticas suaves con $F \cdot F' = 2$ (Teorema 2.4.2).

Posteriormente, aplicamos varias técnicas basadas en sucesiones exactas de Koszul para determinar los grados de un conjunto de generadores del anillo de Cox de elementos generales de las familias de superficies K3 Mori dream de número de Picard tres y cuatro, en donde sabemos que hay 26 familias para número de Picard 3, y 14 familias para número de Picard 4.

Para cumplir con nuestro objetivo planteado, primero aplicamos el algoritmo de Vinberg [Vin75] para calcular los conos efectivo y nef de las superficies (Teorema 3.1.1 y Teorema 4.1.1), dicho algoritmo lo hemos implementado en Magma [BCP97]. Cabe destacar que hasta donde sabemos, en el caso de familias con número de Picard cuatro, el conjunto de las (-2)-curvas de las superficies (cuyas clases generan el cono efectivo) no se conocían. En segundo lugar, identificamos los grados de un conjunto de generadores del anillo de Cox para cada familia en estudio (Teorema 3.2.1 y Teorema 4.2.1), esto es por el Teorema 2.4.2, el Corolario 2.3.4 y otras técnicas que permiten probar que el anillo de Cox no necesita generadores en ciertos grados. Finalmente, probamos un resultado que permite mostrar que el anillo de Cox necesariamente tiene un generador en cierto grado, y con esto probar que el conjunto de grados encontrados en el segundo paso es mínimo en varias de las familias de superficies estudiadas. Además de lo anterior, describimos los modelos proyectivos para todas las superficies K3 Mori dream con número de Picard 4 e identificamos geométricamente los grados de los generadores del anillo de Cox.

Introduction

The *Cox ring* of a normal projective variety X defined over the complex numbers with finitely generated and free divisor class group $\mathrm{Cl}(X)$ is the graded algebra [ADHL15]

$$R(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)).$$

The variety X is called *Mori dream space* when the Cox ring is finitely generated (Definition 2.1.3). Important examples of Mori dream spaces are toric varieties, whose Cox ring is a polynomial ring with a multi-grading which can be explicitly determined in terms of its fan, see [Cox95]. Other important examples are log Fano varieties [BCHM10]. In this context there are two main problems: determine conditions on X such that $R(X)$ is finitely generated, and find an explicit presentation for $R(X)$, i.e. generators and relations for Cox rings. A fundamental property of Mori dream spaces is that any such variety is a GIT quotient of an open Zariski subset of an affine space, the spectrum of $R(X)$, by the action of a quasitorus. This allows to define homogeneous coordinates on X, as in the case of the projective space, and allows a combinatorical approach to certain geometric and arithmetic properties of X, as in the case of toric varieties [ADHL15].

The problem of finding a presentation for the Cox ring of a Mori dream space is interesting and difficult. There exist different techniques for this, which allowed to compute the Cox ring of several classes of special varieties. A pioneering work in this direction is the paper [BP04] by Batyrev and Popov, who identified the generators of the Cox ring of any del Pezzo surface, showing in particular that it is generated by the elements defining the (-1)-curves of the surface if the rank r of the divisor class group satisfies $4 \le r \le 8$. The ideal of relations of the Cox ring of del Pezzo surfaces has been computed in several steps in [STM07, SS07, LM09, TVAV09]. In [CT06] the authors determined the generators of the Cox ring of the blow-up of $\mathbb{P}^n$ in any number of points that lie on a rational normal curve.

More in general, techniques are available to compute Cox rings of special classes of varieties (for example varieties with a torus action, homogeneous spaces, spherical varieties) and to relate the Cox rings of two varieties X, Y obtained one from the other in different ways (for example Y embedded in X satisfying suitable conditions or Y a blow-up of X along an irreducible subvariety contained in the smooth locus), see [ADHL15, Ch. 4] and [HKL16].

This thesis deals with Cox rings of Mori dream K3 surfaces, that is, smooth projective surfaces X over $\mathbb{C}$ with $H^1(X, \mathcal{O}_X) = \{0\}$ and trivial canonical class (Definition 1.4.1) whose Cox ring is finitely generated. Research on Cox rings of K3 surfaces begins in 2009 with the work of Artebani, Hausen, Laface [AHL10] and McKernan [McK10], where the authors proved independently that the Cox ring of a K3 surface is finitely generated if and only if its effective cone is polyhedral, or equivalently if its automorphism group is finite. K3 surfaces with this property have been classified in a series of classical papers [Nik79, Nik84, Nik00, PŠŠ71, Vin07] (see also [ADHL15, §5.1.5]). For Picard number ≥ 3 there is a finite number of families with such property (see Theorem 2.2.5).

In addition, in [AHL10] the authors investigate generators and relations for the Cox ring of K3 surfaces, determining an explicit presentation for certain classes of them. In [Ott13] Ottem also studies generators and relations of Cox ring of K3 surfaces of Picard number two, in particular he describes the Cox ring of some classical examples, such as quartic surfaces containing a line. It should be noted that, before them, Saint-Donat in [SD74] investigated generators and relations of the coordinate ring $\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nH))$, for X a K3 surface and H a very ample divisor of X.

The main objective of this thesis is to develop techniques and computational tools to calculate Cox rings of K3 surfaces. These techniques are then applied to find the degrees of a generating set of Cox ring of Mori dream K3 surfaces of Picard number 3 and 4. As a first step, we extend the known techniques for finding the degrees of generators of the Cox ring. A first result in this direction is the following, which relies on Koszul-type exact sequences.

Proposition 1. (Corollary 2.3.4) *Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$, such that $\cap_{i=1}^3 \operatorname{div}_{E_i}(f_i) = \emptyset$. If $D \in \operatorname{WDiv}(X)$ then the morphism*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \ (g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3,$$

is surjective if $h^1(X, D - E_i - E_j) = 0$ for all distinct $i, j \in \{1, 2, 3\}$ and $h^2(X, D - E_1 - E_2 - E_3) = 0$.

Observe that the surjectivity of the morphism in the statement of Proposition 1 implies that $R(X)$ does not need a generator in degree $[D]$, since all elements of $H^0(X, D)$ can be obtained as polynomials in homogeneous elements of other degrees. This allows to prove the following general theorem on Cox rings of (not necessarily Mori dream) K3 surfaces.

Theorem 1. (Theorem 2.4.2) *Let X be a smooth projective K3 surface over $\mathbb{C}$. Then the degrees of a minimal set of generators of its Cox ring $R(X)$ are either:*

(*i*) *classes of (-2)-curves,*

(*ii*) *classes of nef divisors which are sums of at most three elements of the Hilbert basis of the nef cone (allowing repetitions),*

(*iii*) *or classes of divisors of the form $2(F+F')$ where F, F' are smooth elliptic curves with $F \cdot F' = 2$.*

Afterwards, we apply Theorem 1, Proposition 1 and other techniques based on Koszul exact sequences to determine the degrees of a generating set for the Cox rings of general elements of the families of Mori dream K3 surfaces of Picard number three and four.

In case the Picard number is three, by Theorem 2.2.5 we know that there are 26 families of Mori dream K3 surfaces. For them, we obtain the following result.

Theorem 2. (Theorem 3.2.1) *Let X be a Mori dream K3 surface of Picard number three. The degrees of a set of generators of the Cox ring $R(X)$ are given in Table 5.6. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.*

Looking at Table 5.6 it can be seen that there are six cases in which the Cox ring is generated in seven degrees. In Section 3.3 we describe the geometry of two of these families, namely the families with $\mathrm{Cl}(X)$ isometric to the lattices S_1 and $S_{4,1,1}$ (see Theorem 2.2.5), and we provide a presentation for the Cox ring of a very general member of them. For example, in case the lattice is isometric to S_1 we obtain the following result.

Theorem 3. (Theorem 3.3.2) *Let X be a K3 surface with $\mathrm{Cl}(X) \cong S_1 = (6) \oplus 2A_1$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with three 6-tangent conics C_1, C_2, C_3;*

2. *X can be defined by an equation of the following form in $\mathbb{P}(1,1,1,3)$:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)F_3(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   where F_1, F_2, F_3 are homogeneous of degree 2 and F is homogeneous of degree 3;

3. *the surface has six (-2)-curves: the curves R_{ij}, with $i = 1,2,3$ and $j = 1,2$, such that $\pi(R_{i1}) = \pi(R_{i2}) = C_i$;*

4. *the Cox ring of X has 9 generators: $s_1, \dots, s_6$ defining the (-2)-curves and $s_7, s_8, s_9 \in H^0(\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$;*

5. *for a very general X as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \dots, T_9]/I, \;\; s_i \mapsto T_i,$$

   where the degrees of the generators T_i for $i = 1, \dots, 9$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 0 & -2 & -2 & 0 & -1 & -1 & -1 \\ 0 & -2 & 1 & -3 & 0 & -2 & -1 & -1 & -1 \end{pmatrix}$$

and the ideal I is generated by the following polynomials:

$$T_1T_4T_5 + T_2T_3T_6 - F(T_7, T_8, T_9),$$

$$T_1T_2 - F_1(T_7, T_8, T_9),\ T_3T_4 - F_2(T_7, T_8, T_9),\ T_5T_6 - F_3(T_7, T_8, T_9).$$

These results are contained in the paper [ACDL19]. Projective models for all families of Mori dream K3 surfaces of Picard number three have been recently given by Roulleau ([Rou20b], [Rou20a]).

In case the Picard number is four by Theorem 2.2.5 we know that there are 14 families of Mori dream K3 surfaces. For them, we obtain the following results.

Theorem 4. (Proposition 4.1.1) *Table 5.8 describes the extremal rays and the Hilbert bases of* $\mathrm{Eff}(X)$ *and* $\mathrm{Nef}(X)$ *for each of the* 14 *families of Mori dream K3 surfaces of Picard number four.*

Theorem 5. (Theorem 4.2.1) *Let X be a Mori dream K3 surface of Picard number four such that* $\mathrm{Cl}(X)$ *is not isometric to V_{14}. The degrees of a set of generators of the Cox ring $R(X)$ are given in Table 5.12. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.*

Moreover, we describe projective models for all Mori dream K3 surfaces of Picard number four and we geometrically identify the degrees of the generators of the Cox ring. For example, we prove the following result:

Proposition 2. (Proposition 4.3.6) *Let X be a K3 surface with* $\mathrm{Cl}(X) \cong V_6 = U(3) \oplus A_1 \oplus A_1$. *Then*

1. *there is a double cover* $\pi : X \to \mathbb{P}^2$ *branched along a smooth plane sextic with two* 3*-tangent lines* L_1, L_2 *and two* 6*-tangent conics* C_1, C_2*;*

2. X *can be defined by an equation of the following form in* $\mathbb{P}(1,1,1,3)$*:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)G_1(x_0,x_1,x_2)G_2(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   where $F \in \mathbb{C}[x_0,x_1,x_2]$ *is homogeneous of degree three,* $G_1, G_2 \in \mathbb{C}[x_0,x_1,x_2]$ *are homogeneous of degree two and* $F_1, F_2 \in \mathbb{C}[x_0,x_1,x_2]$ *are homogeneous of degree one;*

3. *the surface has eight* (-2)*-curves: the four curves* R_{ij}*,* $i,j = 1,2$ *such that* $\pi(R_{i1}) = \pi(R_{i2}) = L_i$ *and the four curves* S_{i1}, S_{i2} *such that* $\pi(S_{i1}) = \pi(S_{i2}) = C_i$ *for* $i = 1,2$*;*

4. *the Cox ring of* X *has* 9 *generators:* $s_1,\dots,s_8$ *defining the* (-2)*-curves and* $s_9 \in H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$*;*

5. *for a very general* X *as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1,\dots,T_9]/I,\ s_i \mapsto T_i,$$

   where the degrees of the generators T_i *for* $i = 1,\dots,9$ *are given by the columns of the following matrix*

$$\begin{pmatrix} 0 & -2 & 0 & 0 & -1 & -2 & 0 & -1 & -1 \\ 0 & -2 & -1 & 0 & 0 & -2 & -1 & 0 & -1 \\ 0 & -3 & 0 & 1 & 0 & -2 & -1 & -1 & -1 \\ 1 & -2 & -1 & 0 & -1 & -3 & 0 & 0 & -1 \end{pmatrix}$$

and the ideal I is generated by the following polynomials:

$$T_1T_6 - G_1(T_3T_8, T_5T_7, T_9),$$
$$T_2T_4 - G_2(T_3T_8, T_5T_7, T_9),$$
$$T_1T_2T_3T_5 + T_4T_6T_7T_8 - F(T_3T_8, T_5T_7, T_9).$$

These results are contained in the paper [ACDR20].

The proofs of Theorem 2 and Theorem 5 are obtained by means of three steps. First, we compute the effective and nef cones of the surfaces using a computational program implemented in Magma [BCP97]. To our knowledge, in case the Picard number is four, the set of (-2)-curves of the surfaces (whose classes generate the effective cone) were not known. Secondly, again with the help of several Magma programs, we identify the degrees of a set of generators of $R(X)$ for each family. These programs rely on Theorem 1, Proposition 1 and other techniques which allow to prove that the Cox ring does not need generators in certain degrees. Finally, we prove a result which allows to show that $R(X)$ necessarily has a generator in a certain degree. This result, implemented in Magma, allows to prove that the set of degrees found in the second step is minimal in several cases.

The thesis is organized as follows. In Chapter 1 we recall some concepts and results about projective varieties over the complex numbers which are necessary to subsequently introduce K3 surfaces and Cox rings. The first three sections contain basic definitions and properties of divisors and linear systems, projective surfaces and curves on them, and lattices. Finally, section four deals with K3 surfaces and some of their fundamental properties. In particular, we recall classical results on linear systems on K3 surfaces following [SD74].

Chapter 2 contains preliminaries about Cox rings and Mori dream spaces, focusing on the case of surfaces. Section 2.2 is about Cox rings of surfaces and their classification. In particular, we recall the characterization of Mori dream K3 surfaces [AHL10, Theorem 1] and their classification [ADHL15, Theorem 5.1.5.3]. In Section 2.3 we prove Proposition 1 and, in Section 2.4, we apply it to prove Theorem 1. Moreover, we provide further results showing that $R(X)$ does not need generators in certain special degrees (Lemma 2.4.5, Lemma 2.4.6). Finally, we prove a result showing that the Cox ring necessarily has a generator in a certain degree, once the degrees of a set of generators is known (see Proposition 2.4.9).

Chapter 3 deals with families of Mori dream K3 surfaces with Picard number 3: we compute their effective and nef cones and we prove Theorem 2. In the last section we determine a presentation of $R(X)$ for K3 surfaces with $\mathrm{Cl}(X)$ isometric to either S_1 or $S_{4,1,1}$.

In Chapter 4 we study Mori dream K3 surfaces of Picard number four: we compute their effective and nef cones and we prove Theorem 4. Moreover, in the last section we provide a projective model for each family of Mori dream K3 surfaces with Picard number 4.

Chapter 5 contains the tables with the relevant information about Mori dream K3 surfaces of Picard number three and four: effective cones, nef cones, intersection matrix of (-2)-curves and the intersection properties of a nef and big divisor with minimum self-intersection. Finally, we give the tables containing the degrees of a set of generators of the Cox ring.

In Chapter 6 we briefly present and include the Magma [BCP97] programs used for the proofs of Theorem 3.2.1 and Theorem 4.2.1.

Introducción

El anillo de Cox de una variedad proyectiva normal X definida sobre los números complejos con grupo de clases de divisores $\mathrm{Cl}(X)$ libre y finitamente generado es el álgebra graduada [ADHL15]

$$R(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)).$$

La variedad X es llamada espacio Mori dream cuando el anillo de Cox es finitamente generado (Definición 2.1.3). Importantes ejemplos de espacios Mori dream son variedades tóricas, cuyo anillo de Cox es un anillo de polinomios y su multigradación se puede determinar explícitamente en términos de su fan, ver [Cox95]. Otro importante ejemplo son las variedades de log Fano [BCHM10]. En este contexto hay dos problemas principales: determinar condiciones en X tales que el anillo de Cox $R(X)$ sea finitamente generado, y encontrar una presentación explícita para $R(X)$, es decir determinar generadores y relaciones para el anillo de Cox. Una propiedad fundamental de los espacios de Mori dream es que tal variedad es un cociente GIT de un subconjunto abierto de Zariski de un espacio afín, el espectro de $R(X)$, por la acción de un quasitoro. Esto permite definir coordenadas homogéneas en X, como en el caso del espacio proyectivo, y permite una aproximación a ciertas propiedades geométricas y aritméticas de X, como en el caso de las variedades tóricas [ADHL15].

El problema de encontrar una presentación para anillo de Cox de un espacio de Mori dream es interesante y difícil. Existen varias técnicas diferentes para esto, las cuales han permitido calcular el anillo de Cox de varias clases de variedades especiales. Un trabajo pionero en esta dirección fue el trabajo [BP04] de Batyrev y Popov, quienes identificaron los generadores del anillo Cox de cualquier superficie del Pezzo, mostrando en particular que es generado por los elementos que definen las (-1)-curvas de la superficie si el rango r de su grupo de clases de divisores satisface $4 \leq r \leq 8$. El ideal de las relaciones del anillo de Cox de las superficies del Pezzo se ha calculado en varios pasos en [STM07, SS07, LM09, TVAV09]. En [CT06] los autores determinaron los generadores del anillo Cox de la explosión de $\mathbb{P}^n$ en cualquier número de puntos que se encuentran en una curva normal racional.

En general, hay técnicas para calcular anillos Cox de clases especiales de variedades (por ejemplo: las variedades con acción de un toro, espacios homogéneos, variedades esféricas) y para relacionar los anillos de Cox de dos variedades X, Y (por ejemplo: Y incrustado en X que satisface las condiciones adecuadas o Y una explosión de X a lo largo de una subvariedad irreducible contenida en el locus suave), ver [ADHL15, Cap.4] y [HKL16].

Esta tesis es sobre anillos de Cox de las superficies K3 Mori dream, esto es, las superficies proyectivas suaves X con $H^1(X, \mathcal{O}_X) = \{0\}$ y clase canónica trivial (Definición 1.4.1) cuyo anillo de Cox es finitamente generado. La investigación sobre los anillos de Cox de las superficies K3 comienza en el 2009 con el trabajo de Artebani, Hausen y Laface [AHL10] y McKernan [McK10], donde los autores probaron independientemente que el anillo de Cox de una superficie K3 es finitamente generado si y sólo si su cono efectivo es poliedral, o equivalente si su grupo de automorfismo es finito. Las superficies K3 con esta propiedad han sido clasificadas en una serie de trabajos clásicos [Nik79, Nik84, Nik00, PŠŠ71, Vin07] (ver también [ADHL15, §5.1.5]).

Para superficies con número de Picard ≥ 3 hay un número finito de familias con dicha propiedad (ver Teorema 2.2.5). Además, en [AHL10] los autores investigan generadores y relaciones para el anillo de Cox de dichas superficies, determinando generadores explícitos para ciertas clases de ellos. Hacia el 2012, Ottem [Ott13] también estudia los generadores y las relaciones del anillo de Cox de las superficies K3 con número de Picard dos, en particular él describe el anillo de Cox de algunos ejemplos clásicos, tales como las superficies cuárticas que contienen una línea. Cabe notar que antes que ellos, Saint-Donat [SD74] investigó los generadores y las relaciones del anillo de coordenadas $\oplus_{n\leq 0} H^0(X, \mathcal{O}_X(nH))$, para X una superficie K3 y H un divisor muy amplio en X.

El objetivo principal de esta tesis es desarrollar técnicas y herramientas computacionales para calcular Anillos de Cox de superficies K3. Luego, aplicaremos estas técnicas para encontrar los grados de un conjunto generador de anillo de Cox de superficies K3 Mori dream con número de Picard 3 y 4. Como primer paso, ampliamos las técnicas conocidas para encontrar los grados de generadores del anillo de Cox. Un primer resultado en esta dirección es el siguiente, que se basa en sucesiones exactas de tipo Koszul.

Proposición 1. (Corolario 2.3.4) *Sea X una variedad proyectiva compleja suave, sean E_1, E_2, E_3 divisores efectivos de X y $f_i \in H^0(X, E_i)$, para $i = 1, 2, 3$, tales que $\cap_{i=1}^3 \operatorname{div}_{E_i}(f_i) = \emptyset$. Si $D \in \operatorname{WDiv}(X)$ entonces el morfismo*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D),\ (g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3,$$

es sobreyectivo si $h^1(X, D - E_i - E_j) = 0$ para todos distintos $i, j \in \{1, 2, 3\}$ y $h^2(X, D - E_1 - E_2 - E_3) = 0$.

Observe que la sobreyectividad del morfismo en la Proposición 1 implica que $R(X)$ no necesita un generador en grado $[D]$, ya que todos los elementos de $H^0(X, D)$ pueden obtenerse como polinomios en elementos homogéneos de otros grados. Esto permite probar el siguiente teorema general para los anillos de Cox de superficies K3 (no necesariamente Mori dream).

Teorema 1. (Teorema 2.4.2) *Sea X una superficie K3 proyectiva suave sobre $\mathbb{C}$. Entonces los grados de un conjunto minimal de generadores del anillo de Cox $R(X)$ son:*

(*i*) *clases de (-2)-curvas,*

(*ii*) *clases de divisores nef los cuales son suma de a lo más tres elementos de la base de Hilbert del cono nef (permitiendo repeticiones),*

(*iii*) *o clases de divisores de la forma $2(F + F')$ donde F, F' son curvas elípticas suaves con $F \cdot F' = 2$.*

Posteriormente, aplicamos el Teorema 1, la Proposición 1 y otras técnicas basadas en sucesiones exactas de Koszul para determinar los grados de un conjunto de generadores del anillo de Cox de elementos generales de las familias de superficies K3 Mori dream de número de Picard tres y cuatro.

En el caso de que el número de Picard sea tres, por Teorema 2.2.5 sabemos que hay 26 familias de superficies K3 Mori dream. Para esas superficies, obtenemos el siguiente resultado.

Teorema 2. (Teorema 3.2.1) *Sea X una superficie K3 Mori dream con número de Picard 3. Los grados de un conjunto de generadores del anillo de Cox $R(X)$ es* *dado en la Tabla 5.6. Todos los grados en la tabla son necesarios para generar $R(X)$, excepto posiblemente aquellos marcados con una estrella.*

De la Tabla 5.6 se puede ver que hay seis casos en los que el anillo de Cox se genera con siete grados. En la Sección 3.3 describimos la geometría de dos de estás familias, a saber las familias con $\mathrm{Cl}(X)$ isométrico a los reticulados S_1 y $S_{4,1,1}$ (ver Teorema 2.2.5), además proporcionamos una presentación para el anillo Cox de dichas familias. Por ejemplo, en el caso de la familia con reticulado isométrico a S_1, obtenemos el siguiente resultado.

Teorema 3. (Teorema 3.3.2) *Sea X una superficie K3 con $\mathrm{Cl}(X) \cong S_1 = (6) \oplus A_1^2$. Entonces*

1. *existe un cubrimiento doble $\pi : X \to \mathbb{P}^2$ ramificado a lo largo de una séxtica suave con tres cónicas 6-tangentes C_1, C_2, C_3;*

2. *X se puede definir por una ecuación en $\mathbb{P}(1,1,1,3)$ de la forma:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)F_3(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   donde F_1, F_2, F_3 son polinomios homogéneos de grado 2 y F es un polinomio homogéneo de grado 3;

3. *la superficie tiene seis (-2)-curvas: las curvas R_{ij} con $i = 1,2,3$ y $j = 1,2$ tales que $\pi(R_{i1}) = \pi(R_{i2}) = C_i$;*

4. *el anillo de Cox de X tiene 9 generadores: $s_1, \ldots, s_6$ que definen las (-2)-curvas y $s_7, s_8, s_9 \in H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$;*

5. *para una superficie muy general X como la anterior tenemos el isomorfismo*

$$R(X) \to \mathbb{C}[T_1, \dots, T_9]/I, \;\; s_i \mapsto T_i,$$

donde los grados de los generadores T_i para $i = 1, \dots, 9$ son dados por las columnas de la siguiente matriz

$$\begin{pmatrix} 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 0 & -2 & -2 & 0 & -1 & -1 & -1 \\ 0 & -2 & 1 & -3 & 0 & -2 & -1 & -1 & -1 \end{pmatrix}$$

y el ideal I es generado por los siguientes polinomios:

$$T_1T_4T_5 + T_2T_3T_6 - F(T_7, T_8, T_9),$$

$$T_1T_2 - F_1(T_7, T_8, T_9), \;\; T_3T_4 - F_2(T_7, T_8, T_9), \;\; T_5T_6 - F_3(T_7, T_8, T_9).$$

Estos resultados están contenidos en el artículo [ACDL19]. Los modelos proyectivos para todas las familias de las superficies K3 Mori dream con número de Picard tres, ha sido presentado recientemente por Roulleau ([Rou20b], [Rou20a]).

En el caso de número de Picard cuatro, por Teorema 2.2.5 sabemos que hay 14 familias de superficies K3 Mori dream. Para ellas, obtenemos los siguientes resultados.

Teorema 4. (Proposición 4.1.1) *La Tabla 5.8 describe los rayos extremales y las bases de Hilbert de los conos* $\mathrm{Eff}(X)$ *y* $\mathrm{Nef}(X)$ *para cada una de las 14 familias de superficies K3 Mori dream de número de Picard cuatro.*

Teorema 5. (Teorema 4.2.1) *Sea X una superficie K3 Mori dream de número de*

Picard cuatro, tal que $\mathrm{Cl}(X)$ *no es isométrico al reticulado* V_{14}*. Los grados de un conjunto de generadores del anillo de Cox* $R(X)$ *son dados en la Tabla 5.12. Todos los grados en la tabla son necesarios para generar* $R(X)$*, excepto posiblemente aquellos marcado con una estrella.*

Además, describimos los modelos proyectivos para todas las superficies K3 Mori dream con número de Picard 4 e identificamos geométricamente los grados de los generadores del anillo de Cox. Por ejemplo, demostramos el siguiente resultado para una de las familia de superficies K3 con $\mathrm{Cl}(X) \cong V_6$:

Proposición 2. (Proposición 4.3.6) *Sea* X *una superficie K3 con* $\mathrm{Cl}(X) \cong V_6 = U(3) \oplus A_1 \oplus A_1$*. Entonces*

1. *existe un cubrimiento doble* $\pi : X \to \mathbb{P}^2$ *ramificado a lo largo de una séxtica suave plana con dos rectas* 3*-tangentes* L_1, L_2 *y dos cónicas* 6*-tangentes* C_1, C_2*;*

2. *un elemento general de* X *se define por una ecuación en* $\mathbb{P}(1,1,1,3)$*:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)G_1(x_0,x_1,x_2)G_2(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   donde $F \in \mathbb{C}[x_0,x_1,x_2]$ *es homogéneo de grado tres,* $G_1, G_2 \in \mathbb{C}[x_0,x_1,x_2]$ *son homogéneos de grado dos y* $F_1, F_2 \in \mathbb{C}[x_0,x_1,x_2]$ *son homogéneos de grado uno;*

3. *la superficie tiene ocho* (-2)*-curvas: las cuatro curvas* R_{ij}*,* $i,j = 1,2$ *tales que* $\pi(R_{i1}) = \pi(R_{i2}) = L_i$ *y las cuatro curvas* S_{i1}, S_{i2} *tales que* $\pi(S_{i1}) = \pi(S_{i2}) = C_i$ *para* $i = 1,2$*;*

4. *el anillo de Cox de* X *tiene* 9 *generadores:* $s_1, \ldots, s_8$ *definiendo las* (-2)*-curvas y* $s_9 \in H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$*;*

5. *el anillo de Cox de una superficie general X como la descrita anteriormente, cumple con el isomorfismo*

$$R(X) \to \mathbb{C}[T_1, \dots, T_9]/I, \ s_i \mapsto T_i,$$

donde los grados de los generadores T_i para $i = 1, \dots, 9$ son dados por las columnas de la siguiente matriz

$$\begin{pmatrix} 0 & -2 & 0 & 0 & -1 & -2 & 0 & -1 & -1 \\ 0 & -2 & -1 & 0 & 0 & -2 & -1 & 0 & -1 \\ 0 & -3 & 0 & 1 & 0 & -2 & -1 & -1 & -1 \\ 1 & -2 & -1 & 0 & -1 & -3 & 0 & 0 & -1 \end{pmatrix}$$

y el ideal I es generado por los siguientes polinomios:

$$T_1T_6 - G_1(T_3T_8, T_5T_7, T_9),$$
$$T_2T_4 - G_2(T_3T_8, T_5T_7, T_9),$$
$$T_1T_2T_3T_5 + T_4T_6T_7T_8 - F(T_3T_8, T_5T_7, T_9).$$

Estos resultados están contenidos en el artículo [ACDR20].

Las demostraciones del Teorema 2 y el Teorema 5 se obtienen mediante tres pasos. Primero, calculamos los conos efectivos y nef de las superficies usando un programa computacional implementado en Magma [BCP97]. Hasta donde sabemos, en el caso de que el número de Picard sea cuatro, el conjunto de las (-2)-curvas de las superficies (cuyas clases generan el cono efectivo) no se conocían. En segundo lugar, nuevamente con la ayuda de varios programas de Magma, identificamos los grados de un conjunto de generadores de $R(X)$ para cada familia.

Estos programas se basan en el Teorema 1, la Proposición 1 y otras técnicas que permiten probar que el anillo de Cox no necesita generadores en ciertos grados. Finalmente, probamos un resultado que permite mostrar que $R(X)$ necesariamente tiene un generador en cierto grado. Este resultado, implementado en Magma, permite probar que el conjunto de grados encontrados en el segundo paso es mínimo en varios casos.

La tesis está organizada de la siguiente manera.

En el Capítulo 1 recordamos algunos conceptos y resultados sobre variedades proyectivas sobre los números complejos, los cuales son necesarios para introducir posteriormente las superficies K3 y los anillos de Cox. Las primeras tres secciones contienen definiciones básicas y propiedades de divisores y sistemas lineales, superficies proyectivas y curvas sobre ellas, y reticulados. Finalmente, la sección 1.4 trata sobre las superficies K3 y algunas de sus propiedades fundamentales. En particular, recordamos resultados clásicos de sistemas lineales en superficies K3 siguiendo el trabajo [SD74].

El Capítulo 2 contiene preliminares sobre anillos Cox y espacios de Mori dream, centrándonos en el caso de las superficies. La Sección 2.2, se trata sobre anillos de Cox de superficies y su clasificación. En particular, recordamos la caracterización de superficies K3 Mori dream [AHL10, Teorema 1] y su clasificación [ADHL15, Teorema 5.1.5.3]. En la Sección 2.3 demostramos la proposición 1 y, en la Sección 2.4, aplicamos esto para demostar el Teorema 1. Además, proporcionamos resultados adicionales que muestran que $R(X)$ no necesita generadores en ciertos grados especiales (Lemma 2.4.5, Lemma 2.4.6). Finalmente, demostramos un resultado que muestra que el anillo de Cox necesariamente tiene un generador en cierto grado, una vez que se conocen los grados de un conjunto de generadores (ver Proposición 2.4.9).

El Capítulo 3 se trata de las familias de superficies K3 Mori dream con número de Picard 3: calculamos sus conos efectivos y nef, y demostramos el Teorema 2.

En la última sección determinamos una presentación de $R(X)$ para superficies K3 con $\mathrm{Cl}(X)$ isométrico a S_1 o $S_{4,1,1}$.

En el Capítulo 4 estudiamos las superficies K3 Mori dream con número de Picard cuatro: calculamos sus conos efectivos y nef, y probamos el Teorema 4. Además, en la última sección proporcionamos un modelo proyectivo para cada familia de superficies K3 Mori dream con número de Picard 4.

El Capítulo 5 contiene las tablas con la información relevante acerca de las superficies K3 Mori dream con número de Picard tres y cuatro: conos efectivos, conos nef, matrices de intersección de (-2)-curvas y las propiedades de intersección de un divisor nef y grande con mínima auto-intersección. Finalmente, damos las tablas que contienen los grados de un conjunto de generadores del anillo de Cox.

En el Capítulo 6 presentamos brevemente e incluimos los programas de Magma [BCP97] que usamos para las demostraciones del Teorema 3.2.1 y el Teorema 4.2.1.

Chapter 1

Projective varieties and morphisms

1.1 Divisors and linear systems

In this section X will denote a normal projective variety over the field of complex numbers.

Definition 1.1.1. A *Weil divisor* D of X is a formal finite sum $D = \sum_{i=1}^{n} a_i Y_i$, where $a_i \in \mathbb{Z}$ and the Y_i's are irreducible closed hypersurfaces of X. The *support* of D is the union of the hypersurfaces Y_i such that $a_i \neq 0$. The divisor D is *effective* if $a_i \geq 0$ for all $i = 1, \ldots, n$.

The set of Weil divisors with the sum is a free abelian group which is denoted by $\mathrm{Div}(X)$. It contains the subgroup $\mathrm{PDiv}(X)$, whose elements are the principal divisors, i.e. divisors of the form $\mathrm{div}(f)$, where f is a non-zero rational function on X and $\mathrm{div}(f)$ denotes the associated divisor of zeroes and poles (see [Har77, Chapter II, §6]).

Definition 1.1.2. Two divisors D and D' of X are *linearly equivalent*, denoted by $D \sim D'$, if $D' - D$ is principal.

Definition 1.1.3. The *divisor class group* of X is the quotient group

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/\mathrm{PDiv}(X).$$

We will call its elements *classes* and we will denote by $[D] \in \mathrm{Cl}(X)$ the class of a divisor D.

In case X is locally factorial, i.e. all its local rings are UFD, it can be proved that any Weil divisor is locally principal, which means that there exists an open covering $\{U_i\}_{i\in I}$ of X such that $D_{|U_i}$ is principal for all $i \in I$. In particular this holds if X is smooth. Weil divisors which are locally principal are called *Cartier divisors*. We now recall the relation between the divisor class group and the Picard group of the variety.

Definition 1.1.4. The *Picard group* of X is set of isomorphism classes of invertible sheaves on X equipped with the tensor product.

Given any Weil divisor D on X we can associate to it the sheaf $\mathcal{O}_X(D)$ defined by

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* : (\mathrm{div}(f) + D)_{|U} \geq 0\} \cup \{0\},$$

where $U \subseteq X$ is an open subset and $\mathbb{C}(X)^*$ denotes the group of non-zero rational functions of X. If D is a Cartier divisor then $\mathcal{O}_X(D)$ is an invertible sheaf, since locally a section of $\mathcal{O}_X(D)$ is of the form $\frac{g}{f}$, where $g \in \mathcal{O}_X$ and $\mathrm{div}(f) = D$. This defines a map between the group of Cartier divisors modulo linear equivalence and $\mathrm{Pic}(X)$ which can be proved to be an isomorphism [Har77, Proposition 6.15, Chapter II]. By the previous remark, if X is locally factorial, then there is an isomorphisms $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$. We recall that the Picard group of X is also isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^*)$ [Har77, Exercise 4.5, Chapter III].

In what follows, given any sheaf $\mathcal{F}$ on X we will denote by

- $H^i(X,\mathcal{F})$ the *i-th cohomology group* of the sheaf $\mathcal{F}$, and by $h^i(X,\mathcal{F})$ its dimension as a complex vector space; if $D \in \mathrm{Div}(X)$ we will also use the short-hand notations $H^i(X,D) = H^i(X,\mathcal{O}_X(D))$ and $h^i(X,D) = h^i(X,\mathcal{O}_X(D))$;

- $\chi(\mathcal{F}) := \sum_{i=0}^{\dim(X)} h^i(X,\mathcal{F})$, the *Euler-Poincaré characteristic* of the sheaf $\mathcal{F}$.

A very important exact sequence of sheaves is the following [Har77, §5, Appendix B].

Theorem 1.1.5. *(The exponential sequence) Let X be a compact complex manifold X. The exponential map $e : \mathcal{O}_X \to \mathcal{O}_X^*$ given by $e(f) := e^{2\pi i f}$ is locally surjective and its kernel consists of locally constant integer-valued functions, i.e. there is the following exact sequence:*

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{e} \mathcal{O}_X^* \longrightarrow 1,$$

where $\mathcal{O}_X$ and $\mathcal{O}_X^*$ denote the sheaves of holomorphic functions and of holomorphic invertible functions on X respectively.

The previous short exact sequence gives rise to a long exact cohomology sequence:

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathbb{Z}_X) \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X^*) \longrightarrow H^1(X,\mathbb{Z}_X) \longrightarrow \\ \longrightarrow H^1(X,\mathcal{O}_X) \xrightarrow{\mathrm{e}} H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathbb{Z}_X) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow ... \end{aligned}$$

Since X is compact, the analogous of Liouville's Theorem implies that the only global holomorphic functions on X are constant, so that $H^0(X,\mathcal{O}_X) \cong \mathbb{C}$ and $H^0(X,\mathcal{O}_X^*) \cong \mathbb{C}^*$. Thus we also have the exact sequence:

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \xrightarrow{e} H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X). \qquad (1.1)$$

By the exactness of the sequence, the image of the above exponential map is isomorphic to the quotient of $H^1(X, \mathcal{O}_X)$ by the image of the subgroup $H^1(X, \mathbb{Z})$. The group $T = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ is a divisible group, and Hodge theory shows that $H^1(X, \mathbb{Z})$ is a lattice in $H^1(X, \mathcal{O}_X)$, so T has a natural structure of complex torus. The image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$ is the *Néron-Severi group* of X, denoted by $\mathrm{NS}(X)$, it is a finitely generated group.

Then, we have the short exact sequence

$$0 \longrightarrow T \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

In particular, observe that $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$ if and only if $q(X) := h^1(X, \mathcal{O}_X) = 0$.

We now recall how to associate a rational map to the linear system of a divisor and the definition of certain cones of divisors. Given a Weil divisor D we recall that the map

$$H^0(X, \mathcal{O}_X(D)) - \{0\} \to \mathrm{Div}(X), \quad f \mapsto \mathrm{div}(f) + D$$

defines a bijection between $\mathbb{P}(H^0(X, \mathcal{O}_X(D))$ and the set of effective divisors linearly equivalent to D, which is denoted by $|D|$ and called *complete linear system associated to* D. The *base locus* $\mathrm{Bs}(D)$ is the set of points $p \in X$ which belong to the support of any divisor in $|D|$. The linear system $|D|$ is *base point free* if its base locus is empty. Chosen a basis $f_0, \dots, f_N$ of $H^0(X, \mathcal{O}_X(D))$ one can define a map

$$\varphi_{|D|} : X \backslash \mathrm{Bs}(D) \to \mathbb{P}^N, \ p \mapsto [f_0(p) : \cdots : f_N(p)].$$

If $|D|$ is base point free, this map defines a morphism $\varphi_{|D|} : X \to \mathbb{P}^N$. We also define the *stable base locus* of D as the intersection of the base loci of mD for $m \in \mathbb{Z}, m \geq 1$.

In the following statement we will denote by $D \cdot C$ the intersection number between a divisor and a curve (see [Laz04, §1.1.C] and §1.2 for a definition in the case of surfaces).

Definition 1.1.6. Let X be a smooth projective variety over the field of complex numbers such that $q(X) = h^1(X, \mathcal{O}_X) = 0$.

(i) The *effective cone* $\mathrm{Eff}(X)$ is the cone generated by classes of effective divisors in $\mathrm{Cl}(X)_{\mathbb{Q}} := \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(ii) A divisor $D \in \mathrm{Div}(X)$ is *ample* if there exists a positive integer m such that $\varphi_{|mD|}$ is an embedding. The *ample cone* $\mathrm{Ample}(X)$ is the cone generated by classes of ample divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.

(iii) A divisor $D \in \mathrm{Div}(X)$ is *nef* if $D \cdot C \geq 0$ for any curve C of X. The *nef cone* $\mathrm{Nef}(X)$ is the cone generated by classes of nef divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.

(iv) A divisor $D \in \mathrm{Div}(X)$ is *big* if there exists a positive integer m such that the image of the rational map $\varphi_{|mD|}$ has dimension $\dim(X)$.

(v) A divisor $D \in \mathrm{Div}(X)$ is *semiample* if there exists a positive integer m such that $|mD|$ is base point free. The *semiample cone* $\mathrm{SAmple}(X)$ is the cone generated by classes of semiample divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.

(vi) A divisor $D \in \mathrm{Div}(X)$ is *movable* if its stable base locus has codimension at least two in X. The *movable cone* $\mathrm{Mov}(X)$ is the cone generated by classes of movable divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.

By the Nakai-Moishezon-Kleiman criterion [Laz04, Theorem 1.2.23] and Kleiman's theorem [Laz04, Theorem 1.4.23] the nef cone is the closure of the ample cone and the ample cone is clearly contained in the effective cone. Moreover semiample divisors are nef. Thus

$$\mathrm{Ample}(X) \subseteq \mathrm{SAmple}(X) \subseteq \mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}(X)}.$$

Moreover the following inclusions clearly hold:

$$\mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}(X).$$

In case X is smooth, a distinguished class in $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$ is the following.

Definition 1.1.7. Let X be a smooth projective variety over the complex numbers of dimension n. The *canonical sheaf* of X is $\omega_X = \wedge^n \Omega_X$, where Ω_X denotes the sheaf of rational forms of X. It is an invertible sheaf. A *canonical divisor* of X is any Cartier divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$.

Finally, we recall a well known vanishing theorem [Laz04, Theorem 4.3.1].

Theorem 1.1.8 (Kawamata-Viehweg vanishing theorem)**.** *Let X be a smooth projective variety of dimension n over the complex numbers and let D be a nef and big divisor on X. Then $h^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

1.2 Algebraic Surfaces

In this section we will give some necessary preliminaries on complex surfaces, as a reference we suggest [Bea96, Chapter 1] and [Har77, Chapter 5]. In this thesis by *surface* we mean a normal projective variety of dimension two over the complex numbers and a *curve* on it will be an irreducible closed subvariety of dimension one.

The divisor class group of a smooth surface always carries a symmetric bilinear pairing, called *intersection form*, which can be defined as follows (see [Har77, Chapter

5, pag. 368]).

Theorem 1.2.1. *Let X be a smooth surface. There is a unique pairing*

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}, \;\; (C,\; C') \mapsto C \cdot C'$$

such that

(*i*) *if C and C' are smooth curves meeting transversally, then $C \cdot C' = \#(C \cap C')$,*

(*ii*) *it is symmetric: $C \cdot C' = C' \cdot C$,*

(*iii*) *it is additive: $(C_1 + C_2) \cdot C' = C_1 \cdot C' + C_2 \cdot C'$,*

(*iv*) *it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1 \cdot C' = C_2 \cdot C'$.*

By property (*iv*), this defines a symmetric bilinear form on $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$. The bilinear form is actually given by (see [Bea96, Chapter 1]):

$$C \cdot C' := \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-C')) + \chi(\mathcal{O}_X(-C - C')).$$

If C and C' are two distinct curves on X we have

$$C \cdot C' = \dim H^0(X, \mathcal{O}_{C \cap C'}).$$

Moreover, if $C \subseteq X$ is a smooth curve and C' is any divisor, then

$$(C \cdot C') = \deg \mathcal{O}_C(C').$$

We recall that a divisor D is nef if $D \cdot C \geq 0$ for all curves C on X. By the Nakai-Moishezon-Kleiman criterion [Har77, Theorem 1.10, Chapter V], a divisor D

on a surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all curves C on X. We observe that the nef cone is the dual of the effective cone with respect to the intersection form of X. Moreover, a nef divisor on a surface is big if and only if $D^2 > 0$ ([Laz04, Theorem 2.2.16]).

The following are fundamental formulas which give the Euler-Poincaré characteristic of an invertible sheaf in terms of intersection properties [Har77, Theorem 1.6 and Remark 1.6.1, Chapter V]. In both results X denotes a smooth surface.

Theorem 1.2.2 (Riemann-Roch theorem)**.** *Let $D \in \mathrm{Div}(X)$, then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D^2 - D \cdot K_X) + \chi(\mathcal{O}_X).$$

Theorem 1.2.3 (Noether's formula)**.**

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)),$$

where $e(X) = \sum_{i=0}^{4}(-1)^i \mathrm{rk} H^i(X, \mathbb{Z})$ denotes the topological Euler characteristic of X.

Finally, we recall the following formula which relates the canonical divisor of a surface to that of a curve in it [Har77, Proposition 1.5, Chapter V].

Theorem 1.2.4 (Adjunction formula)**.** *If C is a smooth curve on a smooth surface X, then a canonical divisor for C is given by $K_C = (K_X + C)|_C$. In particular $\deg(K_C) = 2g(C) - 2 = C^2 + C \cdot K_X$.*

1.3 Lattices

Definition 1.3.1. A *lattice* is a finitely generated and free abelian group L equipped with a symmetric bilinear form $b : L \times L \to \mathbb{Z}$. We will denote by q the associated quadratic form. The lattice L is *even* if $q(x) = b(x, x) \in 2\mathbb{Z}$ for any $x \in L$.

Basic invariants of a lattice L are its *rank*, defined as the dimension of the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$, and its *signature*, defined to be the triple of numbers of positive, negative and zero eigenvalues of the extension $q_{\mathbb{R}}$ of the quadratic form q to $L \otimes_{\mathbb{Z}} \mathbb{R}$. The lattice is called positive definite, negative definite, indefinite or non-degenerate if the same holds for the quadratic form $q_{\mathbb{R}}$.

Definition 1.3.2. Let L_1 and L_2 be lattices. An *isometry* between them is a bijective homomorphism of abelian groups $\phi : L_1 \to L_2$ such that

$$q_2(\phi(x), \phi(y)) = q_1(x, y), \text{ for any } x, y \in L_1,$$

where q_i is the quadratic form of L_i, $i = 1, 2$.

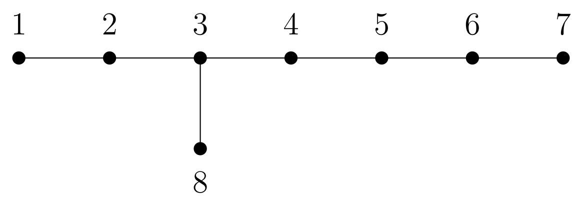
A lattice is usually represented by means of the matrix of its bilinear form with respect to a basis, that matrix is the *Gram matrix*. Recall that a lattice is *unimodular* if the determinant of a Gram matrix of q with respect to a basis is ± 1.

Example 1.3.3. Let U be the lattice associated to the Gram matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The lattice U has rank 2, is unimodular and hyperbolic, i.e. of signature $(1, 1)$.

Example 1.3.4. To any Dynkin diagram of type A_m, D_n, E_6, E_7 or E_8 ($m \geq 1$, $n \geq 4$) one can associate a negative definite lattice in the following way: each vertex of the diagram gives a generator v_i of the lattice with $b(v_i, v_j) = -2$ if $i = j$, $b(v_i, v_j) = 1$ if $i \neq j$ and v_i, v_j are joined by an edge, and $b(v_i, v_j) = 0$ otherwise. For example, the lattice E_8 is the negative definite unimodular lattice of rank eight associated to the following Dynkin diagram



and its Gram matrix is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Given two lattices L_1 and L_2, we denote by $L_1 \oplus L_2$ their direct sum. This is a lattice with respect to the product $(x_1, x_2) \cdot (y_1, y_2) := x_1 \cdot y_1 + x_2 \cdot y_2$, with $x_i, y_i \in L_i$, $i = 1, 2$. Moreover, we denote by L^n the direct sum of n copies of L, and by $L(m)$ the lattice obtained multiplying by m all entries of the Gram matrix of L.

We recall the following theorem by J. Milnor [Mil58].

Theorem 1.3.5 (Milnor's Theorem). *Let L be an indefinite unimodular lattice. If L is even, then $L \cong E_8(\pm 1)^m \oplus U^n$ for some m and n integers. If L is odd, then $L \cong (1)^m \oplus (-1)^n$ for some m and n integers.*

1.4 K3 surfaces

In this section we will recall the definition of K3 surface and some of its basic properties, as a reference we followed [Huy16, Chapter 1 and Chapter 2].

Definition 1.4.1. A *K3 surface* is a smooth surface X such that $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = \{0\}$.

Example 1.4.2. A smooth quartic hypersurface $X \subset \mathbb{P}^3$ is a $K3$ surface (see [Huy16, Example 1.3 (i), Chapter 1]). If X has simple surface singularities, then its minimal resolution is a K3 surface, see [BHPVdV04, Chapter II, §8].

Example 1.4.3. The double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth curve $C \subset \mathbb{P}^2$ of degree six is a K3 surface (see [Huy16, Example 1.3 (iv), Chapter 1]).

If $\varphi : Y \to \mathbb{P}^2$ is a double cover branched along a curve $B \subset \mathbb{P}^2$ with ADE singularities then Y has simple surface singularities and its minimal resolution X is a K3 surface. For more details see [BHPVdV04, Chapter III, §7].

Remarks 1.4.4. Let X be a K3 surface, we have that:

(i) The vector space of holomorphic 2-forms on X is one dimensional.

In fact, since $K_X \sim 0$, there exists a meromorphic 2-form ω of X such that $\operatorname{div}(\omega) = 0$, i.e. ω is holomorphic and has no zeros. If ω' is any holomorphic 2-form, then $\operatorname{div}(\omega'/\omega) = \operatorname{div}(\omega') - \operatorname{div}(\omega) = \operatorname{div}(f)$, where f is a global holomorphic function on X. Since X is compact, f is a constant, thus ω' is a scalar multiple of ω.

(ii) By Serre duality [Har77, Corollary 7.7, Chapter III]

$$h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X)) = h^0(X, \mathcal{O}_X) = 1.$$

Since $h^1(X, \mathcal{O}_X) = 0$, this implies that the Euler characteristic of the structure sheaf $\mathcal{O}_X$ is 2.

(iii) We have that $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ since X is connected and $H^1(X, \mathbb{Z}) = \{0\}$ by the exponential sequence 1.1, since $h^1(X, \mathcal{O}_X) = 0$. Moreover, by Poincaré duality [BHPVdV04, §11, Chapter I] $\mathrm{rk} H^3(X, \mathbb{Z}) = 0$.

(iv) By Noether's formula (Theorem 1.2.3):

$$2 = \chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)) = \frac{1}{12}e(X),$$

thus $e(X) = 24$. On the other hand $e(X) = 2 + \mathrm{rk} H^2(X, \mathbb{Z})$. Thus $H^2(X, \mathbb{Z})$ has rank 22.

(v) The Riemann-Roch formula (Theorem 1.2.2) for an effective divisor D on a K3 surface is

$$h^0(X, D) - h^1(X, D) = 2 + \frac{1}{2}D^2$$

since $h^2(X, D) = h^0(X, -D) = 0$ by Serre duality.

(vi) Adjunction formula (Theorem 1.2.4) for a K3 surface says that $K_C = C_{|C}$ for a smooth curve C. Thus

$$\deg(K_C) = 2g(C) - 2 = C^2.$$

A (-2)-*curve* on a K3 surface is a smooth curve C with $C^2 = -2$. By the previous formula such curves have genus zero.

Given two non-negative integers p, q with $p + q \leq 4$, let $H^{p,q}(X)$ the Dolbeault cohomology group and $h^{p,q}(X)$ its dimension [BHPVdV04, Chapter I, §12]. We recall that by Dolbeault's isomorphism

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p),$$

where Ω_X^p denotes the sheaf of holomorphic p-forms of X. Given a $K3$ surface X the Hodge decomposition theorem [BHPVdV04, Corollary 13.4, Chapter I] gives:

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

where $H^{0,2}(X) = \overline{H^{2,0}(X)}$. We have that: $h^{2,0}(X) = \dim H^0(X, \Omega_X^2) = 1$, $h^{0,2}(X) = h^{2,0}(X)$ and and $h^{1,1}(X) = 20$, because we have seen that $H^2(X, \mathbb{C})$ has dimension 22.

Observe that $H^2(X, \mathbb{C})$ is equipped with a quadratic form coming from the cup product defined on singular cohomology of X. This product can be written in terms of differential forms as

$$(w_1 \cdot w_2) \mapsto w_1 \cdot w_2 := \int_X w_1 \wedge w_2,$$

where w_1 and w_2 are closed 2-forms on X, and the bilinear form restricts to a bilinear form in $H^2(X,\mathbb{Z})$. It can be proved that $H^2(X,\mathbb{Z})$ has no torsion and the intersection product defines a lattice structure on it which is even, unimodular and of signature $(3,19)$ [Huy16, Proposition 3.5, Chapter 1]. By Milnor's Theorem 1.3.5 a lattice with such properties is unique up to isomorphism, thus one obtains the following.

Proposition 1.4.5. *Let X be a K3 surface. Then the cohomology group $H^2(X,\mathbb{Z})$ equipped with the intersection product is a lattice isometric to the K3 lattice:*

$$L_{K3} = U \oplus U \oplus U \oplus E_8 \oplus E_8 = U^3 \oplus E_8^2.$$

It follows from the exponential sequence (1.1) that the Picard group of a K3 surface is isomorphic to the Néron-Severi group $\mathrm{NS}(X) \subset H^2(X,\mathbb{Z})$, in particular it inherits a lattice structure, which can be proved to be the same as the intersection product defined in Theorem 1.2.1. More precisely the following holds (see [Huy16, Proposition 2.4 and §3.3, Chapter 1]).

Proposition 1.4.6. *Let X be a K3 surface. Then*

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X) = H^2(X,\mathbb{Z}) \cap \omega^{\perp},$$

where $\mathbb{C}\omega = H^{2,0}(X)$. Moreover, the intersection form on $\mathrm{Pic}(X)$ is even, non-degenerate, of rank $0 \leq \rho(X) \leq 20$ and of signature $(1, \rho(X)-1)$.

We now recall some classical results on linear systems on K3 surfaces following [SD74], [Huy16, Chapter 2] and [KL07].

Proposition 1.4.7. *Let X be a K3 surface and D be a non-zero effective divisor such that $|D|$ is base point free. Then two cases can occur:*

(*i*) *$D^2 > 0$ and the general element of $|D|$ is a smooth curve of genus $\frac{D^2}{2} + 1$ and $h^1(X, D) = 0$, or*

(*ii*) *$D^2 = 0$ and $D \sim kE$, where $k \geq 1$ is an integer and E is a smooth curve of genus one.*

Let D be a divisor as in case (i) of the previous Proposition. Observe that $h^0(X, D) = \frac{D^2}{2} + 2$ by Riemann-Roch Theorem. Thus it defines a morphism

$$\varphi_{|D|} : X \to \mathbb{P}^{\frac{D^2}{2}+1}$$

and two cases can occur (see [SD74, §4,5,6] for more detailed descriptions):

(i') $|D|$ is *non-hyperelliptic*, i.e. its generic member is not hyperelliptic and $\varphi_{|D|}$ is a birational morphism onto a surface of degree D^2. If D is ample then $\varphi_{|D|}$ is an isomorphism onto its image, otherwise $\varphi_{|D|}$ contracts all (-2)-curves orthogonal to D and the image of the morphism is a surface with rational double points at the image of such curves.

(i'') $|D|$ is *hyperelliptic*, i.e. its generic member is hyperelliptic and $\varphi_{|D|}$ is a degree two morphism onto a rational surface of degree $\frac{D^2}{2}$. As before the image surface has rational double points at the images of the (-2)-curves orthogonal to D.

The following result characterizes hyperelliptic nef divisors [SD74, Theorem 5.2]

Proposition 1.4.8. *Let X be a K3 surface and D be a non-zero effective divisor such that $|D|$ is base point free. Then $|D|$ is hyperelliptic if and only if either $D^2 = 2$, or there is a smooth elliptic curve F such that $D \cdot F = 2$, or $D \sim 2B$ for a smooth curve B with $B^2 = 2$.*

In case (ii) of Proposition 1.4.7 the morphism $\varphi_{|E|} : X \to \mathbb{P}^1$ is an *elliptic fibration*, i.e. a surjective morphism such that the general fibre is a smooth genus one curve. It can be proved that a K3 surface admits an elliptic fibration if and only if there is a non-trivial divisor D with $D^2 = 0$ [Huy16, Prop. 1.3, Chapter 11].

The following result describes the base locus of linear systems on K3 surfaces. A divisor is called *primitive* if it is not a positive multiple of another divisor.

Proposition 1.4.9. *Let X be a smooth projective K3 surface and D be a non-zero effective divisor on X. Then*

(i) *$|D|$ has no base points outside its fixed components;*

(ii) *if D is nef, then $|D|$ is base point free unless there exist a smooth elliptic curve F, a smooth rational curve E and an integer $k \geq 2$ with*

$$D \sim kF + E \text{ and } F \cdot E = 1;$$

(iii) *if D is nef, then $h^1(X, \mathcal{O}_X(D)) = 0$ unless $D \sim kF$, where F is a smooth elliptic curve and $k \geq 2$, in which case $h^1(X, \mathcal{O}_X(kF)) = k - 1$.*

Proof. For (i) see [SD74, Corollary 3.2], (ii) follows from [SD74, §2.7] and for (iii) see [KL07, Theorem]. $\square$

Corollary 1.4.10. *Let X be a K3 surface and D be a non-zero effective divisor on X. If D is nef, then $|2D|$ is base point free. Moreover $|D|$ is base point free if $D^2 = 0$.*

Proof. By Proposition 1.4.9 a nef divisor D which is not base point free is linearly equivalent to a divisor of the form $kF + E$ with $F^2 = 0, E^2 = -2$, $F \cdot E = 1$ and an integer $k \geq 2$. In particular $D \cdot F = kF^2 + E \cdot F = 1$, i.e. the divisor must be primitive, and $D^2 = E^2 + 2kE \cdot F = 2k - 2 \geq 2$. The above implies that a nef divisor D with $D^2 = 0$ is base point free, and that $|2D|$ can not have base locus since otherwise it would have intersection one with a smooth elliptic curve F', while $2D \cdot F'$ is an even number. This proves the statement. $\square$

Corollary 1.4.11. *Let X be a K3 surface such that none of its elliptic fibrations has a section. Then any effective non-zero nef divisor on X is base point free.*

Proof. Assume that D is a nef divisor with non-empty base locus. By Proposition 1.4.9 D is linearly equivalent to a sum involving two smooth curves F, E such that $F^2 = 0, E^2 = -2, F \cdot E = 1$. The morphism associated to $|F|$ defines an elliptic fibration $\varphi_{|F|} : X \to \mathbb{P}^1$ (see [SD74]) and E is a section of it, since $F \cdot E = 1$. $\square$

Remark 1.4.12. If F is a smooth elliptic curve and $k \geq 2$ is an integer, then $h^0(X, \mathcal{O}_X(kF)) = k + 1$ by the Riemann-Roch formula and Proposition 1.4.9 part (iii). This implies that $|kF| = k|F|$, i.e. each element of $|kF|$ is the union of k curves linearly equivalent to F. If $D = kF + E$, where F, E are as in Proposition 1.4.9 and $k \geq 2$ is an integer, then $|D|$ has base locus E, thus each element of $|D|$ is the union of E and an element of $|kF| = k|F|$.

We now explain two algorithms we have implemented in Magma (see Section 6.2) to find the set of (-2)-curves and to compute the cohomology of divisors of a Mori dream K3 surface.

The programs in Section 6.2 compute a set of fundamental roots for the Picard lattice $\mathrm{Cl}(X)$ of a Mori dream K3 surface given its intersection matrix Q. Such set of roots can be assumed to be the set of classes of (-2)-curves of X up to an isometry of $\mathrm{Cl}(X)$ (see [Huy16, Corollary 2.11]). The algorithm is essentially the one known as “Vinberg’s algorithm” [Vin75]. The steps are the following:

- Fix a class $\alpha \in \mathrm{Cl}(X)$ with $\alpha^2 > 0$.

- Find all classes $w \in \mathrm{Cl}(X) \cap \alpha^\perp$ with self-intersection -2 (there are finitely many of them since the restriction of Q to $\alpha^\perp$ is negative definite) and let L be a list of such classes, which is a root system.

- If L is not empty, fix a set $L^+ \subseteq L$ of positive roots as follows: choose randomly an integral combination H of the vectors in L having non zero intersection with all of them and let L^+ be the set of vectors in L having positive intersection with H.

- Construct the list R_0 of simple roots in L^+ inductively as follows: let $L_0^+ \subseteq L^+$ be the set of vectors having minimal intersection m_0 with H and, once L_i^+ is given for some $i \geq 0$, define $L_{i+1}^+ \subseteq L^+$ as the set of vectors having intersection $m_0 + i + 1$ with H and non-negative intersection with all vectors in L_k^+ for $0 \leq k \leq i$. The process stops when $m_0 + n = \max\{v \cdot H : v \in L^+\}$ and the set of simple roots in L^+ is $R_0 := \cup_{0 \leq i \leq n} L_i^+$.

- Construct a set of fundamental roots for $\mathrm{Cl}(X)$ inductively as follows: let R_0 be as in the previous item ($= \emptyset$ if L is empty) and define R_{i+1} as the union of R_i with the set of classes $w \in \mathrm{Cl}(X)$ such that $w^2 = -2$, $w \cdot \alpha = i + 1$ and having non-negative intersection with all the elements of R_i.

- The set R_n is a set of fundamental roots of $\mathrm{Cl}(X)$ if the following property holds for the convex polyhedral cone $\mathcal{C}$ generated by the vectors in R_n: the intersection matrix of the vectors generating any facet of $\mathcal{C}$ is negative semidefinite.

Section 6.1 contains a program which computes the dimensions of cohomology groups $h^0(X, w)$ and $h^1(X, w)$ of a class $w \in \mathrm{Cl}(X)$ of a K3 surface given the classes of (-2)-curves and the intersection matrix of $\mathrm{Cl}(X)$. In case w is effective and nef,

$h^0(X, w)$ and $h^1(X, w)$ can be computed by means of Proposition 1.4.9 part (iii) and the Riemann-Roch formula. In case w is effective and not nef, we find a (-2)-curve having negative intersection with w and we remove it from w. Repeating this operation we find a nef class w' such that $h^0(X, w') = h^0(X, w)$. Finally, $h^1(X, w)$ can be computed by means of the Riemann-Roch formula.

Chapter 2

Cox rings

This chapter starts with some preliminaries on Cox rings and Mori dream spaces based on [ADHL15] and focused on the case of surfaces. Moreover, in section 3 we provide a new technique for computing the degrees of a generating set of the Cox ring and in section 4 we give an application of it to K3 surfaces.

2.1 Preliminaries

We will define the Cox ring for a normal projective algebraic variety with some additional assumptions. For a more general definition see [ADHL15, §1.4].

Definition 2.1.1. Let X be a normal projective variety defined over $\mathbb{C}$ whose divisor class group $\mathrm{Cl}(X)$ is finitely generated and free. Let K be a subgroup of $\mathrm{Div}(X)$ such that the canonical morphism $K \to \mathrm{Cl}(X)$, which associates to D its class $[D]$, is an isomorphism. The *Cox ring of* X is the $\mathbb{C}$-algebra

$$R(X) := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)),$$

where the multiplication is given by the multiplication of rational functions of X.

Observe that $R(X)$ is a K-graded algebra over $\mathbb{C}$, i.e. it has a direct sum decomposition into complex vector spaces

$$R(X)_D := \Gamma(X, \mathcal{O}_X(D)),$$

where $D \in K$, such that $R(X)_{D_1} \cdot R(X)_{D_2} \subseteq R(X)_{D_1+D_2}$. An element $f \in R(X)$ is *homogeneous* if it belongs to $R(X)_D$ for some $D \in K$, and in this case its *degree* is $\deg(f) = [D]$.

Proposition 2.1.2. *The Cox ring does not depend on the choice of the subgroup K of* $\mathrm{Div}(X)$ *mapping isomorphically to* $\mathrm{Cl}(X)$*, up to isomorphisms of graded algebras.*

Proof. Let K, K' be two subgroups of $\mathrm{Div}(X)$ mapping isomorphically to $\mathrm{Cl}(X)$ and denote by $R(X)^K$ and $R(X)^{K'}$ the Cox rings associated to them as in Definition 2.1.1. Let $D_1, \dots, D_r$ be a basis of K and let $f_1, \dots, f_r \in \mathbb{C}(X)^*$ be such that $D_i' = D_i - \mathrm{div}(f_i)$, $i = 1, \dots, r$, form a basis of K'. Let $\eta : K \to \mathbb{C}(X)^*$ be the group homomorphism defined by $D_i \mapsto f_i$. An isomorphism of graded algebras is thus given by

$$\begin{array}{ll} \tilde{\psi} : K \to K' & D \mapsto D - \mathrm{div}(\eta(D)), \\ \psi : R(X)^K \to R(X)^{K'} & f \in R(X)^K_D \mapsto f\eta(D) \in R(X)^{K'}_{\tilde{\psi}(D)}. \end{array}$$

$\square$

In the following, to simplify notation, we usually identify K with $\mathrm{Cl}(X)$ that is, given $w \in \mathrm{Cl}(X)$, we denote by $R(X)_w$ the homogeneous component $\Gamma(X, \mathcal{O}_X(D))$ of $R(X)$, where $D \in K$ and $[D] = w$.

Observe that given any effective divisor E of X there is a unique $D \in K$ such that $[D] = [E]$ and a homogeneous element $f \in R(X)_{[E]}$, unique up to a constant, such that $\mathrm{div}(f) + D = E$. We will say that $f \in R(X)_{[E]}$ is a *defining section of* E.

Definition 2.1.3. A variety X as in Definition 2.1.1 whose Cox ring $R(X)$ is a finitely generated $\mathbb{C}$-algebra is called *Mori dream space*.

The first examples of Mori dream spaces are toric varieties, whose Cox rings are known to be polynomial rings whose generators correspond bijectively to the rays of a fan defining the variety [ADHL15, §2.1].

One important reason why Mori dream spaces are interesting is the following construction, which shows that Mori dream spaces with free divisor class group can be obtained as quotients of an open subset of an affine variety by the action of a torus (see [ADHL15, §1.6.3]). If $R(X)$ is finitely generated one can consider the associated affine variety $\overline{X} = \mathrm{Spec}\, R(X)$, called *total coordinate space*. Since $R(X)$ is K-graded, $\overline{X}$ admits the action of the torus $H = \mathrm{Spec}\, \mathbb{C}[K] \cong (\mathbb{C}^*)^r$, where $r = \mathrm{rank}\, \mathrm{Cl}(X)$. There exists an open subset $\hat{X}$ of $\overline{X}$ such that H_X acts on $\hat{X}$ and there exists a morphism $p : \hat{X} \to X$ which is a good quotient for the H-action. The locus $\overline{X} \backslash \hat{X}$ is called *irrelevant locus* of X and has codimension at least 2 in $\overline{X}$. Its defining ideal in $R(X)$ is given by

$$\mathcal{I}_{irr}(X) = \sqrt{(R_w)},$$

where $w \in \mathrm{Cl}(X)$ is an ample class, $R_w := \oplus_{n>0} R(X)_{nw}$ and (R_w) denotes the ideal generated by R_w in $R(X)$.

The following gives a first necessary condition for the finite generation of the Cox ring.

Proposition 2.1.4. *Let* $\{f_i,\ i \in I\}$ *be a set of homogeneous generators of* $R(X)$*. Then the class of any effective divisor is a linear combination with non-negative integer* *coefficients of the degrees* $\{\deg(f_i) : i \in I\}$*. In particular*

$$\mathrm{Eff}(X) = \mathrm{cone}(\deg(f_i) : i \in I)$$

and the effective cone of a Mori dream space is polyhedral.

Proof. Since $R(X)$ is finitely generated by $\{f_1, \dots, f_r\}$ with $\deg(f_i) = [E_i] \in \mathrm{Cl}(X)$ for $i = 1, \dots, r$. Let E be an effective divisor and $f \in R(X)_{[E]}$, then

$$f = p(f_1, \dots, f_r) = \sum_{i=1}^{n} \alpha_i m_i(f_1, \dots, f_r),$$

where p is a polynomial in r variables with coefficients in $\mathbb{C}$ and $m_1, \dots, m_n$ are its monomials. Observe that each monomial is homogeneous because it is the product of homogeneous elements, and $m_i(f_1, \dots, f_r)$ has $\deg(m_i) = [E]$. Let $m_1(f_1, \dots, f_r) = f_1^{a_1} \cdots f_r^{a_r}$ with a_i non-negative integers. Since $f_1^{a_1} \cdots f_r^{a_r} \in R(X)_{[E]}$ by definition of multiplication in $R(X)$ we have to

$$\deg(f_1^{a_1} \cdots f_r^{a_r}) = a_1 \deg(f_1) + \cdots + a_r \deg(f_r).$$

Then

$$[E] = \deg(m_1(f_1, \dots, f_r)) = \deg(f_1^{a_1} \cdots f_r^{a_r}) = \sum_{i=1}^{r} a_i \deg(f_i) = \sum_{i=1}^{r} a_i [E_i].$$

□

A set of homogeneous generators $\{f_i : i \in I\}$ of $R(X)$ is a *minimal generating set* when each f_i can not be expressed as a polynomial in the remaining elements f_j. In what follows we will say that the Cox ring $R(X)$ *has a generator in degree* $w \in \mathrm{Cl}(X)$

if each minimal generating set of $R(X)$ contains a nontrivial element of $R(X)_w$.

Corollary 2.1.5. *If D is an effective divisor such that $[D]$ is an element of the Hilbert basis of $\operatorname{Eff}(X) \subseteq \operatorname{Cl}(X)_{\mathbb{Q}}$, then the Cox ring $R(X)$ has a generator in degree $[D]$. In particular, if D is an integral effective divisor of X with $h^0(X, D) = 1$, then $R(X)$ has a generator in degree $[D]$.*

Proof. Let f_i, $i \in I$ be a minimal homogeneous generating set of $R(X)$. Since $[D]$ belongs to the Hilbert basis of the effective cone (see [MS00, Definition 7.17]), then $[D] = \deg(f_i)$ for some $i \in I$ by Proposition 2.1.4. □

A *small birational map* between two normal projective varieties X and Y is a rational map $X \to Y$ which defines an isomorphism $U \to V$, where $U \subseteq X$ and $V \subseteq Y$ are open subsets whose complements in X and Y have codimension at least two. Observe that in this case X and Y have isomorphic Cox rings (see [LM09, Example 3.3] for an example of two smooth 3-folds which are not isomorphic and have isomorphic Cox rings).

The following is a general characterization of Mori dream spaces [AHL10, Theorem 2.3].

Theorem 2.1.6. *Let X be a normal projective variety with finitely generated divisor class group. Then the following are equivalent.*

(i) The Cox ring $R(X)$ is finitely generated.

(ii) The effective cone $\operatorname{Eff}(X) \subset \operatorname{Cl}(X)_{\mathbb{Q}}$ is polyhedral and there are small birational maps $\pi_i : X \to X_i$, where $i = 1, \dots, r$, such that each semiample cone $\operatorname{SAmple}(X_i)$ is polyhedral and one has

$$\operatorname{Mov}(X) = \pi_1^* \operatorname{SAmple}(X_1) \cup \cdots \cup \pi_r^* \operatorname{SAmple}(X_r).$$

2.2 Cox rings of surfaces

In the case of surfaces Theorem 2.1.6 implies the following. We recall that a variety is $\mathbb{Q}$-*factorial* if the group of Cartier divisors has finite index in $\mathrm{Div}(X)$.

Theorem 2.2.1. *Let X be a $\mathbb{Q}$-factorial projective surface such that $\mathrm{Cl}(X)$ is finitely generated and free. Then the following are equivalent:*

(*i*) *The Cox ring $R(X)$ is finitely generated.*

(*ii*) *The effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ is polyhedral and $\mathrm{Nef}(X) = \mathrm{SAmple}(X)$.*

As observed before, a Mori dream space in general is not determined by its Cox ring, however in the case of surfaces the following holds.

Corollary 2.2.2. *Let X, Y be two normal projective surfaces with finitely generated and free divisor class group whose Cox rings are isomorphic as graded algebras and finitely generated, then X and Y are isomorphic.*

Proof. We will identify $R := R(X)$ with $R(Y)$ as graded algebras. As explained above, X and Y are GIT quotients of open subsets $\hat{X}$ and $\hat{Y}$ of $\mathrm{Spec}(R)$ by the action of the torus $H = \mathrm{Spec}\,\mathbb{C}[K]$ induced by the K-grading of R. Moreover, the complements of $\hat{X}$ and $\hat{Y}$ are zero sets of ideals I_X, I_Y (the irrelevant ideals) in R, each defined by the choice of an ample class in K. Since X and Y are normal complete surfaces, the semiample cone (whose interior is the ample cone) equals the moving cone [ADHL15, Theorem 4.3.3.5]. On the other hand the moving cone can be computed only in terms of the Cox ring R [ADHL15, Proposition 3.3.2.3]. This implies that an ample class for X is ample for Y as well, thus $\hat{X} = \hat{Y}$ and finally $X \cong Y$. □

The classification of Mori dream surfaces, even in the smooth case, is still open. An important invariant in this context is the *anticanonical Iitaka dimension* which is defined as

$$k(-K_X) := \max_{n \in N} \dim(\varphi_{-nK_X}(X)),$$

if $N = \{n \in \mathbb{Z} : n \geq 0, h^0(-nK_X) > 0\} \neq \{0\}$ and $-\infty$ if $N = \{0\}$. Clearly $k(-K_X) \in \{0, 1, 2, -\infty\}$. Observe that if $k(-K_X) \geq 1$, then X is rational by Castelnuovo's rationality criterion [Har77, Theorem 6.2, Chapter V]. We recall some known results according to this invariant. In case the anticanonical Iitaka dimension is positive, the following holds (see [TVAV09] for i. and [AL11] for ii.).

Theorem 2.2.3. *Let X be a smooth projective rational surface with $q(X) = 0$. Then*

(*i*) *if $k(-K_X) = 2$, then X is a Mori dream space ;*

(*ii*) *if $k(-K_X) = 1$, then X is a Mori dream space if and only if its effective cone is polyhedral or, equivalently, if X contains finitely many (-1)-curves (smooth rational curves with self-intersection -1).*

If X is rational, $k(-K_X) = 0$ and $-K_X$ is nef, then X is not a Mori dream space, since $-K_X$ is a nef divisor which is not semiample. On the other hand, there exist examples of Mori dream rational surfaces with $k(-K_X) = 0$ and $-K_X$ not nef, see [LT13]. The case $k(-K_X) = -\infty$ is still unexplored except for some examples. In case $k(-K_X) = 0$ and K_X is numerically trivial, X is a K3 surface or an Enriques surfaces (i.e. a smooth surface with $q(X) = 0$, $2K_X \sim 0$ and $K_X \not\sim 0$) by [Har77, Theorem 6.3, Chapter V]. The following theorem characterizes Mori dream K3 surfaces [AHL10, Theorem 1]. The analogous result holds for Enriques surfaces [AHL10, Theorem 2.10] (in this case $\mathrm{Cl}(X)$ has torsion).

Theorem 2.2.4. *Let X be a projective K3 surface. Then the following statements are equivalent.*

(i) *X is a Mori dream surface.*

(ii) *The effective cone $\operatorname{Eff}(X) \subseteq \operatorname{Cl}_{\mathbb{Q}}(X)$ is polyhedral.*

(iii) *The automorphism group of X is finite.*

Moreover, if the Picard number is at least three, then (i) is equivalent to the property that X contains a finite non-zero number of smooth rational curves. In this case, these curves are (-2)-curves and their classes generate the effective cone.

Proof. We have that (i) implies (ii) by Proposition 2.1.4. Viceversa, by Theorem 2.2.1 X is a Mori dream space if every nef divisor is semiample. This property of nef divisors follows from Corollary 1.4.10. For the equivalence of (ii) and (iii) and the last statement see [Kov94, Remark 7.2] and [PŠŠ71, §7, Corollary]. □

K3 surfaces with the properties in the previous Theorem have been classified in a series of papers by Piatetski-Shapiro and Shafarevich, Nikulin and Vinberg [Nik79, Nik84, Nik00, PŠŠ71, Vin07]. Thus we have an explicit classification of Mori dream K3 surfaces [ADHL15, Theorem 5.1.5.3].

Theorem 2.2.5. *Let X be a projective K3 surface with $\rho(X) = \operatorname{rk}\operatorname{Cl}(X)$. Then X is a Mori dream surface if and only if one of the following occurs:*

(i) $\rho(X) = 1$.

(ii) $\rho(X) = 2$ *and $\operatorname{Cl}(X)$ contains a class w with $w^2 \in \{-2, 0\}$.*

(*iii*) $\rho(X) = 3$ *and* $\mathrm{Cl}(X)$ *is isometric to one of the* 26 *lattices of* [Nik84]*:*

$$
\begin{aligned}
S'_{4,1,2} &= \langle 2e_1 + e_3, e_2, 2e_3 \rangle,\\
S_{k,1,1} &= \langle ke_1, e_2, e_3 \rangle, \;\; k \in \{4,5,6,7,8,10,12\},\\
S_{1,k,1} &= \langle e_1, ke_2, e_3 \rangle, \;\; k \in \{2,3,4,5,6,9\},\\
S_{1,1,k} &= \langle e_1, e_2, ke_3 \rangle, \;\; k \in \{1,2,3,4,6,8\},
\end{aligned}
$$

where the intersection matrix of e_1, e_2, e_3 *is* $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix}$, *and*

$$S_1 = (6) \oplus A_1^2, \qquad S_2 = (36) \oplus A_2, \qquad S_3 = (12) \oplus A_2,$$

$$S_4 = \begin{bmatrix} 6 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \qquad S_5 = (4) \oplus A_2, \qquad S_6 = \begin{bmatrix} 14 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}.$$

(*iv*) $\rho(X) = 4$ *and* $\mathrm{Cl}(X)$ *is isometric to one of the* 14 *lattices of* [Vin07]*:*

$$V_1 = (8) \oplus A_1^3, \qquad V_2 = (-4) \oplus (4) \oplus A_2, \qquad V_3 = (4) \oplus A_3,$$

$$V_{4,\ldots,7} = U(k) \oplus A_1^2, \;\; k \in \{1,2,3,4\}, \qquad V_{8,\ldots,11} = U(k) \oplus A_2, \;\; k \in \{1,2,3,6\},$$

$$V_{12} = \begin{bmatrix} 0 & -3 \\ -3 & 2 \end{bmatrix} \oplus A_2, V_{13} = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}, V_{14} = \begin{bmatrix} 12 & -2 & 0 & 0 \\ -2 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

(*v*) $5 \leq \rho(X) \leq 19$ *and* $\mathrm{Cl}(X)$ *is isometric to one of the following lattices:*

ρ	*Lattices*
5	$U \oplus A_1^3,\ U(2) \oplus A_1^3,\ U \oplus A_1 \oplus A_2,\ U \oplus A_3,\ U(4) \oplus A_1^3,$ $(6) \oplus A_2^2,\ (4) \oplus D_4,\ (8) \oplus D_4,\ (16) \oplus D_4$
6	$U \oplus D_4,\ U(2) \oplus D_4,\ U \oplus A_1^4,\ U(2) \oplus A_1^4,\ U \oplus A_1^2 \oplus A_2,$ $U \oplus A_2^2,\ U \oplus A_1 \oplus A_3,\ U \oplus A_4,\ U(4) \oplus D_4,\ U(3) \oplus A_2^2$
7	$U \oplus D_4 \oplus A_1,\ U \oplus A_1^5,\ U(2) \oplus A_1^5,\ U \oplus A_1 \oplus A_2^2,$ $U \oplus A_1^2 \oplus A_3,\ U \oplus A_2 \oplus A_3,\ U \oplus A_1 \oplus A_4,\ U \oplus A_5,\ U \oplus D_5$
8	$U \oplus D_6,\ U \oplus D_4 \oplus A_1^2,\ U \oplus A_1^6,\ U(2) \oplus A_1^6,\ U \oplus A_2^3,\ U \oplus A_3^2,$ $U \oplus A_2 \oplus A_4,\ U \oplus A_1 \oplus A_5,\ U \oplus A_6,\ U \oplus A_2 \oplus D_4,\ U \oplus A_1 \oplus D_5,$ $U \oplus E_6$
9	$U \oplus E_7,\ U \oplus D_6 \oplus A_1,\ U \oplus D_4 \oplus A_1^3,\ U \oplus A_1^7,\ U(2) \oplus A_1^7,$ $U \oplus A_7,\ U \oplus A_3 \oplus D_4,\ U \oplus A_2 \oplus D_5,\ U \oplus D_7,\ U \oplus A_1 \oplus E_6$
10	$U \oplus E_8,\ U \oplus D_8,\ U \oplus E_7 \oplus A_1,\ U \oplus D_4 \oplus D_4,\ U \oplus D_6 \oplus A_1^2,$ $U(2) \oplus D_4 \oplus D_4,\ U \oplus D_4 \oplus A_1^4,\ U \oplus A_1^8,\ U \oplus A_2 \oplus E_6$
11	$U \oplus E_8 \oplus A_1,\ U \oplus D_8 \oplus A_1,\ U \oplus D_4 \oplus D_4 \oplus A_1,\ U \oplus D_4 \oplus A_1^5$
12	$U \oplus E_8 \oplus A_1^2,\ U \oplus D_8 \oplus A_1^2,\ U \oplus D_4 \oplus D_4 \oplus A_1^2,\ U \oplus A_2 \oplus E_8$
13	$U \oplus E_8 \oplus A_1^3,\ U \oplus D_8 \oplus A_1^3,\ U \oplus A_3 \oplus E_8$
14	$U \oplus E_8 \oplus D_4,\ U \oplus D_8 \oplus D_4,\ U \oplus E_8 \oplus A_1^4$
15	$U \oplus E_8 \oplus D_4 \oplus A_1$
16	$U \oplus E_8 \oplus D_6$
17	$U \oplus E_8 \oplus E_7$
18	$U \oplus E_8^2$
19	$U \oplus E_8^2 \oplus A_1$

Given a Mori dream space, a natural problem is to find a presentation of its

Cox ring, or at least to determine the degrees of its generators and relations. There exist several different techniques for this, which allowed to compute the Cox ring of several classes of special varieties. A pioneer work in this direction was the paper [BP04] by Batyrev and Popov, who identified the generators of the Cox ring of any del Pezzo surface, showing in particular that it is generated by the elements defining the (-1)-curves of X if the rank of the class group is $4 \leq r \leq 8$. The ideal of relations of the Cox ring of del Pezzo surfaces has been computed in several steps in [STM07, SS07, LM09, TVAV09].

In [CT06] the authors determined the generators of the Cox ring of the blow-up of $\mathbb{P}^n$ in any number of points that lie on a rational normal curve. More in general, techniques are available to compute Cox rings of special classes of varieties (for example varieties with a torus action, homogeneous spaces, spherical varieties) and to relate the Cox rings of two varieties X, Y obtained one from the other in different ways (for example Y embedded in X satisfying suitable conditions or Y a blow-up of X along an irreducible subvariety contained in the smooth locus), see [ADHL15, Chapter 4] and [HKL16].

This thesis aims at developing techniques for computing Cox rings of Mori dream K3 surfaces. We recall some known results.

For the following result see [AHL10, Proposition 3.4].

Proposition 2.2.6. *Let X be a projective K3 surface, $w \in \mathrm{Cl}(X)$ be the class of a smooth irreducible curve $D \subset X$ and $R_{[D]} := \bigoplus_{n\in\mathbb{N}} R(X)_{nw}$.*

(i) If D is not hyperelliptic, then $R_{[D]}$ is generated in degree one.

(ii) If $D^2 = 2$, then $R_{[D]}$ is generated in degrees one and three.

(iii) If D is hyperelliptic and $D^2 > 2$, then $R_{[D]}$ is generated in degrees one and two.

Let X be a K3 surface with $\rho(X) = 1$, D be a curve whose class generates the ample cone of X and $\varphi_{|D|} : X \to \mathbb{P}^n$ be the associated morphism. The previous Proposition implies that the Cox ring of a K3 surface with $\rho(X) = 1$ is isomorphic to the homogeneous coordinate ring of $X \subset \mathbb{P}^n$ if $D^2 > 2$ and, in case $D^2 = 2$, it is isomorphic to the $\mathbb{Z}$-graded ring $\mathbb{C}[T_1, \dots, T_4]/(T_4^2 - f(T_1, T_2, T_3))$, where $\deg(T_i) = 1$ for $i = 1, 2, 3$, $\deg(T_4) = 3$ and $f = 0$ defines the plane sextic curve which is the branch locus of the double covering $\varphi_{|D|} : X \to \mathbb{P}^2$ [ADHL15, Theorem 5.3.2.1].

In [AHL10] the authors gave results about the computation of Cox rings of K3 surfaces of Picard number $\rho(X) \geq 2$. In case $\rho(X) = 2$ the authors identified the degrees of the generators of $R(X)$ when the effective cone is generated by a $\mathbb{Z}$-basis w_1, w_2 of $\mathrm{Cl}(X)$ and give information about the ideal of relations when $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geq 3$. Moreover, for any Picard number, the authors studied K3 surfaces with a non-symplectic involution, i.e. an automorphism $\iota : X \to X$ of order two with $\iota^*\omega_X = -\omega_X$, where $H^{2,0}(X) = \mathbb{C}\omega_X$. They determined the Cox ring of all generic K3 surfaces with a non-symplectic involution when $2 \leq \rho(X) \leq 5$. Moreover, they computed the Cox ring of K3 surfaces that are general double covers of del Pezzo surfaces.

The following result gives the Cox rings of the generic K3 surfaces of Picard number three and four admitting a non-symplectic involution (we will say that a K3 surface is generic in the sense of the definition that appears in [ADHL15, Section 5.3.3]). We recall that the quotient of any such involution, if it has fixed points, is a smooth rational surface. As usual, $\mathbb{F}_n$ denotes the n-th Hirzebruch surface and we denote by $\mathrm{Bl}_i(\mathbb{F}_n)$ a blow-up of $\mathbb{F}_n$ at i general points, $i = 1, 2$.

Proposition 2.2.7 (Proposition 5.3.4.1 and Proposition 5.3.4.2 [ADHL15])**.** *Let X be a generic K3 surface with $\rho(X) = 3, 4$ admitting a non-symplectic involution ι*

acting trivially on the Picard group and let $\pi : X \to Y = X/(\iota)$ be the associated double cover. Then the Cox ring $\mathcal{R}(X)$ is given as follows.

(*i*) *For $\rho(X) = 3$ one has $R(X) = \mathbb{C}[T_1, \dots, T_6]/(T_6^2 - f)$ where $f \in \mathbb{C}[T_1, \dots, T_6]$ is a prime polynomial and the degree of T_i is the i-th column of*

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \textit{if} \quad Y = \mathrm{Bl}_1(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 5 \end{bmatrix} \quad \textit{if} \quad Y = \mathrm{Bl}_1(\mathbb{F}_4).$$

In the first case $\mathrm{Pic}(X) \cong U(2) \oplus A_1$ and the fixed locus of ι is a smooth curve of genus 8. In the second case $\mathrm{Pic}(X) \cong U \oplus A_1$ and the fixed locus of ι is the disjoint union of a smooth rational curve and a smooth curve of genus 9.

(*ii*) *For $\rho(X) = 4$ one has $R(X) = \mathbb{C}[T_1, \dots, T_7]/(T_7^2 - f)$ where $f \in \mathbb{C}[T_1, \dots, T_7]$ is a prime polynomial and the degree of T_i is the i-th column of*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \quad \textit{if} \quad Y = \mathrm{Bl}_2(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 4 \end{bmatrix} \quad \textit{if} \quad Y = \mathrm{Bl}_2(\mathbb{F}_4).$$

In the first case $\mathrm{Pic}(X) \cong U(2) \oplus A_1^2$ and the fixed locus of ι is a smooth curve of genus 7. In the second case $\mathrm{Pic}(X) \cong U \oplus A_1^2$ and the fixed locus of ι is the disjoint union of a smooth rational curve and a smooth curve of genus 8.

Note that the lattices in the previous theorem correspond to the lattices $S_{1,1,1}$ and $S_{1,1,2}$ in Theorem 2.2.5: $U \oplus A_1 \cong S_{1,1,1}$ and $U(2) \oplus A_1 \cong S_{1,1,2}$ (see [Nik84]).

In [Ott13] J.C. Ottem gives a new proof of the finite generation of the Cox ring of K3 surfaces when the effective cone is rational polyhedral and developed a technique that allows to compute the Cox ring of several examples of K3 surfaces of Picard number 2, such as quartic surfaces that contain a line, quartic surfaces that contain two plane conics and double coverings of $\mathbb{F}_4$.

2.3 Koszul type sequences

In this section we will present several techniques which allow one to show that the Cox ring of a projective variety has no generators in a certain degree. Given $f \in \mathbb{C}(X)^*$, we denote by $\operatorname{div}_E(f)$ the divisor $\operatorname{div}(f) + E$.

Theorem 2.3.1. *Let X be a smooth complex projective variety, $E_1, E_2, \dots, E_n$ be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, \dots, n$, such that $\bigcap_{i=1}^n \operatorname{div}_{E_i}(f_i) = \emptyset$. Let*

$$\mathbb{K}_0 := \mathcal{O}_X,\ \mathbb{K}_i := \bigoplus_{1 \le j_1 < \cdots < j_i \le n} \mathcal{O}_X(-E_{j_1} - \cdots - E_{j_i}),\ i = 1, \dots, n.$$

Then there is an exact sequence of sheaves:

$$0 \longrightarrow \mathbb{K}_n \xrightarrow{d_n} \mathbb{K}_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow \mathbb{K}_1 \xrightarrow{d_1} \mathbb{K}_0 \longrightarrow 0, \qquad (2.1)$$

where $d_1(u_j) = f_j u_0$ for $j = 1, \dots, n$ and

$$d_i(u_{j_1 \cdots j_i}) = \sum_{r=1}^{i} (-1)^{r+1} f_{i_r} u_{j_1 \cdots j_{r-1} \hat{j}_r j_{r+1} \cdots j_i},\ i = 2, \dots, n,$$

where $u_{j_1 \cdots j_i}$ is a generator of $\mathcal{O}_X(-E_{j_1} - \cdots - E_{j_i})$ as $\mathcal{O}_X$-module.

Proof. It is enough to prove exactness at any local ring $\mathcal{O}_x$, $x \in X$. Given $x \in X$, let

$R = \mathcal{O}_x$ and $s_i \in R$ be the image of $f_i u_0$. We can assume that $s_1 \in R$ is a unit since $\cap_{i=1}^n \mathrm{div}_{E_i}(f_i) = \emptyset$. Let $E = R^n$ and let $\varphi_1 : E \to R$, $\varphi(e_i) = s_i$. The sequence (2.1) is the Koszul complex $K.(\varphi)$ [Lan02, Chapter XXI, p. 852]:

$$0 \longrightarrow \wedge^n E \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_3} \wedge^2 E \xrightarrow{\varphi_2} E \xrightarrow{\varphi_1} R \longrightarrow 0.$$

We will denote by $H_p K(s_1, \ldots, s_n)$ the p-th homology group of the complex. Observe that $H_0 K(s_1, \ldots, s_n) = \{0\}$ since it is isomorphic to $R/(s_1, \ldots, s_n)$ and s_1 is a unit.

We now prove that all homology groups with $p > 0$ vanish by induction on n. If $n = 1$ the sequence is exact since φ_1 is the multiplication by s_1, which is a unit. Now we assume exactness for $n - 1$. By [Lan02, Theorem 4.5 a), Chapter XXI] there is an exact sequence of Koszul homology groups:

$$\begin{array}{ccccc} H_p K(s_1, \ldots, s_{n-1}) \longrightarrow & H_p K(s_1, \ldots, s_{n-1}) \longrightarrow & H_p K(s_1, \ldots, s_n) \longrightarrow \\ \cdots & \cdots & \cdots \\ H_1 K(s_1, \ldots, s_n) \longrightarrow & H_0 K(s_1, \ldots, s_{n-1}) \longrightarrow & H_0 K(s_1, \ldots, s_{n-1}) \longrightarrow . \end{array}$$

For all $p > 0$ the group $H_p K(s_1, \ldots, s_n)$ is between two groups which are zero by induction (or by the previous remark on $H_0 K$), thus it is zero. □

Considering the case of two or three disjoint divisors, we obtain the following results.

Corollary 2.3.2. *Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$, such that $\mathrm{div}_{E_1}(f_1) \cap \mathrm{div}_{E_2}(f_2) = \emptyset$. If $D \in \mathrm{WDiv}(X)$ is such that $h^1(X, D - E_1 - E_2) = 0$, then there is a surjective* *morphism*

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D), \quad (g_1, g_2) \mapsto g_1 f_1 + g_2 f_2.$$

Proof. Consider the exact sequence of sheaves obtained tensoring sequence (2.1) with $\mathcal{O}_X(D)$:

$$0 \longrightarrow \mathcal{O}_X(D - E_1 - E_2) \longrightarrow \mathcal{O}_X(D - E_1) \oplus \mathcal{O}_X(D - E_2) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Taking the associated long exact sequence in cohomology one obtains the statement.

□

Remark 2.3.3. More generally there is an exact sequence of sheaves [Bea96, Lemma I.5]:

$$0 \to \mathcal{O}_X(D - E_1 - E_2) \to \mathcal{O}_X(D - E_1) \oplus \mathcal{O}_X(D - E_2) \to \mathcal{O}_X \to \mathcal{O}_{E_1 \cap E_2} \to 0.$$

Corollary 2.3.4. *Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$, such that $\cap_{i=1}^3 \operatorname{div}_{E_i}(f_i) = \emptyset$. If $D \in \operatorname{WDiv}(X)$ then the morphism*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \ (g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3,$$

is surjective if one of the following occurs:

(i) $h^1(X, D - E_i - E_j) = 0$ *for all distinct* $i, j \in \{1, 2, 3\}$ *and* $h^2(X, D - E_1 - E_2 - E_3) = 0$.

(ii) $h^1(X, D) = 0$, $h^p(X, D - E_i - E_j) = 0$ *for* $p = 1, 2$ *and for all distinct*

$i, j \in \{1,2,3\}$, and $h^2(X, D-E_1-E_2-E_3) = \sum_{i=1}^{3} h^1(X, D-E_i)$.

Proof. After tensoring with $\mathcal{O}_X(D)$, the exact sequence in Theorem 2.3.1 can be split into two short exact sequences:

$$0 \to \mathcal{O}_X(D-E_1-E_2-E_3) \xrightarrow{d_3} \oplus_{i<j}\mathcal{O}_X(D-E_i-E_j) \xrightarrow{d_2} \mathrm{Im}(d_2) \to 0 \tag{2.2}$$

and

$$0 \to \mathrm{Im}(d_2) \xrightarrow{i} \oplus_{k=1}^{3}\mathcal{O}_X(D-E_k) \xrightarrow{d_1} \mathcal{O}_X(D) \to 0, \tag{2.3}$$

where i is the inclusion morphism. These give rise to the following exact sequences in cohomology:

$$\oplus_{i<j}H^1(X, D-E_i-E_j) \to H^1(X, \mathrm{Im}(d_2)) \to H^2(X, D-E_1-E_2-E_3) \to \oplus_{i<j}H^2(X, D-E_i-E_j)$$

$$\oplus_{k=1}^{3}H^0(X, D-E_k) \xrightarrow{\phi} H^0(X, D) \xrightarrow{\phi'} H^1(X, \mathrm{Im}(d_2)) \xrightarrow{\phi''} \oplus_{k=1}^{3}H^1(X, D-E_k) \to H^1(X, D).$$

If (i) holds, by the first sequence we obtain that $H^1(X, \mathrm{Im}(d_2)) = 0$, then by the second sequence the morphism ϕ is surjective.
On the other hand, if (ii) holds, since $h^1(X, D) = 0$ then ϕ'' is surjective by the second sequence. Moreover by the first sequence we have that

$$\dim(H^1(X, \mathrm{Im}(d_2))) = h^2(X, D-E_1-E_2-E_3) = \dim(\oplus_{k=1}^{3}H^1(X, D-E_k)),$$

then ϕ'' is an isomorphism and one obtains the statement. □

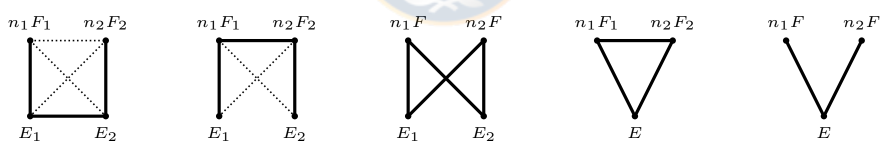
Remark 2.3.5. Observe that the surjectivity of the morphism in both Corollary 2.3.2 and Corollary 2.3.4 implies that $R(X)$ is not generated in degree $[D]$, since any element of $H^0(X, D)$ can be written as a polynomial in homogeneous elements of other degrees.

2.4 Computing Cox rings of K3 surfaces

We start proving a consequence of Corollary 2.3.4 for K3 surfaces. We will denote by $\mathrm{BNef}(X)$ the Hilbert basis of the nef cone $\mathrm{Nef}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$.

Lemma 2.4.1. *Let X be a smooth projective K3 surface over $\mathbb{C}$ and let N_1, N_2, N_3 be nef divisors on X not linearly equivalent to zero. Then there exist N_1', N_2', N_3' nef, effective and non zero divisors such that $N_1 + N_2 + N_3 \sim N_1' + N_2' + N_3'$ and $N_1' \cap N_2' \cap N_3' = \emptyset$, unless $N_1 + N_2 + N_3 \sim 3(2F + E)$ or $N_1 + N_2 + N_3 \sim 2(2F + E) + F$, where F is a smooth elliptic curve, E is a smooth rational curve and $F \cdot E = 1$.*

Proof. If at least two of the three nef divisors N_i are base point free, then the statement clearly holds. If two of the N_i's, say N_1 and N_2, are not base point free then by Proposition 1.4.9 (ii) we can assume $N_1 \sim n_1F_1 + E_1$ and $N_2 \sim n_2F_2 + E_2$, where F_i is a smooth curve of genus one, E_i is a (-2)-curve with $F_i \cdot E_i = 1$ and $n_i \geq 2$ for $i = 1, 2$. There are five possible cases for the intersection graph of F_1, F_2, E_1, E_2, described in the following picture.



In the first case $E_1 \cdot E_2 > 0$, so in the remaining four cases we assume $E_1 \cdot E_2 \leq 0$. When $F_1 \sim F_2$ we denote it by F, and similarly for E_1, E_2. In each case we will explain how N_1', N_2' and N_3' can be chosen. The nefness of such divisors can be easily checked computing intersections. Moreover, base point freeness can be proved using Proposition 1.4.9 (ii) and Remark 1.4.12.

In the first case we take $N_1' \sim n_1F_1 + (n_2 - 1)F_2 + E_1 + E_2$, $N_2' \sim F_2$ and $N_3' \sim N_3$. Observe that N_1' and N_2' are both base point free, thus we conclude by the first remark in the proof.

In the second case, if $F_1 \cdot E_2 > 0$ we choose N_1' and N_2' as in the first case, and similarly if $F_2 \cdot E_1 > 0$. Otherwise we take $N_1' \sim N_1$, $N_2' \sim N_2$ and observe that the intersection of N_1 with N_2 is that of (two divisors linearly equivalent to) n_1F_1 and n_2F_2, which are both base point free, and thus it will not happen at the base locus of N_3.

In the third case we take $N_1' \sim (n_1 + n_2 - 1)F + E_1 + E_2$, $N_2' \sim F$ and $N_3' \sim N_3$. As before one concludes observing that N_1' and N_2' are base point free.

In the fourth case we take $N_1' \sim F_1 + F_2 + E$, $N_2' \sim (n_1 - 1)F_1 + (n_2 - 1)F_2 + E$ and $N_3' \sim N_3$. As before, N_1' and N_2' are both base point free.

So we are lead to consider only the fifth case. If N_3 has base locus then we can conclude applying the previous arguments to a pair N_3, N_j with $j \in \{1, 2\}$ unless $N_i \sim n_iF + E$ for $i = 1, 2, 3$. Moreover we can assume $n_1 = n_2 = n_3 = 2$ since otherwise one can take $N_1' \sim (n_1 + n_2 + n_3 - 5)F + E$, $N_2' \sim 4F + 2E$ and $N_3' \sim F$, with the last two divisors base point free. Thus we are left with the case $N_1 + N_2 + N_3 \sim 3(2F + E)$. Finally, assume that N_3 is base point free. If $N_3 \cdot E = 0$ we can take $N_i' \sim N_i$ for $i = 1, 2, 3$, since up to linear equivalence N_1 and N_2 can be chosen to meet only along E. If $N_3 \cdot E > 0$ we take $N_1' \sim (n_1 + n_2 - 2)F + E$, $N_2' \sim F$ and $N_3' \sim F + E + N_3$. Observe that N_2' is base point free and N_3' is base point free unless N_3 is a multiple of F. Thus we reduce to the case $N_1 \sim n_1F + E$, $N_2 \sim n_2F + E$ and $N_3 \sim mF$ with $m \geq 1$. Moreover we can assume that $n_1 = n_2 = 2$ and $m = 1$, since otherwise we can take $N_1' \sim N_2' \sim F$ and $N_3' \sim (n_1 + n_2 + m - 2)F + 2E$. Thus we are left with the case $N_1 + N_2 + N_3 \sim 2(2F + E) + F$. $\square$

Theorem 2.4.2. *Let X be a smooth projective K3 surface over $\mathbb{C}$. Then the degrees of a minimal set of generators of its Cox ring $R(X)$ are either:*

(*i*) *classes of* (-2)*-curves,*

(*ii*) *classes of nef divisors which are sums of at most three elements of the Hilbert basis of the nef cone (allowing repetitions),*

(*iii*) *or classes of divisors of the form* $2(F+F')$ *where* F, F' *are smooth elliptic curves with* $F \cdot F' = 2$*.*

Proof. By Corollary 2.1.5 $R(X)$ has a generator in all the degrees of the (-2)-curves. Assume now that D is a divisor which is not nef. If X contains no (-2)-curves, then the nef cone equals the closure of the effective cone [Huy16, Corollary 1.7, Chapter 8], thus D is not effective. On the other hand, if X contains (-2)-curves, then there exists a (-2)-curve C such that $D \cdot C < 0$ (see [Kov94, Theorem 1 and Theorem 2]). This implies that C is contained in the base locus of $|D|$, so that the multiplication map $H^0(X, D-C) \to H^0(X, D)$ by a non-zero element of $H^0(X, C)$ is surjective. Thus, unless D is a (-2)-curve itself, $R(X)$ has no generators in degree $[D]$. In what follows we assume D to be nef.

Assume that $D \sim \sum_{i=1}^{r} a_i N_i$, where the a_i's are positive integers with $\sum_{i=1}^{r} a_i \geq 4$ and $[N_i] \in \mathrm{BNef}(X)$. By the hypothesis on D we can find three effective nef divisors N_1, N_2, N_3 such that $D - \sum_{i=1}^{3} N_i$ is nef and not linearly equivalent to zero. Moreover, by Lemma 2.4.1, the divisors N_i can be chosen with $N_1 \cap N_2 \cap N_3 = \emptyset$, unless D is of the following types: $D \sim 4(2F+E)$ or $D \sim 3(2F+E)+F$, where F is a smooth elliptic curve and E is a (-2)-curve with $F \cdot E = 1$. Both cases are considered in Lemma 2.4.4. In what follows we assume that D is not of these types.

Let $A_{ij} := D - N_i - N_j$, with distinct $i, j \in \{1,2,3\}$. The divisors A_{ij} are nef, thus $h^1(X, A_{ij}) = 0$ by Proposition 1.4.9 part (*iii*), unless $A_{ij} \sim kF$, where F is a smooth elliptic curve and $k \geq 2$ is an integer. Moreover $h^2(X, D - N_1 - N_2 - N_3) = h^0(X, N_1 + N_2 + N_3 - D) = 0$ since $D - N_1 - N_2 - N_3$ is an effective non zero divisor.

Thus, unless $A_{ij} \sim kF$, we conclude by Corollary 2.3.4 part (i) with $E_i = N_i$ for $i = 1, 2, 3$, and Remark 2.3.5.

We now consider the case $A_{ij} \sim kF$, that is $D \sim N_i + N_j + kF$ with $k \geq 2$ and F as above. We have that $h^1(X, D - 2F) = 0$ unless $D \sim 2F + \ell F'$, where F' is a smooth elliptic curve and $\ell \geq 2$. This case is considered in Lemma 2.4.5, which shows that $R(X)$ is not generated in degree $[D]$ unless $D \sim 2(F + F')$ with $F \cdot F' = 2$.

Assuming that $D \not\sim 2F + \ell F'$ where F, F' are as above, we now prove that either $h^1(X, D - F - N_i)$ or $h^1(X, D - F - N_j)$ is zero. Assume on the contrary that these are both non zero. Then

$$N_j + (k-1)F \sim D - F - N_i \sim \ell_i F_i,$$

where the first equivalence is due to the assumption on D and the second one is due to Proposition 1.4.9 part (ii), with F_i a smooth elliptic curve and $\ell_i \geq 2$. By Remark 1.4.12 any element of $|\ell_i F_i|$ is a union of curves linearly equivalent to F_i This implies that $F \sim F_i$ and $N_j \sim (\ell_i - k + 1)F$. The same argument for $D - F - N_j$ gives that $N_i \sim (\ell_j - k + 1)F$. Thus $D \sim rF$ with $r \geq 4$. By Lemma 2.4.5 in this case $R(X)$ is not generated in degree $[D]$.

Thus we can assume that $h^1(X, D - F - N_i) = 0$. Taking $E_1, E_2 \in |F|$ distinct and $E_3 \in |N_i|$ we can thus conclude applying Corollary 2.3.4 part (i). Observe that $h^2(X, D - E_1 - E_2 - E_3) = h^2(N_j + (k-2)F) = 0$ since $N_j + (k-2)F$ is an effective non zero divisor. $\square$

Remark 2.4.3. Proposition 6.5 i) [AHL10] gives an example for a K3 surface X whose Cox ring has a generator in a degree as in case iii) of the previous theorem. The surface X is a K3 surface with $\mathrm{Cl}(X) \cong U(2)$, which has a non-symplectic involution i with $X/(i) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, in Theorem 3.2.1 there are examples for the case ii), i.e generators of $R(X)$ whose degree is the sum of two or three elements of the Hilbert basis of the nef cone (allowing repetitions). In this way, we can say that the previous theorem is optimal.

Lemma 2.4.4. *Let X be a K3 surface and let $D = 4(2F + E)$ or $D = 3(2F + E) + F$, where F is a smooth elliptic curve and E is a (-2)-curve with $F \cdot E = 1$. Then $R(X)$ has no generators in degree $[D]$.*

Proof. Assume that $D = 4(2F + E)$. Let $N_1 \sim 2(2F + E)$, $N_2 \sim F$ and $N_3 \sim E$. Since N_1 and N_2 are base point free by Proposition 1.4.9, then we can assume that $N_1 \cap N_2 \cap N_3 = \emptyset$. Observe that $h^1(D - N_1 - N_2) = h^1(3F + 2E) = 0$ since there is an exact sequence

$$H^1(3F + E) \to H^1(3F + 2E) \to H^1((3F + 2E)_{|E}) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = \{0\}$$

and $h^1(3F + E) = 0$ by Proposition 1.4.9, (iii). The same result implies that $h^1(D - N_2 - N_3) = h^1(7F + 3E) = 0$ and $h^1(D - N_1 - N_3) = h^1(4F + E) = 0$. Finally $h^2(D - N_1 - N_2 - N_3) = h^2(3F + E) = h^0(-3F - E) = 0$.

Now assume that $D = 3(2F + E) + F$. Let $N_1 \sim 5F + 2E$ and $N_2 \sim N_3 \sim F$. Since $N_2 \sim N_3$ is base point free, we can assume that $N_1 \cap N_2 \cap N_3 = \emptyset$. Observe that $D - N_1 - N_2 \sim F + E$, $D - N_2 - N_3 \sim 5F + 3E$ and $D - N_1 - N_2 - N_3 \sim E$. We have that $h^1(F + E) = h^1(5F + 3E) = 0$ by [SD74, Lemma 2.2] since both divisors are effective and connected. Moreover $h^2(E) = h^0(-E) = 0$.

In both the previous cases we conclude by Corollary 2.3.4, part (i) with $E_i = N_i$ for $i = 1, 2, 3$, and Remark 2.3.5. $\square$

Lemma 2.4.5. *Let X be a K3 surface and let $D = aF + bF'$, where F, F' are two smooth elliptic curves of X which are not linearly equivalent and a, b are integers with*

$a \geq b \geq 0$. *Then* $R(X)$ *has no generators in degree* $[D]$ *if one of the following holds:*

(i) $b = 0$ *and* $a \geq 2$*;*

(ii) $a \geq 3$*;*

(iii) $a = b = 2$ *and* $F \cdot F' > 2$.

Proof. If $D = aF$ with $a \geq 2$, then $R(X)$ is not generated in degree $[D]$ since $H^0(X, aF)$ is the a-th symmetric power of $H^0(X, F)$, see Remark 1.4.12.

Now assume $b > 0$ and $a \geq 3$. We now prove that $h^1(X, D - 2F) = 0$. If $h^1(X, D - 2F)$ is not zero, then by Proposition 1.4.9 *(ii)*

$$D - 2F \sim (a-2)F + bF' \sim rF'',$$

where $r \geq 2$ is an integer and F'' is a smooth elliptic curve. The previous relation implies that $((a-2)F + bF')^2 = 0$ and thus, since $a > 2$, $F \cdot F' = 0$. Since F, F' are fibers of elliptic fibrations, this means that $F \sim F'$, contradicting our hypothesis. Applying Corollary 2.3.2 with $E_1, E_2 \in |F|$ distinct we conclude that $R(X)$ has no generators in degree $[D]$.

We finally consider the case $a = b = 2$, that is when $D = 2(F + F')$, and $F \cdot F' > 2$. Thus $(F+F')^2 = 2F \cdot F' > 2$. Moreover, if E is any elliptic curve, then $(F+F') \cdot E > 2$. This is due to the fact that the fibers F, E of two distinct elliptic fibrations have intersection number at least two, since otherwise F would be mapped isomorphically to $\mathbb{P}^1$ by the morphism associated to $|E|$. This implies that $F + F'$ is not hyperelliptic by Proposition 1.4.8. By Proposition 2.2.6 (i) $R(X)_{[F+F']}$ is generated in degree one, in particular $R(X)$ is not generated in degree $[D]$. □

The following two results show that $R(X)$ is not generated in certain special degrees.

Lemma 2.4.6. *Let X be a K3 surface and $D = F + D'$ be a nef divisor, where F is nef with $F^2 = 0$ and D' is very ample. Assume that $F \sim E_1 + E_2$, where E_1, E_2 are (-2)-curves, and that the image of the natural map*

$$\phi : H^0(D - E_1) \oplus H^0(D - E_2) \to H^0(D)$$

has codimension two. Then $R(X)$ has no generator in degree $[D]$.

Proof. Observe that F defines an elliptic fibration $\varphi_{|F|} : X \to \mathbb{P}^1$ and E_1, E_2 are the components of a reducible fiber of $\varphi_{|F|}$. Thus E_1, E_2 intersect at two points p, q, which could be infinitely near. Let $V_{p,q} \subset H^0(D)$ be the subspace of sections vanishing at p and q. Since the image of ϕ has codimension two, then it coincides with $V_{p,q}$. Since D' is very ample there are two sections $s_1, s_2 \in H^0(D')$ such that $s_1(p)$ and $s_1(q)$ are not zero and such that $s_2(p) = 0$ and $s_2(q) \neq 0$. Let $t \in H^0(F)$ be a section not vanishing on $E_1 + E_2$. The sections $s_1 t$ and $s_2 t$, together with $V_{p,q}$, generate $H^0(D)$. □

Lemma 2.4.7. *Let X be a K3 surface, D be a nef and base point free divisor with $D^2 = 2$ and i be the covering involution of the associated double cover $\varphi_{|D|}$. The Cox ring $R(X)$ has no generator in degree $[3D]$ if there exists a (-2)-curve E which is not invariant for i and such that $3D - E$ is effective and base point free.*

Proof. Observe that $\mathrm{Sym}^3 H^0(D)$ is a codimension one subspace of $H^0(3D)$ and is the invariant subspace for the action of i^* on $H^0(3D)$. Since $3D - E$ is effective and base point free, then there exists a non-constant section in $H^0(3D)$ of the form ss_E, where $s_E \in H^0(E)$ and $s \in H^0(3D - E)$ is not divisible by $i^*(s_E)$. Such section is not i-invariant, thus it generates $H^0(3D)$ together with $\mathrm{Sym}^3 H^0(D)$. □

The following results, on the other hand, allow one to show under certain conditions that $R(X)$ has a generator in a certain degree.

Lemma 2.4.8. *Let X be a K3 surface and let $D = E_1 + E_2 + E_3$ be a base point free divisor, where E_1, E_2, E_3 are (-2)-curves such that $h^1(E_i + E_j) = 0$ for all distinct i, j. Then the natural map*

$$\psi : \bigoplus_{i=1,2,3} H^0(D - E_i) \to H^0(D)$$

is not surjective. Moreover, if $E_1 \cap E_2 \cap E_3 = \emptyset$ are disjoint, then the image of ψ has codimension one.

Proof. If $E_1 \cap E_2 \cap E_3$ is not empty, then ψ is clearly not surjective, since D is base point free. On the other hand, if $E_1 \cap E_2 \cap E_3$ is empty, then we can consider the associated Koszul exact sequence of sheaves in Theorem 2.3.1, which gives rise to the two short exact sequences (2.2) and (2.3). The first sequence, using the fact that $h^0(E_i) = 1, h^1(E_i) = h^2(E_i) = 0$ for $i = 1, 2, 3$ and $h^1(\mathcal{O}_X) = 0$, $h^2(\mathcal{O}_X) = 1$, gives $h^0(\mathrm{Im}(d_2)) = 2$ and $h^1(\mathrm{Im}(d_2)) = 1$. Using this and the fact that $h^1(E_i + E_j) = 0$ in the second sequence, we find that the image of ψ has codimension one in $H^0(D)$. $\square$

Proposition 2.4.9. *Let $G = \{w_0, \ldots, w_r\} \subset \mathrm{Cl}(X)$ containing the degrees of a homogeneous generating set of $R(X)$ and let $w_0 \in G$ be such that the associated linear system is base point free. Then $R(X)$ has a generator in degree $w_0 \in G$ if one of the following holds:*

- *(i) any linear combination $w_0 = \sum_{i=1}^r a_i w_i$ with $a_i \in \mathbb{Z}, a_i \geq 0$ contains the class of a (-2)-curve in its support;*
- *(ii) any linear combination $w_0 = \sum_{i=1}^r a_i w_i$ with $a_i \in \mathbb{Z}, a_i \geq 0$ contains in its support one of the classes of two (-2)-curves E_1, E_2 with $E_1 \cdot E_2 > 0$;*

(iii) $w_0 = w_1 + w_2 + w_3$, *where* w_i *are the classes of three* (-2)*-curves with* $h^1(w_i + w_j) = 0$ *for all distinct* i, j, *and any linear combination* $w_0 = \sum_{i=1}^r a_i w_i$ *with* $a_i \in \mathbb{Z}, a_i \geq 0$ *contains one among* w_1, w_2, w_3 *in its support.*

Proof. Let L be the linear system associated to w_0, let S be the subspace of $H^0(w_0)$ whose elements are polynomials in elements of $H^0(w_i)$, $i = 1, \ldots, r$, and let L_S be the corresponding subspace of L. The hypothesis in (i) says that any divisor in L_S contains E in its support. Similarly, the hypothesis in (ii) says that any divisor in L_S contains $E_1 \cap E_2$ in its support. Since L is base point free, this implies that $L \neq L_S$, so that $R(X)$ has a generator in degree w_0. Finally, if the hypothesis in (iii) holds, S is equal to the image of ψ in Lemma 2.4.8. On the other hand, since w_0 satisfies the hypothesis of Lemma 2.4.8, ψ is not surjective, i.e. $S \neq H^0(w_0)$. Thus $R(X)$ has a generator in degree w_0. □

We conclude this section recalling a result by Ottem [Ott13, Proposition 2.2].

Proposition 2.4.10. *Let* X *be a smooth projective K3 surface. Let* A *and* B *be nef divisors on* X *such that* $|B|$ *is base point free. Then the multiplication map*

$$H^0(X, A) \otimes H^0(X, B) \to H^0(X, A + B)$$

is surjective if $h^1(X, A - B) = h^1(X, A) = 0$ *and* $h^2(X, A - 2B) = 0$.

Chapter 3

K3 surfaces of Picard number three

By Theorem 2.2.5 there are 26 families of K3 surfaces with Picard number three whose general member has finitely generated Cox ring. These families have been identified and studied by V.V. Nikulin in [Nik84]. In this chapter we will determine the degrees of a generating set of $R(X)$ for each such family. For the Picard lattices of the families, we will use the notation in Theorem 2.2.5. The results in this Chapter are contained in the article [ACDL19].

3.1 Effective and nef cones

By Corollary 2.1.5 the Cox ring has a generator in each degree $w \in \mathrm{Cl}(X)$ in the Hilbert basis of $\mathrm{Eff}(X)$. As a first step, we determine the effective cone and the nef cone for all families. Observe that in [Nik84] the author already computed the set of (-2)-curves of each family.

Proposition 3.1.1. *Table 5.2 describes the extremal rays and the Hilbert bases of* $\mathrm{Eff}(X)$ *and* $\mathrm{Nef}(X)$ *for each of the* 26 *families of Mori dream K3 surfaces of Picard number three.*

Proof. A set of fundamental roots for the lattice $\mathrm{Cl}(X)$ is obtained by means of the algorithm described in section 1.4, implemented in the Magma (see section 6.2). The nef cone is thus obtained as the dual of the effective cone with respect to the intersection form of $\mathrm{Cl}(X)$. □

In the tables we will adopt this notation: $\mathrm{Eff}(X)$ is the effective cone and $\mathrm{BEff}(X)$ is its Hilbert basis, $E(X)$ is the list of classes of (-2)-curves, $\mathrm{Nef}(X)$ is the nef cone and $\mathrm{BNef}(X)$ is its Hilbert basis, $N(X)$ is the list of the extremal rays of $\mathrm{Nef}(X)$.

Remark 3.1.2. The classes of smallest positive self-intersection in the Hilbert basis of the nef cone define interesting projective models for Mori dream K3 surfaces of Picard number three. These models have been described in the recent papers [Rou20b] and [Rou20a].

3.2 Generators of $R(X)$

We now state and prove the main result of this chapter.

Theorem 3.2.1. *Let X be a Mori dream K3 surface of Picard number three. The degrees of a set of generators of the Cox ring $R(X)$ is given in Table 5.6. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.*

Proof. By Theorem 2.4.2 it is enough to consider the classes of (-2)-curves, nef classes which are sums of at most three elements of the Hilbert basis of the nef cone, and classes of divisors of the form $2(F+F')$, where F, F' are smooth elliptic curves with $F \cdot F' = 2$.

The classes of (-2)-curves and the Hilbert basis of the nef cone have been determined in Proposition 3.1.1. The only case where there exist two smooth elliptic curves F, F' with $F \cdot F' = 2$ is case 22, i.e. the family of K3 surfaces with Picard lattice isometric to $S_{1,1,2} \cong U(2) \oplus A_1$, whose Cox ring has been computed in [AHL10, Proposition 6.6, i)] (see Remark 3.3.6).

By Proposition 1.4.9 the linear system of any nef divisor of X is base point free unless there exists a smooth elliptic curve F and a (-2)-curve E such that $E \cdot F = 1$. This happens only for the case 21, i.e. the family of K3 surfaces with Picard lattice isometric to $S_{1,1,1} \cong U \oplus A_1$, whose Cox ring has been computed in [AHL10, Proposition 6.6, ii)] (see Remark 3.3.6). For the following arguments we will exclude the cases 21 and 22.

Given the list L of all nef classes which are sums of at most three elements of the Hilbert basis of the nef cone, we analyse it using the techniques in section 2.3, with the help of a Magma program described in section 6.3. More precisely, these are the main steps. We consider the following four sets:

$$T1 := \{\{A, B\} : A, B \in E(X) \cup \mathrm{BNef}(X),\ A \cdot B = 0\},$$

$$T2 := \{\{E_1, E_2, E_3\} : E_i \in E(X) \cup \mathrm{BNef}(X),\ E_3 \notin E(X)\},$$

$$T3 := \{(A, B) : A, B \in \mathrm{BNef}(X), h^1(A - B) = h^1(A) = h^0(2B - A) = 0\},$$

$$T4 := \{3A : A \in \mathrm{BNef}(X),\ A^2 \neq 2\} \cup \{2A : A \in \mathrm{BNef}(X), A \text{ is not hyperelliptic or } A^2 = 2\}.$$

We apply the following tests to any element $D \in L$:

Test 1. checks whether $h^1(X, D - A - B) = 0$ for some $\{A, B\} \in T1$. If this holds, then $R(X)$ has no generator in degree $[D]$ by Corollary 2.3.2 and Test 1 returns false.

Test 2. checks whether there exists $\{E_1, E_2, E_3\} \in T2$ such that $h^1(X, D-E_i-E_j) = 0$ for all $i, j \in \{1,2,3\}$ and $h^2(X, D-E_1-E_2-E_3) = 0$. If this holds, then $R(X)$ has no generator in degree $[D]$ by Corollary 2.3.4 and Test 2 returns false.

Test 3. checks whether D can be written as a sum $A+B$, where $(A,B) \in T3$. If this holds, then $R(X)$ has no generator in degree $[D]$ by Proposition 2.4.10 and Test 3 returns false.

Test 4. checks whether $[D] \in T4$. If this holds, then $R(X)$ has no generator in degree $[D]$ by Proposition 2.2.6 and Test 4 returns false.

Test 5. if D is a sum of two elements of $\mathrm{BNef}(X)$, it checks whether D satisfies the hypotheses of Lemma 2.4.6. If this holds, then $R(X)$ has no generator in degree $[D]$ and Test 5 returns false.

Test 6. if D is a sum of three elements of $\mathrm{BNef}(X)$, it checks the same property of Test 3, where A is a sum of two elements in $\mathrm{BNef}(X)$ and $B \in \mathrm{BNef}(X)$. If this holds, then $R(X)$ has no generator in degree $[D]$ by Proposition 2.4.10 and Test 6 returns false.

Let G be the set containing the degrees of all (-2)-curves and the degrees in L for which the tests are true. In order to determine which such degrees are necessary to generate $R(X)$, we apply Proposition 2.4.9 (which is implemented in the Magma function `Minimal`, see Section 6.4). Finally, we apply Lemma 2.4.7 to show that generators in certain degrees of type $[3D]$ are not necessary. □

3.3 Some special cases

Looking at Table 5.6 one can see that there are two cases where $R(X)$ is generated in six degrees (these are $S_{1,1,1}$ and $S_{1,1,2}$, see Remark 3.3.6) and other cases where it is generated in seven degrees: $S_1, S_5, S_{4,1,1}, S_{1,3,1}, S_{1,1,3}$ and $S_{1,1,4}$. In this section we describe the geometry of the families S_1 and $S_{4,1,1}$ and we provide a presentation for the Cox ring of a very general member of them. We expect that similar techniques can provide a presentation of $R(X)$ also in the remaining cases.

We observe that when we say that the surface X is very general, means that the coefficients of the equation that defines X belong to the complement of the union of countably many proper closed subsets of the parameter space.

Example 3.3.1 (Case S_1)**.** Let X be a K3 surface with $\mathrm{Cl}(X) \cong S_1 = (6) \oplus 2A_1$. We denote the natural generators of $\mathrm{Cl}(X)$ by e_1, e_2, e_3 with

$$e_1^2 = 6, \;\; e_2^2 = e_3^2 = -2, \;\; e_i \cdot e_j = 0 \;\text{ for } i \neq j.$$

By Proposition 3.1.1 the classes of the (-2)-curves can be taken to be:

$$\begin{aligned} f_1 &= e_2 & f_2 &= 2e_1 - 3e_2 - 2e_3 \\ f_3 &= e_3 & f_4 &= 2e_1 - 2e_2 - 3e_3 \\ f_5 &= e_1 - 2e_2 & f_6 &= e_1 - 2e_3. \end{aligned}$$

The Hilbert basis of the effective cone contains, besides the previous classes, the ample class $h = e_1 - e_2 - e_3$. We now determine a presentation for $R(X)$, in particular we show that it is a complete intersection.

Theorem 3.3.2. *Let X be a K3 surface with* $\mathrm{Cl}(X) \cong S_1 = (6) \oplus 2A_1$*. Then*

1. *there is a double cover* $\pi : X \to \mathbb{P}^2$ *branched along a smooth plane sextic with three* 6*-tangent conics* C_1, C_2, C_3*;*

2. X *can be defined by an equation of the following form in* $\mathbb{P}(1,1,1,3)$*:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)F_3(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   where F_1, F_2, F_3 *are homogeneous of degree* 2 *and* F *is homogeneous of degree* 3*;*

3. *the surface* X *contains six* (-2)*-curves: the curves* R_{ij}*, with* $i = 1,2,3$ *and* $j = 1,2$*, such that* $\pi(R_{i1}) = \pi(R_{i2}) = C_i$*;*

4. *the Cox ring of* X *has* 9 *generators:* $s_1, \ldots, s_6$ *defining the* (-2)*-curves and* $s_7, s_8, s_9 \in H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$*;*

5. *for a very general* X *as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \ldots, T_9]/I, \;\; s_i \mapsto T_i,$$

   where the degrees of the generators T_i *for* $i = 1, \ldots, 9$ *are given by the columns of the following matrix*

$$\begin{pmatrix} 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 0 & -2 & -2 & 0 & -1 & -1 & -1 \\ 0 & -2 & 1 & -3 & 0 & -2 & -1 & -1 & -1 \end{pmatrix}$$

   and the ideal I *is generated by the following polynomials:*

$$T_1T_4T_5 + T_2T_3T_6 - F(T_7,T_8,T_9),$$
$$T_1T_2 - F_1(T_7,T_8,T_9), \;\; T_3T_4 - F_2(T_7,T_8,T_9), \;\; T_5T_6 - F_3(T_7,T_8,T_9).$$

Proof. Let $h = e_1 - e_2 - e_3 \in \mathrm{Cl}(X)$, with the previous notation. Observe that $h^2 = 2$, h is ample and the associated linear system is base point free by Lemma 1.4.9, thus it defines a degree two covering $\pi : X \to \mathbb{P}^2$ ramified along a smooth sextic curve $B = \{\psi(x_0, x_1, x_2) = 0\}$. Since $2h = f_1 + f_2 = f_3 + f_4 = f_5 + f_6$, the image by π of the six (-2)-curves of X are three smooth conics $Q_1, Q_2, Q_3 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(Q_i)$ is the union of two smooth rational curves for each $i = 1, 2, 3$. Let D be the union of the conics Q_1, Q_2, Q_3. By looking at the intersection graph of the (-2)-curves one can see that $\pi_{|\pi^{-1}(D)} : \pi^{-1}(D) \to D$ is trivial, thus by Lemma 3.3.3 there exists a plane cubic $C = \{F(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$ such that $B \cdot D = 2C \cdot D$. Let $F_i = 0$ be an equation for Q_i, $i = 1, 2, 3$. Consider the pencil

$$G_{\lambda_0,\lambda_1}(x_0, x_1, x_2) := \lambda_0 F_1(x_0, x_1, x_2) F_2(x_0, x_1, x_2) F_3(x_0, x_1, x_2) + \lambda_1 F(x_0, x_1, x_2)^2$$

and let $\bar{\lambda}_0, \bar{\lambda}_1 \in \mathbb{C}$ be such that the curve $V = \{G_{\bar{\lambda}_0,\bar{\lambda}_1}(x_0, x_1, x_2) = 0\}$ intersects B in 19 distinct points, i.e. in $B \cap D$ and one more point. By Bezout's theorem, since B and V intersect in at least 37 points counting multiplicity and B is irreducibile, then $\psi(x_0, x_1, x_2) = \alpha G_{\bar{\lambda}_0,\bar{\lambda}_1}(x_0, x_1, x_2)$ for some $\alpha \in \mathbb{C}^*$. This proves the first part of the statement.

The degrees of the generators of $R(X)$ are given in Theorem 3.2.1. Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \ldots, s_6$ defining the (-2)-curves of X and a basis s_7, s_8, s_9 of $H^0(h)$. The last three relations are obvious, due to the fact that $s_1 s_2$ defines the preimage of the conic Q_1, similarly for the other two cases. Observe that $h^0(3h) = 11$, $\dim(\mathrm{Sym}^3 H^0(h)) = 10$ and $s_1 s_4 s_5, s_2 s_3 s_6 \in H^0(3h)$. Moreover, $\mathrm{Sym}^3 H^0(h)$ is the invariant subspace of $H^0(3h)$ for the natural action of the covering involution i of π. Since $s_1 s_4 s_5 + s_2 s_3 s_6$ is invariant, then it belongs to $\mathrm{Sym}^3 H^0(h)$. Given that $x_3 + F = 0$ and $x_3 - F = 0$ define the two preimages of the curve D, each mapping isomorphically onto D by π, thus each of them is the union of three smooth rational curves mapping to Q_1, Q_2, Q_3. Up to a renumbering the sections s_i we can assume that $x_3 + F = s_1 s_4 s_5$ and $x_3 - F = s_2 s_3 s_6$.

This gives the first relation, $(x_3+F)(x_3-F) = F_1 F_2 F_3$ implies that $x_3+F = s_1 s_4 s_5$ and $x_3 - F = s_2 s_3 s_6$, then $2F = s_1 s_4 s_5 - s_2 s_3 s_6$.

We will now prove that I is prime. Let g_1, g_2, g_3, g_4 be the generators of I (in the order given in the statement), let $X_0 = \mathbb{C}^9$, $X_i = V(g_1, \dots, g_i) \subset \mathbb{C}^9$ for $i = 1, 2, 3$ and L_i be the linear system on X_i generated by the divisors cut out by the monomials of g_{i+1}, for $i = 0, 1, 2, 3$. The key remark is that, by the generality assumptions on F and F_1, F_2, F_3, the zero set of g_{i+1} is the general element of the linear system L_i (up to a coordinate change in the variables s_i for g_1). The linear system L_i has no components in its base locus and is not composed with a pencil for each $i = 1, 2, 3$, since it can be easily checked that its subsystem generated by the monomials of g_{i+1} in s_7, s_8, s_9 already satisfies both properties. It follows that X_i is irreducible by Bertini's first theorem [Laz04, Theorem 3.3.1], i.e. I is prime. We recall that $R(X)$ is an integral domain [ADHL15, Theorem 1.5.1.1] and has Krull dimension equal to $\dim(X) + \operatorname{rank} \operatorname{Cl}(X) = 5$ [ADHL15, §1.6]. Since the ring $\mathbb{C}[T_1, \dots, T_9]/I$ is an integral domain, it has Krull dimension 5 and it surjects onto $R(X)$, then it is isomorphic to $R(X)$. $\square$

The following is well-known, see for example [Ver83, Proposition 1.7, Chapter 3].

Lemma 3.3.3. *Let $B \subset \mathbb{P}^2$ be a smooth plane curve of degree 6, let $\pi : X \to \mathbb{P}^2$ be the $2:1$ cover of $\mathbb{P}^2$ branched along B, and let $D \subset \mathbb{P}^2$ be a curve not containing components of B. The restriction of the cover:*

$$\pi|_{\pi^{-1}(D)} : \pi^{-1}(D) \to D$$

is trivial if and only if there exists a curve $C \subset \mathbb{P}^2$ of degree 3 such that $B \cdot D = 2C \cdot D$.

Note that, in the previous Lemma the restriction of the cover: $\pi|_{\pi^{-1}(D)} : \pi^{-1}(D) \to D$ is trivial if $\pi^{-1}(D)$ is the union of two irreducible curves C_1 and C_2 such that the restriction of π to each of them is an isomorphism $\pi|_{C_i} : C_i \to D$.

Example 3.3.4 (Case $S_{4,1,1}$)**.** Let X be a K3 surface with $\mathrm{Cl}(X) \cong S_{4,1,1}$, whose intersection matrix is

$$\begin{bmatrix} -32 & 0 & 4 \\ 0 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix}.$$

By Proposition 3.1.1 the classes of the (-2)-curves can be taken to be:

$$f_1 = e_2, \quad f_2 = e_3, \quad f_3 = e_1 + 3e_2 + 4e_3.$$

The Hilbert basis of the nef cone contains the ample class $h := e_1 + 4e_2 + 5e_3$, with $h^2 = 6$ and $h \cdot f_i = 2$ for $i = 1, 2, 3$, and three classes of elliptic fibrations $h_1 = e_2 + e_3$, $h_2 = e_1 + 3e_2 + 5e_3$ and $h_3 = e_1 + 4e_2 + 4e_3$.

Theorem 3.3.5. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong S_{4,1,1}$. Then*

1. *X can be defined as the zero set in $\mathbb{P}^4$ of two equations of the following form*

$$F(x_0, \dots, x_4) = 0, \; F_1(x_0, \dots, x_3)F_2(x_0, \dots, x_3)F_3(x_0, \dots, x_3) + x_4 G(x_0, \dots, x_4) = 0,$$

   where F, G are homogeneous of degree 2 and F_1, F_2, F_3 are homogeneous of degree 1;

2. *X contains three (-2)-curves defined by $F = F_i = x_4 = 0$ for $i = 1, 2, 3$;*

3. *the Cox ring of a very general X as before has seven generators: s_1, s_2, s_3 defining the (-2)-curves, t_1, t_2, t_3 defining the curves of equation $F = F_i = G = 0$ (of degree h_i, $i = 1, 2, 3$) and t be the restriction of a section in $H^0(\mathcal{O}_{\mathbb{P}^4}(1))$ independent from F_1, F_2, F_3;*

4. *for a very general X as before we have an isomorphism*

$$\begin{aligned} &R(X) \to \mathbb{C}[T_1, \ldots, T_7]/I \\ &\quad s_1 \mapsto T_1,\ s_2 \mapsto T_2,\ s_3 \mapsto T_3,\ t_1 \mapsto T_4,\ t_2 \mapsto T_5,\ t_3 \mapsto T_6,\ t \mapsto T_7, \end{aligned}$$

where the degrees of the generators T_i for $i = 1, \ldots, 7$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 1 & 3 & 4 & 4 \\ 0 & 1 & 4 & 1 & 5 & 4 & 5 \end{pmatrix}$$

and the ideal I is generated by the polynomials:

$$F(T_1T_2T_3, T_1T_5, T_2T_6, T_3T_4, T_7),$$
$$T_4T_5T_6 + G(T_1T_2T_3, T_1T_5, T_2T_6, T_3T_4, T_7).$$

Proof. Let $h := e_1 + 4e_2 + 5e_3 \in \mathrm{Cl}(X)$, with the previous notation. Observe that h is ample and not hyperelliptic, thus it defines an embedding of X in $\mathbb{P}^4$ as complete intersection of a quadric and a cubic hypersurface. Observe that

$$h = f_1 + h_2 = f_2 + h_3 = f_3 + h_1 = f_1 + f_2 + f_3, \quad 2h = h_1 + h_2 + h_3.$$

This means that X has three reducible hyperplane sections, which are union of a conic and an elliptic curve of degree 4. The three conics are contained in a hyperplane H and the three elliptic curves are contained in a quadric K. Up to a coordinate change we can assume that $H = \{x_4 = 0\}$. Each conic is contained in a plane, thus $X \cap H = \{x_4 = \tilde{F} = F_1F_2F_3 = 0\}$, where $\tilde{F}, F_i \in \mathbb{C}[x_0, x_1, x_2, x_3]$ are homogeneous, $\deg(\tilde{F}) = 2$ and $\deg(F_i) = 1$. This implies that X has an equation as in the statement with $F = \tilde{F} \pmod{x_4}$. Observe that the equation of the quadric K is $G = 0$. By Theorem 3.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1,\ f_2,\ f_3,\ h,\ h_1,\ h_2,\ h_3.$$

Let s_i be a generator of $H^0(f_i)$ for $i = 1, 2, 3$. Moreover let s_1s_2, t_1 be a basis of $H^0(h_1)$, s_2s_3, t_2 be a basis of $H^0(h_2)$ and s_1s_3, t_3 be a basis of $H^0(h_3)$. Observe that $h^0(h) = 5$ and $H^0(h)$ contains the subspace S generated by $s_1s_2s_3, s_1t_2, s_2t_3, s_3t_1$. Given a linear combination of such sections with coefficients $\alpha_1, \dots, \alpha_4 \in \mathbb{C}$ and evaluating it at a point p where $s_1(p) = s_2(p) = 0$ gives $\alpha_4 s_3 t_1(p) = 0$. By the generality assumption the three conics $s_i = 0$ do not have a common intersection, thus $s_3(p) \neq 0$. Moreover $t_1(p) \neq 0$ since t_1 defines a section of the elliptic fibration associated to h_1 which is distinct from $s_1s_2 = 0$. Thus $\alpha_4 = 0$. The same argument for the pairs s_1, s_3 and s_2, s_3 gives $\alpha_2 = \alpha_3 = 0$. Thus $\dim(S) = 4$. This implies that a set of generators for $R(X)$ is given by $s_1, s_2, s_3, t_1, t_2, t_3, t$, where $t \in H^0(h)$ with $t \notin S$. Among these generators there are relations of the following form:

$$F(T_1T_2T_3, T_1T_5, T_2T_6, T_3T_4, T_7) = 0,\ T_4T_5T_6 + G(T_1T_2T_3, T_1T_5, T_2T_6, T_3T_4, T_7) = 0,$$

where the first relation comes from the equation of the quadric containing X, while the second relation comes from the equation of the cubic containing X. In fact, using the geometric description of item 3 we can assume $F_1 = s_1t_2, F_2 = s_2t_3, F_3 = s_3t_1$ and $x_4 = s_1s_2s_3$, and replacing in the second equation of item 1, we obtain the second relation. It can be proved with the same type of argument used in the proof of Theorem 3.3.2 that the ideal I is prime for general F, F_i, G. Thus $\mathbb{C}[T_1, \dots, T_7]/I \cong R(X)$. $\square$

Remark 3.3.6 (Cases $S_{1,1,1}$ and $S_{1,1,2}$)**.** Generic K3 surfaces X with Picard lattices isometric to $U \oplus A_1 \cong S_{1,1,1}$ and $U(2) \oplus A_1 \cong S_{1,1,2}$ carry a non-symplectic involution ι acting trivially on their Picard group. The quotient surface $Y = X/\iota$ is the blow-up of a Hirzebruch surface $\mathbb{F}_4$ at one point in the first case and the blow-up of a smooth quadric surface at one point in the second case. A presentation of their Cox rings has been computed (see Proposition 2.2.7 in Section 2.2). In both cases a set of generators of $R(X)$ is given by the pull-back of a set of generators of $R(Y)$ together with elements defining the irreducible components of the fixed locus of ι (two in the first case and one in the second case).

Chapter 4

K3 surfaces of Picard number four

By Theorem 2.2.5 there are 14 families of K3 surfaces with Picard number four whose general member has finitely generated Cox ring. These families have been identified and studied by Vinberg in [Vin07]. In this chapter we will determine the degrees of a generating set of $R(X)$ for each such family, except when X is a K3 surface with $\mathrm{Cl}(X) \cong V_{14}$. The results in this Chapter are contained in the article [ACDR20].

4.1 Effective and nef cones

We start computing Hilbert bases of the effective cone and of the nef cone for all Mori dream K3 surfaces of Picard number 4. According to the author's knowledge, the following result is new.

Theorem 4.1.1. *Table 5.8 describes the extremal rays and the Hilbert bases of* $\mathrm{Eff}(X)$ *and of* $\mathrm{Nef}(X)$ *for each of the* 14 *families of Mori dream K3 surfaces of Picard number four.*

Proof. A set of fundamental roots for the lattice $\mathrm{Cl}(X)$ is obtained by means of the algorithm described in section 1.4, implemented in the Magma program Find -2 (see section 6.2). The nef cone is thus obtained as the dual of the effective cone with respect to the intersection form of $\mathrm{Cl}(X)$. $\square$

In the tables we will adopt the same notation we used for Picard number 3 in Chapter 3.

4.2 Generators of $R(X)$

Theorem 4.2.1. *Let X be a Mori dream K3 surface of Picard number four such that $\mathrm{Cl}(X)$ is not isometric to V_{14}. The degrees of a set of generators of the Cox ring $R(X)$ are given in Table 5.12. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.*

Proof. By Theorem 2.4.2 it is enough to consider the classes of (-2)-curves, nef classes which are sums of at most three elements of the Hilbert basis of the nef cone, and classes of divisors of the form $2(F+F')$, where F, F' are smooth elliptic curves with $F \cdot F' = 2$.

The classes of (-2)-curves and the Hilbert basis of the nef cone have been determined in Theorem 4.1.1.

By Proposition 1.4.9 the linear system of any nef divisor is base point free unless there exists a smooth elliptic curve F and a (-2)-curve E such that $E \cdot F = 1$. This happens only for the families of K3 surfaces with Picard lattice isometric to $V_4 = U \oplus 2A_1$ and $V_8 = U \oplus A_2$. In case $\mathrm{Pic}(X) \cong V_4$ the Cox ring has been computed in [AHL10, Proposition 6.7, ii)] (see Proposition 2.2.7). Moreover, there are two cases where the K3 surface has two smooth elliptic curves F, F' with $F \cdot F' = 2$, the families of K3 surfaces with Picard lattice isometric to $V_5 = U(2) \oplus 2A_1$ and $V_9 = U(2) \oplus A_2$.

In case $\mathrm{Pic}(X) \cong V_5$ the Cox ring has been computed in [AHL10, Proposition 6.7, i)] (see Proposition 2.2.7). The cases where $\mathrm{Pic}(X) \cong V_8$ or V_9 correspond to families $\mathcal{F}_8$ and $\mathcal{F}_9$ respectively (which we will study in section 4.3 below). For the following arguments we exclude the above cases, which will be described in the next section.

Given the list L of all nef classes which are sums of at most three elements of the Hilbert basis of the nef cone, we analyse it using the techniques in section 2.3, with the help of a Magma program described in section 6.3.

As in the proof of Theorem 3.2.1, we can make a list containing the degrees of all (-2)-curves and the degrees in L for which the tests are true, we apply Proposition 2.4.9 to determine which degrees of the list are necessary, and we apply Lemma 2.4.7 to show that generators in certain degrees of type $[3D]$ are not necessary.

For more details, see the proof of the Theorem 3.2.1, as it proceeds analogously. □

4.3 Geometry and projective models

In this section we will provide a projective model for each family of Mori dream K3 surfaces of Picard number 4 and we identify geometrically the degrees of the generators of the Cox ring. We will call $\mathcal{F}_i$ the family of K3 surfaces whose Picard lattice is isometric to V_i, with the notation in Theorem 2.2.5.

The family $\mathcal{F}_1$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_1 = (8) \oplus 3A_1$. By Theorem 4.1.1 X contains 12 (-2)-curves. Moreover, the Hilbert basis of the nef cone of X contains 51 classes, with six classes of elliptic fibrations:

$$\mathrm{BNef}[1],\ \mathrm{BNef}[6],\ \mathrm{BNef}[11],\ \mathrm{BNef}[27],\ \mathrm{BNef}[29],\ \mathrm{BNef}[35].$$

Each elliptic fibration is without sections and has two fibers of type $\tilde{A}_1$ (see [BH-PVdV04, Chapter V, §7]).

Proposition 4.3.1. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_1 = (8) \oplus 3A_1$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with six 6-tangent conics $C_1, \dots, C_6$;*
2. *the surface X has twelve (-2)-curves: the curves R_{ij}, $i = 1, \dots, 6$, $j = 1, 2$, such that $\pi(R_{i1}) = \pi(R_{i2}) = C_i$;*
3. *the Cox ring of X has 15 generators $s_1, \dots, s_{15}$, where $s_1, \dots, s_{12}$ are defining sections of the (-2)-curves and s_{13}, s_{14}, s_{15} is a basis of $H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$.*

Proof. Let $f_1, \dots, f_{12}$ be the classes of the (-2)-curves (see Table 5.8) and let $h = \mathrm{BNef}[4]$. Then $h^2 = 2$ and $h \cdot f_i = 2$ for all i, thus h is ample and the associated linear system is base point free by Corollary 1.4.11. Thus h defines a morphism $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic B. Since

$$2h = f_7 + f_{10} = f_4 + f_{12} = f_8 + f_9 = f_3 + f_6 = f_5 + f_{11} = f_1 + f_2,$$

the image by π of the twelve (-2)-curves of X are six smooth conics $C_1, \dots, C_6 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(C_i)$ is the union of two smooth rational curves R_{ij} with $j = 1, 2$, for any $i = 1, \dots, 6$. This implies that the conics are tangent to B at 6 points.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_{12}, \; h.$$

Observe that h is an element of the Hilbert basis of the effective cone of X (see

Table 5.8). Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \ldots, s_{12}$ defining the (-2)-curves of X and a basis s_{13}, s_{14}, s_{15} of $H^0(h)$. □

The family $\mathcal{F}_2$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_2 = (4) \oplus (-4) \oplus A_2$. By Theorem 4.1.1 X contains six (-2)-curves. The Hilbert basis of the nef cone contains 35 classes, with four classes of elliptic fibrations:

$$\mathrm{BNef}[5], \ \mathrm{BNef}[11], \ \mathrm{BNef}[23], \ \mathrm{BNef}[29].$$

Each elliptic fibration is without sections and has one singular fiber of type $\tilde{A}_2$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.2. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_2 = (4) \oplus (-4) \oplus A_2$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with two 3-tangent lines L_1, L_2 and one 6-tangent conic C;*

2. *the surface has six (-2)-curves: the curves R_{ij}, $i, j = 1, 2$ such that $\pi(R_{i1}) = \pi(R_{i2}) = L_i$ and the curves S_1, S_2 such that $\pi(S_1) = \pi(S_2) = C$;*

3. *the Cox ring has at least 23 generators: $s_1, \ldots, s_6$ defining the (-2)-curves, $s_7, \ldots, s_{10}$ defining each a smooth fiber of the four elliptic fibrations, $s_{11} \in H^0(h)$ independent from the elements defining $\pi^{-1}(L_1), \pi^{-1}(L_2)$, $s_{12}, \ldots, s_{23}$ whose degrees are elements of the Hilbert basis of the nef cone with self-intersections 4 (for $i = 12, \ldots, 15$), 10 (for $i = 16, \ldots, 19$) and 12 (for $i = 20, \ldots, 23$).*

Proof. Let $f_1, \dots, f_6$ be the classes of the (-2)-curves (see Table 5.8) and $h = \mathrm{BNef}[15]$. Then $h^2 = 2$, $h \cdot f_1 = h \cdot f_3 = 2$ and $h \cdot f_i = 1$ for $i \neq 1, 3$. Thus h is ample and the associated linear system $|h|$ is base point free by Corollary 1.4.11. Thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic B. Since $h = f_2 + f_5 = f_4 + f_6$ and $2h = f_1 + f_3$, the image by π of the six (-2)-curves of X are a smooth conic $C \subseteq \mathbb{P}^2$ and two smooth lines $L_1, L_2 \subseteq \mathbb{P}^2$. The last statement follows from Theorem 4.2.1. Observe that we clearly need one generator for each (-2)-class, one generator for each elliptic fibration (since one of its fibers is reducible) and one generator in degree h . In the remaining degrees, we can only conclude that there is at least one generator by Proposition 2.4.9. □

The family $\mathcal{F}_3$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_3 = (4) \oplus A_3$. By Theorem 4.1.1 X contains five (-2)-curves and the Hilbert basis of the nef cone of X contains 10 classes. The surface has a unique elliptic fibration, defined by the class BNef[3], which has no sections and has two reducible fibers of type $\tilde{A}_1$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.3. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_3 = (4) \oplus A_3$. Then*

1. *there is a minimal resolution $\varphi : X \to Y$ of a double cover $\pi : Y \to \mathbb{P}^2$ branched along a plane sextic with a double point p with two bitangent lines L_1, L_2 passing through p;*

2. *the surface has five (-2)-curves: the exceptional divisor E over the singular point $\pi^{-1}(p)$ and four curves R_{ij}, with $i, j = 1,\ 2$, where $\pi\varphi(R_{ij}) = L_i$;*

3. *the Cox ring of X has at least ten generators: $s_1, \dots, s_5$ defining the (-2)-curves, $s_6 \in H^0(h)$ and $s_7, \dots, s_{10}$ whose degree is a class in the Hilbert basis*

of the nef cone with self-intersection 4.

Proof. Let $f_1, \ldots, f_5$ be the classes of the (-2)-curves (see Table 5.8) and $h = \mathrm{BNef}[2]$. Then $h^2 = 2$, $h \cdot f_5 = 0$ and $h \cdot f_i = 1$ for $i \neq 5$. Thus h is nef. By Corollary 1.4.11 the linear system associated to h is base point free and thus defines a morphism $\psi : X \to \mathbb{P}^2$ which contracts the (-2)-curve with class f_5 to a point $p \in \mathbb{P}^2$ and is branched along a plane sextic B with one node at p. Since $h = f_1+f_4+f_5 = f_2+f_3+f_5$ the image by ψ of the four (-2)-curves of classes $f_1, \ldots, f_4$ are two lines L_1, L_2 passing through p and tangent to B in two more points.

By Theorem 4.2.1 a minimal set of generators of the Cox ring $R(X)$ has the following degrees:

$$f_1, \ldots, f_5,\ h,\ h_1, h_2, h_3, h_4,$$

where $h_i = \mathrm{BNef}[i]$ with $i = 1, 4, 7, 9$. Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \ldots, s_5$ defining the (-2)-curves of X and an element s_6 such that $s_1s_4s_5, s_2s_3s_5, s_6$ is a basis of $H^0(h)$. In the remaining degrees $h_1, \ldots, h_4$ we can only conclude that there is at least one generator by Proposition 2.4.9. □

The family $\mathcal{F}_4$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_4 = U \oplus 2A_1$. By Theorem 4.1.1 X contains five (-2)-curves and the Hilbert basis of the nef cone of X contains five classes. The surface has a unique elliptic fibration, defined by the class $\mathrm{BNef}[5]$, which has a section and two reducible fibers of type $\tilde{A}_1$ (by [BHPVdV04, Chapter V, §7]).

In this case, the Cox ring has been computed in [AHL10, Proposition 6.7, ii)] (see Proposition 2.2.7), where X is described as a double cover of the Hirzebruch surface $\mathbb{F}_4$ blown-up at two general points p, q. We will use the following notation for curves in $\mathbb{F}_4$: F_1, F_2 are the two fibers of the projection $\mathbb{F}_4 \to \mathbb{P}^1$ passing through p and q,

S_1 is the curve in $\mathbb{F}_4$ with $S_1^2 = -4$ and S_2 is a curve passing through p and q with $S_2^2 = 4$ and $S_1 \cdot S_2 = 0$. We will call S_1 also its pull-back in the blow-up of $\mathbb{F}_4$ at p, q.

We recall these properties in the following proposition:

Proposition 4.3.4. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_4 = U \oplus 2A_1$. Then*

1. *X is a double cover $\pi : X \to Y$, where Y is a blow-up of $\mathbb{F}_4$ at two general points p, q, branched along S_1 and a smooth curve B of genus 8;*

2. *the surface has five (-2)-curves, which are the preimages by π of: S_1, the proper transforms of F_1, F_2 and the two exceptional divisors over p and q;*

3. *the Cox ring of X has 7 generators: $s_1, \dots, s_5$ defining the (-2)-curves, s_6 defining the preimage of the proper transform of S_2 and s_7 defining $\pi^{-1}(B)$;*

4. *for a very general X as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \dots, T_7]/I, \;\; s_i \mapsto T_i,$$

where the degrees of the generators T_i for $i = 1, \dots, 7$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & -1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the ideal I is generated by the following polynomial

$$T_7^2 - F(T_1, \dots, T_6),$$

where $F(T_1,\dots,T_6)=f(T_1^2,T_2,\dots,T_6)$ *and* f *is the defining polynomial of* B *in the Cox ring of* Y*.*

The family $\mathcal{F}_5$

Let X be a K3 surface with $\mathrm{Cl}(X)\cong V_5=U(2)\oplus 2A_1$. By Theorem 4.1.1 X contains six classes of (-2)-curves. The Hilbert basis of the nef cone of X contains five classes, three of them are classes of elliptic fibrations:

$$\mathrm{BNef}[3],\ \mathrm{BNef}[4],\ \mathrm{BNef}[5]$$

without sections and each having two reducible fibers of type $\tilde{A}_1$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.5. *Let* X *be a K3 surface with* $\mathrm{Cl}(X)\cong V_5=U(2)\oplus 2A_1$*. Then*

1. *there is a minimal resolution* $\varphi: X\to Y$ *of a double cover* $\pi: Y\to\mathbb{P}^2$ *branched along a plane sextic* B *with three nodes* p_1,p_2,p_3*;*

2. *the surface has six* (-2)*-curves: the exceptional divisors* E_1,E_2,E_3 *over the three nodes* p_1,p_2,p_3 *and the curves* R_1,R_2,R_3 *such that* $\pi\varphi(R_i)$ *is the line through* p_j,p_k *with* $j,k\neq i$*;*

3. *the Cox ring of* X *has* 7 *generators:* $s_1,\dots,s_6$ *defining the* (-2)*-curves and* s_7 *defining the ramification curve;*

4. *for a very general* X *as before we have an isomorphism*

$$R(X)\to\mathbb{C}[T_1,\dots,T_7]/I,\ s_i\mapsto T_i,$$

where the degrees of the generators T_i for $i = 1, \dots, 7$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 & -2 \\ -1 & -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and the ideal I is generated by the following polynomial:

$$T_7^2 - f(T_1T_3T_5, T_2T_5T_6, T_1T_4T_6),$$

where $f(x_0, x_1, x_2)$ is a polynomial which defines the plane sextic B.

Proof. Let $f_1, \dots, f_5$ be the classes of the (-2)-curves (see Table 5.8) and let $h =$ BNef[1]. Then $h^2 = 2$, $h \cdot f_i = 0$ for $i = 1, 5, 6$ and $h \cdot f_i = 2$ for $i = 2, 3, 4$. Thus h is nef. By Corollary 1.4.11 the linear system associated to h is base point free and thus defines a degree two morphism to $\mathbb{P}^2$ branched along a plane sextic B which contracts the (-2)-curves of classes f_1, f_5, f_6. Since such classes have zero intersection, then the three (-2)-curves are disjoint, thus are mapped to three distinct points $p_1, p_2, p_3 \in \mathbb{P}^2$. The plane sextic B has three nodes at p_1, p_2, p_3. Moreover, since $h = f_1 + f_3 + f_5 = f_2 + f_5 + f_6 = f_1 + f_4 + f_6$, the (-2)-curves with classes f_2, f_3, f_4 are mapped to the three lines passing through p_2, p_3, p_1, p_2 and p_1, p_3 respectively.

As in the previous case, the Cox ring has been computed in [AHL10, Proposition 6.7, i)] (see Proposition 2.2.7), where X is also described as the double cover of a smooth quadric surface $\mathbb{F}_0$ blown-up at two general points. □

The family $\mathcal{F}_6$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_6 = U(3) \oplus 2A_1$. By Theorem 4.1.1 X contains eight (-2)-curves. The Hilbert basis of the nef cone of X contains 19 classes, four of them are classes of elliptic fibrations:

$$\mathrm{BNef}[5],\ \mathrm{BNef}[8],\ \mathrm{BNef}[18], \mathrm{BNef}[19].$$

Each elliptic fibration has no sections and has two fibers of type $\tilde{A}_1$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.6. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_6 = U(3) \oplus A_1 \oplus A_1$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with two 3-tangent lines L_1, L_2 and two 6-tangent conics C_1, C_2;*

2. *X can be defined by an equation of the following form in $\mathbb{P}(1,1,1,3)$:*

$$x_3^2 = F_1(x_0,x_1,x_2)F_2(x_0,x_1,x_2)G_1(x_0,x_1,x_2)G_2(x_0,x_1,x_2) + F(x_0,x_1,x_2)^2,$$

   where $F \in \mathbb{C}[x_0,x_1,x_2]$ is homogeneous of degree three, $G_1, G_2 \in \mathbb{C}[x_0,x_1,x_2]$ are homogeneous of degree two and $F_1, F_2 \in \mathbb{C}[x_0,x_1,x_2]$ are homogeneous of degree one;

3. *the surface has eight (-2)-curves: the four curves R_{ij}, $i,j = 1,2$ such that $\pi(R_{i1}) = \pi(R_{i2}) = L_i$ and the four curves S_{i1}, S_{i2} such that $\pi(S_{i1}) = \pi(S_{i2}) = C_i$ for $i = 1,2$;*

4. *the Cox ring of X has 9 generators: $s_1, \dots, s_8$ defining the (-2)-curves and $s_9 \in H^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$;*

5. *for a very general X as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \dots, T_9]/I, \ s_i \mapsto T_i,$$

where the degrees of the generators T_i for $i = 1, \dots, 9$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & -2 & 0 & 0 & -1 & -2 & 0 & -1 & -1 \\ 0 & -2 & -1 & 0 & 0 & -2 & -1 & 0 & -1 \\ 0 & -3 & 0 & 1 & 0 & -2 & -1 & -1 & -1 \\ 1 & -2 & -1 & 0 & -1 & -3 & 0 & 0 & -1 \end{pmatrix}$$

and the ideal I is generated by the following polynomials:

$$T_1T_6 - G_1(T_3T_8, T_5T_7, T_9),$$
$$T_2T_4 - G_2(T_3T_8, T_5T_7, T_9),$$
$$T_1T_2T_3T_5 + T_4T_6T_7T_8 - F(T_3T_8, T_5T_7, T_9).$$

Proof. Let $f_1, \dots, f_8$ be the classes of the (-2)-curves (see Table 5.8) and $h =$ BNef[15]. Then $h^2 = 2$, $h \cdot f_i = 2$ for $i = 1, 2, 4, 6$ and $h \cdot f_i = 1$ for $i = 3, 5, 7, 8$. Thus h is ample. By Corollary 1.4.11 the associated linear system is base point free and thus defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic. Since $h = f_3 + f_8 = f_5 + f_7$ and $2h = f_1 + f_6 = f_2 + f_4$ the image by π of the eight (-2)-curves of X are two 3-tangent lines L_1, L_2 and two 6-tangent conics $C_1, C_2 \subseteq \mathbb{P}^2$.

The proof of item 2. is similar to the proof of Theorem 3.3.2.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_8,\ h.$$

Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \dots, s_8$ defining the (-2)-curves of X and a section s_9 such that s_3s_8, s_5s_7, s_9 is a basis of $H^0(h)$. The first two relations are obvious, due to the fact that s_1s_6 and s_2s_4 define the preimage of the conics $C_1, C_2 \subseteq \mathbb{P}^2$ defined by $C_1 := \{G_1 = 0\}$ and $C_2 := \{G_2 = 0\}$. Observe that, $h^0(3h) = 11$, $\dim(\mathrm{Sym}^3 H^0(h)) = 10$ and $s_1s_2s_3s_5, s_4s_6s_7s_8 \in H^0(3h)$. Moreover, $\mathrm{Sym}^3 H^0(h)$ is the invariant subspace of $H^0(3h)$ for the natural action of the covering involution i of π. Since $s_1s_2s_3s_5 + s_4s_6s_7s_8$ is invariant, then it belongs to $\mathrm{Sym}^3 H^0(h)$. The last relation, follows from the fact that $(x_3 + F)(x_3 - F) = F_1F_2G_1G_2$ thus we can assume $x_3 + F = s_1s_2s_3s_5$, $x_3 - F = s_4s_6s_7s_8$, so that $2F = s_1s_2s_3s_5 - s_4s_6s_7s_8$ up to rescaling the generators s_i we obtain the last relation. It can be proved with the same type of argument used in the proof of Theorem 3.3.2 that the ideal I is prime for general F, G_1, G_2. Since $R(X)$ is an integral domain of dimension $\dim(X) + \operatorname{rank}\mathrm{Cl}(X) = 6$, then $\mathbb{C}[T_1, \dots, T_9]/I \cong R(X)$.

$\square$

The family $\mathcal{F}_7$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_7 = U(4) \oplus A_1 \oplus A_1$. By Theorem 4.1.1 X contains eight (-2)-curves and the Hilbert basis of the nef cone of X contains 15 classes, with six classes of elliptic fibrations BNef[4], BNef[7], BNef[9], BNef[13], BNef[14] and BNef[15]. Each elliptic fibration is without sections and has two fibers of type $\tilde{A}_1$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.7. *Let X be a K3 surface with* $\mathrm{Cl}(X) \cong V_7 = U(4) \oplus A_1 \oplus A_1$. *Then*

1. *X is isomorphic to a smooth quartic surface in $\mathbb{P}^3$ having four hyperplane sections which are the union of two conics;*

2. *a general X can be defined by an equation of the form*

$$G_1(x_1, \dots, x_3)G_2(x_0, \dots, x_3) + F_1(x_0, \dots, x_3)F_2(x_0, \dots, x_3)F_3(x_0, \dots, x_3)F_4(x_0, \dots, x_3) = 0,$$

   where $G_1, G_2 \in \mathbb{C}[x_0, \dots, x_3]$ are homogeneous of degree two and $F_i \in \mathbb{C}[x_0, \dots, x_3]$ are homogeneous of degree one for $i = 1, 2, 3, 4$;

3. *the surface has eight (-2)-curves: the eight conics;*

4. *the Cox ring of X has 8 generators: the sections $s_1, \dots, s_8$ defining the (-2)-curves, where s_6s_7, s_3s_5, s_1s_8 and s_2s_4 define the four reducible hyperplane sections;*

5. *for a very general X as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \dots, T_8]/I, \ s_i \mapsto T_i,$$

   where the degrees of the generators T_i for $i = 1, \dots, 8$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and the ideal I is generated by the following polynomials:

$$T_1T_2T_3T_6 - G_1(T_6T_7, T_3T_5, T_1T_8, T_2T_4),$$
$$T_4T_5T_7T_8 - G_2(T_6T_7, T_3T_5, T_1T_8, T_2T_4).$$

Proof. Let $f_1, \dots, f_8$ be the classes of the (-2)-curves and let $h = \text{BNef}[11]$. Then $h^2 = 4$ and $h \cdot f_i = 2$ for all i. Thus h is ample. Moreover, h is not hyperelliptic by Proposition 1.4.8. By Corollary 1.4.11 the associated linear system $|h|$ is base point free, thus it defines an embedding of X in $\mathbb{P}^3$ as a smooth quartic surface. Since $h = f_1 + f_8 = f_2 + f_4 = f_3 + f_5 = f_6 + f_7$ and $h \cdot f_i = 2$ for all i, then X has four hyperplane sections which decompose in the union of two conics. We will denote by C_i the conic whose class is f_i. Observe that $f_1 + f_2 + f_3 + f_6 = f_4 + f_5 + f_7 + f_8 = 2h$. This means that the four conics C_1, C_2, C_3, C_6 are contained in a quadric. The same holds for the conics C_4, C_5, C_7, C_8.

Consider the family $\mathcal{F}$ of quartic surfaces Y in $\mathbb{P}^3$ defined by an equation of the form $G_1G_2 + F_1F_2F_3F_4 = 0$, where $G_1, G_2 \in \mathbb{C}[x_0, \dots, x_3]$ are homogeneous polynomials of degree two and $F_i \in \mathbb{C}[x_0, \dots, x_3]$ are homogeneous polynomials of degree one for $i = 1, 2, 3, 4$. Observe that the intersection of Y with any hyperplane $\ell_i = 0$ is the union of two conics C_{i1}, C_{i2} defined by $G_1 = 0$ and $G_2 = 0$. In particular, for a general choice of the polynomials $G_1, G_2, F_1, \dots, F_4$ we have $C_{i1} \cdot C_{i2} = 4$, $C_{ij} \cdot C_{i'j} = 2$ for $i \neq i'$ and $j = 1, 2$ and $C_{ij} \cdot C_{i'j'} = 0$ for $i \neq i'$ and $j \neq j'$. An easy computation shows that the classes of $C_{11} + C_{21}, C_{12} + C_{32}, C_{21}, C_{32}$ generate the Picard lattice. Thus a general member of $\mathcal{F}$ belongs to the family $\mathcal{F}_7$. Moreover, since the dimension of such family of hypersurfaces is 16, the general element of $\mathcal{F}_7$ belongs to $\mathcal{F}$.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_8, \ h.$$

Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \dots, s_8$ defining the (-2)-curves of X. Observe that in the equation of an element of $\mathcal{F}$ the four polynomials $F_1, \dots, F_4$ can be taken to be independent, so that they generate $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. This implies that generically $s_1s_8, s_2s_4, s_3s_5, s_6s_7$ generate $H^0(h)$. Thus a new generator in degree $H^0(h)$ is not necessary. The two relations are, due to the fact that $s_1s_2s_3s_6$ and $s_4s_5s_7s_8$ define the quadrics $Q_1, Q_2 \in \mathbb{P}^3$ defined by $Q_1 := \{G_1 = 0\}$ and $Q_2 := \{G_2 = 0\}$.

It can be proved with the same type of argument used in the proof of Theorem 3.3.2 that the ideal I is prime for general G_1, G_2. Thus $\mathbb{C}[T_1, \dots, T_8]/I \cong R(X)$, since it is an integral domain of dimension $\dim R(X) = \dim(X) + \operatorname{rank} \operatorname{Cl}(X) = 6$. $\square$

The family $\mathcal{F}_8$

Let X be a K3 surface with $\operatorname{Cl}(X) \cong V_8 = U \oplus A_2$. By Theorem 4.1.1 X contains four (-2)-curves and the Hilbert basis of the nef cone of X contains five classes. The class BNef[5] defines an elliptic fibration having a section and one fiber of type $\tilde{A}_2$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.8. *Let X be a K3 surface with $\operatorname{Cl}(X) \cong V_8 = U \oplus A_2$. Then*

1. *there is a degree two morphism $\varphi : X \to \mathbb{P}^4$, whose image is a cone over a rational normal cubic in $\mathbb{P}^3$, which factors through a degree two morphism $\mu : X \to \mathbb{F}_3$ branched along the union of the smooth rational curve E with $E^2 = -3$ and a reduced curve B intersecting E at one point p and the general*

fiber of $\mathbb{F}_3$ at 3 points;

2. *the surface has four (-2)-curves: $\mu^{-1}(E)$, $\mu^{-1}(p)$ and two smooth rational curves mapping to the fiber of $\mathbb{F}_3$ through p;*

3. *the Cox ring $R(X)$ has at least 8 generators.*

Proof. Let $f_1, \ldots, f_4$ be the classes of the (-2)-curves, where f_3 is the class of the section of the elliptic fibration defined by BNef[5], and $h = \mathrm{BNef}[3]$. Then $h^2 = 6$, $h \cdot f_i = 1$ for $i = 1, 4$ and $h \cdot f_i = 0$ for $i = 2, 3$. Since $h \cdot \mathrm{BNef}[5] = 2$ then the associated linear system is hyperelliptic by Proposition 1.4.8. Moreover, it is base point free by Proposition 1.4.9 ii), since $h \neq 4\,\mathrm{BNef}[5] + f_3$. Let φ be the associated morphism. Since $h = 3\,\mathrm{BNef}[5] + 2f_3 + f_2$ where $\mathrm{BNef}[5] \cdot f_2 = 0$, $\mathrm{BNef}[5] \cdot f_3 = 1$ and $f_2 \cdot f_3 = 1$, then h satisfies the hypothesis of [SD74, Proposition 5.7, ii)] thus $\varphi(X)$ is a cone over a rational normal twisted cubic in $\mathbb{P}^3$. Moreover, by [SD74, (5.9.2)] we have the description of φ given in the statement.

To determine the degrees of the generators of the Cox ring, we notice that by Proposition 1.4.9 ii), the linear system associated to a nef divisor $D \sim k\,\mathrm{BNef}[5] + f_3$, for $k \geq 2$, is not base point free, and the only divisor in the Hilbert basis of the nef cone with this property is BNef[4]. We apply all Tests excluding from the sets $T_1, \cdots, T_5$ the element BNef[4].

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_4,\ h_1,\ h_2,\ h_3,\ h_5.$$

Clearly $R(X)$ must contain the sections $s_1, \ldots, s_4$ defining the (-2)-curves and a section in degree $h_5 := \mathrm{BNef}[5]$. By Proposition 1.4.8, the classes $h_i := \mathrm{BNef}[i]$ for $i = 1,\ 2,\ 5$ are non-hyperelliptic in the Hilbert basis of the nef cone, and $h_3 := \mathrm{BNef}[3]$

is hyperelliptic, also we have to $h_1^2 = h_2^2 = 12$, $h_3^2 = 6$ and $h_5^2 = 0$. □

The family $\mathcal{F}_9$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_9 = U(2) \oplus A_2$. By Theorem 4.1.1 X contains four classes of (-2)-curves. The Hilbert basis of the nef cone contains seven classes, two of them defining elliptic fibrations BNef[6] and BNef[7] without sections and with one fiber of type $\tilde{A}_2$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.9. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_9 = U(2) \oplus A_2$. Then*

1. *there is a minimal resolution $\varphi : X \to Y$ of a double cover $\pi : Y \to \mathbb{P}^2$ branched along a plane sextic B having two nodes p_1, p_2 and such that the line L through the nodes is tangent to B at one point;*
2. *the surface has four (-2)-curves: two curves R_1, R_2 such that $\pi\varphi(R_i) = L$, $i = 1, 2$ and two curves E_1, E_2 with $\pi\varphi(E_i) = p_i$, $i = 1, 2$;*
3. *the Cox ring $R(X)$ has at least 10 generators.*

Proof. Let $f_1, \ldots, f_4$ be the classes of the (-2)-curves and $h = \mathrm{BNef}[5]$. We have that $h^2 = 2$, $h \cdot f_i = 1$ for $i = 1, 4$ and $h \cdot f_i = 0$ for $i = 2, 3$. By Corollary 1.4.11 the associated linear system is base point free and thus defines a degree two morphism $\pi : X \to \mathbb{P}^2$ which contracts the (-2)-curves of classes f_2, f_3. Since $f_2 \cdot f_3 = 0$, the branch locus of π is a plane sextic B with two nodes at $p, q \in \mathbb{P}^2$. Moreover $f_1 + f_2 + f_3 + f_4 = h$ and $f_1 \cdot f_4 = 1$. Thus the (-2)-curves of classes f_1, f_4 are mapped to a line L passing through p, q and tangent to B at one more point.

By Theorem 2.4.2 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_4,$$

$$\mathrm{BNef}[1],\ \mathrm{BNef}[2],\ \mathrm{BNef}[3],\ \mathrm{BNef}[4],\ \mathrm{BNef}[6],\ \mathrm{BNef}[7],\ 2(\mathrm{BNef}[6]+\mathrm{BNef}[7]),$$

where $\mathrm{BNef}[1], \mathrm{BNef}[2], \mathrm{BNef}[3], \mathrm{BNef}[4]$ are non-hyperelliptic by Proposition 1.4.8 with $\mathrm{BNef}[1]^2 = \mathrm{BNef}[2]^2 = 12$, $\mathrm{BNef}[3]^2 = \mathrm{BNef}[4]^2 = 10$ and also the special case $2(\mathrm{BNef}[6]+\mathrm{BNef}[7])$, where the classes $\mathrm{BNef}[6], \mathrm{BNef}[7]$ define the two elliptic fibrations of X, with $\mathrm{BNef}[6]\cdot\mathrm{BNef}[7] = 2$.

Then, applying the tests (described in the proof of the Theorem 3.2.1) in these degrees we find that the degrees of the generators are those in the previous set excluding the special case. □

The family $\mathcal{F}_{10}$

Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_{10} = U(3)\oplus A_2$. By Theorem 4.1.1 X contains four (-2)-curves. The Hilbert basis of the nef cone of X contains five classes, four of them, $\mathrm{BNef}[2], \mathrm{BNef}[3], \mathrm{BNef}[4]$ and $\mathrm{BNef}[5]$, defining elliptic fibrations without sections and with one fiber of type $\tilde{A}_2$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.10. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_{10} = U(3)\oplus A_2$. Then*

1. *X is isomorphic to a smooth quartic surface in $\mathbb{P}^3$ having one hyperplane section which is the union of four lines;*

2. *a general X can be defined by an equation of the form*

$$F_0(x_0,\ldots,x_3)G(x_0,\ldots,x_3)+F_1(x_0,\ldots,x_3)F_2(x_0,\ldots,x_3)F_3(x_0,\ldots,x_3)F_4(x_0,\ldots,x_3) = 0,$$

   where $F_i \in \mathbb{C}[x_0,\ldots,x_3]$ are homogeneous of degree one for $i = 0,\ldots,4$ and $G \in \mathbb{C}[x_0,\ldots,x_3]$ is homogeneous of degree three;

3. *the surface has four (-2)-curves: the four lines;*

4. *the Cox ring of X has 8 generators: $s_1, \dots, s_4$ defining the (-2)-curves and $s_5, \dots, s_8$ defining each a smooth fiber of one of the elliptic fibrations of X;*

5. *for a very general X as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \dots, T_8]/I, \ s_i \mapsto T_i,$$

where the degrees of the generators T_i for $i = 1, \dots, 8$ are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 \end{pmatrix}$$

and the ideal I is generated by the following polynomials:

$$T_1T_2T_3T_4 - F_0(T_1T_5, T_4T_6, T_2T_7, T_3T_8),$$
$$T_5T_6T_7T_8 - G(T_1T_5, T_4T_6, T_2T_7, T_3T_8).$$

Proof. Let $f_1, \dots, f_4$ be the classes of the (-2)-curves and let $h = \mathrm{BNef}[1]$. Then $h^2 = 4$ and $h \cdot f_i = 1$ for all i. Thus h is ample and is non-hyperelliptic by Proposition 1.4.8. By Corollary 1.4.11 the linear system associated to h is base point free, thus it defines an embedding of X in $\mathbb{P}^3$ as a smooth quartic surface. Observe that $h = f_1 + f_2 + f_3 + f_4$ with $h \cdot f_i = 1$ for all i. This means that X has one hyperplane section which is the union of four lines.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_4,\ h,\ e_1,\ e_2,\ e_3,\ e_4,$$

where e_1, e_2, e_3, e_4 define the four elliptic fibrations of X. Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \dots, s_4$ defining the (-2)-curves and generators $s_5, \dots, s_8$ defining smooth fibers of the elliptic fibrations. Moreover, observe that

$$h = e_1 + f_1 = e_2 + f_4 = e_3 + f_2 = e_4 + f_3.$$

An argument similar to the one in the proof of Proposition 4.3.7 shows that X can be defined by an equation of the form $F_0G + F_1F_2F_3F_4 = 0$ in $\mathbb{P}^3$, where F_i are homogeneous of degree one for $i = 0, \dots, 4$ and g of degree 3. Since $F_1, \dots, F_4$ can be chosen to be independent, then $s_1s_5, s_4s_6, s_2s_7, s_3s_8$ are a basis of $H^0(h)$. Thus a generator in degree h is not necessary. The first relation is due to the fact that the hyperplane section $F_0 = 0$ is the union of the four lines F_i for $i = 1, 2, 3, 4$, that correspond to the four (-2)-curves. The second relation is due to the fact that the four elliptic curves $s_i = 0$, $i = 5, \dots, 8$ are cut out by the cubic $G = 0$.

It can be proved with the same type of argument used in the proof of Theorem 3.3.2 that the ideal I is prime for general G, F_0. Thus $\mathbb{C}[T_1, \dots, T_8]/I \cong R(X)$, since it is an integral domain of dimension $\dim R(X) = \dim(X) + \operatorname{rank} \operatorname{Cl}(X) = 6$. $\square$

The family $\mathcal{F}_{11}$

Let X be a K3 surface with $\operatorname{Cl}(X) \cong V_{11} = U(6) \oplus A_2$. By Theorem 4.1.1 X contains six (-2)-curves. The Hilbert basis of the nef cone of X contains 27 classes, eight of them defining elliptic fibrations without sections and with one fiber of type $\tilde{A}_2$ (by

[BHPVdV04, Chapter V, §7]).

Proposition 4.3.11. *Let X be a K3 surface with* $\mathrm{Cl}(X) \cong V_{11} = U(6) \oplus A_2$*. Then*

1. *X is isomorphic to a smooth quartic surface in $\mathbb{P}^3$ having three reducible hyperplane sections which are the union of two conics;*
2. *X contains six (-2)-curves: the six conics;*
3. *the Cox ring of X has at least 20 generators.*

Proof. Let $f_1, \dots, f_6$ be the classes of the (-2)-curves and $h = \mathrm{BNef}[25]$. Then $h^2 = 4$, $h \cdot f_i = 2$ for all i and by Proposition 1.4.8 it is non-hyperelliptic. By Corollary 1.4.11 the associated linear system is base point free. thus it defines an embedding of X in $\mathbb{P}^3$ as a smooth quartic surface. Observe that $h = f_1 + f_5 = f_2 + f_4 = f_3 + f_6$. This means that X has three reducible hyperplane sections which are the union of two conics.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_6,\ h_1, \dots, h_{14},\ h^*,\ h_1^*, \dots, h_{12}^*$$

where h_i and h_i^* are classes in the Hilbert basis of the nef cone such that

$$h_i \in \{\mathrm{BNef}[j] : j = 4, 7, 9, 11, 13, 15, 16, 18, 20, 21, 23, 24, 26, 27\}$$

$$h_i^* \in \{\mathrm{BNef}[j] : j = 1 - 3, 5, 6, 8, 10, 12, 14, 17, 19, 22, 25\}.$$

By Proposition 1.4.8, the classes h_i and h_i^* are non-hyperelliptic for all i, and if $v := \mathrm{BNef}[i]$ we have

$v^2 = 0$ for $i = 4, 7, 13, 16, 18, 21, 26, 27$

$v^2 = 6$ for $i = 9, 11, 15, 20, 23, 24$

$v^2 = 10$ for $i = 1, 2, 3, 5, 6, 8, 10, 12, 14, 17, 19, 22.$

By the minimality test (Proposition 2.4.9) the degrees not marked with a star are necessary to generate $R(X)$. Thus $R(X)$ has at least 20 generators. $\square$

The family $\mathcal{F}_{12}$

Let X be a K3 surface with

$$\mathrm{Cl}(X) \cong V_{12} = \begin{bmatrix} 0 & -3 \\ -3 & 2 \end{bmatrix} \oplus A_2.$$

By Theorem 4.1.1 X contains six classes of (-2)-curves. The Hilbert basis of the nef cone of X contains 33 classes, two of them defining elliptic fibrations (BNef[30] and BNef[31]) without sections and with one fiber of type $\tilde{A}_2$ (by [BHPVdV04, Chapter V, §7]).

Proposition 4.3.12. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_{12}$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines L_1, L_2, L_3;*

2. *a general X can be defined by an equation of the form*

$$F_1(x_0, x_1, x_2)F_2(x_0, x_1, x_2)F_3(x_0, x_1, x_2)G_1(x_0, x_1, x_2) + G_2(x_0, x_1, x_2)^2 = 0,$$

where $F_i \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous of degree one for $i = 1, 2, 3$ and $G_1, G_2 \in \mathbb{C}[x_0x_1x_2]$ are homogeneous of degree three;

3. *the surface has six* (-2)*-curves: the curves* R_{ij}*,* $i = 1, 2, 3$*,* $j = 1, 2$*, such that* $\pi(R_{i1}) = \pi(R_{i2}) = L_i$*;*

4. *the Cox ring of* X *has eight generators:* $s_1, \ldots, s_6$ *defining the* (-2)*-curves and* s_7, s_8 *defining smooth fibers of the two elliptic fibrations of* X*;*

5. *for a very general* X *as before we have an isomorphism*

$$R(X) \to \mathbb{C}[T_1, \ldots, T_8]/I, \ s_i \mapsto T_i,$$

where the degrees of the generators T_i *for* $i = 1, \ldots, 8$ *are given by the columns of the following matrix*

$$\begin{pmatrix} -1 & 0 & -1 & 0 & 0 & -1 & -2 & -1 \\ -2 & 0 & -3 & 1 & 0 & -2 & -3 & -3 \\ 1 & 0 & 1 & 0 & -1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 & 3 & 0 \end{pmatrix}$$

and the ideal I *is generated by the following polynomials:*

$$T_7T_8 - G_1(T_1T_2, T_3T_4, T_5T_6),$$
$$T_1T_2T_3T_4T_5T_6 - G_2(T_1T_2, T_3T_4, T_5T_6).$$

Proof. Let $f_1, \ldots, f_6$ be the classes of the (-2)-curves and $h = \mathrm{BNef}[33]$. Then $h^2 = 2$, $h \cdot f_i = 1$ for all i. Thus h is ample. By Corollary 1.4.11 the associated linear system is base point free, thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic B. Since $h = f_1 + f_2 = f_3 + f_4 = f_5 + f_6$, the image by π of the six (-2)-curves of X are three lines $L_1, L_2, L_3 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(L_i)$ is the union of two smooth rational curves for each $i = 1, \ldots, 3$. This implies that the lines L_i are 3-tangent to B.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_6,\ h^*,\ e_1, e_2$$

where e_1 and e_2 are the two elliptic fibrations of X. Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \ldots, s_6$ defining the (-2)-curves of X and one section defining a smooth fiber for each elliptic fibration.

By a similar argument used in the Theorem 3.3.2. general X with Picard lattice isometric to V_{12} is a double cover of $\mathbb{P}^2$ branched along a smooth plane sextic B defined by an equation of the form $F_1F_2F_3G_1 + G_2^2 = 0$, where $F_1, F_2, F_3 \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous of degree one and $G_1, G_2 \in \mathbb{C}[x_0, x_1, x_2]$ of degree 3. Since F_1, F_2, F_3 can be chosen to be independent, then s_1s_2, s_3s_4, s_5s_6 give a basis of $H^0(h)$. Thus a generator of $R(X)$ in degree h is not necessary.

It can be proved with the same type of argument used in the proof of Theorem 3.3.2 that the ideal I is prime for general G_1, G_2. Thus $\mathbb{C}[T_1, \ldots, T_8]/I \cong R(X)$, since it is an integral domain of dimension $\dim R(X) = \dim(X) + \operatorname{rank} \operatorname{Cl}(X) = 6$. □

The family $\mathcal{F}_{13}$

Let X be a K3 surface with

$$\operatorname{Cl}(X) \cong V_{13} = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}.$$

By Theorem 4.1.1 X contains six (-2)-curves. The Hilbert basis of the nef cone of X contains 39 classes of positive self-intersection. Thus X has no elliptic fibrations.

Proposition 4.3.13. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong V_{13}$. Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines L_1, L_2 and L_3;*
2. *the surface has six (-2)-curves: the six curves R_{ij}, $i = 1,\ 2,\ 3$, $j = 1,\ 2$ such that $\pi(R_{i1}) = \pi(R_{i2}) = L_i$;*
3. *the Cox ring of X has at least 24 generators.*

Proof. Let $f_1, \dots, f_6$ be the classes of the (-2)-curves and $h = \mathrm{BNef}[1]$. Then $h^2 = 2$ and $h \cdot f_i = 1$ for all i. Thus h is ample and the associated linear system is base point free by Corollary 1.4.11. Thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic B. Since $h = f_3 + f_5 = f_1 + f_2 = f_4 + f_6$, the image by π of the six (-2)-curves of X are three lines $L_1, L_2, L_3 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(L_i)$ is the union of two smooth rational curves for each $i = 1, 2, 3$. This implies that L_1, L_2, L_3 are 3-tangent to B.

By Theorem 4.2.1 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \dots, f_6,\ h^*,\ h_1, h_2, \dots, h_{18},$$

where $h_1,\ h_2, \dots,\ h_{18}$ are classes in the Hilbert basis of the nef cone, non-hyperelliptic, with self-intersection 4 (six of them), 26 (six of them) and 28 (six of them). By the minimality test (Proposition 2.4.9) $R(X)$ has a generator in all the above degrees, except possibly for h. $\square$

The family $\mathcal{F}_{14}$

Let X be a K3 surface with

$$\mathrm{Cl}(X) \cong V_{14} = \begin{bmatrix} 12 & -2 & 0 & 0 \\ -2 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2. \end{bmatrix}$$

By Theorem 4.1.1 X contains eight classes of (-2)-curves. The Hilbert basis of the nef cone of X contains 111 classes, none of them defining elliptic fibrations.

Proposition 4.3.14. *Let X be a K3 surface with* $\mathrm{Cl}(X) \cong V_{14}$. *Then*

1. *there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with one* 3*-tangent line L and three* 6*-tangent conics C_1, C_2 and C_3;*
2. *the surface has eight* (-2)*-curves: the curves R_{ij}, $i = 1, 2, 3$ such that $\pi(R_{i1}) = \pi(R_{i2}) = C_i$ and the curves S_1, S_2 such that $\pi(S_1) = \pi(S_2) = L$;*
3. *the Cox ring of X has at least* 71 *generators of which 8 are sections that define the* (-2)*-curves and the degree of the others are elements of the Hilbert basis of the nef cone.*

Proof. Let $f_1, \ldots, f_8$ be the classes of the (-2)-curves and $h = \mathrm{BNef}[8]$. Then $h^2 = 2$, $h \cdot f_i = 2$ for $i = 1, \ldots, 5, 8$ and $h \cdot f_i = 1$ for $i = 6, 7$. By Corollary 1.4.11 the associated linear system is base point free, thus defines a double cover of $\mathbb{P}^2$ branched along a smooth plane sextic B. Since $h = f_6 + f_7$, and $2h = f_4 + f_5 = f_2 + f_8 = f_1 + f_3$, the image by π of the eight (-2)-curves of X are three smooth conics $C_1, C_2, C_3 \subseteq \mathbb{P}^2$ and one line $L \subseteq \mathbb{P}^2$, such that $\pi^{-1}(C_i)$, $i = 1, 2, 3$ and $\pi^{-1}(L)$ is the union of two smooth rational curves.

To obtain item 3. we apply Test 1 and Proposition 2.4.9 (which is implemented in the Magma function `Minimal`), see Section 6.3 and Section 6.4. □

We note that in this case we are not looking at classes of nef divisors which are sums of two or three elements of the Hilbert basis of the nef cone, because we are unable to compute a minimal generating set of $R(X)$ for computational reasons (the Hilbert basis of the nef cone contains 111 elements).

Chapter 5

Tables

This chapter contains the tables with all the relevant information about Mori dream K3 surfaces of Picard number three and four: Picard lattice, effective cone and its Hilbert basis, nef cone and its Hilbert basis. Moreover, we provide a nef and big divisor in the Hilbert basis of nef cone with minimum self-intersection in each family of K3 surfaces and its intersection properties with (-2)-curves. Finally, we will give the degrees of a set of generators of the Cox ring $R(X)$.

We recall that in the tables we will adopt this notation: $\mathrm{Cl}(X)$ denotes the Picard lattice of the surface, $\mathrm{Eff}(X)$ is the effective cone and $\mathrm{BEff}(X)$ is its Hilbert basis, $E(X)$ is the set of generators of the extremal rays of the effective cone (i.e. the set of classes of the (-2)-curves), $\mathrm{Nef}(X)$ is the nef cone and $\mathrm{BNef}(X)$ is its Hilbert basis, $N(X)$ is the set of generators of the extremal rays of $\mathrm{Nef}(X)$.

5.1 K3 surfaces of Picard number three

In Table 5.1 we give the intersection matrix of (-2)-curves, Table 5.2 describes $E(X)$, $\mathrm{BEff}(X)$, $N(X)$ and $\mathrm{BNef}(X)$ for each family of Mori dream K3 surfaces of Picard number 3.

In Tables 5.3 and 5.4 we give the Hilbert basis of the nef cone of X, when the Picard lattice is isometric to S_2, S_6, $S_{1,9,1}$, $S_{1,1,6}$ or $S_{1,1,8}$.

For each family of Mori dream K3 surfaces of Picard number three, in Table 5.5 we give a nef and big class $H \in \mathrm{BNef}(X)$ of minimal self-intersection and its intersection properties with the (-2)-curves E_i.

In Table 5.6 we give the degrees of a set of generators of the Cox ring $R(X)$. We recall that all degrees in Table 5.6 are necessary to generate $R(X)$, except possibly for those marked with a star.

Table 5.1: Intersection matrix of (-2)-curves for $\varrho(X) = 3$.

$N°$	Lattice	intersection matrix of (-2)-curves
1	S_1	$\begin{bmatrix} -2 & 0 & 4 & 0 & 6 & 4 \\ 0 & -2 & 0 & 4 & 4 & 6 \\ 4 & 0 & -2 & 6 & 0 & 4 \\ 0 & 4 & 6 & -2 & 4 & 0 \\ 6 & 4 & 0 & 4 & -2 & 0 \\ 4 & 6 & 4 & 0 & 0 & -2 \end{bmatrix}$
2	S_2	$\begin{bmatrix} -2 & 1 & 7 & 1 & 10 & 7 \\ 1 & -2 & 1 & 7 & 7 & 10 \\ 7 & 1 & -2 & 10 & 1 & 7 \\ 1 & 7 & 10 & -2 & 7 & 1 \\ 10 & 7 & 1 & 7 & -2 & 1 \\ 7 & 10 & 7 & 1 & 1 & -2 \end{bmatrix}$
3	S_3	$\begin{bmatrix} -2 & 1 & 4 & 1 \\ 1 & -2 & 1 & 4 \\ 4 & 1 & -2 & 1 \\ 1 & 4 & 1 & -2 \end{bmatrix}$
4	S_4	$\begin{bmatrix} -2 & 1 & 3 & 1 \\ 1 & -2 & 1 & 6 \\ 3 & 1 & -2 & 1 \\ 1 & 6 & 1 & -2 \end{bmatrix}$
5	S_5	$\begin{bmatrix} -2 & 0 & 1 & 3 \\ 0 & -2 & 3 & 1 \\ 1 & 3 & -2 & 0 \\ 3 & 1 & 0 & -2 \end{bmatrix}$
6	S_6	$\begin{bmatrix} -2 & 1 & 5 & 1 & 6 & 5 \\ 1 & -2 & 0 & 9 & 5 & 11 \\ 5 & 0 & -2 & 11 & 1 & 9 \\ 1 & 9 & 11 & -2 & 5 & 0 \\ 6 & 5 & 1 & 5 & -2 & 1 \\ 5 & 11 & 9 & 0 & 1 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
7	$S'_{4,1,2}$	$\begin{bmatrix} -2 & 2 & 2 & 6 \\ 2 & -2 & 6 & 2 \\ 2 & 6 & -2 & 2 \\ 6 & 2 & 2 & -2 \end{bmatrix}$
8	$S_{4,1,1}$	$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix}$
9	$S_{5,1,1}$	$\begin{bmatrix} -2 & 2 & 2 & 18 \\ 2 & -2 & 3 & 2 \\ 2 & 3 & -2 & 2 \\ 18 & 2 & 2 & -2 \end{bmatrix}$
10	$S_{6,1,1}$	$\begin{bmatrix} -2 & 4 & 2 & 2 \\ 4 & -2 & 2 & 2 \\ 2 & 2 & -2 & 10 \\ 2 & 2 & 10 & -2 \end{bmatrix}$
11	$S_{7,1,1}$	$\begin{bmatrix} -2 & 5 & 2 & 5 & 2 & 16 \\ 5 & -2 & 2 & 5 & 16 & 2 \\ 2 & 2 & -2 & 16 & 26 & 26 \\ 5 & 5 & 16 & -2 & 2 & 2 \\ 2 & 16 & 16 & 2 & -2 & 26 \\ 16 & 2 & 26 & 2 & 26 & -2 \end{bmatrix}$
12	$S_{8,1,1}$	$\begin{bmatrix} -2 & 2 & 6 & 2 \\ 2 & -2 & 2 & 6 \\ 6 & 2 & -2 & 2 \\ 2 & 6 & 2 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
13	$S_{10,1,1}$	$\begin{bmatrix} -2 & 8 & 8 & 18 & 22 & 22 & 2 & 2 \\ 8 & -2 & 18 & 8 & 22 & 2 & 2 & 22 \\ 8 & 18 & -2 & 8 & 2 & 22 & 22 & 2 \\ 18 & 8 & 8 & -2 & 2 & 2 & 22 & 22 \\ 22 & 22 & 2 & 2 & -2 & 18 & 38 & 18 \\ 22 & 2 & 22 & 2 & 18 & -2 & 18 & 38 \\ 2 & 2 & 22 & 22 & 38 & 18 & -2 & 18 \\ 2 & 22 & 2 & 22 & 18 & 38 & 18 & -2 \end{bmatrix}$
14	$S_{12,1,1}$	$\begin{bmatrix} -2 & 10 & 14 & 2 & 2 & 10 \\ 10 & -2 & 2 & 2 & 14 & 10 \\ 14 & 2 & -2 & 10 & 10 & 2 \\ 2 & 2 & 10 & -2 & 10 & 14 \\ 2 & 14 & 10 & 10 & -2 & 2 \\ 10 & 10 & 2 & 14 & 2 & -2 \end{bmatrix}$
15	$S_{1,2,1}$	$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix}$
16	$S_{1,3,1}$	$\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix}$
17	$S_{1,4,1}$	$\begin{bmatrix} -2 & 3 & 1 & 2 \\ 3 & -2 & 2 & 1 \\ 1 & 2 & -2 & 11 \\ 2 & 1 & 11 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
18	$S_{1,5,1}$	$\begin{bmatrix} -2 & 4 & 1 & 2 & 11 & 23 \\ 4 & -2 & 2 & 6 & 2 & 14 \\ 1 & 2 & -2 & 14 & 23 & 66 \\ 2 & 6 & 14 & -2 & 4 & 2 \\ 11 & 2 & 23 & 4 & -2 & 1 \\ 23 & 14 & 66 & 2 & 1 & -2 \end{bmatrix}$
19	$S_{1,6,1}$	$\begin{bmatrix} -2 & 2 & 1 & 5 \\ 2 & -2 & 5 & 1 \\ 1 & 5 & -2 & 2 \\ 5 & 1 & 2 & -2 \end{bmatrix}$
20	$S_{1,9,1}$	$\begin{bmatrix} -2 & 10 & 8 & 2 & 10 & 26 & 2 & 8 & 26 \\ 10 & -2 & 2 & 8 & 10 & 2 & 26 & 26 & 8 \\ 8 & 2 & -2 & 1 & 26 & 25 & 37 & 46 & 37 \\ 2 & 8 & 1 & -2 & 26 & 37 & 25 & 37 & 46 \\ 10 & 10 & 26 & 26 & -2 & 8 & 8 & 2 & 2 \\ 26 & 2 & 25 & 37 & 8 & -2 & 46 & 37 & 1 \\ 2 & 26 & 37 & 25 & 8 & 46 & -2 & 1 & 37 \\ 8 & 26 & 46 & 37 & 2 & 37 & 1 & -2 & 25 \\ 26 & 8 & 37 & 46 & 2 & 1 & 37 & 25 & -2 \end{bmatrix}$
21	$S_{1,1,1}$	$\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
22	$S_{1,1,2}$	$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$
23	$S_{1,1,3}$	$\begin{bmatrix} -2 & 3 & 0 & 2 \\ 3 & -2 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 2 & 0 & 6 & -2 \end{bmatrix}$
24	$S_{1,1,4}$	$\begin{bmatrix} -2 & 0 & 2 & 4 \\ 0 & -2 & 4 & 2 \\ 2 & 4 & -2 & 0 \\ 4 & 2 & 0 & -2 \end{bmatrix}$
25	$S_{1,1,6}$	$\begin{bmatrix} -2 & 6 & 2 & 6 & 6 & 2 \\ 6 & -2 & 6 & 2 & 2 & 6 \\ 2 & 6 & -2 & 18 & 0 & 16 \\ 6 & 2 & 18 & -2 & 16 & 0 \\ 6 & 2 & 0 & 16 & -2 & 18 \\ 2 & 6 & 16 & 0 & 18 & -2 \end{bmatrix}$
26	$S_{1,1,8}$	$\begin{bmatrix} -2 & 2 & 0 & 8 & 8 & 16 & 14 & 18 \\ 2 & -2 & 8 & 0 & 16 & 8 & 18 & 14 \\ 0 & 8 & -2 & 14 & 2 & 18 & 8 & 16 \\ 8 & 0 & 14 & -2 & 18 & 2 & 16 & 8 \\ 8 & 16 & 2 & 18 & -2 & 14 & 0 & 8 \\ 16 & 8 & 18 & 2 & 14 & -2 & 8 & 0 \\ 14 & 18 & 8 & 16 & 0 & 8 & -2 & 2 \\ 18 & 14 & 16 & 8 & 8 & 0 & 2 & -2 \end{bmatrix}$

Table 5.2: Effective and Nef cone for K3 surfaces with $\varrho(X) = 3$.

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
1	S_1	$(0,1,0)$, $(0,0,1)$, $(1,-2,0)$, $(1,0,-2)$, $(2,-3,-2)$, $(2,-2,-3)$	$E(X) \cup \{(1,-1,-1)\}$	$(1,0,0)$, $(2,-3,0)$, $(2,0,-3)$, $(4,-6,-3)$, $(4,-3,-6)$, $(5,-6,-6)$	$(1,-1,-1)$, $(1,-1,0)$, $(1,0,-1)$, $(1,0,0)$, $(2,-3,-1)$, $(2,-3,0)$, $(2,-1,-3)$, $(2,0,-3)$, $(3,-4,-3)$, $(3,-3,-4)$, $(4,-6,-3)$, $(4,-3,-6)$, $(5,-6,-6)$
2	S_2	$(0,1,0)$, $(0,0,1)$, $(1,-5,-3)$, $(1,-3,-5)$, $(2,-9,-8)$, $(2,-8,-9)$	$E(X) \cup \{(1,-4,-4)\}$	$(1,0,0)$, $(5,-24,-12)$, $(5,-12,-24)$, $(13,-60,-48)$, $(13,-48,-60)$, $(17,-72,-72)$	See Table 5.3
3	S_3	$(0,1,0)$, $(0,0,1)$, $(1,-3,-2)$, $(1,-2,-3)$	$E(X)$	$(1,0,0)$, $(3,-8,-4)$, $(3,-4,-8)$, $(5,-12,-12)$	$(1,-2,-2)$, $(1,-2,-1)$, $(1,-1,-2)$, $(1,-1,-1)$, $(1,0,0)$, $(2,-5,-4)$, $(2,-5,-3)$, $(2,-4,-5)$, $(2,-3,-5)$, $(3,-8,-4)$, $(3,-7,-7)$, $(3,-4,-8)$, $(5,-12,-12)$
4	S_4	$(0,1,0)$, $(0,-1,-1)$, $(1,-1,1)$, $(2,1,3)$	$E(X)$	$(3,-1,-2)$, $(7,-9,2)$, $(13,9,18)$, $(17,1,22)$	$(1,-1,0)$, $(1,0,0)$, $(1,0,1)$, $(2,-2,1)$, $(2,-1,-1)$, $(2,1,2)$, $(3,-1,-2)$, $(3,-1,3)$, $(3,1,4)$, $(3,2,4)$, $(4,-5,1)$, $(4,0,5)$, $(7,-9,2)$, $(8,5,11)$, $(10,1,13)$, $(13,9,18)$, $(17,1,22)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
5	S_5	$(0,1,0)$, $(1,-1,-2)$, $(0,0,1)$, $(1,-2,-1)$	$E(X)$	$(1,0,0)$, $(3,-4,-4)$, $(3,-4,-2)$, $(3,-2,-4)$	$(1,-1,-1),(1,0,0)$, $(2,-2,-1),(2,-1,-2)$, $(3,-4,-4),(3,-4,-3)$, $(3,-4,-2),(3,-3,-4)$, $(3,-2,-4)$
6	S_6	$(0,-1,0)$, $(0,0,-1)$, $(1,3,1)$, $(2,3,5)$, $(2,5,4)$, $(3,6,7)$	$E(X)$ $\cup$ $\{(1,2,2)\}$	$(3,-1,-2)$, $(5,13,4)$, $(17,31,40)$, $(19,23,46)$, $(25,65,42)$, $(41,89,90)$	See Table 5.3
7	$S'_{4,1,2}$	$(0,1,0)$, $(2,3,1)$, $(0,1,1)$, $(2,3,2)$	$E(X)$ $\cup$ $\{(1,2,1)\}$	$(0,2,1)$, $(2,4,1)$, $(2,4,3)$, $(4,6,3)$	$(0,2,1),(1,2,1)$, $(1,3,1),(1,3,2)$, $(2,4,1),(2,4,3)$, $(3,5,2),(3,5,3)$, $(4,6,3)$
8	$S_{4,1,1}$	$(0,1,0)$, $(0,0,1)$, $(1,3,4)$	$E(X)$	$(0,1,1)$, $(1,3,5)$, $(1,4,4)$	$(0,1,1),(1,3,5)$, $(1,4,4),(1,4,5)$
9	$S_{5,1,1}$	$(0,1,0)$, $(0,0,1)$, $(1,4,5)$, $(4,15,24)$	$E(X)$	$(0,1,1)$, $(1,5,5)$, $(4,15,25)$, $(5,19,29)$	$(0,1,1),(1,4,6)$, $(1,5,5),(1,5,6)$, $(2,8,13),(3,12,17)$, $(4,15,25),(5,19,29)$, $(5,19,30)$
10	$S_{6,1,1}$	$(0,0,1)$, $(1,5,6)$, $(2,9,14)$, $(0,1,0)$	$E(X)$	$(0,1,1)$, $(1,6,6)$, $(2,9,15)$, $(3,14,20)$	$(0,1,1),(1,5,7)$, $(1,5,8),(1,6,6)$, $(1,6,7),(2,9,15)$, $(2,10,13),(3,14,20)$, $(3,14,21)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
11	$S_{7,1,1}$	$(0,0,1)$, $(1,6,7)$, $(0,1,0)$, $(3,16,24)$, $(4,21,34)$, $(6,33,46)$	$E(X)$ $\cup$ $\{(2,11,16)\}$	$(0,1,1)$, $(1,7,7)$, $(4,21,35)$, $(7,37,58)$, $(7,39,53)$, $(9,49,70)$	$(0,1,1),(1,6,8)$, $(1,6,9),(1,7,7)$, $(1,7,8),(2,11,16)$, $(2,11,17),(2,11,18)$, $(2,12,15),(3,16,25)$, $(3,16,26),(3,17,23)$, $(4,21,35),(4,22,31)$, $(4,23,30),(5,27,40)$, $(5,28,38),(6,32,49)$, $(7,37,58),(7,37,59)$, $(7,38,55),(7,39,53)$, $(8,43,64),(9,49,70)$, $(10,55,77)$
12	$S_{8,1,1}$	$(0,0,1)$, $(1,6,9)$, $(1,7,8)$, $(0,1,0)$	$E(X)$	$(0,1,1)$, $(1,6,10)$, $(1,8,8)$, $(2,13,17)$	$(0,1,1),(1,6,10)$, $(1,7,9),(1,7,10)$, $(1,8,8),(1,8,9)$, $(2,13,17),(2,13,18)$, $(2,14,17)$
13	$S_{10,1,1}$	$(1,9,10)$, $(4,32,43)$, $(0,0,1)$, $(3,23,34)$, $(2,15,24)$, $(6,47,66)$, $(4,33,42)$, $(0,1,0)$	$E(X)$ $\cup$ $\{(1,8,11)\}$	$(0,1,1)$, $(1,10,10)$, $(2,15,25)$, $(5,38,58)$, $(5,42,52)$, $(8,65,85)$, $(9,70,100)$, $(10,79,109)$	$(0,1,1),(1,8,11)$, $(1,8,12),(1,8,13)$, $(1,9,11),(1,10,10)$, $(1,10,11),(2,15,25)$, $(2,17,21),(3,23,35)$, $(3,26,31),(4,31,45)$, $(5,38,58),(5,38,59)$, $(5,40,54),(5,41,53)$, $(5,42,52),(7,54,79)$, $(7,55,77),(8,65,85)$, $(9,70,100),(9,72,97)$, $(9,74,95),(10,79,109)$, $(13,102,143)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
14	$S_{12,1,1}$	$(0,1,0)$, $(1,9,14)$, $(2,19,26)$, $(0,0,1)$, $(1,11,12)$, $(2,20,25)$	$E(X) \cup \{(1,10,13)\}$	$(0,1,1)$, $(1,9,15)$, $(1,12,12)$, $(3,28,40)$, $(3,31,37)$, $(4,39,51)$	$(0,1,1),(1,9,15)$, $(1,10,13),(1,10,14)$, $(1,10,15),(1,11,13)$, $(1,12,12),(1,12,13)$, $(2,19,27),(2,21,25)$, $(3,28,40),(3,28,41)$, $(3,29,39),(3,30,38)$, $(3,31,37),(3,32,37)$, $(4,39,51),(5,48,65)$, $(5,50,63)$
15	$S_{1,2,1}$	$(1,0,0)$, $(0,0,1)$, $(0,1,1)$	$E(X)$	$(0,1,2)$, $(4,3,8)$, $(4,5,8)$	$(0,1,2),(1,1,2)$, $(2,2,5),(2,3,5)$, $(4,3,8),(4,5,8)$
16	$S_{1,3,1}$	$(1,0,0)$, $(0,0,1)$, $(0,1,2)$	$E(X)$	$(0,1,3)$, $(1,1,2)$, $(2,1,4)$	$(0,1,3),(1,1,2)$, $(1,1,3),(2,1,4)$
17	$S_{1,4,1}$	$(1,0,0)$, $(0,1,3)$, $(0,0,1)$, $(3,3,8)$	$E(X)$	$(0,1,4)$, $(4,3,8)$, $(8,3,16)$, $(24,29,80)$	$(0,1,4),(1,1,3)$, $(1,1,4),(2,1,4)$, $(2,1,5),(2,3,9)$, $(3,2,6),(4,3,8)$, $(4,4,11),(4,5,14)$, $(5,2,10),(7,8,22)$, $(8,3,16),(14,17,47)$, $(24,29,80)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
18	$S_{1,5,1}$	$(1,0,0)$, $(0,1,4)$, $(0,0,1)$, $(4,3,10)$, $(5,6,21)$, $(16,16,55)$	$E(X)$ $\cup$ $\{(2,2,7)\}$	$(0,1,5)$, $(5,3,10)$, $(5,7,25)$, $(10,3,20)$, $(20,19,65)$, $(190,197,680)$	$(0,1,5),(1,1,4)$, $(1,1,5),(1,2,8)$, $(2,1,4),(2,1,5)$, $(2,1,6),(2,2,7)$, $(2,3,11),(3,1,6)$, $(3,2,7),(5,3,10)$, $(5,7,25),(6,2,13)$, $(6,5,17),(7,8,28)$, $(9,9,31),(10,3,20)$, $(13,12,41),(14,15,52)$, $(20,19,65),(25,25,86)$, $(30,31,107),(46,47,162)$, $(51,53,183),(118,122,421)$, $(190,197,680)$
19	$S_{1,6,1}$	$(1,0,0)$, $(1,1,4)$, $(0,0,1)$, $(0,1,5)$	$E(X)$	$(0,1,6)$, $(2,1,4)$, $(4,1,8)$, $(4,7,32)$	$(0,1,6),(1,1,5)$, $(1,1,6),(1,2,10)$, $(2,1,4),(2,1,5)$, $(2,1,6),(2,1,7)$, $(2,2,9),(2,3,14)$, $(2,4,19),(3,1,6)$, $(3,1,7),(3,4,18)$, $(3,5,23),(4,1,8)$, $(4,7,32)$
20	$S_{1,9,1}$	$(2,1,6)$, $(0,1,8)$, $(0,0,1)$, $(1,0,0)$, $(4,5,34)$, $(3,6,43)$, $(7,6,39)$, $(5,8,56)$, $(8,8,53)$	$E(X)$ $\cup$ $\{(1,1,7)$, $(2,3,21)$, $(3,3,20)\}$	$(0,1,9)$, $(3,1,6)$, $(3,7,51)$, $(6,1,12)$, $(9,7,45)$, $(9,13,90)$, $(12,13,87)$, $(42,73,516)$, $(78,73,480)$	See Table 5.4

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
21	$S_{1,1,1}$	$(-1,0,0)$, $(0,1,0)$, $(1,0,1)$	$E(X)$	$(1,1,1)$, $(1,2,2)$, $(2,3,4)$	$(1,1,1)$, $(1,2,2)$, $(2,3,4)$
22	$S_{1,1,2}$	$(0,-1,-1)$, $(1,2,2)$, $(0,3,2)$	$E(X)$	$(0,2,1)$, $(1,1,1)$, $(1,4,3)$	$(0,2,1)$, $(1,1,1)$, $(1,4,3)$
23	$S_{1,1,3}$	$(0,-2,-1)$, $(1,6,3)$, $(2,3,2)$, $(0,5,2)$	$E(X)$	$(0,3,1)$, $(3,9,5)$, $(3,24,11)$, $(6,12,7)$	$(0,3,1),(1,4,2)$, $(1,9,4),(2,6,3)$, $(3,7,4),(3,9,5)$, $(3,14,7),(3,19,9)$, $(3,24,11),(4,9,5)$, $(6,12,7)$
24	$S_{1,1,4}$	$(0,-3,-1)$, $(1,2,1)$, $(0,7,2)$, $(1,12,4)$	$E(X)$	$(0,4,1)$, $(2,8,3)$, $(2,14,5)$, $(2,28,9)$	$(0,4,1),(1,6,2)$, $(1,9,3),(1,16,5)$, $(2,8,3),(2,11,4)$, $(2,14,5),(2,21,7)$, $(2,28,9)$
25	$S_{1,1,6}$	$(0,5,1)$, $(2,3,1)$, $(3,16,4)$, $(1,0,0)$, $(4,15,4)$, $(0,1,0)$	$E(X)$ $\cup$ $\{(1,4,1)\}$	$(0,6,1)$, $(3,3,1)$, $(3,6,1)$, $(3,21,5)$, $(6,18,5)$, $(15,66,17)$	See Table 5.4
26	$S_{1,1,8}$	$(3,4,1)$, $(1,0,0)$, $(4,9,2)$, $(0,1,0)$, $(4,15,3)$, $(0,7,1)$, $(3,16,3)$, $(1,12,2)$	$E(X)$ $\cup$ $\{(1,6,1),(2,5,1)$, $(2,11,2),(3,10,2)\}$	$(0,8,1)$, $(4,4,1)$, $(4,8,1)$, $(4,28,5)$, $(4,56,9)$, $(8,24,5)$, $(20,40,9)$, $(20,88,17)$	See Table 5.4

Table 5.3: BNef(X) for Cl(X) $= S_2$, S_6.

$N°$	BNef(X)
2	$(1,-4,-4),(1,-4,-3),(1,-4,-2),(1,-3,-4),(1,-3,-3),(1,-3,-2),$ $(1,-2,-4),(1,-2,-3),(1,-2,-2),(1,-2,-1),(1,-1,-2),(1,-1,-1),(1,0,0),$ $(2,-9,-7),(2,-9,-6),(2,-9,-5),(2,-7,-9),(2,-6,-9),(2,-5,-9),(3,-14,-10),$ $(3,-14,-9),(3,-14,-8),(3,-14,-7),(3,-13,-12),(3,-12,-13),(3,-10,-14),$ $(3,-9,-14),(3,-8,-14),(3,-7,-14),(4,-19,-11),(4,-19,-10),(4,-18,-15),$ $(4,-15,-18),(4,-11,-19),(4,-10,-19),(5,-24,-12),(5,-23,-18),(5,-22,-20),$ $(5,-21,-21),(5,-20,-22),(5,-18,-23),(5,-12,-24),(6,-27,-23),(6,-23,-27),$ $(7,-32,-26),(7,-30,-29),(7,-29,-30),(7,-26,-32),(8,-37,-29),(8,-29,-37),$ $(9,-41,-34),(9,-38,-38),(9,-34,-41),(10,-46,-37),(10,-37,-46),(11,-47,-46),$ $(11,-46,-47),(13,-60,-48),(13,-55,-55),(13,-48,-60),(17,-72,-72)$
6	$(1,0,0),(1,1,0),(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,0,-1),(2,4,1),$ $(2,5,2),(2,5,3),(3,-1,-2),(3,4,7),(3,5,7),(3,7,2),(3,7,6),(4,8,9),$ $(4,10,3),(4,10,7),(5,6,12),(5,7,12),(5,13,4),(5,13,5),(5,13,6),(5,13,7),$ $(5,13,8),(6,11,14),(6,13,13),(7,18,12),(8,13,19),(9,19,20),(11,19,26),$ $(11,22,25),(11,24,24),(12,15,29),(13,25,30),(14,25,33),(15,28,35),$ $(15,39,25),(17,31,40),(19,23,46),(25,54,55),(25,65,42),(41,89,90)$

Table 5.4: BNef(X) for Cl(X) $= S_{1,9,1}$, $S_{1,1,6}$ and $S_{1,1,8}$.

$N°$	BNef(X)
20	$(0,1,9), (1,1,7), (1,1,8), (1,1,9), (1,2,15), (1,3,23), (2,1,7), (2,1,8), (2,1,9), (2,1,10), (2,3,21), (2,4,29), (2,5,37), (3,1,6), (3,1,7), (3,1,8), (3,1,9), (3,1,10), (3,2,13), (3,3,20), (3,7,51), (4,1,8), (4,1,9), (4,1,10), (4,1,11), (5,1,10), (5,1,11), (5,3,19), (5,4,26), (5,9,64), (6,1,12), (6,8,55), (7,5,32), (7,8,54), (7,11,77), (8,15,107), (9,7,45), (9,13,90), (10,9,59), (10,13,89), (10,17,120), (11,11,73), (11,13,88), (12,13,87), (12,19,133), (13,23,163), (15,25,176), (16,29,206), (17,15,98), (18,17,112), (18,31,219), (19,19,126), (20,33,232), (21,37,262), (23,39,275), (25,23,151), (26,25,165), (26,45,318), (29,51,361), (31,53,374), (32,29,190), (33,31,204), (34,33,218), (34,59,417), (40,37,243), (41,39,257), (42,73,516), (48,45,296), (55,51,335), (56,53,349), (63,59,388), (78,73,480)$
25	$(0,6,1), (1,4,1), (1,5,1), (1,6,1), (1,9,2), (2,4,1), (2,5,1), (2,6,1), (3,3,1), (3,4,1), (3,5,1), (3,6,1), (3,7,2), (3,21,5), (4,20,5), (5,19,5), (6,18,5), (7,36,9), (8,35,9), (9,34,9), (11,51,13), (12,50,13), (15,66,17)$
26	$(0,8,1), (1,6,1), (1,7,1), (1,8,1), (1,13,2), (1,20,3), (2,5,1), (2,6,1), (2,7,1), (2,8,1), (2,11,2), (2,18,3), (2,25,4), (2,32,5), (3,5,1), (3,6,1), (3,7,1), (3,8,1), (3,10,2), (3,23,4), (3,30,5), (3,37,6), (3,44,7), (4,4,1), (4,5,1), (4,6,1), (4,7,1), (4,8,1), (4,28,5), (4,35,6), (4,42,7), (4,49,8), (4,56,9), (5,9,2), (5,27,5), (6,14,3), (6,26,5), (7,19,4), (7,25,5), (8,13,3), (8,24,5), (8,43,8), (9,18,4), (9,42,8), (10,23,5), (10,41,8), (11,28,6), (11,40,8), (12,22,5), (12,58,11), (13,27,6), (13,57,11), (14,32,7), (14,56,11), (16,31,7), (16,73,14), (17,36,8), (17,72,14), (20,40,9), (20,88,17)$

Table 5.5: Intersection of a nef and big divisor H with (-2)-curves for $\varrho(X) = 3$.

$N°$	$\mathrm{Cl}(X)$	H	Intersection properties
1	S_1	(1,-1,-1)	$H^2 = 2,\ H \cdot E_i = 2,\ i = 1, 2, 3, 4, 5, 6$
2	S_2	(1,-4,-4)	$H^2 = 4,\ H \cdot E_i = 4,\ i = 1, 2, 3, 4, 5, 6$
3	S_3	(1,-2,-2)	$H^2 = 4,\ H \cdot E_i = 2,\ i = 1, 2, 3, 4$
4	S_4	(1,0,1)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_3 = 1,\ H \cdot E_2 = H \cdot E_4 = 2$
5	S_5	(1,-1,-1)	$H^2 = 2,\ H \cdot E_i = 1,\ i = 1, 2, 3, 4$
6	S_6	(1,2,2)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_5 = 2,\ H \cdot E_i = 3,\ i = 2, 3, 4, 6$
7	$S'_{4,1,2}$	(1,2,1)	$H^2 = 2,\ H \cdot E_i = 2,\ i = 1, 2, 3, 4$
8	$S_{4,1,1}$	(1,4,5)	$H^2 = 6,\ H \cdot E_i = 2,\ i = 1, 2, 3$
9	$S_{5,1,1}$	(1,4,6)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_4 = 4,\ H \cdot E_2 = H \cdot E_3 = 1$
10	$S_{6,1,1}$	(1,5,7)	$H^2 = 4,\ H \cdot E_1 = H \cdot E_2 = 2,\ H \cdot E_3 = H \cdot E_4 = 4$
11	$S_{7,1,1}$	(2,11,16)	$H^2 = 6, H \cdot E_1 = H \cdot E_2 = H \cdot E_4 = 4$ $H \cdot E_3 = H \cdot E_5 = H \cdot E_6 = 10$
12	$S_{8,1,1}$	(1,7,9)	$H^2 = 8,\ H \cdot E_i = 4,\ i = 1, 2, 3, 4$
13	$S_{10,1,1}$	(1,8,11)	$H^2 = 2, H \cdot E_i = 4,\ i = 1, 2, 3, 4$ $H \cdot E_i = 6,\ i = 5, 6, 7, 8$
14	$S_{12,1,1}$	(1,10,13)	$H^2 = 6,\ H \cdot E_i = 6,\ i = 1, 2, 3, 4, 5, 6$
15	$S_{1,2,1}$	(1,1,2)	$H^2 = 2,\ H \cdot E_1 = 0,\ H \cdot E_2 = H \cdot E_3 = 1,$
16	$S_{1,3,1}$	(1,1,3)	$H^2 = 4,\ H \cdot E_1 = H \cdot E_2 = 1,\ H \cdot E_3 = 2$
17	$S_{1,4,1}$	(1,1,3)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_2 = 1,\ H \cdot E_3 = H \cdot E_4 = 3$

$N°$	$\mathrm{Cl}(X)$	H	Intersection properties
18	$S_{1,5,1}$	(2,2,7)	$H^2 = 2, H \cdot E_1 = H \cdot E_5 = 3,$ $H \cdot E_2 = H \cdot E_4 = 2,$ $H \cdot E_3 = H \cdot E_6 = 8$
19	$S_{1,6,1}$	(1,1,5)	$H^2 = 6,\ H \cdot E_i = 3,\ i = 1,2,3,4$
20	$S_{1,9,1}$	(1,1,7)	$H^2 = 4, H \cdot E_1 = H \cdot E_2 = 4,$ $H \cdot E_3 = H \cdot E_4 = 5,$ $H \cdot E_6 = H \cdot E_7 = 14,$ $H \cdot E_8 = H \cdot E_9 = 17, H \cdot E_5 = 10$
21	$S_{1,1,1}$	(1,2,2)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_2 = 0,\ H \cdot E_3 = 1$
22	$S_{1,1,2}$	(1,4,3)	$H^2 = 2,\ H \cdot H_2 = H \cdot H_3 = 0,\ H \cdot H_1 = 2$
23	$S_{1,1,3}$	(1,4,2)	$H^2 = 2, H \cdot E_1 = H \cdot E_2 = 1,$ $H \cdot E_3 = H \cdot E_4 = 2$
24	$S_{1,1,4}$	(1,9,3)	$H^2 = 4,\ H \cdot E_i = 2,\ i = 1,2,3,4$
25	$S_{1,1,6}$	(1,4,1)	$H^2 = 2, H \cdot E_1 = H \cdot E_2 = 2,$ $H \cdot E_i = 4, i = 3,4,5,6$
26	$S_{1,1,8}$	(1,6,1)	$H^2 = 6, H \cdot E_1 = H \cdot E_7 = 10,$ $H \cdot E_2 = H \cdot E_8 = 6,$ $H \cdot E_3 = H \cdot E_5 = 12,$ $H \cdot E_4 = H \cdot E_6 = 4,$

Table 5.6: Degrees of a set of generators of $R(X)$ for $\varrho(X) = 3$.

$N°$	$\mathrm{Cl}(X)$	Degrees of generators of $R(X)$
1	S_1	BEff
2	S_2	E, BNef
3	S_3	E, BNef
4	S_4	E, BNef
5	S_5	$E, \mathrm{BNef}[i], i = 1, 2, 5$
6	S_6	$E, \mathrm{BNef}[i], i = 1-7, 10, 11^*, 12, 13, 15, 16, 18-20, 25, 27, 28, 30, 33, 34, 38, 40^*, 41, 42^*, 43^*$
7	$S'_{4,1,2}$	E, BNef
8	$S_{4,1,1}$	E. BNef
9	$S_{5,1,1}$	E, BNef
10	$S_{6,1,1}$	E, BNef
11	$S_{7,1,1}$	E, BNef
12	$S_{8,1,1}$	E, BNef
13	$S_{10,1,1}$	$E, \mathrm{BNef}, 3\,\mathrm{BNef}[2]$ $\mathrm{BNef}[2] + \mathrm{BNef}[i],\ i = 1^*, 6^*, 8^*, 13^*, 17^*, 20^*, 21^*, 24^*$
14	$S_{12,1,1}$	$E \cup \mathrm{BNef}$
15	$S_{1,2,1}$	E, BNef
16	$S_{1,3,1}$	E, BNef
17	$S_{1,4,1}$	$E, \mathrm{BNef}[i], i = 1-12, 13^*, 14, 15^*$

$N°$	$\mathrm{Cl}(X)$	Degrees of generators of $R(X)$
18	$S_{1,5,1}$	$E, \mathrm{BNef}[i], i = 1-17, 18^*, 19-26, 27^*$
19	$S_{1,6,1}$	E, BNef
20	$S_{1,9,1}$	$E, \mathrm{BNef}, \mathrm{BNef}[11] + \mathrm{BNef}[i], i = 20^*, 34^*, 43^*$ $\mathrm{BNef}[2] + \mathrm{BNef}[i], i = 11^*, 12^*, 13^*, 20^*, 29^*, 33^*$
21	$S_{1,1,1}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[3], \mathrm{BNef}[2] + \mathrm{BNef}[3]$
22	$S_{1,1,2}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[2], \mathrm{BNef}[1] + \mathrm{BNef}[2] + \mathrm{BNef}[3]$
23	$S_{1,1,3}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[2], \mathrm{BNef}[6]$
24	$S_{1,1,4}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[3], \mathrm{BNef}[7]$
25	$S_{1,1,6}$	$E, \mathrm{BNef}[i],\ i = 1, 2, 3, 5, 6, 9, 13-17$
26	$S_{1,1,8}$	$E, \mathrm{BNef}[i],\ i = 1, 2, 3, 5, 7, 11, 15, 19, 20, 24, 29, 34, 35, 38, 39, 41$ $\mathrm{BNef}[2] + \mathrm{BNef}[i],\ i = 7^*, 11^*, 19^*$ $\quad \mathrm{BNef}[19] + \mathrm{BNef}[i],\ i = 7^*, 11^*$

5.2 K3 surfaces of Picard number four

In Table 5.7 we give the intersection matrix of (-2)-curves, in Table 5.8 we give $E(X)$, $\mathrm{BEff}(X)$, $N(X)$ and $\mathrm{BNef}(X)$ for each family of Mori dream K3 surfaces of Picard number four.

In Tables 5.9 and 5.10 we give the Hilbert basis of the nef cone of X when the lattice $\mathrm{Cl}(X)$ is isometric to V_1, V_2, V_{13} or V_{14}.

For each family of Mori dream K3 surfaces of Picard number three, in Table 5.11 we give a nef and big class $H \in \mathrm{BNef}(X)$ of minimal self-intersection and its intersection properties with the (-2)-curves E_i.

In Table 5.12 we give the degrees of a set of generators of the Cox ring $R(X)$ when $\mathrm{Cl}(X)$ is not isometric to V_{14} All degrees in the Table 5.12 are necessary to generate $R(X)$, except possibly for those marked with a star. In case $\mathrm{Cl}(X) \cong V_{14}$ we give a subset of the degrees of a minimal generating set of $R(X)$.

Table 5.7: Intersection matrix of (-2)-curves for $\varrho(X) = 4$.

$N°$	Lattice	intersection matrix of (-2)-curves
1	V_1	$\begin{bmatrix} -2 & 6 & 0 & 0 & 2 & 4 & 0 & 0 & 4 & 4 & 2 & 4 \\ 6 & -2 & 4 & 4 & 2 & 0 & 4 & 4 & 0 & 0 & 2 & 0 \\ 0 & 4 & -2 & 4 & 0 & 6 & 2 & 0 & 4 & 2 & 4 & 0 \\ 0 & 4 & 4 & -2 & 4 & 0 & 0 & 2 & 2 & 4 & 0 & 6 \\ 2 & 2 & 0 & 4 & -2 & 4 & 0 & 4 & 0 & 4 & 6 & 0 \\ 4 & 0 & 6 & 0 & 4 & -2 & 2 & 4 & 0 & 2 & 0 & 4 \\ 0 & 4 & 2 & 0 & 0 & 2 & -2 & 4 & 0 & 6 & 4 & 4 \\ 0 & 4 & 0 & 2 & 4 & 4 & 4 & -2 & 6 & 0 & 0 & 2 \\ 4 & 0 & 4 & 2 & 0 & 0 & 0 & 6 & -2 & 4 & 4 & 2 \\ 4 & 0 & 2 & 4 & 4 & 2 & 6 & 0 & 4 & -2 & 0 & 0 \\ 2 & 2 & 4 & 0 & 6 & 0 & 4 & 0 & 4 & 0 & -2 & 4 \\ 4 & 0 & 0 & 6 & 0 & 4 & 4 & 2 & 2 & 0 & 4 & -2 \end{bmatrix}$
2	V_2	$\begin{bmatrix} -2 & 1 & 6 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 3 & 0 \\ 6 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 0 & 3 \\ 1 & 3 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 3 & 1 & -2 \end{bmatrix}$
3	V_3	$\begin{bmatrix} -2 & 0 & 0 & 2 & 1 \\ 0 & -2 & 2 & 0 & 1 \\ 0 & 2 & -2 & 0 & 1 \\ 2 & 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 & -2 \end{bmatrix}$
4	V_4	$\begin{bmatrix} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 & 0 \\ 2 & 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
5	V_5	$\begin{bmatrix} -2 & 0 & 2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 2 & 2 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & 2 & 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 & 0 & -2 \end{bmatrix}$
6	V_6	$\begin{bmatrix} -2 & 4 & 2 & 0 & 2 & 6 & 0 & 0 \\ 4 & -2 & 2 & 6 & 2 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 6 & 0 & -2 & 0 & 4 & 2 & 2 \\ 2 & 2 & 1 & 0 & -2 & 0 & 3 & 0 \\ 6 & 0 & 0 & 4 & 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 3 & 2 & -2 & 1 \\ 0 & 0 & 3 & 2 & 0 & 2 & 1 & -2 \end{bmatrix}$
7	V_7	$\begin{bmatrix} -2 & 2 & 2 & 0 & 0 & 2 & 0 & 4 \\ 2 & -2 & 2 & 4 & 0 & 2 & 0 & 0 \\ 2 & 2 & -2 & 0 & 4 & 2 & 0 & 0 \\ 0 & 4 & 0 & -2 & 2 & 0 & 2 & 2 \\ 0 & 0 & 4 & 2 & -2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 2 & 2 & 4 & -2 & 2 \\ 4 & 0 & 0 & 2 & 2 & 0 & 0 & -2 \end{bmatrix}$
8	V_8	$\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$
9	V_9	$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$

$N°$	Lattice	intersection matrix of (-2)-curves
10	V_{10}	$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$
11	V_{11}	$\begin{bmatrix} -2 & 1 & 1 & 1 & 4 & 1 \\ 1 & -2 & 1 & 4 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 4 \\ 1 & 4 & 1 & -2 & 1 & 1 \\ 4 & 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 4 & 1 & 1 & -2 \end{bmatrix}$
12	V_{12}	$\begin{bmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 & 3 \\ 1 & 0 & 0 & 1 & 3 & -2 \end{bmatrix}$
13	V_{13}	$\begin{bmatrix} -2 & 3 & 0 & 1 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 1 & 0 & 1 & -2 & 0 & 3 \\ 1 & 0 & 3 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & 1 & -2 \end{bmatrix}$
14	V_{14}	$\begin{bmatrix} -2 & 0 & 6 & 0 & 4 & 1 & 1 & 4 \\ 0 & -2 & 4 & 4 & 0 & 1 & 1 & 6 \\ 6 & 4 & -2 & 4 & 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & -2 & 6 & 1 & 1 & 0 \\ 4 & 0 & 0 & 6 & -2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 & -2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & -2 & 1 \\ 4 & 6 & 0 & 0 & 4 & 1 & 1 & -2 \end{bmatrix}$

Table 5.8: Effective and Nef cone for K3 surfaces with $\varrho(X) = 4$.

N°	Cl(X)	$E(X)$	BEff(X)	$N(X)$	BNef(X)
1	V_1	$(1,-1,0,-2)$, $(1,-1,2,0)$, $(0,0,-1,0)$, $(2,-2,2,-3)$, $(0,1,0,0)$, $(2,-2,3,-2)$, $(1,0,1,-2)$, $(1,-2,0,-1)$, $(1,0,2,-1)$, $(1,-2,1,0)$, $(2,-3,2,-2)$, $(0,0,0,1)$	$E(X)$ $\cup$ $\{(1,-1,1,-1)\}$	$(1,-2,0,0)$, $(1,0,0,-2)$, $(1,0,0,0)$, $(1,0,2,0)$, $(3,-4,0,-4)$, $(3,-4,2,-4)$, $(3,-4,4,-2)$, $(3,-4,4,0)$, $(3,-2,4,-4)$, $(3,0,4,-4)$, $(5,-8,4,-4)$, $(5,-4,4,-8)$, $(5,-4,8,-4)$, $(7,-8,8,-8)$	See Table 5.9
2	V_2	$(1,1,1,0)$, $(0,0,0,1)$, $(-1,1,1,0)$, $(0,0,-1,-1)$, $(0,1,1,-1)$, $(0,1,2,1)$	$E(X)$	$(-3,6,4,-4)$, $(-3,6,8,4)$, $(-1,1,0,0)$, $(-1,3,4,0)$, $(1,1,0,0)$, $(1,3,4,0)$, $(3,6,4,-4)$, $(3,6,8,4)$	See Table 5.9
3	V_3	$(1,0,2,1)$, $(0,0,0,-1)$, $(1,1,2,2)$, $(0,1,0,0)$, $(0,-1,-1,0)$	$E(X)$	$(1,0,0,0)$, $(1,1,2,1)$, $(2,-1,2,1)$, $(2,1,2,3)$, $(3,0,4,4)$	$(1,0,0,0)$, $(1,0,1,1)$, $(1,1,2,1)$, $(2,-1,2,1)$, $(2,0,2,1)$, $(2,1,2,2)$, $(2,1,2,3)$, $(3,0,4,3)$, $(3,0,4,4)$, $(3,1,4,4)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
4	V_4	$(0,0,0,-1)$, $(-1,0,1,0)$, $(1,-1,0,0)$, $(-1,0,0,1)$, $(0,0,-1,0)$	$E(X)$	$(-2,-2,0,1)$, $(-2,-2,1,0)$, $(-2,-2,1,1)$, $(-1,-1,0,0)$, $(-1,0,0,0)$	$(-2,-2,0,1)$, $(-2,-2,1,0)$, $(-2,-2,1,1)$, $(-1,-1,0,0)$, $(-1,0,0,0)$
5	V_5	$(0,-1,0,1)$, $(0,-1,1,0)$, $(0,0,0,-1)$, $(-1,0,1,0)$, $(-1,0,0,1)$, $(0,0,-1,0)$	$E(X)$	$(-1,-1,0,1)$, $(-1,-1,1,0)$, $(-1,-1,1,1)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-1,-1,0,1)$, $(-1,-1,1,0)$, $(-1,-1,1,1)$, $(-1,0,0,0)$, $(0,-1,0,0)$
6	V_6	$(0,0,0,1)$, $(-2,-2,-3,-2)$, $(0,-1,0,-1)$, $(0,0,1,0)$, $(-1,0,0,-1)$, $(-2,-2,-2,-3)$, $(0,-1,-1,0)$, $(-1,0,-1,0)$	$E(X)$	$(-4,-4,-6,-3)$, $(-4,-4,-3,-6)$, $(-3,-2,-3,-3)$, $(-2,-3,-3,-3)$, $(-2,-2,-3,0)$, $(-2,-2,0,-3)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-4,-4,-6,-3),(-4,-4,-3,-6)$, $(-3,-3,-4,-3),(-3,-3,-3,-4)$, $(-3,-2,-3,-3),(-3,-2,-3,-2)$, $(-3,-2,-2,-3),(-2,-3,-3,-3)$, $(-2,-3,-3,-2),(-2,-3,-2,-3)$, $(-2,-2,-3,-1),(-2,-2,-3,0)$, $(-2,-2,-1,-3),(-2,-2,0,-3)$, $(-1,-1,-1,-1),(-1,-1,-1,0)$, $(-1,-1,0,-1),(-1,0,0,0)$, $(0,-1,0,0)$

$N°$	$\mathrm{Cl}(X)$	$E(X)$	$\mathrm{BEff}(X)$	$N(X)$	$\mathrm{BNef}(X)$
7	V_7	$(0,-1,0,1)$, $(0,0,0,-1)$, $(-1,-1,-2,1)$, $(-1,-1,-1,2)$, $(0,0,1,0)$, $(-1,0,0,1)$, $(0,-1,-1,0)$, $(-1,0,-1,0)$	$E(X)$	$(-2,-1,-2,2)$, $(-1,-2,-2,2)$, $(-1,-1,-2,0)$, $(-1,-1,0,2)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-2,-2,-3,2),(-2,-2,-2,3)$, $(-2,-1,-2,1),(-2,-1,-2,2)$, $(-2,-1,-1,2),(-1,-2,-2,1)$, $(-1,-2,-2,2),(-1,-2,-1,2)$, $(-1,-1,-2,0),(-1,-1,-1,0)$, $(-1,-1,-1,1),(-1,-1,0,1)$, $(-1,-1,0,2),(-1,0,0,0)$, $(0,-1,0,0)$
8	V_8	$(0,0,0,1)$, $(-1,0,1,0)$, $(1,-1,0,0)$, $(0,0,-1,-1)$	$E(X)$	$(-3,-3,1,-1)$, $(-3,-3,2,1)$, $(-1,-1,0,0)$, $(-1,0,0,0)$	$(-3,-3,1,-1),(-3,-3,2,1)$, $(-2,-2,1,0),(-1,-1,0,0)$, $(-1,0,0,0)$
9	V_9	$(0,0,0,1)$, $(0,-1,1,0)$, $(-1,0,1,0)$, $(0,0,-1,-1)$	$E(X)$	$(-3,-3,2,-2)$, $(-3,-3,4,2)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-3,-3,2,-2),(-3,-3,4,2)$, $(-2,-2,1,-1),(-2,-2,2,1)$, $(-1,-1,1,0),(-1,0,0,0)$, $(0,-1,0,0)$
10	V_{10}	$(0,0,0,-1)$, $(0,-1,1,1)$, $(-1,0,1,1)$, $(0,0,-1,0)$	$E(X)$	$(-1,-1,1,2)$, $(-1,-1,2,1)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-1,-1,1,1),(-1,-1,1,2)$, $(-1,-1,2,1),(-1,0,0,0)$, $(0,-1,0,0)$

$N°$	Cl(X)	$E(X)$	BEff(X)	$N(X)$	BNef(X)
11	V_{11}	$(0,0,0,-1)$, $(0,-1,1,1)$, $(-1,-1,3,2)$, $(-1,0,1,1)$, $(-1,-1,2,3)$, $(0,0,-1,0)$	$E(X)$	$(-3,-2,6,6)$, $(-2,-3,6,6)$, $(-2,-1,2,4)$, $(-2,-1,4,2)$, $(-1,-2,2,4)$, $(-1,-2,4,2)$, $(-1,0,0,0)$, $(0,-1,0,0)$	$(-3,-3,7,7), (-3,-2,5,6)$, $(-3,-2,6,5), (-3,-2,6,6)$, $(-2,-3,5,6), (-2,-3,6,5)$, $(-2,-3,6,6), (-2,-2,3,5)$, $(-2,-2,4,5), (-2,-2,5,3)$, $(-2,-2,5,4), (-2,-1,2,3)$, $(-2,-1,2,4), (-2,-1,3,2)$, $(-2,-1,3,3), (-2,-1,4,2)$, $(-1,-2,2,3), (-1,-2,2,4)$, $(-1,-2,3,2), (-1,-2,3,3)$, $(-1,-2,4,2), (-1,-1,1,1)$, $(-1,-1,1,2), (-1,-1,2,1)$, $(-1,-1,2,2), (-1,0,0,0)$, $(0,-1,0,0)$
12	V_{12}	$(-1,-2,1,2)$, $(0,0,0,-1)$, $(-1,-3,1,1)$, $(0,1,0,0)$, $(0,0,-1,0)$, $(-1,-2,2,1)$	$E(X)$	$(-7,-15,9,9)$, $(-5,-12,3,6)$, $(-5,-12,6,3)$, $(-4,-6,3,6)$, $(-4,-6,6,3)$, $(-2,-3,0,0)$, $(-2,-3,3,3)$, $(-1,-3,0,0)$	$(-7,-15,9,9), (-5,-12,3,6)$, $(-5,-12,6,3), (-5,-11,6,6)$, $(-4,-9,3,5), (-4,-9,4,5)$, $(-4,-9,5,3), (-4,-9,5,4)$, $(-4,-6,3,6), (-4,-6,6,3)$, $(-3,-7,3,3), (-3,-6,2,4)$, $(-3,-6,3,4), (-3,-6,4,2)$, $(-3,-6,4,3), (-3,-6,4,4)$, $(-3,-5,2,4), (-3,-5,4,2)$, $(-2,-5,1,2), (-2,-5,2,1)$, $(-2,-4,1,2), (-2,-4,2,1)$, $(-2,-3,0,0), (-2,-3,1,1)$, $(-2,-3,1,2), (-2,-3,2,1)$, $(-2,-3,2,2), (-2,-3,2,3)$, $(-2,-3,3,2), (-2,-3,3,3)$, $(-1,-3,0,0), (-1,-2,0,0)$, $(-1,-2,1,1)$

$N°$	Cl(X)	$E(X)$	BEff(X)	$N(X)$	BNef(X)
13	V_{13}	$(0,0,0,-1)$, $(1,0,0,1)$, $(0,0,-1,0)$, $(1,1,0,0)$, $(1,0,1,0)$, $(0,-1,0,0)$	$E(X)$	$(2,-1,-1,-1)$, $(5,1,1,1)$, $(6,-3,-3,4)$, $(6,-3,4,-3)$, $(6,4,-3,-3)$, $(8,-4,3,3)$, $(8,3,-4,3)$, $(8,3,3,-4)$	See Table 5.9
14	V_{14}	$(1,0,-1,-2)$, $(0,-1,1,-1)$, $(1,2,-1,0)$, $(2,2,-3,-2)$, $(0,0,1,0)$, $(0,0,-1,1)$, $(1,1,0,-2)$, $(2,3,-3,-1)$	$E(X)$	$(2,-3,2,-1)$, $(7,-3,-8,-11)$, $(7,12,-8,4)$, $(8,3,8,-19)$, $(13,3,-2,-29)$, $(13,18,-2,-14)$, $(17,12,-28,-16)$, $(17,27,-28,-1)$, $(22,27,-38,-11)$, $(23,18,-22,-34)$, $(23,33,-22,-19)$, $(28,33,-32,-29)$	See Table 5.10

Table 5.9: BNef(X) for Cl(X) $= V_1$, V_2 and V_{13}.

$N°$	BNef(X)
1	$(1,-2,0,0),(1,-1,0,-1),(1,-1,0,0),(1,-1,1,-1),(1,-1,1,0),(1,0,0,-2),(1,0,0,-1),$ $(1,0,0,0),(1,0,1,-1),(1,0,1,0),(1,0,2,0),(2,-3,0,-2),(2,-3,1,-2),$ $(2,-3,2,-1),(2,-3,2,0),(2,-2,0,-3),(2,-2,1,-3),(2,-2,3,-1),(2,-2,3,0),$ $(2,-1,2,-3),(2,-1,3,-2),(2,0,2,-3),(2,0,3,-2),(3,-5,2,-2),(3,-4,0,-4),$ $(3,-4,1,-4),(3,-4,2,-4),(3,-4,3,-3),(3,-4,4,-2),(3,-4,4,-1),(3,-4,4,0),$ $(3,-3,3,-4),(3,-3,4,-3),(3,-2,2,-5),(3,-2,4,-4),(3,-2,5,-2),(3,-1,4,-4),$ $(3,0,4,-4),(4,-6,3,-4),(4,-6,4,-3),(4,-4,3,-6),(4,-4,6,-3),(4,-3,4,-6),$ $(4,-3,6,-4),(5,-8,4,-4),(5,-6,5,-6),(5,-6,6,-5),(5,-5,6,-6),(5,-4,4,-8),$ $(5,-4,8,-4),(7,-8,8,-8)$
2	$(-3,6,4,-4),(-3,6,8,4),(-2,4,3,-2),(-2,4,5,2),(-1,1,0,0),(-1,2,1,-1),$ $(-1,2,2,0),(-1,2,2,1),(-1,3,2,-2),(-1,3,3,-1),(-1,3,4,0),(-1,3,4,1),$ $(-1,3,4,2),(0,1,0,0),(0,1,1,0),(0,2,1,-1),(0,2,2,1),(0,3,2,-2),$ $(0,3,3,-1),(0,3,4,0),(0,3,4,1),(0,3,4,2),(1,1,0,0),(1,2,1,-1),$ $(1,2,2,0),(1,2,2,1),(1,3,2,-2),(1,3,3,-1),(1,3,4,0),(1,3,4,1),$ $(1,3,4,2),(2,4,3,-2),(2,4,5,2),(3,6,4,-4),(3,6,8,4)$
13	$(1,0,0,0),(2,-1,-1,-1),(2,-1,-1,0),(2,-1,-1,1),(2,-1,0,-1),(2,-1,0,0),$ $(2,-1,0,1),(2,-1,1,-1),(2,-1,1,0),(2,0,-1,-1),(2,0,-1,0),(2,0,-1,1),$ $(2,0,0,-1),(2,0,1,-1),(2,1,-1,-1),(2,1,-1,0),(2,1,0,-1),(3,-1,1,1),$ $(3,0,0,1),(3,0,1,0),(3,1,-1,1),(3,1,0,0),(3,1,1,-1),(4,0,1,1),$ $(4,1,0,1),(4,1,1,0),(5,-2,-2,3),(5,-2,3,-2),(5,1,1,1),(5,3,-2,-2),$ $(6,-3,-3,4),(6,-3,2,2),(6,-3,4,-3),(6,2,-3,2),(6,2,2,-3),(6,4,-3,-3),$ $(8,-4,3,3),(8,3,-4,3),(8,3,3,-4)$

Table 5.10: BNef(X) for $\mathrm{Cl}(X) = V_{14}$.

$N°$	BNef(X)
14	$(1,-1,0,-1),(1,-1,1,-1),(1,0,-1,-1),(1,0,0,-2),(1,0,0,-1),$ $(1,0,0,0),(1,0,1,-2),(1,1,-1,-1),(1,1,-1,0),(1,1,0,-1),$ $(2,-3,2,-1),(2,-2,1,-1),(2,-1,-2,-3),(2,-1,-1,-3),(2,0,-2,-3),$ $(2,0,-1,-4),(2,1,-3,-2),(2,1,-2,-3),(2,1,0,-4),(2,1,1,-4),$ $(2,2,-3,-1),(2,2,-1,0),(2,2,0,-3),(2,3,-3,0),(2,3,-2,-1),$ $(2,3,-2,0),(2,3,-2,1),(2,3,-1,-1),(3,-1,-3,-5),(3,0,-4,-4),$ $(3,1,-1,-6),(3,1,2,-7),(3,1,3,-7),(3,2,-2,-5),(3,2,2,-6),$ $(3,3,-5,-2),(3,3,-4,-3),(3,3,-2,-4),(3,4,-5,-1),(3,4,-4,-2),$ $(3,4,-2,-3),(3,4,-1,-3),(3,5,-4,1),(3,5,-3,1),(4,1,0,-9),$ $(4,1,1,-9),(4,3,-6,-4),(4,3,-5,-5),(4,4,-4,-5),(4,4,1,-6),$ $(4,5,-4,-4),(4,5,0,-5),(4,6,-6,-1),(4,6,-5,-2),(5,1,-1,-11),$ $(5,2,-7,-6),(5,2,-2,-10),(5,3,-8,-5),(5,4,-5,-7),(5,6,-8,-3),$ $(5,6,-7,-4),(5,7,-5,-4),(5,7,-2,-5),(5,7,-1,-5),(5,8,-8,0),$ $(5,8,-7,0),(6,-2,-7,-9),(6,5,-10,-5),(6,7,-7,-6),(6,9,-10,-1),$ $(6,10,-7,3),(7,-3,-8,-11),(7,3,6,-16),(7,4,-5,-12),(7,5,-11,-7),$ $(7,5,-6,-11),(7,8,-12,-4),(7,9,-12,-3),(7,10,-6,-6),(7,10,-5,-6),$ $(7,11,-11,-1),(7,12,-8,4),(8,3,8,-19),(8,7,-8,-11),(8,8,-13,-6),(8,11,-13,-3),$ $(8,11,-8,-7),(9,7,-9,-13),(9,10,-10,-10),(9,11,-15,-5),(9,11,-10,-9),$ $(9,13,-9,-7),(10,10,-11,-12),(10,13,-11,-9),(11,3,-2,-24),(11,13,-13,-11),$ $(11,15,-2,-12),(13,3,-2,-29),(13,18,-2,-14),(14,10,-23,-13),(14,22,-23,-1),$ $(17,12,-28,-16),(17,27,-28,-1),(18,22,-31,-9),(19,15,-18,-28),(19,27,-18,-16),$ $(22,27,-38,-11),(23,18,-22,-34),(23,27,-26,-24),(23,33,-22,-19),(28,33,-32,-29)$

Table 5.11: Intersection of a nef and big divisor H with (-2)-curves for $\varrho(X) = 4$.

$N°$	$\mathrm{Cl}(X)$	H	Intersection properties
1	V_1	(1,-1,1,-1)	$H^2 = 2,\ H \cdot E_i = 2,\ i = 1, \ldots, 12$
2	V_2	(0,1,1,0)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_3 = 2$ $H \cdot E_i = 1,\ i = 2, 4, 5, 6$
3	V_3	(1,0,1,1)	$H^2 = 2,\ H \cdot E_5 = 0$ $H \cdot E_i = 1,\ i = 1, 2, 3, 4$
4	V_4	(-2,-2,1,1)	$H^2 = 4,\ H \cdot E_1 = H \cdot E_5 = 2$ $H \cdot E_i = 0,\ i = 2, 3, 4$
5	V_5	(-1,-1,0,1)	$H^2 = 2,\ H \cdot E_i = 0,\ i = 1, 5, 6$ $H \cdot E_i = 2,\ i = 2, 3, 4$
6	V_6	(-1,-1,-1,-1)	$H^2 = 2,\ H \cdot E_i = 2,\ i = 1, 2, 4, 6$ $H \cdot E_i = 1,\ i = 3, 5, 7, 8$
7	V_7	(-1.-1.-1.1)	$H^2 = 4,\ H \cdot E_i = 2,\ i = 1, \ldots, 8$
8	V_8	(-2,-2,1,0)	$H^2 = 6,\ H \cdot E_1 = H \cdot E_4 = 1$ $H \cdot E_2 = H \cdot E_3 = 0$
9	V_9	(-1,-1,1,0)	$H^2 = 2,\ H \cdot E_1 = H \cdot E_4 = 1$ $H \cdot E_2 = H \cdot E_3 = 0$
10	V_{10}	(-1,-1,1,1)	$H^2 = 4,\ H \cdot E_i = 1,\ i = 1, 2, 3, 4$
11	V_{11}	(-1,-1,2,2)	$H^2 = 4,\ H \cdot E_i = 2,\ i = 1, \ldots, 6$
12	V_{12}	(1,-2,1,1)	$H^2 = 2,\ H \cdot E_i = 1,\ i = 1, \ldots, 6$
13	V_{13}	(1,0,0,0)	$H^2 = 2,\ H \cdot E_i = 1,\ i = 1, \ldots, 6$
14	V_{14}	(1,1,-1,-1)	$H^2 = 2,\ H \cdot E_i = 2,\ i = 1, 2, 3, 4, 5, 8$ $H \cdot E_6 = H \cdot E_7 = 1$

Table 5.12: Degrees of a set of generators of $R(X)$ of $\varrho(X) = 4$.

$N°$	$\mathrm{Cl}(X)$	Degrees of generators of $R(X)$
1	V_1	BEff
2	V_2	$E, \mathrm{BNef}[i], i = 1-5, 7, 11, 14, 15, 20, 23, 25, 29, 32-35$
3	V_3	$E, \mathrm{BNef}[i], i = 1, 2, 4, 7, 9$
4	V_4	$E, \mathrm{BNef}[3], \mathrm{BNef}[3] + \mathrm{BNef}[4]$
5	V_5	$E, \mathrm{BNef}[1] + \mathrm{BNef}[2]$
6	V_6	$E, \mathrm{BNef}[15]$
7	V_7	E
8	V_8	$E, \mathrm{BNef}[i], i = 1, 2, 3, 5$
9	V_9	$E, \mathrm{BNef}[i], i = 1-4, 6, 7$
10	V_{10}	$E, \mathrm{BNef}[i], i = 2-5$
11	V_{11}	$E, \mathrm{BNef}[i], i = 4, 7, 9, 11, 13, 15, 16, 18, 20, 21, 23, 24, 26, 27$ $\mathrm{BNef}[i]^*, i = 1-3, 5, 6, 8, 10, 12, 14, 17, 19, 22, 25$
12	V_{12}	$E, \mathrm{BNef}[30], \mathrm{BNef}[31]$
13	V_{13}	$E, \mathrm{BNef}[i], i = 1^*, 7, 9, 12, 14, 16, 17, 27, 28, 30-39$
14	V_{14}	Contains the degrees in Table 5.13

Table 5.13: Degrees of generators of the Cox ring of the family $\mathcal{F}_{14}$.

$N°$	degrees of generators
14	$(1,0,-1,-2),(0,-1,1,-1),(1,2,-1,0),(2,2,-3,-2),(0,0,1,0),$ $(0,0,-1,1),(1,1,0,-2),(2,3,-3,-1),(1,-1,0,-1),(1,0,-1,-1),$ $(1,0,0,0),(1,1,-1,-1),(1,1,-1,0),(2,-3,2,-1),(2,-2,1,-1),$ $(2,-1,-2,-3),(2,1,-3,-2),(2,1,0,-4),(2,2,-3,-1),(2,2,0,-3),$ $(2,3,-3,0),(2,3,-2,1),(3,0,-4,-4),(3,1,2,-7),(3,2,-2,-5),$ $(3,2,2,-6),(3,3,-5,-2),(3,3,-2,-4),(3,4,-5,-1),(3,4,-2,-3),$ $(3,5,-4,1),(4,1,0,-9),(4,4,-4,-5),(4,5,-4,-4),(4,5,0,-5),$ $(5,2,-2,-10),(5,3,-8,-5),(5,7,-2,-5),(5,8,-8,0),(6,-2,-7,-9),$ $(6,5,-10,-5),(6,9,-10,-1),(6,10,-7,3),(7,-3,-8,-11),(7,3,6,-16),$ $(7,5,-6,-11),(7,8,-12,-4),(7,9,-12,-3),(7,10,-6,-6),(7,12,-8,4),$ $(8,3,8,-19),(8,7,-8,-11),(8,11,-8,-7),(9,10,-10,-10),(9,11,-10,-9),$ $(11,3,-2,-24),(11,15,-2,-12),(13,3,-2,-29),(13,18,-2,-14),$ $(14,10,-23,-13),(14,22,-23,-1),(17,12,-28,-16),(17,27,-28,-1),$ $(18,22,-31,-9),(19,15,-18,-28),(19,27,-18,-16),(22,27,-38,-11),$ $(23,18,-22,-34),(23,27,-26,-24),(23,33,-22,-19),(28,33,-32,-29)$

Chapter 6

Magma Programs

In this chapter we briefly present and include the Magma [BCP97] programs used for the proofs of Theorem 3.2.1 and Theorem 4.2.1. We always assume that a basis of the Picard lattice has been fixed, so that $\mathrm{Cl}(X)$ is identified to $\mathbb{Z}^{\varrho(X)}$ and the intersection form to a matrix Q. The Magma programs are organized in four libraries:

- `LSK3Lib.m`: library for linear systems on K3 surfaces
- `Find-2.m`: library for computing the set of (-2)-curves of a Mori dream K3 surface
- `TestLib.m`: library containing the test functions described in the proof of Theorem 3.2.1
- `MinimalLib.m`: library containing functions which check the minimality of a generating set of the Cox ring

Here is a link to a folder containing the Magma codes and the following databases:

- `K3Rank3.txt`: contains the intersection matrix of $\mathrm{Cl}(X)$ and the list of classes of (-2)-curves for all Mori dream K3 surfaces of Picard number 3

- `K3Rank4.txt`: contains the intersection matrix of $\mathrm{Cl}(X)$ and the list of classes of (-2)-curves for all Mori dream K3 surfaces of Picard number 4
- `Gen(K3Rank3)`: contains the output of the function `gen` for Mori dream K3 surfaces of Picard number 3
- `Gen(K3Rank4)`: contains the output of the function `gen` for Mori dream K3 surfaces of Picard number 4.

In the following sections we briefly describe the functions contained in each of the libraries.

6.1 `LSK3Lib.m`

In this section we describe the functions for linear systems on K3 surfaces contained in the library `LSK3Lib.m`:

- `qua`: returns the intersection product of two vectors given the matrix Q.
- `h01`, `h0`, `h1`: compute h^0 and h^1 of a divisor on a K3 surface. given the set of classes of (-2)-curves and Q, by means of the algorithm described in Section 1.4.
- `Eff` and `HBEff`: compute the effective cone and a Hilbert basis of it.
- `Nef` and `HBNef`: compute the nef cone and a Hilbert basis of it.
- `Hyperelliptic`: checks whether a divisor on a K3 surface is hyperelliptic.
- `IsNef`: checks whether a divisor on a K3 surface is nef.
- `IsVAmple`: checks whether a divisor on a K3 surface is very ample.

Program 6.1

```
qua:=  function(A,B,Q)
 K  :=  CoefficientRing(Q);
 n  :=  Nrows(Q);
 return(Matrix(K,1,n,Eltseq(A))*Q*Matrix(K,n,1,Eltseq(B)))[1,1];
 end  function;
```

Program 6.2

```
function h01(D,neg,Q)
 L := ToricLattice(#Eltseq(D));
 B := L!Eltseq(D);
  if B eq Zero(L) then return [1,0]; end if;
 h := qua(B,B,Q)/2+2;
 Eff := Cone([L!Eltseq(v) : v in neg]);
  if B notin Eff then
   if -B notin Eff then return [0,-h];
   else
     v := h01(-B,neg,Q);
     return [0,v[2]];
   end if;
  end if;
 repeat
  m,i := Min([qua(B,E,Q) : E in neg]);
  C := neg[i];
   if m lt 0 then B := B - L!Eltseq(C); end if;
 until m ge 0;
   if B eq Zero(L) then return [1,1-h];
   else
     if qua(B,B,Q) eq 0 then  d := Gcd(Eltseq(B));
       s := d+1;
     else
        s := qua(B,B,Q)/2+2;
     end if;
 return [s,s - h];
   end if;
end function;
```

Program 6.3

```
h0 := function(D,neg,Q)
 return h01(D,neg,Q)[1];
end function;
```

Program 6.4

```
h1 := function(D,neg,Q)
 return h01(D,neg,Q)[2];
end function;
```

Program 6.5

```
Eff := function(neg)
 L := ToricLattice(#Eltseq(neg[1]));
 return Cone([L!Eltseq(v) : v in neg]);
end function;
```

Program 6.6

```
HBEff := function(neg)
 return HilbertBasis(Eff(neg));
end function;
```

Program 6.7

```
Nef := function(neg,Q)
 L := ToricLattice(#Eltseq(neg[1]));
 return Cone([L!Eltseq(v) : v in Rays(Dual(Eff(neg)*Q))]);
end function;
```

Program 6.8

```
HBNef := function(neg,Q)
 return HilbertBasis(Nef(neg,Q));
end function;
```

Program 6.9

```
Hyperelliptic := function(D,hb,Q)
 L := ToricLattice(#Eltseq(D));
 D := L!Eltseq(D);
 C0 := [w : w in hb | qua(w,w,Q) eq 0];
 C1 := [w : w in hb | qua(w,w,Q) eq 2];
  for p in C0 do
     if qua(D,D,Q) gt 0 and qua(D,p,Q) eq 2
       then return true, D;
     end if;
  end for;
  for p in C1 do
     if qua(D,D,Q) gt 0 and D eq 2*p
      then return true, D;
     end if;
  end for;
  if qua(D,D,Q) eq 2
     then return true, D;
  end if;
 return false;
end function;
```

Program 6.10

```
IsNef := function(D,neg,Q)
 if Min([qua(D,E,Q) : E in neg]) ge 0 then
    return true;
 end if;
 return false;
end function;
```

Program 6.11

```
IsVAmple := function(D,neg,hb,Q)
 if Min([qua(D,E,Q) : E in neg]) gt 0 and
    Hyperelliptic(D,hb,Q) eq false then
    return true;
  end if;
 return false;
end function;
```

6.2 Find-2.m

In this section we describe the functions used for computing the set of (-2)-curves of a Mori dream K3 surface.

- `qua`: same as in Section 6.1 using the algorithm described in Section 1.4, these functions be contained in the library `Find-2.`

  `Pts`: given a diagonal matrix $D \in \mathrm{GL}(n, \mathbb{Q})$ with $D_{1,1} > 0$ and $D_{i,i} < 0$ for $i \neq 1$, $B \in \mathrm{GL}(n, \mathbb{Q})$ and a non-negative integer m returns the list `Pts(D,B,m)` of vectors $y \in \mathbb{Q}^n$ such that $y_1 = m, y^T D y = -2$ and $B^T y \in \mathbb{Z}^n$.

- `Test`: given an intersection matrix and a list of vectors with self-intersection -2, it computes the cone $\mathcal{C}$ generated by the vectors and returns true if, for any facet F of $\mathcal{C}$, the intersection matrix of the vectors generating F is negative semidefinite.

- `FindEff`: given the intersection matrix Q of the Picard lattice of a Mori dream K3 surface, it returns a set of fundamental roots of the lattice using the algorithm described in Section 1.4. To simplify the computation of the sets R_i, the program first finds a diagonal matrix $D \in \mathrm{GL}(n, \mathbb{Q})$ as in function `Pts` and $B \in \mathrm{GL}(n, \mathbb{Q})$ such that $D = BQB^T$. Thus $\alpha = dB^{-1}e_1$ with $d = |\mathrm{Det}(B)|$, and the sets R_i, $i \geq 0$, are found using the function `Pts(D,B,m)` with $m = \frac{i}{d}$. When the function `Test` applied to the list of vectors returns true, the program returns R_n.

Next we give the programs:

Program 6.12

```
Pts := function(D,B,m)
 n := Nrows(D);
 L := <>;
 Append(~L,[m]);
 M := Transpose(B)^(-1);
 for i in [2..n-1] do
  d := Lcm([Denominator(p) : p in Eltseq(M[i])]);
  u := Floor(d*Sqrt((D[1,1]*m^2+2)/-D[i,i]));
  Append(~L,[k/d : k in [-u..u]]);
 end for;
 lis := [[p[i] : i in [1..#L]] : p in CartesianProduct(L)];
 pts := [];
 for v in lis do
  a := (&+[D[i,i]*v[i]^2 : i in [1..#v]]+2)/(-D[n,n]);
  if a eq 0 then
    Append(~pts,v cat [0]);
  else
   b,r := IsSquare(a);
   if b then
    Append(~pts,v cat [r]);
    Append(~pts,v cat [-r]);
   end if;
  end if;
 end for;
 return pts;
end function;
```

Program 6.13

```
Test := function(Q,ll)
  if #ll eq 0 then return false; end if;
 n := Nrows(Q);
 C := Cone([Eltseq(v) : v in ll]);
  if Dimension(C) lt n then
    return false;
  else
    L := <>;
      for F in Facets(C) do
        ra := Rays(F);
        Append(~L,Matrix(#ra,#ra,[qua(Q,p,q) : p,q in ra]));
      end for;
   return &and[IsNegativeSemiDefinite(M) : M in L];
   end if;
end function;
```

Program 6.14

```
FindEff := function(Q)
 n := Nrows(Q);
 D,B := OrthogonalizeGram(Q);
 B := Matrix(Rationals(),B);
 k := [i : i in [1..n] | D[i,i] gt 0][1];
 if k ne 1 then
  J := PermutationMatrix(Rationals(),Sym(n)!(1,k));
  B := Transpose(J)*B;
  D := Transpose(J)*D*J;
 end if;
 A<[x]> := AffineSpace(Rationals(),n);
 pts := [A!p : p in Pts(D,B,0)];
 l1 := {Matrix(1,n,Eltseq(p))*B : p in pts};
 l1 := {p : p in l1 | &and[Denominator(a) eq 1 : a in Eltseq(p)]};
 if #l1 gt 1 then
  repeat
   H := &+[Random(Integers(),10)*C : C in l1];
  until 0 notin [qua(Q,H,C) : C in l1];
  l1 := {}; l2 := {};
  repeat
   u := Min([Abs(qua(Q,H,C)) : C in l1]);
   l2 := l2 join {C : C in l1 | qua(Q,H,C) eq u};
   l1 := l1 diff &join{{v,-v} : v in l2};
   l1 := l1 join {C : C in l2 | &and[qua(Q,C,E) ge 0 : E in l1]};
  until #l1 eq 0;
  l1 := l1;
  l0 := l1;
 end if;
 u := 0;
 d := Abs(Determinant(B));
 repeat
  u := u + 1/d;
  pts := Pts(D,B,u);
  l1 := {Matrix(1,n,Eltseq(p))*B : p in pts};
  l2 := {v : v in l1 | &and[Denominator(p) eq 1 : p in Eltseq(v)]};
  l3 := {v : v in l2 | &and[qua(Q,v,w) ge 0 : w in l1]};
  pairs := {{a,b} : a,b in l3 | a ne b and qua(Q,a,b) lt 0};
  if #pairs ne 0 then for v in pairs do
    w := Eltseq(v);
    if w[1] in Cone([v : v in l0] cat [w[2]]) then l3 := l3 diff {w[1]};
     else l3 := l3 diff {w[2]};
    end if;
   end for;
  end if;
  l1 := l1 join {v : v in l2 | &and[qua(Q,v,w) ge 0 : w in l1]};
 until Test(Q,l1);
 return l1;
end function
```

6.3 `TestLib.m`

This section contains the test functions described in the proof of Theorem 3.2.1, containes in the Magma library `TestLib.m`.

For all the programs below we define `n`:=3 if the Picard number is 3 and `n`:=4 if the Picard number is 4. Moreover, `neg` is the list of classes of (-2)-curves and `hb` is the list of vectors in the Hilbert basis of the nef cone.

- `S`: given $v \in \mathrm{Cl}(X)$, `neg`, `hb` and the matrix Q, it gives the set of all classes w in either `neg` or `hb` which are distinct from v and such that $v - w$ is effective.

- `Ti` (i=1,...,4) and `testi` (i=1,...,6): constructs the set Ti and performs the Test i described in the proof of Theorem 3.2.1.

- `test`: it checks if a divisor passes test1, test2, test3, and test4.

- `gen`: given `neg` and Q, it returns three lists L_1, L_2, L_3 of classes of divisors: L_1 contains the classes in `hb` for which `test` is true, L_2 the sums of two elements in `hb` for which `test` and `test5` are true and L_3 the sums of three elements in `hb` for which `test` and `test6` are true.

Next, we give the programs:

Program 6.15

```
S := function(D,neg,hb,Q)
L := ToricLattice(n);
D := L!Eltseq(D);
neg := [L!Eltseq(v) : v in neg];
hb := [L!Eltseq(v) : v in hb];
eff := Cone(neg);
 return {L!w: w in Set(neg) join Set(hb)| w ne D
 and h0(D-w,neg,Q) gt 0};
end function;
```

Program 6.16

```
T1 := function(neg,hb,Q)
L := ToricLattice(n);
neg := [L!Eltseq(v) : v in neg];
hb :=[L!Eltseq(v) : v in hb];
  return {[a,b]: a,b in Set(neg) join Set(hb)|qua(a,b,Q) eq 0};
end function;
```

Program 6.17

```
T2 := function(neg,hb)
L := ToricLattice(n);
neg := [L!Eltseq(v) : v in neg];
hb :=[L!Eltseq(v) : v in hb];
C := Set(neg) join Set(hb);
  return {A: A in Subsets(C,3)| not
  &and{a in Set(neg): a in A}};
end function;
```

Program 6.18

```
T3 := function(neg,hb,Q)
L := ToricLattice(n);
neg := [L!Eltseq(v) : v in neg];
hb :=[L!Eltseq(v) : v in hb];
  return {[L!a,b]: a,b in Set(hb)| h1(a-b,neg,Q) eq 0
  and h1(a,neg,Q) eq 0 and h0(2*b-a,neg,Q) eq 0};
end function;
```

Program 6.19

```
T4 := function(neg,hb,Q)
C1 := {3*a: a in hb|qua(a,a,Q) ne 2};
C2 := {2*a: a in hb| qua(a,a,Q) eq 2
          or Hyperelliptic(a,hb,Q) eq false};
   return C1 join C2;
end function;
```

Program 6.20

```
test1 := function(D,neg,hb,Q,t1,Sd);
L := ToricLattice(n);
D := L!Eltseq(D);
T1b := [p : p in t1 | &and{a in Sd : a in p}];
   for p in T1b do
       if h1(D-L!p[1]-L!p[2],neg,Q) eq 0 then
            return false, p;
       end if;
   end for;
  return true;
end function;
```

Program 6.21

```
test2 := function(D,neg,hb,Q,t2,Sd)
L := ToricLattice(n);
D := L!Eltseq(D);
T2b:= [A: A in t2|&and{a in Sd: a in A}];
  for E in T2b do
      if &+[h1(D-L!&+S,neg,Q) : S in Subsets(E,2)] eq 0
        and h0(L!&+E-D,neg,Q) eq 0 then
        return false, E;
      end if;
  end for;
 return true;
end function;
```

Program 6.22

```
test3 := function(D,neg,hb,Q,t3,Sd)
L := ToricLattice(n);
D := L!Eltseq(D);
T3b := [p:p in t3|&and{a in Sd: a in p}];
sum := [p:p in T3b| D eq L!(p[1]+p[2])];
  if #sum ge 1 then
    return false, sum;
  end if;
 return true;
end function;
```

Program 6.23

```
test4 := function(D,neg,hb,Q,t4)
L := ToricLattice(n);
D := L!Eltseq(D);
  if D in t4 then
    return false;
  end if;
 return true;
end function;
```

Program 6.24

```
test5 := function(D,neg,hb,Q)
L := ToricLattice(n);
D := L!Eltseq(D);
d := h0(D,neg,Q);
neg := [L!Eltseq(v) : v in neg];
hb := [L!Eltseq(v) : v in hb];
E := {[a,b]: a,b in neg|h0(D-a-b,neg,Q) gt 0
        and h1(D-a-b,neg,Q) eq 0 and qua(a,b,Q) eq 2};
E := {e: e in E|IsVAmple(D-e[1]-e[2],neg,hb, Q)};
 for e in E do
   if d eq h0(D-e[1],neg,Q)+h0(D-e[2],neg,Q)-h0(D-e[1]-e[2],neg,Q)+2
     then return false, e;
   end if;
 end for;
 return true;
end function;
```

Program 6.25

```
test6 := function(D,neg,hb,Q,Sd)
L := ToricLattice(n);
Sums := {L!a+L!b: a,b in Sd| a in Set(hb) and b in Set(hb)};
A := {[L!p,L!q]: p in Sums, q in Sd| q in Set(hb) and L!D eq L!p+L!q};
B := {v: v in A|h1(v[1]-v[2],neg,Q) eq 0 and h1(v[1],neg,Q) eq 0
        and h0(2*v[2]-v[1],neg,Q) eq 0};
  if #B gt 0 then
     return false , B;
  end if;
 return true;
end function;
```

Program 6.26

```
test := function(D,neg,hb,Q,t1,t2,t3,t4,Sd)
 if test1(D,neg,hb,Q,t1,Sd) and test2(D,neg,hb,Q,t2,Sd) and
     test3(D,neg,hb,Q,t3,Sd) and test4(D,neg,hb,Q,t4) then
     return true;
  else
     return false;
 end if;
end function;
```

Program 6.27

```
gen := function(neg,Q)
 hb := HBNef(neg,Q);
 t1 := T1(neg,hb,Q);
 t2 := T2(neg,hb);
 t3 := T3(neg,hb,Q);
 t4 := T4(neg,hb,Q);
 SumOf2 := {&+[a,b] : a,b in hb};
 SumOf3 := {&+[a,b,c] : a,b,c in hb} diff SumOf2;
 L1 := [];
 L2 := [];
 L3 := [];
  for D in hb do
    Sd := S(D,neg,hb,Q);
      if test(D,neg,hb,Q,t1,t2,t3,t4,Sd) then
         Append(~L1,D);
    end if;
  end for;
  for D in SumOf2 do
    Sd := S(D,neg,hb,Q);
      if test(D,neg,hb,Q,t1,t2,t3,t4,Sd) and test5(D,neg,hb,Q) then
         Append(~L2,D);
     end if;
  end for;
  for D in SumOf3 do
    Sd := S(D,neg,hb,Q);
      if test(D,neg,hb,Q,t1,t2,t3,t4,Sd)  and test6(D,neg,hb,Q,Sd) then
        Append(~L3,D);
       end if;
  end for;
 return L1,L2,L3;
end function;
L1,L2,L3 := gen(E,M);
```

6.4 `MinimalLib.m`

This section contains the functions which check the minimality of a generating set of the Cox ring, contained en the Magma library `MinimalLib.m`.

- `RR`: given a matrix $M \in M_{m\times n}(\mathbb{Z})$ and a vector $v \in \mathbb{Z}^m$, it finds all $w \in \mathbb{Z}^n$ with non-negative integral coefficients such that $Mw = v$.

- `SG`: given $v \in \mathrm{Cl}(X)$, `neg` `hb`, Q and a set of classes of divisors G, it finds all $w \in G$ such that $v - w$ is effective.

- `RRD`: it finds all possible ways to write v as a linear combination with non-negative integer coefficients of the classes in `SG(v,neg,hb,Q,G)`.

- `Minimal`: given $v \in \mathrm{Cl}(X)$, `neg`, `hb`, Q and a set of classes of divisors G, it first computes `SG(v,neg,hb,Q,G)`. If the latter is empty, then it returns true. Otherwise, it finds all possible ways of writing v as a non-negative linear combination of elements in G, using the function `RR`, and checks whether one of the hypotheses of Proposition 2.4.9 are satisfied. If one of them is satisfied, it returns true and the list of the (-2)-curves appearing in the statement of Proposition 2.4.9.

For all the programs below we define `d`:=3 if the Picard number is 3 or `d`:=4 if the Picard number is 4. Next, we give the programs:

Program 6.28

```
RR := function(M,v);
n:=Ncols(M);
m:=Nrows(M);
L1:=ToricLattice(n);
L2:=ToricLattice(m);
f:=hom<L1->L2|Transpose(M)>;
C:=ZeroCone(L2);
w:=L2!Eltseq(v);
P:=ConeToPolyhedron(PositiveQuadrant(L1)) meet Polyhedron(C,f,-w);
 return Points(P);
end function;
```

Program 6.29

```
SG := function(D,neg,hb,Q,G)
L := ToricLattice(d);
D := L!Eltseq(D);
neg := [L!Eltseq(v) : v in neg];
hb := [L!Eltseq(v) : v in hb];
eff := Cone(neg);
 return [i: i in [1..#G]| L!Eltseq(G[i]) ne D
            and h0(D-L!G[i],neg,Q) gt 0];
end function;
```

Program 6.30

```
RRD := function(D,neg,hb,Q,G);
A := Matrix([G[i]: i in  SG(D,neg,hb,Q, G)]);
 return RR(Transpose(A),D), SG(D,neg,hb,Q,G);
end function;
```

Program 6.31

```
Minimal := function(D,neg,hb,Q,G);
L:=ToricLattice(d);
I := SG(D,neg,hb,Q,G);
  if #I eq 0 then
    return true;
  end if;
A := Matrix([G[i]: i in I]);
R := [Eltseq(p): p in RR(Transpose(A),D)];
N := [i: i in [1..#I]|G[I[i]] in neg];
B1 := [i: i in N| &*[R[l][i]: l in [1..#R]] ne 0];
B2 := [{i,j}: i,j in N|i ne j and &*[R[l][i]+R[l][j]:
          l in [1..#R]] ne 0 and qua(G[I[i]],G[I[j]],Q) ge 1];
B := {{i,j,k}: i,j,k in N|#{i,j,k} eq 3 and &*[R[l][i]+R[l][j]+R[l][k]:
          l in [1..#R]] ne 0  and L!D eq L!G[I[i]]+L!G[I[j]]+L!G[I[k]]};
B3 := [p: p in B|{h1(L!G[I[a]]+L!G[I[b]],neg,Q): a,b in p|a ne b} eq {0}];
  if #B1 gt 0 or #B2 gt 0 or #B3 gt 0 then
    return true, B1, B2, B3;
  end if;
 return false;
end function;
```

Bibliography

[ACDL19] Michela Artebani, Claudia Correa Deisler, and Antonio Laface, *Cox rings of K3 surfaces of Picard number three* (2019), available at https://arxiv.org/pdf/1909.01267.pdf. ↑11, 20, 70

[ACDR20] Michela Artebani, Claudia Correa Deisler, and Xavier Roulleau, *Mori dream K3 surfaces of Picard number four: projective models and Cox rings, preprint* (2020). ↑13, 22, 82

[ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. ↑6, 7, 14, 15, 16, 23, 44, 46, 49, 51, 54, 55, 77

[AHL10] Michela Artebani, Jürgen Hausen, and Antonio Laface, *On Cox rings of K3 surfaces*, Compos. Math. **146** (2010), no. 4, 964–998. ↑4, 7, 14, 16, 17, 23, 48, 50, 54, 55, 64, 72, 83, 84, 88, 91

[AL11] Michela Artebani and Antonio Laface, *Cox rings of surfaces and the anticanonical Iitaka dimension*, Adv. Math. **226** (2011), no. 6, 5252–5267. ↑50

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. ↑6, 15

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). ↑5, 13, 14, 22, 24, 143

[Bea96] Arnaud Beauville, *Complex algebraic surfaces*, Second, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. ↑30, 31, 59

[BHPVdV04] W. Barth, K Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3), vol. 4, Springer-Verlag, Berlin, 2004. ↑35, 36, 37, 85, 86, 87, 88, 90, 92, 94, 97, 99, 100, 103, 104

[BP04] Victor V. Batyrev and Oleg N. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 2004, pp. 85–103. ↑7, 16, 54

[Cox95] David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. ↑6, 15

[CT06] A. M. Castravet and J. Tevelev, *Hilbert's 14th problem and Cox rings.*, Compos. Math. **6** (2006), 1479–1498. ↑7, 16, 54

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977. ↑25, 26, 27, 30, 31, 32, 36, 50

[HKL16] Jürgen Hausen, Simon Keicher, and Antonio Laface, *Computing Cox rings*, Math. Comp. **85** (2016), no. 297, 467–502. ↑7, 16, 54

[Huy16] Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR3586372 ↑35, 38, 40, 41, 63

[KL07] Andreas Leopold Knutsen and Angelo Felice Lopez, *A sharp vanishing theorem for line bundles on $K3$ or Enriques surfaces*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3495–3498. ↑38, 40

[Kov94] S. Kovács, *The cone of curves of K3 surface*, Math. Ann. **300** (1994), 681–691. ↑51, 63

[Lan02] Serre Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. ↑58

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. MR2095471 ↑29, 30, 32, 77

[LM09] Antonio Laface and Velasco Mauricio, *Picard-graded Betti numbers and the defining ideals of Cox rings.*, J. Algebra **2** (2009), 353–372. ↑7, 16, 48, 54

[LT13] Antonio Laface and Damiano Testa, *Nef and semiample divisors on rational surfaces*, Torsors, étale homotopy and applications to rational points, 2013, pp. 429–446. ↑50

[McK10] James McKernan, *Mori dream spaces*, Jpn. J. Math. **5** (2010), no. 1, 127–151. ↑4, 7, 16

[Mil58] John Milnor, *On simply connected 4-manifolds*, Symposium internacional de topología algebraica International symposium on algebraic topology (1958), 122–128. ↑34

[MS00] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2000. ↑48

[Nik00] Viacheslav V. Nikulin, *A remark on algebraic surfaces with polyhedral Mori cone*, Nagoya Math. J. **157** (2000), 73–92. ↑7, 16, 51

[Nik79] ______, *Quotient-groups of groups of automorphisms of hyperbolic forms of subgroups generated by* 2*-reflections*, Dokl. Akad. Nauk SSSR **248** (1979), no. 6, 1307–1309. ↑7, 16, 51

[Nik84] ______, $K3$ *surfaces with a finite group of automorphisms and a Picard group of rank three*, Trudy Mat. Inst. Steklov. **165** (1984), 119–142. Algebraic geometry and its applications. ↑7, 16, 51, 52, 56, 70

[Ott13] John Christian Ottem, *Cox rings of K3 surfaces with Picard number 2*, J. Pure Appl. Algebra **217** (2013), no. 4, 709–715. ↑8, 17, 57, 69

[PŠŠ71] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type* K3, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572. ↑7, 16, 51

[Rou20a] Xavier Roulleau, *An atlas of K3 surfaces with finite automorphism group* (2020), available at https://arxiv.org/abs/2003.08985. ↑11, 20, 71

[Rou20b] ______, *On the geometry of K3 surfaces with finite automorphism group and no elliptic fibrations* (2020), available at https://arxiv.org/pdf/1909.01909.pdf. ↑11, 20, 71

[SD74] B. Saint-Donat, *Projective models of $K-3$ surfaces*, Amer. J. Math. **96** (1974), 602–639. ↑8, 13, 17, 23, 38, 39, 40, 41, 65, 98

[SS07] V. V. Serganova and A. N. Skorobogatov, *Del Pezzo surfaces and representation theory.*, Algebra Number Theory **4** (2007), 393–419. ↑7, 16, 54

[STM07] Mike Stillman, Damiano Testa, and Velasco Mauricio, *Grobner bases, monomial group actions, and the Cox rings of del Pezzo surfaces*, J. Algebra **2** (2007), 777–801. ↑7, 16, 54

[TVAV09] Damiano Testa, Anthony Várilly-Alvarado, and Mauricio Velasco, *Cox rings of degree one del pezzo surfaces*, Algebra and Number Theory **3** (2009), no. 7, 729–761. ↑7, 16, 50, 54

[Ver83] Alexius Maria Vermeulen, *Weierstrass points of weight two on curves of genus three*, Universiteit van Amsterdam, 1983. ↑77

[Vin07] È. B. Vinberg, *Classification of 2-reflective hyperbolic lattices of rank 4*, Tr. Mosk. Mat. Obs. **68** (2007), 44–76. ↑7, 16, 51, 52, 82

[Vin75] È. B. Vinberg, *Some arithmetical discrete groups in Lobačevskiĭ spaces*, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), 1975, pp. 323–348. MR0422505 ↑5, 41